Sampled-data models for linear and nonlinear systems

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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A Paz

Contents

Abstract

Continuous-time systems are usually modelled by differential equations arising from physical laws. However, the use of these models in practice requires discretisation. In this thesis we consider sampled-data models for linear and nonlinear systems. We study some of the issues involved in the sampling process, such as the *accuracy* of the sampled-data models, the *artifacts* produced by the particular sampling scheme, and the *relations* to the underlying continuous-time system. We review, extend and present new results, making extensive use of the delta operator which allows a clearer connection between a sampled-data model and the underlying continuous-time system.

In the first part of the thesis we consider sampled-data models for linear systems. In this case exact discrete-time representations can be obtained. These models depend, not only on the continuous-time system, but also on the *artifacts* involved in the sampling process, namely, the sample and hold devices. In particular, these devices play a key role in determining the *sampling zeros* of the discrete-time model.

We consider robustness issues associated with the use of discrete-time models for continuous-time system identification from sampled data. We show that, by using *restricted bandwidth* frequency domain maximum likelihood estimation, the identification results are robust to (possible) under-modelling due to the sampling process.

Sampled-data models provide a powerful tool also for continuous-time optimal control problems, where the presence of constraints can make the explicit solution impossible to find. We show how this solution can be *arbitrarily approximated* by an associated sampled-data problem using fast sampling rates. We also show that there is a natural convergence of the *singular structure* of the optimal control problem from discrete- to continuous-time, as the sampling period goes to zero.

In Part II we consider sampled-data models for nonlinear systems. In this case we can only obtain *approximate* sampled-data models. These discrete-time models are simple and accurate in a well defined sense. For deterministic systems, an insightful observation is that the proposed model contains *sampling zero dynamics*. Moreover, these correspond to the same dynamics associated with the asymptotic sampling zeros in the linear case.

The topics and results presented in the thesis are believed to give important insights into the use of sampled-data models to represent linear and nonlinear continuous-time systems.

Symbols and Acronyms

$\langle \circ, \circ \rangle$	: Inner product.
$\sim$	: *Distributed as* (for random variables).
*	: Complex conjugation.
^{T}	: Matrix (or vector) transpose.
γ	: Complex variable associated to the δ-operator.
Δ	: Sampling period.
δ	: Delta operator (forward divided difference).
$\delta(t)$	: Dirac delta or continuous-time impulse function.
$\delta_K[k]$	: Kronecker delta or discrete-time impulse function.
$\mu(t)$	: unitary step function or Heaviside function.
$\mu[k]$	: Discrete-time unitary step function.
$\rho = \frac{d}{dt}$	: Time-derivative operator.
ω	: angular frequency, in [rad/s].
ω_N	: Nyquist frequency, $\omega_N = \frac{\omega_s}{2}$.
ω_s	: Sampling frequency, $\omega_s = \frac{2\pi}{\Delta}$.
A, B, C, D	: State-space matrices in continuous-time.
$A_\delta, B_\delta, C_\delta, D_\delta$	: State-space matrices in discrete-time using the δ-operator.
A_q, B_q, C_q, D_q	: State-space matrices in discrete-time using the shift operator q.
adj	: Adjoint of a matrix.
$\mathcal{C}^n$	: Space of functions whose first n derivatives are continuous.
CAR	: Continuous-time auto regressive.
CT	: Continuous-time.
CTWN	: Continuous-time white noise.
DFT	: Discrete Fourier transform.
DT	: Discrete-time.
DTFT	: Discrete-time Fourier transform.
DTWN	: Discrete-time white noise.
$E\{\cdot\}$	: Expected value.

$\mathcal{F}\{\cdot\}$: (Continuous-time) Fourier transform.
$\mathcal{F}^{-1}\{\cdot\}$: (Continuous-time) Inverse Fourier transform.
$\mathcal{F}_d\{\cdot\}$: Discrete-time Fourier transform.
$\mathcal{F}_d^{-1}\{\cdot\}$: Discrete-time Inverse Fourier transform.
FDML : Frequency domain maximum likelihood.
$G(\rho)$: Deterministic part of a continuous-time system.
$G(s)$: Continuous-time transfer function (s-domain of the Laplace transform).
$G_\delta(\gamma)$: Discrete-time transfer function (γ-domain corresponding to the δ operator).
$G_q(z)$: Discrete-time transfer function (z-domain corresponding to the shift operator q).
GHF : Generalised hold function.
GSF : Generalised sampling filter.
$H(\rho)$: Stochastic part of a continuous-time system.
$h_g(t)$: Impulse response of a generalised hold or sampling function.
$\Im$: Imaginary part of a complex number.
I_n : Identity matrix of dimension n.
IV : Instrumental variables.
$\mathcal{L}\{\cdot\}$: Laplace transform.
$\mathcal{L}^{-1}\{\cdot\}$: Inverse Laplace transform.
ℓ_2 : Space of square summable sequences.
$\mathcal{L}_2$: Space of square integrable functions.
LQ : Linear-quadratic (optimal control problem).
LS : Least squares.
MIFZ(D) : Model incorporating fixed zero (dynamics).
MIMO : Multiple-input multiple-output (system).
MIPZ(D) : Model incorporating parameterised zero (dynamics).
MPC : Model predictive control.
$\mathbb{N}$: Set of natural number (positive integers).
$N(\mu, \sigma^2)$: Normal or Gaussian distribution, with mean μ and variance σ^2.
NMP : Non-minimum phase.
$\mathcal{O}(\Delta^n)$: Function *of order* Δ^n.
PEM : Prediction error methods.
QP : Quadratic Programming.
q : Forward shift operator.
$\Re$: Real part of a complex number.
$\mathbb{R}$: Set of real numbers.
s : Complex variable corresponding to the Laplace transform.
SD : Sampled data.

SDE : Stochastic differential equation.
SDRM : Simple derivative replacement model.
SISO : Single-input single-output (system).
$\mathbb{Z}$: Set of integer number.
$\mathcal{Z}\{\cdot\}$: $\mathcal{Z}$-transform.
$\mathcal{Z}^{-1}\{\cdot\}$: Inverse $\mathcal{Z}$-transform.
z : Complex variable corresponding to the $\mathcal{Z}$-transform.

Chapter 1

Introduction

1.1 Sampling and sampled-data models

Models for continuous-time dynamical systems often arise from the application of physical laws such as conservation of mass, momentum, and energy. These models typically take the form of linear or nonlinear differential equations, where the parameters involved can usually be interpreted in terms of physical properties of the system. In practice, however, these kinds of models are not appropriate to interact with digital devices. In any situation where digital controllers have to act on a real system, this action can be applied (or updated) only at some specific time instants. Similarly, if we are interested in collecting information from signals of a given system, this data can usually only be recorded (and stored) at specific instants. This constitutes nowadays an unavoidable paradigm: continuous-time systems interact with actuators and sensors that are accessible only at discrete-time instants. As a consequence, the **sampling** process of continuous-time systems is a key problem both for estimation and control purposes (Middleton and Goodwin, 1990; Feuer and Goodwin, 1996; Åström and Wittenmark, 1997). In this context, the current thesis considers **sampled-data models** for linear and nonlinear systems. The focus is on describing, in discrete-time, the relationship between the input signals and the samples of the continuous-time system outputs. In particular, we study issues such as the *accuracy* of the sampled-data models, the *artifacts* produced by a particular sampling scheme, and the *relations* to the underlying continuous-time system.

The sampling process for a continuous-time system is represented schematically in Figure 1.1. In this figure we see that there are three basic elements involved in the sampling process. All of these elements play a core role in determining the appropriate discrete-time input-output description:

- The **hold device**, used to generate the continuous-time input $u(t)$ of the system, based on a discrete time sequence u_k defined at specific time instants t_k;

- The **continuous-time system**, defined by a set of linear or nonlinear differential equations, which generates the continuous-time output $y(t)$ from the input $u(t)$, initial conditions, and/or possible unmeasured disturbances; and

- The **sampling device**, which generates an output sequence of samples y_k from the continuous-

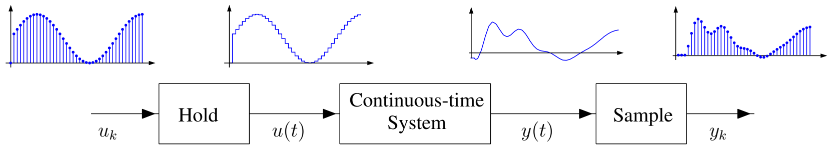


Figure 1.1: Scheme of the sampling process of a continuous-time system.

time output $y(t)$, possibly including some form of *anti-aliasing* filtering prior to instantaneous sampling.

For linear systems, it is possible to obtain **exact** sampled-data models from the sampling scheme shown in Figure 1.1. In particular, given a deterministic continuous-time system, it is possible to obtain a discrete-time model which replicates the sequence of output samples. In the stochastic case, where the input of the system is assumed to be a *continuous-time white noise* process, a sampled-data model can be obtained such that its output sequence has the same second order properties as the continuous-time output at the sampling instants.

However, obtaining sampled-data models for nonlinear systems is a much more difficult task. In fact, these models are, in most cases, either unknown or impossible to compute because of the inherent difficulties in solving nonlinear differential equations, both in the deterministic and stochastic frameworks (Nešić *et al.*, 1999; Kloeden and Platen, 1992). As a consequence, only **approximate** discrete-time models are possible to obtain. In this case, we will typically be interested in sampled-data models that are *accurate* in some well defined sense. The accuracy of these discrete-time descriptions has proven to be a key issue when trying to apply results based on such models. In the context of control design, for example, a controller designed to stabilise an approximate sampled plant model may fail to stabilises the exact discrete-time model, no matter how small the sampling period Δ is chosen (Nešić and Teel, 2004).

Sampled-data models for continuous-time systems are useful in control, simulation and estimation of system parameters (system identification). They are, in fact, the required tool needed to fill the gap between continuous control and the output signals, as seen at the sampling instants. Most of the existing literature regarding discrete-time (and, thus, sampled data) systems has traditionally expressed these models in terms of the shift operator q, and the associated $\mathcal{Z}$-transform. However, when using this kind of models is not easy to relate to the *corresponding* results applicable to the continuous-time case. This is true even when the sampling period is chosen arbitrarily small. The inter-relationship between sampled-data models and their underlying continuous-time counterparts is more easily understood in the unified framework facilitated by use of the *delta operator* (Middleton and Goodwin, 1990):

$$\delta = \frac{q-1}{\Delta} \tag{1.1}$$

There is a considerable amount of literature showing that the use of this operator gives conceptual and numerical advantages over the traditional shift operator q (Middleton and Goodwin, 1986; Middleton and Goodwin, 1990; Goodwin *et al.*, 1992; Li and Gevers, 1993; Neuman, 1993; Gevers and Li, 1993; Mansour, 1993; Premaratne and Jury, 1994; Feuer and Goodwin, 1996; Premaratne *et al.*, 2000; Suchomski, 2001; Suchomski, 2002; Lennartson *et al.*, 2004). In particular, sampled-data models *rewritten* in terms of the delta operator models explicitly include the sampling period Δ in such a way that, when the sampling rate is increased, the underlying continuous-time system representation is recovered. In the same fashion, several discrete- and continuous-time results in control, estimation and system identification can be understood in the same framework when sampled-data models are expressed in terms of divided differences using (1.1).

Even though the use of delta operator models provides a *natural connection* between the discrete and continuous-time domains, one needs to be careful with issues inherent to the sampling process. The use of discrete-time models to represent continuous-time systems implies that there is **loss of information**. In the time domain, the intersample behaviour of signals is unknown, whereas in the frequency domain, high frequency signal components will fold back to low frequencies, making them impossible to distinguish. Thus, for any non-zero sampling period, there will always be a *difference* between the sampled-data model and the underlying continuous-time description. For example, it is well known that discrete-time models will have, in general, more zeros than the original continuous-time system (Åström *et al.*, 1984; Wahlberg, 1988). These extra zeros, sometimes called *sampling zeros*, are a result of the frequency folding effect due to the sampling process. When using delta operator models, these sampling zeros asymptotically converge to infinity as the sampling period goes to zero. However, they play a key role, for example, when using Least Squares for parameter estimation (Larsson and Söderström, 2002).

The problems arising from the sampling process can only be mitigated by appropriate **assumptions**. Among common assumptions we have the use zero-order hold inputs and bandlimited signals (Pintelon and Schoukens, 2001). On the other hand, the presence of sampling zeros (and their asymptotic behaviour) is determined by the continuous-time system relative degree. However, relative degree may be an *ill-defined* characteristic for continuous-time models, easily affected by high frequency modelling errors, even beyond the sampling frequency. Thus, one needs to be careful when relying on assumptions that may be inherently non-robust to the effects of sampling.

In this thesis we will repeatedly highlight the issues and assumptions related to the use of sampled-data models. Examples of these issues are:

- Sampled-data characteristics depend not only on the continuous-time system but also on the sampling process itself. Indeed, for linear systems, the discrete-time poles depend on the continuous-time poles and the sampling period, but the zeros depend on the choice of the hold and the sampling devices.

- The effects of sampling *artifacts*, such as sampling zeros, play an important role in describing accurate sampled-data models, even though they may become negligible as the sampling period goes to zero. This applies both to linear systems, where discrete-time models can be precisely characterised, and nonlinear systems, studied in Part II, where they can only be approximately described.

- Any sampled-data description is based on some kind of model of the true system. However, under-modelling errors will usually arise at high frequencies due to the presence of unmodelled poles, zeros, and/or time delays in the continuous-time system (Goodwin *et al.*, 2001). This means that models usually have to be considered within a *bandwidth of validity* for the continuous-time system.

- For stochastic models, the unknown input is assumed to be a (filtered) continuous-time white noise process. However, this is known to be a mathematical abstraction that does not usually correspond to physical reality (Jazwinski, 1970). It can only be approximated to some desired degree of accuracy by conventional stochastic processes with broad band spectra (Kloeden and Platen, 1992). This means that stochastic systems need to be treated carefully as the non-ideal nature of the noise corresponds to a form of high frequency modelling error.

1.2 Problem statement

In the current thesis, we will study sampled-data models for linear and nonlinear systems. We also explore the use of these models for control and estimation. Our focus throughout is aimed at answering the following kind of questions:

- When using sampled-data models to represent a continuous-time system, what characteristics are inherent to the underlying continuous-time system and what are a consequence of the sampling process itself?
- Is it reasonable to assume that, as the sampling rate is increased, the sampled-data model becomes indistinguishable from the underlying continuous-time system? How does this convergence occur?
- What issues are important when using a discrete-time model to represent a system that actually evolves in continuous-time?
- Can known results on sampling for linear systems be extended, under appropriate conditions, to nonlinear systems?

1.3 Thesis overview

Following this introduction, the contents of the thesis are presented into six chapters, that have been organised in two parts: the first part considers linear systems, and the second part deals with extensions to the nonlinear case. A final chapter presents a summary and conclusions, and two appendices have been included with supporting material.

In more detail, Part I explores sampled data models for linear systems. A key point here is that the resultant discrete-time models are exact, *i.e.*, they are able to exactly replicate the sequence of samples of the system output, in the deterministic case, or its statistical properties, for stochastic systems. We present various extensions of existing results. More importantly, we cast several known results in a different light, thus *opening the door* to the nonlinear case treated in Part II. A more detailed description of the various chapters follows.

In Chapter 2, sampling of deterministic and stochastic linear systems is reviewed. Here we present and extend well-known results, in such a way as to introduce the basic building blocks required in subsequent chapters. Many of the results have been traditionally presented in terms of the q operator and the associated z-domain. Here, we rewrite and extend these results using the δ-operator and the corresponding γ-domain variable. In particular, we obtain a novel representation of the asymptotic sampling zeros for the linear case. This result will prove very useful in the nonlinear context studied later.

In Chapter 3 we explore the *artifacts* produced by the sampling process. It is known that the zeros of the sampled-data model depend on the hold device (for deterministic models) and the output prefilter (in the stochastic case). In this chapter, we will show how these devices can be designed so as to asymptotically assign the sampling zeros. The design procedure depends only on the continuous-time system relative degree. However, high frequency modelling errors can affect the relative degree assumption. Thus, we introduce the concept of *bandwidth of validity*, within which our design procedure is shown to be robust.

In a similar fashion, robustness issues in continuous-time system identification from sampled-data are explored in Chapter 4. In particular, we study robustness problems that may arise when trying to estimate continuous-time system parameters using sampled-data models. We show that some algorithms commonly used are inherently sensitive to assumptions about the inter-sample behaviour of the signals or to the frequency response of the system beyond the sampling frequency. On the other hand, we propose the use of a maximum likelihood estimation in the frequency domain over a *limited bandwidth* of frequencies. We show that this procedure succesfully addresses the robustness issues previously considered, such as sampling zeros in discrete-time models and the presence of high-frequency undermodelling in the continuous-time system.

We conclude Part I, in Chapter 5, by exploring how sampled-data models are utilised in Linear-Quadratic (LQ) constrained optimal control problems. Two results are presented: firstly, it is shown that the solution to the LQ constrained problem, in continuous-time, can be approximated arbitrarily closely by solving an associated sampled-data LQ constrained control problem, using a sufficiently small sampling period. Furthermore, the solution is shown to satisfy the *continuous-time constraints* (*i.e.*, at all time instants) by appropriately scaling the discrete-time constraints (at the sampling instants). Secondly, we revisit the operator factorisation approach to LQ problems. We use this approach to formulate an interesting convergence result: the (finite set of) singular values of a linear operator, associated with the sampled-data model, are shown to converge to (a subset of) the singular values of the continuous-time system operator. The latter result can be applied, for example, in suboptimal control strategies for fast sampling applications, exploiting the singular structure of the system to solve the constrained control problem.

Part II considers sampled-data models for nonlinear systems. A key departure from the linear case is that, in the nonlinear case, exact discrete-time models are usually impossible to obtain. Thus, we present approximate sampled-data models both for deterministic and stochastic systems.

Chapter 6 presents an approximate sampled-data model for deterministic nonlinear systems. The model is simple to obtain. We show that it provides a more accurate description (as a function of the sampling period and the nonlinear relative degree) than do models obtained by simply using Euler integration. An interesting feature of the proposed model is that it includes extra zero dynamics. In fact, these *sampling zero dynamics* turn out to be exactly the same as those that arise (asymptotically) in the linear case. This result gives important new insights into the effect of sampling in nonlinear dynamic systems. As an illustration, the impact of these nonlinear sampling zeros on models used for system identification is explored.

Sampled-data models for stochastic nonlinear systems are studied in Chapter 7. We briefly review the mathematical background of nonlinear stochastic systems by using stochastic differential equation. In particular, we present a sampled-data model based on numerical solution of stochastic differential equations.

Finally, Chapter 8 presents concluding remarks and summarises the topics presented in the thesis. Some possible future lines of research are also proposed.

We also have included two appendices with supporting material for the contents in the thesis. Appendix A presents some useful results about matrices, whereas in Appendix B a brief review of linear operators in Hilbert spaces is presented.

1.4 Thesis contributions

The main contributions of this thesis are believed to be:

Chapter 2. In this chapter we present results on sampled-data models, both in q- and δ-operator frameworks. In particular, a novel characterisation of the asymptotic sampling zeros in the γ-domain (*i.e.*, using the δ-operator) is given. This formulation is given in terms of polynomials in the variable γ which are also shown to satisfy a recursive relationship.

Chapter 3. We analyse the role of sample and hold devices in obtaining sampled-data models. Specifically, it is shown that generalised hold functions (GHF) can be designed to assign the asymptotic location of sampling zeros for deterministic systems. A dual result is presented, namely, a design method to design generalised sampling filters (GSF) to assign the asymptotic sampling zeros of stochastic linear models. Both procedures are shown to depend only on the continuous-time system relative degree.

Chapter 4. We illustrate issues that may arise when trying to identify continuous-time systems from sampled data. We propose a maximum likelihood estimation procedure in the frequency domain restricted to a limited bandwidth, and we show that this procedure is robust to sampling effects.

Chapter 5. We present two asymptotic results regarding the use of sampled-data models in constrained linear quadratic optimal control problems. Firstly, we show that, as the sampling rate increases, the solution of an associated sampled-data control problem with (possibly) tighter constraints converges to the original continuous-time problem. In the second result, we show that the (finite set of) singular values of the discrete-time system operator converges to a subset of the (infinite set of) singular values of the continuous-time system operator, as the sampling period goes to zero.

Chapter 6. We present an approximate sampled-data model for nonlinear deterministic systems. This model is shown to have very insightful features: firstly, it is based on a more accurate approximation than simple Euler integration, and, secondly, it includes *sampling zero dynamics* with no counterpart in continuous time. Moreover, we show that these sampling zero dynamics are exactly the same as the sampling zeros that arise in the linear case.

Chapter 7. We present an approximate sampled-data models for nonlinear stochastic systems. This discrete-time model is based on numerical solution of stochastic differential equations. We show that the accuracy of the obtained model can be precisely characterised. We also unveil connections to the linear case.

1.5 Associated publications

Most of the results presented in this thesis have been published, by the author, in journal and conference papers. The following list details the relevant publications:

- Journal papers:

  1. J.I. Yuz and G.C. Goodwin, *On sampled-data models for nonlinear systems*. Special Issue of the IEEE Transactions on Automatic Control on *System Identification: Linear v/s Nonlinear*, Vol. 50(10), pages 1477–1489, October 2005.
  2. J.I. Yuz, G.C. Goodwin, A. Feuer and J. De Doná, *Control of constrained linear systems using fast sampling rates*. Systems and Control Letters, Vol. 54(10), pages 981–990, October 2005.

- Conference papers:

  1. J.I Yuz, G.C. Goodwin, A. Feuer and J. De Doná, *Singular structure convergence for linear quadratic problems*. European Control Conference (ECC), Cambridge, UK, 2003
  2. J.I. Yuz, G.C. Goodwin and H. Garnier. *Generalised hold functions for fast sampling rates*. 43rd IEEE Conference on Decision and Control (CDC), Nassau, Bahamas, December 2004.
  3. G.C. Goodwin, J.I. Yuz and H. Garnier, *Robustness issues in continuous-time system identification from sampled data*. 16th IFAC World Congress. Prague, Czech Republic, July 2005.
  4. J.I. Yuz and G.C. Goodwin, *Generalised sampling filters and stochastic sampling zeros*. Joint CDC-ECC'05, Seville, Spain, December 2005.
  5. J.I. Yuz and G.C. Goodwin, *Sampled-data models for stochastic nonlinear systems*. To be presented at the 14th IFAC Symposium on System Identification, SYSID 2006. Newcastle, Australia, March 2006.

Previous and parallel work published by the author during the Ph.D. studies are detailed as follows:

- Journal papers

  - G.C. Goodwin, M.E. Salgado and J.I. Yuz, *Performance limitations for linear feedback systems in the presence of plant uncertainty*. IEEE Transactions on Automatic Control, Vol. 48(8), pages 1312–1319, August 2003.
  - J.I. Yuz and G.C. Goodwin, *Loop performance assessment for decentralised control of stable linear systems*. European Journal of Control, Vol. 9(1), pages 116–130, 2003.

  - J.I. Yuz and M.E. Salgado, *From classical to state-feedback-based controllers*. IEEE Control Systems Magazine, vol. 23(4), pages 58–67, August 2003.

- Other publications by the author:

  - M.E. Salgado, J.I. Yuz, and R. Rojas, *Análisis de Sistemas Lineales* (book in Spanish). Pearson – Prentice Hall, Spain, 2005.
  - M.E. Salgado and J.I. Yuz, *State space analysis and system properties*, Chapter 24 in *Handbook on Mechatronics*, edited by R. Bishop. CRC Press, Florida, USA, 2002.
  - J.I. Yuz, M.M. Serón and G.C. Goodwin, *Cheap control performance limitations for linear systems based on the Fröbenius norm*. Technical report EE03018, School of Electrical Engineering, *The* University *of* Newcastle, Australia, 2003.

Part I

Linear Systems

Chapter 2

The sampling process for linear systems

2.1 Overview

In this chapter we review and extend known results on sampled-data models for linear systems. We consider both the deterministic and the stochastic case. Most of the existing discrete-time literature is based on the use of the shift operator q. Here we recast these results using the delta operator, δ. This is particularly useful in the sampled-data framework, since δ-models explicitly include the sampling period Δ. Indeed, the results presented here can be considered as the key building blocks for the chapters that follow.

In this first part of the Thesis we consider only the case of linear systems. For this case it is possible to obtain **exact** sampled-data models. For deterministic systems, we obtain discrete-time models that exactly describe the relationship between an input sequence u_k and the sampled output $y_k = y(k\Delta)$ (Section 2.2). In the case of stochastic systems, we obtain a sampled-data model such that its output sequence y_k has the same second order properties as does the continuous-time output $y(t)$ at the sampling instants (Section 2.4).

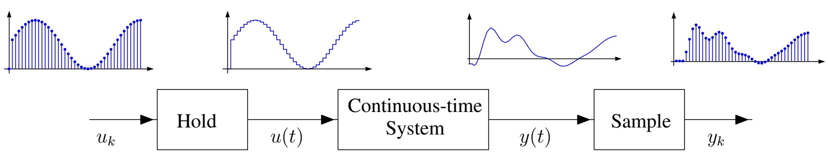


Figure 2.1: Sampling process

We consider the general sampling scheme represented in Figure 2.1. In this figure, we can distinguish three basic elements:

- The **hold device**, used to generate the continuous-time input $u(t)$ of the system based on a discrete time sequence u_k defined at specific time instants t_k.

- The **continuous-time system**, defined by a set of differential equations evolving in continuous-time. Here we restrict our attention to the linear case. Thus, we will describe the system generically as:

$$y(t) = G(\rho)u(t) + H(\rho)\dot{v}(t) \tag{2.1}$$

  where $\rho = \frac{d}{dt}$ is the time-derivative operator, and:

  - $G(\rho)$ is the deterministic part of a continuous-time system, which takes into account the effect of the deterministic input $u(t)$; and
  - $H(\rho)$ is the stochastic part of the system or *noise model*, which describes the part of the output that does not depend on the input $u(t)$. This includes the effect of disturbances and noise, represented as *continuous-time white noise* process $\dot{v}(t)$ filtered through the system $H(\rho)$ (We will give a more mathematically rigorous treatment later).

- The **sampling device**, which gives an output sequence of samples, y_k. This can be obtained taking samples of $y(t)$ either instantaneously or after some form of anti-aliasing filtering.

We will assume that the input updates and the output samples happen at the same time instants t_k, which are integer multiples of a uniform sampling period Δ, *i.e.*, $t_k = k\Delta$.

We will focus our attention to single-input single-output (SISO) systems. Nevertheless, several of the results are expressed using state-space models and, thus, they can be readily extended to the multivariable case.

In the following sections we consider the sampling process for deterministic and stochastic systems. For the linear case, considered in this first part of the thesis, the superposition principle holds. Thus, this apparent separate treatment is well justified. It reflects the fact that the effects of the deterministic and the stochastic part of (2.1) can be considered indenpendently of each other.

2.2 Sampling of deterministic linear systems

In this section we consider the sampling process for the deterministic part of the system (2.1), *i.e.*, we are interested in a discrete-time description of the relationship between the known input sequence u_k and the samples of the output, y_k. Stochastic systems are considered later in Section 2.4 on page 25.

A strictly proper SISO linear time-invariant system can be represented in state-space form as:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.2}$$

$$y(t) = Cx(t) \tag{2.3}$$

where the system state vector is $x(t) \in \mathbb{R}^n$, and the matrices are $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$.

Lemma 2.1 *This system can also be represented as:*

$$Y(s) = G(s)U(s) \tag{2.4}$$

*where $U(s)$ and $Y(s)$ Laplace transforms of $u(t)$ and $y(t)$, respectively, and the system **transfer function** is:*

$$G(s) = C(sI_n - A)^{-1}B \tag{2.5}$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

The transfer function (2.5) can be represented as a quotient of polynomials:

$$G(s) = \frac{F(s)}{E(s)} \tag{2.6}$$

The roots of $F(s)$ and $E(s)$ are the **zeros** and the **poles** of the system, respectively. If in (2.2)–(2.3) the pair (A, B) is completely controllable and (C, A) is completely observable then the state-space model is a **minimal realization** (Kwakernaak and Sivan, 1972). This implies that there are no zero/pole cancellations in (2.6), and, thus, the numerator and denominator polynomials are given by:

$$F(s) = C\,\mathrm{adj}(sI_n - A)B \tag{2.7}$$

$$E(s) = \det(sI_n - A) \tag{2.8}$$

The numerator (2.7) can also be expressed as:

$$F(s) = \det \begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix} \tag{2.9}$$

(See equation (A.16) in Appendix A.)

2.2.1 The hold device

In Figure 2.1 there is a hold device which generates a continuous-time input to the system, $u(t)$, from a sequence of values u_k, defined at specific time instants $t_k = k\Delta$. The most commonly used hold devices are:

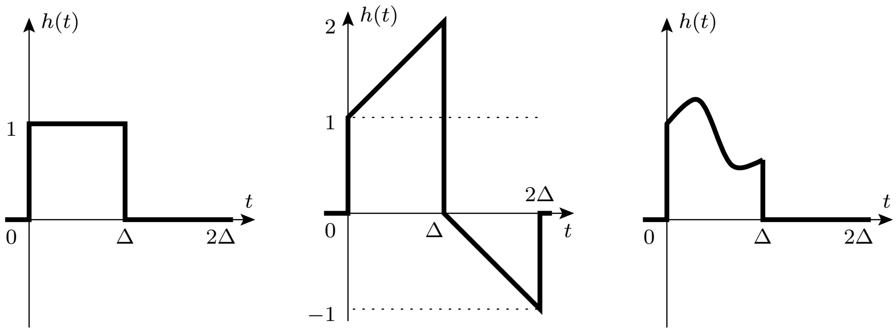
Zero-Order Hold (ZOH) , which simply keeps its output constant between sampling instants, *i.e.*,:

$$u(t) = u_k \qquad ; k\Delta \leq t < (k+1)\Delta \tag{2.10}$$

First-Order Hold (FOH) , which does a linear *extrapolation* using the current and the previous elements of the input sequence, *i.e.*,:

$$u(t) = u_k + \frac{u_k - u_{k-1}}{\Delta}(t - k\Delta) \qquad ; k\Delta \leq t < (k+1)\Delta \tag{2.11}$$

There are, indeed, other and more general options for the hold device, such as, for example, Fractional Order Holds (FROH) (Blachuta, 2001) and Generalised Hold Functions (see Chapter 3). Any of these can be uniquely characterised by their *impulse response* $h(t)$, defined as the continuous-time output (of the hold device) obtained when u_k is the Kronecker delta function. Figure 2.2 shows the *impulse responses* corresponding to the ZOH, FOH, and a more general hold function (Feuer and Goodwin, 1996).



(a) Zero-Order Hold (b) First-Order Hold (c) Generalised Hold

Figure 2.2: Impulse response of some hold devices.

2.2.2 The sampled-data model

There are different ways to obtain the sampled data model corresponding to the sampling scheme in Figure 2.1. The model can be derived directly from the transfer function (2.6) or from the state space model (2.2)–(2.3). The following result allows us to obtain a state-space representation of the sampled-data model when the continuous-time system input is generated using a ZOH.

Lemma 2.2 *If the input of the continuous-time system* (2.2)–(2.3) *is generated from the input sequence* u_k *using a ZOH, then a state-space representation of the resulting sampled-data model is given by:*

$$q\,x_k = x_{k+1} = A_q x_k + B_q u_k \tag{2.12}$$

$$y_k = C x_k \tag{2.13}$$

where the sampled output is $y_k = y(k\Delta)$*, and the matrices are:*

$$A_q = e^{A\Delta} \qquad\qquad B_q = \int_0^{\Delta} e^{A\eta} B d\eta \tag{2.14}$$

Proof. The state evolution starting at $t = t_k = k\Delta$ is given by (Kwakernaak and Sivan, 1972; Åström and Wittenmark, 1997):

$$x(k\Delta + \tau) = e^{A\tau}x(k\Delta) + \int_{k\Delta}^{k\Delta+\tau} e^{A(k\Delta+\tau-\eta)}Bu(\eta)d\eta \tag{2.15}$$

Replacing $\tau = \Delta$, and noticing that $u(\eta) = u_k$, when $k\Delta \leq \eta < k\Delta + \Delta$, we obtain (2.12). Equation (2.13) is obtained directly from the instantaneous relation (2.3).

□

The discrete-time transfer function representation of the sampled-data system can be readily obtained from Lemma 2.2 as:

$$G_q(z) = C(zI_n - A_q)^{-1}B_q \tag{2.16}$$

The last expression is equivalent to the pulse transfer function obtained directly from the continuous-time transfer function, as stated in the following lemma

Lemma 2.3 *The sampled-data transfer function* (2.16) *can be obtained using the inverse Laplace transform of the continuous-time step response, computing its $\mathcal{Z}$-transform, and dividing it by the $\mathcal{Z}$-transform of a discrete-time step:*

$$G_q(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=k\Delta}\right\} \tag{2.17}$$

$$= (1 - z^{-1})\frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{e^{s\Delta}}{z - e^{s\Delta}}\frac{G(s)}{s}ds \tag{2.18}$$

where Δ is the sampling period and $\gamma \in \mathbb{R}$ is such that all poles of $G(s)/s$ have real part less than γ. Furthermore, if the integration path in (2.18) *is closed by a semicircle to the right, we obtain:*

$$G_q(z) = (1 - z^{-1})\sum_{\ell=-\infty}^{\infty}\frac{G((\log z + 2\pi j\ell)/\Delta)}{\log z + 2\pi j\ell} \tag{2.19}$$

Proof. See, for example, Åström and Wittenmark (1997).

□

Expression (2.19), when considered in the frequency domain replacing $z = e^{j\omega\Delta}$, illustrates the well-known aliasing effect: the frequency response of the sampled-data system is obtained by folding of the continuous-time frequency response, *i.e.*,

$$G_q(e^{j\omega\Delta}) = \frac{1}{\Delta}\sum_{\ell=-\infty}^{\infty} H_{ZOH}\left(j\omega + j\frac{2\pi}{\Delta}\ell\right) G\left(j\omega + j\frac{2\pi}{\Delta}\ell\right) \tag{2.20}$$

where $H_{ZOH}(s)$ is the Laplace transform of the ZOH impulse response in Figure 2.2(a), *i.e.*,

$$H_{ZOH}(s) = \frac{1 - e^{-s\Delta}}{s} \tag{2.21}$$

Equation (2.20) can be also derived from (2.16) using the state-space matrices in (2.14) (Feuer and Goodwin, 1996, Lemma 4.6.1).

The sampled-data model for a given continuous-time system depends on the choice of the hold device. As a way of illustration, the following lemma establishes the sampled-data model obtained when, insted of the ZOH considered above, the continuous-time input is generated by a First Order Hold.

Lemma 2.4 *If the continuous-time plant input is generated using a FOH as in* (2.11)*, the corresponding sampled-data model can be represented in the following state-space form*

$$\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A_q & B_q^1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} B_q^2 \\ 1 \end{bmatrix} u_k \tag{2.22}$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} \tag{2.23}$$

where:

$$A_q = e^{A\Delta} \qquad B_q^1 = \int_0^{\Delta}\left(2 - \frac{\eta}{\Delta}\right)e^{A\eta}B\,d\eta \qquad B_q^2 = \int_0^{\Delta}\left(\frac{\eta}{\Delta} - 1\right)e^{A\eta}B\,d\eta \tag{2.24}$$

The discrete-time transfer function can be obtained both from the state-space representation as:

$$G_q(z) = \begin{bmatrix} C & 0 \end{bmatrix} \left(zI_{n+1} - \begin{bmatrix} A_q & B_q^1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B_q^2 \\ 1 \end{bmatrix} \tag{2.25}$$

or from the continuous-time transfer function $G(s)$, i.e.,

$$G_q(z) = (1 - z^{-1})^2 \mathcal{Z} \left\{ \mathcal{L}^{-1} \left\{ \frac{(1 + \Delta s)}{\Delta s^2} G(s) \right\} \right\} \tag{2.26}$$

Proof. See, for example, Hagiwara *et al.* (1993).

□

Sampled-data models obtained using a more general kind of hold device, such as the generalised hold with impulse response as shown 2.2(c), will be discussed in Chapter 3. In fact, we will see that it is possible to arbitrarily assign the discrete-time *sampling* zeros by designing the generalised hold device.

2.2.3 Poles and Zeros

We next consider the relation between the poles and zeros of the sampled-data model (2.16) and the poles and zeros of the continuous-time system (2.5).

The discrete-time model (2.16) can be rewritten as a quotient of polynomials as

$$G_q(z) = \frac{F_q(z)}{E_q(z)} \tag{2.27}$$

where:

$$F_q(z) = C\,\mathrm{adj}(zI_n - A_q)B_q = \det \begin{bmatrix} zI_n - A_q & -B_q \\ C & 0 \end{bmatrix} \tag{2.28}$$

$$E_q(z) = \det(zI_n - A_q) \tag{2.29}$$

where the second equality in (2.28) is a consequence of (A.16) in Appendix A.

The relationship between the poles of the pulse transfer function (2.27) and the underlying continuous-time system can be established from equation (2.14). Indeed, if λ_i is an eigenvalue of A (*i.e.*, a pole of $G(s)$), then $e^{\lambda_i \Delta}$ is an eigenvalue of $A_q = e^{A\Delta}$, and, thus, a pole of $G_q(z)$ (Åström and Wittenmark, 1997).

On the other hand, the relation between the zeros in discrete- and in continuous-time is much more involved as we can see in the numerator polynomial $F_q(z)$. Moreover, the discrete-time transfer function (2.27) will generically have relative degree 1, independent of the relative degree of the continuous-time system. Thus fact, extra zeros appear in the sampled-data model with no continuous-time counterpart. These, so called **sampling zeros**, can be asymptotically characterised as we will study in Section 2.3.

Similar relations between discrete- and continuous-time poles and zeros can be established when using non-ZOH input. For example, if we consider the sampled-data model obtained in Lemma 2.4 for the FOH case, we see that the discrete-time poles are given by the eigenvalues of $e^{A\Delta}$ (as in the ZOH case) plus one pole at the origin $z = 0$. On the other hand, the discrete-time zeros will be generically different that the ones obtained when using a ZOH.

2.2.4 Delta operator models

Most of the results and textbooks concerning discrete-time systems usually describe the models using the shift operator q and and the associated $\mathcal{Z}$-transform, as in Lemma 2.2 (Jury, 1958; Franklin *et al.*, 1990; Åström and Wittenmark, 1997). The relation between a sampled-data model and the underlying continuous-time system can be better expressed by the use of the delta operator (Middleton and Goodwin, 1990; Premaratne and Jury, 1994; Feuer and Goodwin, 1996; Lennartson *et al.*, 2004), both in the (discrete) time and complex variable domain:

$$\delta = \frac{q-1}{\Delta} \qquad \iff \qquad \gamma = \frac{z-1}{\Delta} \tag{2.30}$$

The use of the δ-operator corresponds to a reparameterisation of sampled-data models that allows one to explicitly include the sampling period in the discrete-time description.

Remark 2.5 *The operator δ defined in* (2.30) *is also called (forward) divided difference or (forward) Euler operator. However, it is important to distinguish between the exact sampled-data models expressed in terms of this operator (obtained in the same way as exact shift operator models), and those models obtained by simple Euler integration. The latter are only approximate discrete-time descriptions of the underlying continuous-time system, where time derivatives have been replaced by divided differences.*

The following lemma presents the δ-operator model corresponding to the sampled-data description obtained in Lemma 2.2.

Lemma 2.6 *The discrete-time model* (2.12)–(2.13) *in Lemma 2.2 can be rewritten using the delta operator as:*

$$\delta\, x_k = A_\delta x_k + B_\delta u_k \tag{2.31}$$

$$y_k = C x_k \tag{2.32}$$

where:

$$A_\delta = \frac{e^{A\Delta} - I_n}{\Delta} \qquad B_\delta = \frac{B_q}{\Delta} \tag{2.33}$$

Proof. The expressions follow directly using Lemma 2.2 and the definition of the delta operator:

$$\delta\, x_k = \frac{x_{k+1} - x_k}{\Delta} = \frac{A_q - I_n}{\Delta} x_k + \frac{B_q}{\Delta} u_k \tag{2.34}$$

□

Remark 2.7 *One advantage of using the delta operator model in Lemma 2.6 becomes apparent when considering the case of fast sampling rates. Specifically, as the sampling period go to zero, the matrices of the shift operator model in Lemma 2.2 do not have a relation with their continuous-time counterpart. In fact, from* (2.14)*, we have that:*

$$A_q = e^{A\Delta} = I_n + A\Delta + \ldots \qquad \xrightarrow{\Delta \to 0} I_n \tag{2.35}$$

$$B_q = \int_0^\Delta e^{A\eta} B d\eta \qquad \xrightarrow{\Delta \to 0} 0 \tag{2.36}$$

On the other hand, the matrices used in the delta operator model (2.31) *converge to their continuous-time counterpart (Middleton and Goodwin, 1990):*

$$A_\delta = \frac{e^{A\Delta} - I_n}{\Delta} = A + \frac{\Delta}{2}A^2 + \dots \xrightarrow{\Delta \to 0} A \tag{2.37}$$

$$B_\delta = \frac{1}{\Delta}\int_0^\Delta e^{A\eta} B d\eta \xrightarrow{\Delta \to 0} B \tag{2.38}$$

□

Many results in linear system theory can be understood in a unified framework by the use of the δ-operator. As in the previous remark, the continuous-time case can be readily obtained as the limiting discrete-time case, when the sampling period tends to zero. Moreover, the use of the delta operator has been shown to also provide numerical advantages for computational purposes (Goodwin *et al.*, 1992).

Lemma 2.6 provides a state-space sampled-data model in terms of the δ-operator. As a consequence, we have that the δ-operator transfer function, expressed using the associated complex variable γ in (2.30), is given by:

$$G_\delta(\gamma) = \frac{F_\delta(\gamma)}{E_\delta(\gamma)} \tag{2.39}$$

where:

$$F_\delta(\gamma) = C\,\mathrm{adj}(\gamma I_n - A_\delta)B_\delta = \det\begin{bmatrix} \gamma I_n - A_\delta & -B_\delta \\ C & 0 \end{bmatrix} \tag{2.40}$$

$$E_\delta(\gamma) = \det(\gamma I_n - A_\delta) \tag{2.41}$$

The convergence results in (2.37)–(2.38) imply that:

$$\lim_{\Delta\to 0} F_\delta(\gamma) = F(\gamma) \quad , \quad \lim_{\Delta\to 0} E_\delta(\gamma) = E(\gamma) \quad \text{, and, thus,} \quad \lim_{\Delta\to 0} G_\delta(\gamma) = G(\gamma) \tag{2.42}$$

In this thesis we will typically express sampled-data models using the delta operator format since it allows one to maintain a strong connection with the underlying continuous-time system. However, shift operator models will also be used, where appropriate, either to present results traditionally expressed in this framework, or for the sake of simplicity.

2.3 Asymptotic sampling zeros

As we have already seen in Section 2.2, the poles of a sampled-data model can be readily characterised in terms of the sampling period, Δ, and the continuous-time system poles. However, the relation between the zeros from discrete- to continuous-time is much more involved. Furthermore, the discrete-time model will have, in general, relative degree 1, which implies the presence of extra zeros with no continuous-time counterpart. These are usually called the *sampling zeros* of the discrete-time model.

In this section, we review results concerning the asymptotic behaviour of the zeros in sampled-data models, as the sampling period goes to zero. These results follow the seminal work by Åström *et al.* (1984), where the asymptotic location of the *intrinsic* and *sampling* zeros was first described, for the ZOH case, using shift operator models.

The next result characterises the sampled-data model (and the sampling zeros) corresponding to an n-th order integrator in terms of very specific polynomials.

Lemma 2.8 *(Åström* et al.*, 1984, Lemma 1) For a sampling period Δ, the pulse transfer function corresponding to the n-th order integrator $G(s) = s^{-n}$, is given by:*

$$G_q(z) = \frac{\Delta^n}{n!} \frac{B_n(z)}{(z-1)^n} \tag{2.43}$$

where:

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \ldots + b_n^n \tag{2.44}$$

$$b_k^n = \sum_{\ell=1}^{k} (-1)^{k-\ell} \, \ell^n \binom{n+1}{k-\ell} \tag{2.45}$$

□

Remark 2.9 *The polynomials defined in (2.44)–(2.45) correspond, in fact, to the Euler-Fröbenius polynomials (also called* reciprocal polynomials*) and are known to satisfy several properties (Mårtensson, 1982; Weller* et al.*, 2001):*

1. *Their coefficients can be computed recursively:*

$$b_1^n = b_n^n = 1 \qquad ; \forall n \geq 1 \tag{2.46}$$

$$b_k^n = k b_k^{n-1} + (n-k+1) b_{k-1}^{n-1} \qquad ; k = 2, \ldots, n-1 \tag{2.47}$$

2. *Their roots are always negative real numbers.*

3. *From the symmetry of the coefficients in* (2.45), i.e., $b_k^n = b_{n+1-k}^n$*, it follows that, if $B_n(z_0) = 0$, then $B_n(z_0^{-1}) = 0$.*

4. *They satisfy an interlacing property, namely, every root of the polynomial $B_{n+1}(z)$ lays between every two adjacent roots of $B_n(z)$, for $n \geq 2$.*

5. *The following recursive relation holds:*

$$B_{n+1}(z) = z(1-z){B_n}'(z) + (nz+1)B_n(z) \quad ; \forall n \geq 1 \tag{2.48}$$

where ${B_n}' = \frac{dB_n}{dz}$.

We next list the first of these polynomials:

$$B_1(z) = 1 \tag{2.49}$$

$$B_2(z) = z + 1 \tag{2.50}$$

$$B_3(z) = z^2 + 4z + 1 = (z + 2 + \sqrt{3})(z + 2 - \sqrt{3}) \tag{2.51}$$

$$B_4(z) = z^3 + 11z^2 + 11z + 1 = (z+1)(z + 5 + 2\sqrt{6})(z + 5 - 2\sqrt{6}) \tag{2.52}$$

These polynomials will also play a role in the characterisation of the asymptotic zeros of the sampled output spectrum. In particular, in the stochastic case considered in Section 2.5, the following result will be used.

Lemma 2.10 *The polynomials defined in Lemma 2.8 satisfy the following equation:*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\log z + j2\pi k)^n} = \frac{z\, B_{n-1}(z)}{(n-1)!(z-1)^n} \quad , n \geq 2 \tag{2.53}$$

Proof. As shown by Wahlberg (1988), this result follows from Lemma 2.8 and expression (2.19). □

Remark 2.11 *Throughout the current Thesis, we will see that sampled-data models for n-th order integrator play a very important role in obtaining asymptotic results. Indeed, as the sampling rate increases, a system of relative degree n,* behaves *as an n-th order integrator. This will be a recurrent and insightful interpretation for deterministic and stochastic systems, both in the linear and nonlinear frameworks.*

As stated at the beginning of this chapter, our aim here is not only to review the existing literature regarding sampling of linear systems, but to also extend known results by using the δ-operator. In particular, the next result constitutes one small but novel contribution in this thesis, which recasts Lemma 2.8 in the delta operator framework. This alternative formulation will prove to be particularly useful when considering sampled-data models for nonlinear systems in Chapter 6.

Lemma 2.12 *Given a sampling period Δ, the exact sampled-data model corresponding to the n-th order integrator $G(s) = s^{-n}$, $n \geq 1$, when using a ZOH input, is given by:*

$$G_\delta(\gamma) = \frac{p_n(\Delta\gamma)}{\gamma^n} \tag{2.54}$$

where the polynomial $p_n(\Delta\gamma)$ is given by:

$$p_n(\Delta\gamma) = \det M_n \tag{2.55}$$

and where the matrix M_n is defined by:

$$M_n = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-2}}{(n-1)!} & \frac{\Delta^{n-1}}{n!} \\ -\gamma & 1 & \dots & \frac{\Delta^{n-3}}{(n-2)!} & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & -\gamma & 1 & \frac{\Delta}{2!} \\ 0 & \dots & 0 & -\gamma & 1 \end{bmatrix} \tag{2.56}$$

Proof. The n-th order integrator $G(s) = s^{-n}$ can be represented in the state-space form (2.2)–(2.3), where the matrices take the specific form:

$$A = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.57}$$

The equivalent sampled-data system (2.31)–(2.32) can readily be obtained on noting that, by the Cayley-Hamilton theorem (Horn and Johnson, 1990), any matrix A satisfies its own characteristic equation,*i.e.*, $A^n = 0$. As a consequence, the corresponding exponential matrix (see Appendix A) is readily obtained:

$$e^{A\Delta} = I + A\Delta + \ldots + A^{n-1}\tfrac{\Delta^{n-1}}{(n-1)!} \tag{2.58}$$

Substituting into (2.33), we obtain:

$$A_\delta = \frac{e^{A\Delta} - I}{\Delta} = \begin{bmatrix} 0 & 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-2}}{(n-1)!} \\ 0 & 0 & 1 & \dots & \frac{\Delta^{n-3}}{(n-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & 0 & \dots & & 0 \end{bmatrix} \qquad B_\delta = \int_0^\Delta e^{A\eta} d\eta B = \begin{bmatrix} \frac{\Delta^{n-1}}{n!} \\ \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots \\ \frac{\Delta}{2} \\ 1 \end{bmatrix} \tag{2.59}$$

Note that substituting these matrices into (2.31) and applying the delta transform (Middleton and Goodwin, 1990), with initial conditions equal to zero, we obtain the following set of equations:

$$\begin{bmatrix} \gamma X_1 \\ \gamma X_2 \\ \vdots \\ \gamma X_{n-1} \\ \gamma X_n \end{bmatrix} = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-2}}{(n-1)!} & \frac{\Delta^{n-1}}{n!} \\ 0 & 1 & \dots & \frac{\Delta^{n-3}}{(n-2)!} & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \frac{\Delta}{2!} \\ 0 & & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_n \\ U \end{bmatrix} \tag{2.60}$$

This set of algebraic equations can be solved in terms of the first state, $X_1(\gamma) = Y(\gamma)$:

$$\begin{bmatrix} \gamma Y \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma X_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-1}}{n!} \\ -\gamma & 1 & \dots & \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -\gamma & 1 \end{bmatrix}}_{M_n} \begin{bmatrix} X_2 \\ \vdots \\ X_n \\ U \end{bmatrix} \tag{2.61}$$

Next, using *Cramer's Rule* (Strang, 1988), we can solve the system for the input $U(\gamma)$ in terms of $Y(\gamma)$:

$$U = \frac{\det N}{\det M_n} \tag{2.62}$$

where M_n is defined as in (2.61) (see also (2.56)), and:

$$N = \begin{bmatrix} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-2}}{(n-1)!} & \gamma Y \\ -\gamma & 1 & \dots & \frac{\Delta^{n-3}}{(n-2)!} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & -\gamma & 1 & 0 \\ 0 & \dots & & -\gamma & 0 \end{bmatrix} \tag{2.63}$$

From (2.62), using definition (2.55), and computing the determinant of the matrix N, for example, along the last column, we obtain the inverse sampled-data system transfer function:

$$U(\gamma) = \frac{\gamma^n}{p_n(\Delta\gamma)} Y(\gamma) \quad \Rightarrow \quad G_\delta(\gamma) = \frac{Y(\gamma)}{U(\gamma)} = \frac{p_n(\Delta\gamma)}{\gamma^n} \tag{2.64}$$

□

Remark 2.13 *The above result, though formally equivalent to the known shift domain expressions in Lemma 2.8, describes the results in, what we believe to be, a novel form which differs from the usual format given in the literature (Middleton and Goodwin, 1990; Feuer and Goodwin, 1996). This will prove useful later especially in relation to nonlinear systems.*

Remark 2.14 *The polynomials $p_n(\Delta\gamma)$ in Lemma 2.12, when rewritten in terms of the z-variable using* (2.30)*, correspond to the Euler-Fröbenius polynomials. In fact, the following relation holds:*

$$p_n(\Delta\gamma)\big|_{\gamma=\frac{z-1}{\Delta}} = p_n(z-1) = \frac{B_n(z)}{n!} \tag{2.65}$$

The role of these polynomials in describing pulse transfer function zeros for linear systems was first described by Åström et al. *(1984).*

The first of the Euler-Fröbenius polynomials in the γ-domain (corresponding to those in (2.49)–(2.51), on page 19) are given by:

$$p_1(\Delta\gamma) = 1 \tag{2.66}$$

$$p_2(\Delta\gamma) = 1 + \tfrac{\Delta}{2}\gamma \tag{2.67}$$

$$p_3(\Delta\gamma) = 1 + \Delta\gamma + \tfrac{\Delta^2}{6}\gamma^2 \tag{2.68}$$

Remark 2.15 *Note that, in the γ-domain, the Euler-Fröbenius polynomials are function of the argument $\Delta\gamma$. This means that their roots, in the complex plane of the variable γ, all go to infinity as Δ goes to zero.*

An immediate consequence of Lemma 2.12 is a recursive relation for the polynomials $p_n(\Delta\gamma)$. We first present the following preliminary result.

Lemma 2.16 *For any integer $n \geq 1$, consider the matrix M_n defined in* (2.56) *and* (2.61)*. Then we have:*

$$(M_n)^{-1}\begin{bmatrix}\gamma\\0\\\vdots\\0\end{bmatrix} = \frac{1}{p_n(\Delta\gamma)}\begin{bmatrix}\gamma p_{n-1}(\Delta\gamma)\\ \gamma^2 p_{n-2}(\Delta\gamma)\\ \vdots\\ \gamma^n\end{bmatrix} \tag{2.69}$$

Proof. The left hand side of equation (2.69) corresponds to solving system (2.61) by inverting the matrix M_n, and omitting the output variable $Y(\gamma)$. Thus, in the same way that we solved (2.61) for $U(\gamma)$ in the proof of Lemma 2.12, we can use Cramer's Rule (Strang, 1988) to solve for every state X_ℓ, $\ell = 2,\ldots,n$. This leads to:

$$X_\ell = \frac{\det N_{\ell-1}}{\det M_n} Y \qquad ;\ell = 2,\ldots,n \tag{2.70}$$

where $N_{\ell-1}$ is the matrix obtained by replacing the $(\ell-1)$-th column of M_n by the vector on the left of equation (2.61). Thus:

$$N_{\ell-1} = \left[\begin{array}{ccc|c|ccc}
1 & \cdots & \frac{\Delta^{\ell-3}}{(\ell-2)!} & \gamma & \frac{\Delta^{\ell-1}}{\ell!} & \cdots & \frac{\Delta^{n-1}}{n!}\\ \hline
-\gamma & \ddots & & 0 & \frac{\Delta^{\ell-2}}{(\ell-1)!} & \cdots & \frac{\Delta^{n-2}}{(n-1)!}\\
0 & \ddots & 1 & \vdots & \vdots & \ddots & \vdots\\
\vdots & \ddots & -\gamma & 0 & \frac{\Delta}{2!} & \cdots & \frac{\Delta^{n-\ell+1}}{(n-\ell+2)!}\\ \hline
0 & \cdots & 0 & 0 & 1 & \cdots & \frac{\Delta^{n-\ell}}{(n-\ell+1)!}\\
\vdots & \ddots & \vdots & 0 & -\gamma & \ddots & \frac{\Delta^{n-\ell-1}}{(n-\ell)!}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right] \tag{2.71}$$

Then, computing the determinant along the $(\ell-1)$-th column, we have that:

$$\det N_{\ell-1} = \gamma\,(-1)^{\ell}(\det P)(\det M_{n-\ell+1}) \tag{2.72}$$

where:

$$P = \begin{bmatrix} -\gamma & 1 & \dots & \frac{\Delta^{\ell-4}}{(\ell-3)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & -\gamma & 1 \\ 0 & \dots & 0 & -\gamma \end{bmatrix} \quad \Rightarrow \quad \det P = (-\gamma)^{\ell-2} \tag{2.73}$$

and, from definition (2.55)–(2.56):

$$\det M_{n-\ell+1} = p_{n-\ell+1}(\Delta\gamma) \tag{2.74}$$

Substituting (2.73) and (2.74) in (2.72), we obtain:

$$\det N_{\ell-1} = \gamma^{\ell-1} p_{n-\ell+1}(\Delta\gamma) \tag{2.75}$$

It then follows that the solution of (2.61) is:

$$\begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_n \\ U \end{bmatrix} = (M_n)^{-1} \begin{bmatrix} \gamma \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} Y = \frac{1}{p_n(\Delta\gamma)} \begin{bmatrix} \gamma p_{n-1}(\Delta\gamma) \\ \gamma^2 p_{n-2}(\Delta\gamma) \\ \vdots \\ \gamma^{n-1} p_1(\Delta\gamma) \\ \gamma^n \end{bmatrix} Y \tag{2.76}$$

which is equivalent to equation (2.69).

□

Using the above result, we next present a second novel result, which establishes a recursive relation between the Euler-Fröbenious polynomials in the γ-domain. Note that this recursion, together with (2.48) for the z-domain formulation, may be helpful to compute the polynomial coefficients.

Lemma 2.17 *The polynomials $p_n(\Delta\gamma)$ defined by* (2.55)–(2.56) *satisfy the recursion:*

$$p_0(\Delta\gamma) \triangleq 1 \tag{2.77}$$

$$p_n(\Delta\gamma) = \sum_{\ell=1}^{n} \frac{(\Delta\gamma)^{\ell-1}}{\ell!} p_{n-\ell}(\Delta\gamma) \quad ; n \geq 1 \tag{2.78}$$

and:

$$\lim_{\Delta\to 0} p_n(\Delta\gamma) = 1 \qquad ; \forall n \in \{1, 2, \dots\} \tag{2.79}$$

Proof. From the definition of matrix M_n in (2.56), we have that:

$$M_n = \left[\begin{array}{c|ccc} 1 & \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-1}}{n!} \\ \hline -\gamma & & & \\ \vdots & & M_{n-1} & \\ 0 & & & \end{array}\right] \tag{2.80}$$

The determinant of this matrix can be readily computed, using properties of block matrices (see Appendix A):

$$\det M_n = \det M_{n-1} \times \det \left(1 - \begin{bmatrix} \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-1}}{n!} \end{bmatrix} (M_{n-1})^{-1} \begin{bmatrix} -\gamma \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \tag{2.81}$$

Using definition (2.55) and the preliminary result in Lemma 2.16, we have that:

$$p_n(\Delta\gamma) = p_{n-1}(\Delta\gamma) \left(1 + \begin{bmatrix} \frac{\Delta}{2!} & \dots & \frac{\Delta^{n-1}}{n!} \end{bmatrix} \frac{1}{p_{n-1}(\Delta\gamma)} \begin{bmatrix} \gamma\, p_{n-2}(\Delta\gamma) \\ \vdots \\ \gamma^{n-2}\, p_1(\Delta\gamma) \\ \gamma^{n-1} \end{bmatrix} \right) \tag{2.82}$$

The recursive relation in (2.78) corresponds exactly to (2.82).

Finally, (2.79) readily follows from the recursion (2.78), on noting that:

$$\lim_{\Delta\to 0} p_n(\Delta\gamma) = \lim_{\Delta\to 0} p_{n-1}(\Delta\gamma) = \ldots = \lim_{\Delta\to 0} p_1(\Delta\gamma) = 1 \tag{2.83}$$

□

We next consider the case of a general SISO linear continuous-time system. Again, we are interested in the corresponding discrete-time model when a ZOH input is applied. The relationship between the continuous-time poles and those of the discrete-time model can be easily determined. However, the relationship between the zeros in the continuous and discrete-time domains is much more involved. We consider the asymptotic case as the sampling rate increases.

Lemma 2.18 *(Åström* et al.*, 1984, Theorem 1) Let $G(s)$ be a rational function:*

$$G(s) = \frac{F(s)}{E(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{2.84}$$

*and $G_q(z)$ the corresponding pulse transfer function. Assume that $m < n$,*i.e.*, $G(s)$ strictly proper. Then as the sampling period Δ goes to 0, m zeros of $G_q(z)$ go to 1 as $e^{z_i\Delta}$, and the remaining $n-m-1$ zeros of $G_q(z)$ go to the zeros of the polynomial $B_{n-m}(z)$ defined in Lemma 2.8,* i.e.*,*

$$G_q(z) \xrightarrow{\Delta\approx 0} K\frac{\Delta^{n-m}(z-1)^m B_{n-m}(z)}{(n-m)!(z-1)^n} \tag{2.85}$$

□

The above result was expressed in terms of the shift operator. We can also reexpress the result in the delta operator framework as in the following lemma.

Lemma 2.19 *Consider a SISO linear continuous-time system described by the transfer function* (2.84)*. Given a sampling period Δ, the discrete-time sampled-data model corresponding to this system, for a ZOH input, is given by:*

$$G_\delta(\gamma) = \frac{F_\delta(\gamma)}{E_\delta(\gamma)} \tag{2.86}$$

and, as the sampling period Δ goes to zero:

$$F_\delta(\gamma) \longrightarrow F(\gamma)p_{n-m}(\Delta\gamma) \tag{2.87}$$

$$E_\delta(\gamma) = \prod_{\ell=1}^{n} \left(\gamma - \tfrac{e^{p_\ell \Delta}-1}{\Delta}\right) \longrightarrow E(\gamma) \tag{2.88}$$

Proof. See any of the references (Middleton and Goodwin, 1990; Feuer and Goodwin, 1996).

□

The results presented above consider the case when the continuous-time system input is generated by a ZOH. Similar results can be obtained when a different hold device is utilised. In particular, in Chapter 3 we will see that the zeros (and thus, the asymptotic sampling zeros) of the discrete-time model depend on the particular choice of the hold. As an illustration, we present the following result that characterises the sampling zeros obtained when the continuous-time input to the system is generated using a FOH, in terms of polynomials closely related to the Euler-Fröbenius polynomials defined in Lemma 2.8.

Lemma 2.20 *(Hagiwara* et al.*, 1993, Theorem 2) Consider $G(s)$ a strictly proper transfer function as in* (2.84)*. When using a FOH to generate the continuous-time input, as in* (2.11)*, the pulse transfer function* (2.26) *has, in general, n zeros. Moreover, as the sampling period Δ goes to* 0*:*

$$G_q(z) \xrightarrow{\Delta \approx 0} K \frac{\Delta^{n-m}(z-1)^m C_{n-m}(z)}{(n-m+1)!z(z-1)^n} \tag{2.89}$$

where the sampling zero polynomials, $C_\ell(z)$, are given by:

$$C_\ell(z) = B_{\ell+1}(z) + (\ell+1)(z-1)B_\ell(z) \tag{2.90}$$

□

2.4 Sampling of stochastic linear systems

In this section we consider the sampling of stochastic linear systems described as:

$$y(t) = H(\rho)\dot{v}(t) \tag{2.91}$$

where $\dot{v}(t)$ is a *continuous-time white noise* input process. This kind of models typically describe situations where the input of a system cannot be measured or is unknown. They are sometimes also called, generically, *noise models*.

We will show how a sampled-data model can be obtained from (2.91) that is **exact**, in the sense that the second order properties (*i.e.*, spectrum) of its output sequence are the same as the second order properties of the output of the continuous-time system at the sampling instants.

We first review the relationship between the spectrum of a continuous-time process and the associated discrete-time spectrum of the sequence of samples. We next briefly discuss the difficulties that may arise when dealing with a white noise process in continuous-time. Then we show how sampled-data models can be obtained for the system (2.91). Finally, we characterise the asymptotic sampling zeros that appear in the sampled spectrum, in a similar way that we did for the deterministic case in the previous section.

2.4.1 Spectrum of a sampled process

Let us consider a stationary continuous-time stochastic process $y(t)$, with zero mean and covariance function:

$$r_y(\tau) = E\{y(t+\tau)y(t)\} \tag{2.92}$$

The associated spectral density, or **spectrum**, of this process is given by the Fourier transform of the covariance function (2.92), *i.e.*,

$$\Phi_y(\omega) = \mathcal{F}\{r_y(\tau)\} = \int_{-\infty}^{\infty} r_y(\tau)e^{j\omega\tau}d\tau \qquad ; \omega \in (-\infty, \infty) \tag{2.93}$$

If we instantaneously sample the continuous-time signal, with sampling period Δ, we obtain the sequence $y_k = y(k\Delta)$. The covariance of this sequence, $r_y^d[\ell]$, is equal to the continuous-time signal covariance *at the sampling instants*:

$$r_y^d[\ell] = E\{y_{k+\ell}\, y_k\} = E\{y(k\Delta + \ell\Delta)\, y(k\Delta)\} = r_y(\ell\Delta) \tag{2.94}$$

The power spectral density of the sampled signal is given by the Discrete-Time Fourier Transform (DTFT) of the covariance function, namely:

$$\Phi_y^d(\omega) = \Delta \sum_{k=-\infty}^{\infty} r_y^d[k]e^{-j\omega k\Delta} \qquad ; \omega \in \left[\tfrac{-\pi}{\Delta}, \tfrac{-\pi}{\Delta}\right] \tag{2.95}$$

Remark 2.21 *Note that we have used the DTFT as defined in Feuer and Goodwin (1996), which includes the sampling period Δ as a scaling factor. As a consequence, the DTFT defined this way converges to the continuous-time Fourier transform as the sampling period Δ goes to zero.*

Remark 2.22 *The continuous and discrete-time spectral densities, in* (2.93) *and* (2.95) *respectively, are **real** functions of the frequency ω. However, to make the connections to the deterministic case apparent, we will sometimes express the continuous-time spectrum in terms of the complex variables $s = j\omega$,* i.e.,

$$\Phi_y(\omega) = \Phi_y(j\omega) = \Phi_y(s)\big|_{s=j\omega} \qquad \textit{(CT spectrum)} \tag{2.96}$$

and the discrete-time spectrums in terms of $z = e^{j\omega\Delta}$ or $\gamma = \gamma_\omega = \frac{e^{j\omega\Delta}-1}{\Delta}$, for shift and delta operator models, respectively, i.e.,

$$\Phi_y^d(\omega) = \Phi_y^q(e^{j\omega\Delta}) = \Phi_y^\delta(\gamma_\omega) \tag{2.97}$$

where:

$$\Phi_y^q(e^{j\omega\Delta}) = \Phi_y^q(z)\big|_{z=e^{j\omega\Delta}} \qquad \textit{(q-domain DT spectrum)} \tag{2.98}$$

$$\Phi_y^\delta(\gamma_\omega) = \Phi_y^\delta(\gamma)\big|_{\gamma=\frac{e^{j\omega\Delta}-1}{\Delta}} \qquad \textit{(δ-domain DT spectrum)} \tag{2.99}$$

The following lemma relates the spectrum of the sampled sequence to the spectrum of the original continuous-time process.

Lemma 2.23 *Let us consider a stochastic process $y(t)$, with spectrum given by* (2.93)*, together with its sequence of samples $y_k = y(k\Delta)$, with discrete-time spectrum given by* (2.95)*. Then the following relationship holds:*

$$\Phi_y^d(\omega) = \sum_{\ell=-\infty}^{\infty} \Phi_y\left(\omega + \tfrac{2\pi}{\Delta}\ell\right) \tag{2.100}$$

Proof. The discrete-time spectrum (2.95) can be rewritten in terms of the inverse (continuous-time) Fourier transform of the covariance function:

$$\Phi_y^d(\omega) = \Delta \sum_{k=-\infty}^{\infty} r_y(k\Delta)e^{-j\omega k\Delta} = \Delta \sum_{k=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_y(\eta)e^{j\eta k\Delta}d\eta\right] e^{-j\omega k\Delta} \tag{2.101}$$

$$= \frac{\Delta}{2\pi}\int_{-\infty}^{\infty} \Phi_y(\eta)\left[\sum_{k=-\infty}^{\infty} e^{j(\eta-\omega)k\Delta}\right] d\eta \tag{2.102}$$

The sum of complex exponentials can be rewritten as an infinite sum of Dirac impulses spread every $\frac{2\pi}{\Delta}$ (Feuer and Goodwin, 1996; Oppenheim and Schafer, 1999), *i.e.*,

$$\sum_{k=-\infty}^{\infty} e^{j(\eta-\omega)k\Delta} = \frac{2\pi}{\Delta}\sum_{\ell=-\infty}^{\infty} \delta\left(\eta-\omega-\tfrac{2\pi}{\Delta}\ell\right) \tag{2.103}$$

Using (2.103) in (2.102), the result is obtained:

$$\Phi_y^d(\omega) = \int_{-\infty}^{\infty} \Phi_y(\eta)\sum_{\ell=-\infty}^{\infty} \delta\left(\eta-\omega-\tfrac{2\pi}{\Delta}\ell\right)d\eta = \sum_{\ell=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} \Phi_y(\eta)\delta\left(\eta-\omega-\tfrac{2\pi}{\Delta}\ell\right)d\eta}_{\Phi_y\left(\omega+\frac{2\pi}{\Delta}\ell\right)} \tag{2.104}$$

□

Equation (2.100) reflects the well-known consequence of the sampling process: the aliasing effect. For deterministic systems, an analogue result was obtained in (2.20). In the stochastic case considered here, the discrete-time spectrum is obtained by folding high frequency components of the continuous-time spectrum.

2.4.2 Continuous-time white noise

The input $\dot{v}(t)$ to the system (2.91) is modelled as zero mean *white noise* process in continuous-time. This means that it is a stochastic process that satisfies the following two conditions:

1. $E\{\dot{v}(t)\} = 0$, for all t; and
2. $\dot{v}(t)$ is independent of $\dot{v}(s)$, *i.e.*, $E\{\dot{v}(t)\dot{v}(s)\} = 0$, for all $t \neq s$.

However, if we look for a stochastic process with continuous paths that satisfies the previous two conditions, this happens to be equal to zero in the mean square sense, *i.e.*, $E\{\dot{v}(t)^2\} = 0$, for all t (Åström, 1970). This points to the source of some difficulties since the process $\dot{v}(t)$ does not exist in a meaningful sense. Indeed, equation (2.109) (below) should actually be written as a stochastic differential equation (Øksendal, 2003):

$$dx = A\,x\,dt + Bdv \tag{2.105}$$

where dv are independent increments, Gaussian distributed, of a process $v(t)$. This corresponds to a *Wiener* process that satisfies the following properties (Kallianpur, 1980):

1. It has zero mean, *i.e.*, $E\{v(t)\} = 0$, for all t ;
2. Its increments are independent, *i.e.*, $E\{(v(t_1) - v(t_2))(v(s_1) - v(s_2))\} = 0$, for all $t_1 > t_2 > s_1 > s_2 \geq 0$; and

3. For every s and t, $s \leq t$, $v(t) - v(s)$ has a Gaussian distribution with zero mean and variance $E\{(v(t) - v(s))^2\} = \sigma_v^2 |t - s|$

This process is not differentiable everywhere. However, if we define the *continuous-time white noise process* (CTWN) $\dot{v}(t)$ formally as its derivative, then the first two conditions on $v(t)$ correspond to the previous conditions required for the process $\dot{v}(t)$. Note also that the third condition above implies that CTWN will have infinite variance:

$$E\{dv\, dv\} = E\{(v(t + dt) - v(t))^2\} = \sigma_v^2 dt \quad \Rightarrow \quad E\{\dot{v}^2\} = \infty \tag{2.106}$$

Remark 2.24 *Indeed, a continuous-time white noise process is a mathematical abstraction and does not physically exist (Jazwinski, 1970), but it can be approximated to any desired degree of accuracy by conventional stochastic processes with broad band spectra (Kloeden and Platen, 1992).*

Equation (2.106) suggests that one may consider σ_v^2 as the **incremental variance** of the Wiener process $v(t)$. Moreover, we can think of $\dot{v}(t)$ as a *generalised* process, introducing a Dirac delta function to define its covariance structure:

$$r_{\dot{v}}(t - s) = E\{\dot{v}(t)\, \dot{v}(s)\} = \sigma_v^2 \delta(t - s) \tag{2.107}$$

In the frequency domain, σ_v^2 corresponds, in fact, to the **power spectral density** of $\dot{v}(t)$ (Feuer and Goodwin, 1996), which is constant for all frequencies:

$$\Phi_{\dot{v}}(\omega) = \int_{-\infty}^{\infty} r_{\dot{v}}(\tau) e^{-j\omega\tau} d\tau = \sigma_v^2 \qquad \forall \omega \in (-\infty, \infty) \tag{2.108}$$

2.4.3 A stochastic sampled-data model

In the sequel we will assume that the the process $y(t)$ in (2.91) does not contain any unfiltered white noise components. In practice, this can be guaranteed by the use of an anti-aliasing filter. As a consequence, we assume that $H(\rho)$ in (2.91) is a strictly proper transfer function that can be represented in state space form as:

$$\frac{dx(t)}{dt} = Ax(t) + B\dot{v}(t) \tag{2.109}$$

$$y(t) = Cx(t) \tag{2.110}$$

where the system state vector is $x(t) \in \mathbb{R}^n$, the matrices are $A \in \mathbb{R}^{n \times n}$ and $B, C^T \in \mathbb{R}^n$, and the input $\dot{v}(t)$ is a CTWN process with (constant) spectral density σ_v^2.

In the previous subsection we briefly discussed the characteristics required for the CTWN input to the stochastic model (2.91). We have seen that a rigorous treatment of the state-space model (2.109)–(2.110) requires that one represents the system as a (set of) stochastic differential equation (SDE). However, for linear systems we will obtain the same results if we proceed formally considering the system as simply driven by the CTWN input (see also Remark 7.4 on page 126).

The following result gives us the sampled-data model when considering instantaneous sampling of the output (2.110).

Lemma 2.25 *Consider the stochastic system defined in state-space form* (2.109)–(2.110)*, where the input $\dot{v}(t)$ is a CTWN process with (constant) spectral density σ_v^2. When the output $y(t)$ is instantaneously sampled, with sampling period Δ, an equivalent discrete-time model is given by:*

$$\delta x_k = A_\delta\, x_k + v_k \tag{2.111}$$

$$y_k = C x_k \tag{2.112}$$

where $A_\delta = \frac{1}{\Delta}(e^{A\Delta} - I_n)$, and the sequence v_k is a discrete-time white noise (DTWN) process, with zero mean and covariance structure given by:

$$E\{v_k\, v_\ell^T\} = \Omega_\delta\, \frac{\delta_K[k-\ell]}{\Delta} \tag{2.113}$$

where:

$$\Omega_\delta = \frac{\sigma_v^2}{\Delta} \int_0^\Delta e^{A\eta} B B^T e^{A^T\eta} d\eta \tag{2.114}$$

Proof. The proof (when using shift operator models) can be found, for example, in (Söderström, 2002). Arguing as in (2.15), we have that:

$$x_{k+1} = e^{A\Delta} x_k + \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B\dot{v}(\eta) d\eta \quad \Rightarrow \quad \delta x_k = \frac{e^{A\Delta} - I_n}{\Delta} x_k + v_k \tag{2.115}$$

where the noise sequence is:

$$v_k = \frac{1}{\Delta} \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B\dot{v}(\eta)\, d\eta \tag{2.116}$$

The covariance of v_k is now given by:

$$E\{v_k\, v_\ell^T\} = \frac{1}{\Delta^2} E\left\{ \left(\int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B\dot{v}(\eta) d\eta \right) \left(\int_{\ell\Delta}^{\ell\Delta+\Delta} e^{A(\ell\Delta+\Delta-\xi)} B\dot{v}(\xi) d\xi \right)^T \right\} \tag{2.117}$$

$$= \frac{1}{\Delta^2} \int_{k\Delta}^{k\Delta+\Delta} \int_{\ell\Delta}^{\ell\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B\, \underbrace{E\{\dot{v}(\eta)\dot{v}(\xi)\}}_{\sigma_v^2 \delta(\eta-\xi)} B^T e^{A^T(\ell\Delta+\Delta-\xi)} d\xi d\eta \tag{2.118}$$

The double integral above is non-zero only when $k = \ell$ and $\eta = \xi$. Thus, we obtain:

$$E\{v_k\, v_\ell^T\} = \frac{\sigma_v^2}{\Delta^2} \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B B^T e^{A^T(k\Delta+\Delta-\eta)} d\eta\ \delta_K[k-\ell] \tag{2.119}$$

Upon changing variables in the integral, the above expression can be shown to be equivalent to (2.113)–(2.114).

□

Remark 2.26 *Matrix Ω_δ is in fact the (constant) spectral density of the noise vector v_k, as can be seen applying discrete-time Fourier transform to* (2.113)*:*

$$\mathcal{F}_d \left\{ \Omega_\delta\, \frac{\delta_K[k]}{\Delta} \right\} = \Delta \sum_{k=-\infty}^{\infty} \Omega_\delta\, \frac{\delta_K[k]}{\Delta} e^{-j\omega k\Delta} = \Omega_\delta \qquad ;\ \omega \in \left[-\tfrac{\pi}{\Delta}, \tfrac{\pi}{\Delta}\right] \tag{2.120}$$

Remark 2.27 *Note that the previous result allows us to recover the continuous-time stochastic description* (2.109)*, as the sampling period* Δ *goes to zero. In particular, the covariance* (2.113) *corresponds (in continuous-time) to the covariance of the vector process* $B\dot{v}(t)$ *in* (2.109)*, as it can be readily seen on noting that:*

$$\lim_{\Delta\to 0} \Omega_\delta = \sigma_v^2 BB^T = \Omega_c \tag{2.121}$$

$$\lim_{\Delta\to 0} \tfrac{1}{\Delta}\delta_K[k-\ell] = \delta(t_k - t_\ell) \tag{2.122}$$

Given continuous-time system, Lemma 2.25 provides a sampled-data model expressed in terms of the δ-operator. A corresponding shift operator model can readily be obtained rewriting (2.111) as:

$$q\, x_k = x_{k+1} = A_q\, x_k + \tilde{v}_k \tag{2.123}$$

where $\tilde{v}_k = v_k\,\Delta$ and, as before, $A_q = 1 + A_\delta\Delta$. Note that, for this model, the covariance structure of the noise sequence is given by:

$$E\{\tilde{v}_k\, \tilde{v}_\ell^T\} = \Delta^2 E\{v_k\, v_\ell^T\} = \Delta\,\Omega_\delta\,\delta_K[k-\ell] = \Omega_q\,\delta_K[k-\ell] \tag{2.124}$$

where we have defined $\Omega_q = \Omega_\delta\,\Delta$.

Remark 2.28 *As noticed by Farrell and Livstone (1996), the matrix* Ω_q *in* (2.124) *can be computed solving the discrete-time Lyapunov equation (see Appendix A):*

$$\Omega_q = P - A_q P A_q^T \tag{2.125}$$

or, equivalently, in the δ*-domain:*

$$\Omega_\delta = A_\delta P + P A_\delta^T + \Delta A_\delta P A_\delta^T \tag{2.126}$$

where P *satisfies the continuous-time Lyapunov equation* $AP + PA^T + \Omega_c = 0$*, for stable systems, or* $AP + PA^T - \Omega_c = 0$*, for anti-stable systems. For Lemma 2.25 we have, in particular,* $\Omega_c = \sigma_v^2 BB^T$*.*

The sampled-data model (2.111)–(2.112) is driven by a vector white noise process v_k. The covariance of this process is determined by the matrix Ω_δ in (2.114), which will generically be *full rank* (Söderström, 2002). We will go further and describe the sampled process $y_k = y(k\Delta)$ as the output of a sampled-data model driven by a *single* scalar noise source. This can be achieved by, first, obtaining the discrete-time spectrum of the sampled sequence y_k, and then performing **spectral factorisation** (Anderson and Moore, 1979).

The output spectrum of the sampled-data model is given in the following result.

Lemma 2.29 *The output spectrum* $\Phi_y^d(\omega)$ *of the sampled-data model* (2.111)–(2.112) *can be obtained as:*

$$\Phi_y^\delta(\gamma_\omega) = C(\gamma_\omega I_n - A_\delta)^{-1}\Omega_\delta(\gamma_\omega^* I_n - A_\delta^T)^{-1}C^T \tag{2.127}$$

where $\gamma_\omega = \frac{1}{\Delta}(e^{j\omega\Delta} - 1)$ *and* * *denote complex conjugation. Using* (2.123)*, this spectrum can be equivalently obtained as:*

$$\Phi_y^q(e^{j\omega\Delta}) = \Delta C(e^{j\omega\Delta} I_n - A_q)^{-1}\Omega_q(e^{-j\omega\Delta} I_n - A_q^T)^{-1}C^T \tag{2.128}$$

Proof. The result in the delta domain readily follows considering (2.111)–(2.112). These equations define a discrete-time linear system with a vector input v_k and output y_k. The output spectrum is then given by (Middleton and Goodwin, 1990):

$$\Phi_y^d(\omega) = H_\delta(\gamma_\omega)\Phi_v^d(\omega)H_\delta(\gamma_\omega^*)^T \tag{2.129}$$

where $H_\delta(\gamma_\omega) = C(\gamma_\omega I_n - A_\delta)^{-1}$, and the spectrum of the input noise is $\Phi_v^d(\omega) = \Omega_\delta$ (see Remark 2.26). Equation (2.128) follows from the relations $\gamma_\omega = \frac{e^{j\omega\Delta}-1}{\Delta}$, $A_\delta = \frac{A_q-1}{\Delta}$, and $\Omega_\delta = \frac{\Omega_q}{\Delta}$.

□

Remark 2.30 *The previous lemma allows us to obtain an expression for the discrete-time spectrum of the sequence of output samples. Additionally, this spectrum can be used to obtain a stochastic sampled-data model by utilising spectral factorisation.*

Next we present examples showing how stochastic sampled-data models can be obtained utilising the previous results.

Example 2.31 *Let us consider the first order continuous-time auto-regressive (CAR) system:*

$$\frac{dy(t)}{dt} - a_0 y(t) = b_0 \dot{v}(t) \tag{2.130}$$

where $a_0 < 0$ and $\dot{v}(t)$ is a CTWN process of unitary spectral density, i.e., $\sigma_v^2 = 1$. *A suitable state-space model can readily be obtained as:*

$$\frac{dx(t)}{dt} = a_0 x(t) + b_0 \dot{v}(t) \tag{2.131}$$

$$y(t) = x(t) \tag{2.132}$$

A sampled-data model for this system is readily obtained:

$$q\,x_k = e^{a_0\Delta}x_k + \tilde{v}_k \qquad \delta x_k = \left(\frac{e^{a_0\Delta}-1}{\Delta}\right)x_k + v_k \tag{2.133}$$

$$y_k = x_k \qquad y_k = x_k \tag{2.134}$$

where $\tilde{v}_k$ and v_k are DTWN processes with variance Ω_q and $\frac{\Omega_\delta}{\Delta}$, respectively. Note that these variances are not very useful when considering the sampling period Δ tending to zero. If we compute them, for example, using Remark 2.28, we can see that they are badly scaled:

$$\Omega_q = \Delta\Omega_\delta = b_0^2\frac{(e^{2a_0\Delta}-1)}{2a_0} \quad \xrightarrow{\Delta\to 0} \quad 0 \tag{2.135}$$

$$\frac{\Omega_\delta}{\Delta} = b_0^2\frac{(e^{2a_0\Delta}-1)}{2a_0\Delta^2} \quad \xrightarrow{\Delta\to 0} \quad \infty \tag{2.136}$$

On the other hand, as noticed in Remark 2.27, the ***spectral density*** *Ω_δ converges naturally to its continuous-time counterpart:*

$$\Omega_\delta = b_0^2\frac{(e^{2a_0\Delta}-1)}{2a_0\Delta} \quad \xrightarrow{\Delta\to 0} \quad b_0^2 \tag{2.137}$$

□

In the previous example, a stochastic sampled-data model was obtained for first order systems in terms of a single scalar noise source. For higher order systems, Lemma 2.25 gives a sampled-data model in terms of a vector input v_k. However, as described above, we can obtain an **equivalent** sampled-data model, with a single scalar noise source as input, using spectral factorisation. The output of this system has the same second order statistics, *i.e.*, the same discrete-time spectrum (2.128), as the original sampled-data model.

In the following example, we illustrate how a sampled-data model can be obtained for a second order system by spectral factorisation of the sampled output spectrum.

Example 2.32 *Consider the second order continuous-time auto-regressive (CAR) system:*

$$\frac{d^2y(t)}{dt} + a_1\frac{dy(t)}{dt} + a_0 y(t) = \dot{v}(t) \tag{2.138}$$

where $\dot{v}(t)$ is CTWN process of unitary spectral density, i.e., $\sigma_v^2 = 1$.

A state-space model is given by:

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{v}(t) \tag{2.139}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{2.140}$$

For this system, even though is possible to compute the corresponding sampled-data model (2.111)–(2.112) *and the spectral density of the noise* (2.114)*, the expressions are more involved than for the first order case in Example 2.31. Moreover, the sampled-data model obtained would depend on two noise sources.*

Using (2.128)*, and after some long calculations, we see that the discrete-time output spectrum has the form:*

$$\Phi_y^q(z) = K\frac{z(b_2z^2 + b_1z + b_0)}{(z - e^{\lambda_1\Delta})(z - e^{\lambda_2\Delta})(1 - e^{\lambda_1\Delta}z)(1 - e^{\lambda_2\Delta}z)} \tag{2.141}$$

where λ_1 and λ_2 are the continuous-time system poles, and:

$$b_2 = (\lambda_1 - \lambda_2)\left[e^{(\lambda_1+\lambda_2)\Delta}(\lambda_2 e^{\lambda_1\Delta} - \lambda_1 e^{\lambda_2\Delta}) + \lambda_1 e^{\lambda_1\Delta} - \lambda_2 e^{\lambda_2\Delta}\right] \tag{2.142}$$

$$b_1 = \left[(\lambda_1 + \lambda_2)(e^{2\lambda_1\Delta} - e^{2\lambda_2\Delta}) + (\lambda_1 - \lambda_2)(e^{2(\lambda_1+\lambda_2)\Delta} - 1)\right] \tag{2.143}$$

$$b_0 = b_2 \tag{2.144}$$

$$K = \frac{\Delta}{2\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)} \tag{2.145}$$

If we perform spectral factorisation on the sampled spectrum (2.141) *we can obtain a sampled-data model in terms of only one noise source,* i.e.,

$$\Phi_y^q(z) = H_q(z)H_q(z^{-1}) \tag{2.146}$$

where:

$$H_q(z) = \frac{\sqrt{K}(c_1z + c_0)}{(z - e^{\lambda_1\Delta})(z - e^{\lambda_2\Delta})} \tag{2.147}$$

The expression for the numerator coefficients (and, thus, of the only sampling zero) of the latter discrete-time model are involved. However, it is possible to obtain an asymptotic characterisation of *this sampled-data model as the sampling period goes to zero, in a similar fashion as was done for the deterministic case. We will revisit this model in Example 2.39 in Section 2.5.*

□

Lemma 2.25 allows us to obtain a sampled-data model for a stochastic system when its output is instantaneously sampled. However, if the output $y(t)$ contains a white noise component this approach becomes impractical because the resulting sequence of samples has infinite variance (Åström, 1970; Feuer and Goodwin, 1996; Söderström, 2002). To overcome this difficulty, the output of the system has to be prefiltered before sampling. In particular, the following result presents the sampled-data model when we use an **integrating filter** (also called averaging filter) before sampling.

Lemma 2.33 *Consider the stochastic system:*

$$\frac{dx(t)}{dt} = Ax(t) + B\dot{v}(t) \tag{2.148}$$

$$y(t) = Cx(t) + \dot{w}(t) \tag{2.149}$$

where $\dot{v}(t)$ and $\dot{w}(t)$ are CTWN processes such that:

$$E\left\{ \begin{bmatrix} \dot{v}(t) \\ \dot{w}(t) \end{bmatrix} \begin{bmatrix} \dot{v}(s) \\ \dot{w}(s) \end{bmatrix}^T \right\} = \Omega_c \delta(t-s) \tag{2.150}$$

where $\Omega_c \geq 0$. If the output of the system is sampled using the integrating filter:

$$\bar{y}_k = \bar{y}(k\Delta) = \frac{1}{\Delta}\int_{k\Delta-\Delta}^{k\Delta} y(\tau)d\tau \tag{2.151}$$

then the following sampled-data model is obtained:

$$\delta x_k = A_\delta \, x_k + v_k \tag{2.152}$$

$$\bar{y}_{k+1} = C_{IF} x_k + w_k \tag{2.153}$$

where:

$$A_\delta = \frac{e^{A\Delta} - I_n}{\Delta} \qquad ; \qquad C_{IF} = \frac{1}{\Delta} C \int_0^\Delta e^{A\eta} d\eta \tag{2.154}$$

and the DTWN sequences in (2.152)–(2.153) *have the following covariance structure:*

$$E\left\{ \begin{bmatrix} v_k \\ w_k \end{bmatrix} \begin{bmatrix} v_\ell \\ w_\ell \end{bmatrix}^T \right\} = \begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix} \frac{\delta_K[k-\ell]}{\Delta} \tag{2.155}$$

$$\begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix} = \frac{1}{\Delta}\int_0^\Delta e^{\bar{A}\eta} \bar{B} \Omega_c \bar{B}^T e^{\bar{A}^T \eta} d\eta \tag{2.156}$$

where:

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \tag{2.157}$$

Proof. The proof is based on the key observation that, if we *rename* the output of the system as $y(t) = \frac{dz(t)}{dt}$, then, substituting in (2.151), we obtain:

$$\bar{y}_{k+1} = \frac{1}{\Delta}\int_{k\Delta}^{k\Delta+\Delta} \frac{dz(\tau)}{d\tau}d\tau = \frac{z(k\Delta+\Delta) - z(k\Delta)}{\Delta} = \delta z_k \tag{2.158}$$

If we rewrite the continuous-time system (2.148)–(2.149), using the matrices in (2.157), as:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \bar{A}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \bar{B}\begin{bmatrix} \dot{v}(t) \\ \dot{w}(t) \end{bmatrix} \tag{2.159}$$

we can then apply the result in Lemma 2.25. More details of the proof can be found in (Feuer and Goodwin, 1996, Lemma 6.4.1).

□

Remark 2.34 *The discrete-time model* (2.152)–(2.153) *can also be expressed, in terms of the shift operator q, as:*

$$x_{k+1} = A_q\, x_k + \tilde{v}_k \tag{2.160}$$

$$\bar{y}_{k+1} = C_{IF}x_k + w_k \tag{2.161}$$

where $A_q = I_n + A_\delta\Delta$, $\tilde{v}_k = v_k\Delta$, *and:*

$$E\left\{\begin{bmatrix} \tilde{v}_k \\ w_k \end{bmatrix}\begin{bmatrix} \tilde{v}_\ell \\ w_\ell \end{bmatrix}^T\right\} = \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix}\delta_K[k-\ell] \tag{2.162}$$

where:

$$\Omega_q = \Delta\Omega_\delta \quad ; \quad \Sigma_q = \Sigma_\delta \quad ; \quad \Gamma_q = \frac{1}{\Delta}\Gamma_\delta \tag{2.163}$$

We can now obtain the discrete-time spectrum corresponding to the sampled output of the integrating filter scheme. This is similar in spirit to Lemma 2.29 for the instantaneous sampling case.

Lemma 2.35 *The output spectrum of the sampled-data model* (2.152)–(2.153) *is given by:*

$$\Phi_{\bar{y}}^{\delta}(\gamma_\omega) = \begin{bmatrix} C_{IF}(\gamma_\omega I_n - A_\delta)^{-1} & 1 \end{bmatrix}\begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix}\begin{bmatrix} (\gamma_\omega^* I_n - A_\delta^T)^{-1}C_{IF}^T \\ 1 \end{bmatrix} \tag{2.164}$$

where $\gamma_\omega = \frac{1}{\Delta}(e^{j\omega\Delta} - 1)$ *and where* * *denotes complex conjugation. Equivalently, if the shift operator model* (2.160)–(2.161) *is utilised, then the output spectrum can be expressed as:*

$$\Phi_{\bar{y}}^{q}(e^{j\omega\Delta}) = \Delta\begin{bmatrix} C_{IF}(e^{j\omega\Delta}I_n - A_q)^{-1} & 1 \end{bmatrix}\begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix}\begin{bmatrix} (e^{-j\omega\Delta}I_n - A_q^T)^{-1}C_{IF}^T \\ 1 \end{bmatrix} \tag{2.165}$$

Proof. The proof follows the same lines as that of Lemma 2.29. Equations (2.152)–(2.153) define a discrete-time linear system with vector inputs v_k and w_k, and output y_k. The output spectrum is then given by (Middleton and Goodwin, 1990):

$$\Phi_{\bar{y}}^{d}(\omega) = H_\delta(\gamma_\omega)\begin{bmatrix} \Phi_v^d(\omega) & \Phi_{vw}^d(\omega) \\ (\Phi_{vw}^d(\omega))^T & \Phi_w^d(\omega) \end{bmatrix}H_\delta(\gamma_\omega^*)^T \tag{2.166}$$

where:

$$H_\delta(\gamma_\omega) = \begin{bmatrix} C_{IF}(\gamma_\omega I_n - A_\delta)^{-1} \\ 1 \end{bmatrix} \tag{2.167}$$

and the spectrum of the input noise are obtained from (2.155), *i.e.*,

$$\begin{bmatrix} \Phi_v^d(\omega) & \Phi_{vw}^d(\omega) \\ (\Phi_{vw}^d(\omega))^T & \Phi_w^d(\omega) \end{bmatrix} = \begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix} \tag{2.168}$$

Equation (2.165) follows from the relations $\gamma_\omega = \frac{e^{j\omega\Delta}-1}{\Delta}$, $A_\delta = \frac{A_q - 1}{\Delta}$, and the matrix relations in (2.163).

□

In Lemmas 2.29 and 2.35 we have given expressions for the output spectrum corresponding to discrete-time models when sampling instantaneously and when using an integrating pre-filter, respectively. Sampled-data models can be obtained from these spectra by performing spectral factorisation (as in Example 2.32). The following classic result shows how this spectral factorisation can be performed by using Kalman filtering, in order to obtain an **innovations model**. The single noise source that appears as input to this model is known as the innovations sequence.

Lemma 2.36 *Consider a state-space discrete-time model as in* (2.160)–(2.161). *Then the following* ***innovations model*** *is equivalent, in the sense that the outputs of the two models share the same second order properties:*

$$z_{k+1} = A_q z_k + K_q e_k \tag{2.169}$$

$$\bar{y}_{k+1} = C_{IF} z_k + e_k \tag{2.170}$$

where e_k is a discrete-time white noise sequence with covariance matrix:

$$E\{e_k^2\} = \Gamma_q + C_{IF} P C_{IF}^T \tag{2.171}$$

The Kalman gain K_q is given by:

$$K_q = (A_q P C_{IF}^T + \Sigma_q)(\Gamma_q + C_{IF} P C_{IF}^T)^{-1} \tag{2.172}$$

where P is the state covariance matrix given by the discrete-time algebraic Riccati equation:

$$A_q P A_q^T - P - K_q(\Gamma_q + C_{IF} P C_{IF}^T) K_q^T + \Omega_q = 0 \tag{2.173}$$

Proof. See, for example, (Anderson and Moore, 1979).

□

2.5 Asymptotic sampling zeros of the output spectrum

In the previous section we have seen that the output spectrum of the sampled-data model contains *sampling zeros* which have no counterpart in the underlying continuous-time system. Similar to the deterministic case, these zeros can be asymptotically characterised.

The following two results appear in (Wahlberg, 1988). They characterise the asymptotic sampling zeros of the output spectrum in the case of instantaneous sampling and when using an integrating pre-filter.

Lemma 2.37 *(Wahlberg, 1988, Theorem 3.1) Consider the instantaneous sampling of the continuous-time process* (2.91)*. We then have that:*

$$\Phi_y^d(\omega) \xrightarrow{\Delta \to 0} \Phi_y(\omega) \tag{2.174}$$

uniformly in s, on compact subsets. Moreover, let $\pm z_i$, $i = 1, \ldots, m$ be the $2m$ zeros of $\Phi_y(s)$, and $\pm p_i$, $i = 1, \ldots, n$ its $2n$ poles. Then:

- *$2m$ zeros of $\Phi_y^d(z)$ will converge to 1 as $e^{\pm z_i \Delta}$;*
- *The remaining $2(n-m)-1$ will converge to the zeros of $zB_{2(n-m)-1}(z)$ as Δ goes to zero; and*
- *The $2n$ poles of $\Phi_y^d(z)$ equal $e^{\pm p_i \Delta}$, and will hence go to 1 as Δ goes to zero.*

Proof. The proof follows from the fact that, for large $|s|$, the continuous-time spectrum $\Phi_y(s)$ can be approximated by a $2(n-m)$-th order integrator. Then the sampled data spectrum can be obtained from the infinite sum (2.100). Using Lemma 2.10, we then have that:

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\log z + j2\pi k)^{2(n-m)}} = \frac{z\, B_{2(n-m)-1}(z)}{(2(n-m)-1)!(z-1)^{2(n-m)}} \tag{2.175}$$

The remaining details of the proof can be found in (Wahlberg, 1988) □

Lemma 2.38 *(Wahlberg, 1988, Theorem 3.2) Consider the averaging sampling of the continuous-time process* (2.91)*. Then the results of Lemma 2.37 essentially apply save that the remaining $2(n-m)$ zeros of $\Phi_y^d(z)$ will converge to the zeros of $B_{2(n-m)}(z)$ as Δ goes to zero.*

Proof. See (Wahlberg, 1988).

□

Example 2.39 *Consider again the second order CAR system in Example 2.32. The discrete-time spectrum* (2.141) *was obtained for the case of instantaneous sampling of the output $y(t)$. Exact expressions for the sampling zeros of this spectrum are quite involved. However, performing a Taylor series expansion of the numerator we have that:*

$$Kz(b_2 z^2 + b_1 z + b_0) = \frac{\Delta^4}{3!} z(z^2 + 4z + 1) + \mathcal{O}(\Delta^5) \tag{2.176}$$

which, asymptotically as Δ goes to zero, is consistent with Lemma 2.37, noting that $B_3(z) = z^2 + 4z + 1$ as in (2.51)*.*

The asymptotic sampled spectrum can be obtained as:

$$\begin{aligned} \Phi_y^q(z) &= \frac{\Delta^4}{6} \frac{(z + 4 + z^{-1})}{(z - e^{\lambda_1 \Delta})(z - e^{\lambda_2 \Delta})(z^{-1} - e^{\lambda_1 \Delta})(z^{-1} - e^{\lambda_2 \Delta})} \\ &= \frac{\Delta^4}{6(2 - \sqrt{3})} \frac{(z + 2 - \sqrt{3})}{(z - e^{\lambda_1 \Delta})(z - e^{\lambda_2 \Delta})} \frac{(z^{-1} + 2 - \sqrt{3})}{(z^{-1} - e^{\lambda_1 \Delta})(z^{-1} - e^{\lambda_2 \Delta})} \end{aligned} \tag{2.177}$$

Then, the spectrum can be written as $\Phi_y^q(z) = H_q(z) H_q(z^{-1})$, where:

$$H_q(z) = \frac{\Delta^2}{3 - \sqrt{3}} \frac{(z + 2 - \sqrt{3})}{(z - e^{\lambda_1 \Delta})(z - e^{\lambda_2 \Delta})} \tag{2.178}$$

The corresponding δ-operator model can be obtained by changing variable $z = 1+\gamma\Delta$. This yields the following discrete-time model:

$$H_\delta(\gamma) = \frac{1 + \frac{1}{3-\sqrt{3}}\Delta\gamma}{\left(\gamma - \frac{e^{\lambda_1\Delta}-1}{\Delta}\right)\left(\gamma - \frac{e^{\lambda_2\Delta}-1}{\Delta}\right)} \tag{2.179}$$

which clearly converges to the underlying continuous-time system (2.138)*, as the sampling period goes to zero.*

□

2.6 Summary

In this chapter we have reviewed the fundamental concepts underlying the analysis of sampled-data models for continuous-time linear systems. Known results have been summarised and extensions using the δ-operator have been presented.

For deterministic systems, we have presented a sampled-data model which exactly describes the samples of the system continuous-time output. Also a characterisation of the poles and the zeros of the discrete-time model has been given. Well-known results regarding the presence and convergence of *sampling zeros* have been reviewed and extended. In particular, we have presented two novel results: the characterisation of the *asymptotic sampling zero* polynomials in Lemma 2.12, and the recursive relation for these polynomials, in Lemma 2.17. The given formulation is actually an alternative way of writing (in the δ domain) the usual characterisation in terms of the Euler-Fröbenius polynomials. The alternative form will prove to be a key enabling result in the nonlinear case presented in Chapter 6.

Corresponding results have also been given for stochastic systems. Our analysis has concentrated on the sampled output spectrum, which contains also sampling zeros. These zeros can be asymptotically characterised as the sampling period goes to zero. Sampled-data models for this kind of systems can be obtained by spectral factorisation of this discrete-time output spectrum. As a consequence, these models are *exact* in the sense that the output has the same second order properties (covariance) as does the continuous-time system output *at the sampling instants*.

Chapter 3

Generalised sample and hold devices

3.1 Overview

In the previous chapter we have shown that sampled-data models are determined, not only by the underlying continuous-time system, but also by the sampling process itself. The poles of the sampled-data model depend only on the continuous-time poles and the sampling period. However, the zeros also depend on the *artifacts* of the sampling process, namely, how the continuous-time input is generated and how the output samples are obtained. In fact, for deterministic system, different models arise when using a zero- or a first-order hold to generate the continuous-time input (Section 2.2). In a similar fashion, different stochastic sampled-data models are obtained when different filters are used before instantaneous sampling (Section 2.4).

In this chapter we study the effect of the sample and hold devices in a more general setting. In particular, we describe any hold or sampling device by its *impulse response* (Feuer and Goodwin, 1996). In this framework, ZOH and FOH are particular cases of, so called, **generalised hold functions** (GHF). Similarly, instantaneous sampling and averaging (or integrating) sampling can be included as particular cases of **generalised sampling filters** (GSF).

For deterministic systems, it is well-known that, given a continuous-time system and a sampling period Δ, the GHF can be used to shift the zeros of the corresponding sampled-data model (Kabamba, 1987). However, the use of the GHF can give misleading results when essential characteristics of the continuous-time system, such as non-minimum phase (NMP) behaviour, are *artificially removed* (Zhang and Zhang, 1994; Feuer and Goodwin, 1994). The discrete-time sequences in this case may differ significantly from the underlying continuous-time signals, making the sampled-data model not a good description of the continuous-time system. Here we propose a hold design that deals only with the effects of sampling in the discrete-time model. We thus focus in the, so called, *sampling zeros*.

For the stochastic case, a dual result holds. It is well known that the discrete-time description of the system depends on the prefilter used prior to sampling (instantaneously) the system output (Wahlberg, 1988). Indeed, we have shown in Section 2.5 that different sampled-data models, with different asymptotic *sampling zeros*, arise when the output samples are obtained using (or not using) an integrating filter before instantaneous sampling.

Other duality results between sample and hold devices have previously been highlighted. For example, in (Feuer and Goodwin, 1996), an optimal sampled-data control problem, using a ZOH input, is shown to be dual to an optimal state estimation problem, when using an integrating filter on the system output.

In this chapter, we consider the use of generalised holds and sampling filters to deal with sampling zeros **only**. We begin, in Section 3.2, by reviewing the use of GHF in the sampling process of deterministic systems. Then, in Section 3.3, we present a GHF design procedure that places the *sampling zeros* asymptotically at the origin, as the sampling period tends to zero. Section 3.4 presents the GSF and explores its role in the sampling process for stochastic systems. Then, in Section 3.5, we show how a GSF can be designed to assign the sampling zeros of the output spectrum, and, thus, of the corresponding stochastic sampled-data model, to the origin as the sampling frequency increases.

The design procedures presented in this chapter are independent of the particular system, both for the deterministic and stochastic cases, depending only on the system relative degree. Furthermore, the results obtained are asymptotic, as the sampling period goes to zero. In Section 3.6, we explore the robustness of these design procedures to, both, non-zero sampling periods and high frequency errors in the continuous-time system model.

3.2 Generalised hold functions

In this section we study the role of the hold device in obtaining sampled-data models for deterministic systems. Here we consider a more general settling than in Chapter 2, where only zero- and first-order holds were studied.

A linear hold device can be completely characterised by its *impulse response*, $h_g(t)$ (Feuer and Goodwin, 1996). This function is the continuous-time signal generated by the hold device when its (discrete-time) input is a Kronecker delta function:

$$u_k = \delta_K[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad \Rightarrow \qquad u(t) = h_g(t) \tag{3.1}$$

Figure 3.2 schematically represents a GHF and its impulse response. Zero- and first-order holds can be understood as particular cases of GHFs. Indeed, their impulse responses are shown in Figure 2.2 on page 14.

Note that, given an input sequence u_k, the continuous-time signal generated by the hold is given by:

$$u(t) = \sum_{k=-\infty}^{\infty} h_g(t - k\Delta) u_k \tag{3.2}$$

Assumption 3.1 *For the sake of simplicity, we will restrict our analysis to the class of GHFs whose impulse response has support on one sampling interval,* i.e., $h_g(t) = 0$, *for all* $t \notin [0, \Delta)$.

The previous assumption excludes from our analysis, for example, the FOH in Figure 2.2(b). However, the next result shows that the class of GHFs considered here provides enough *freedom* to arbitrarily assign the zeros of the sampled-data model.

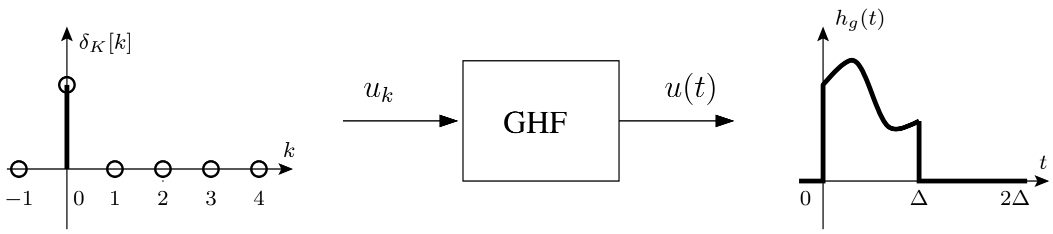


Figure 3.1: Schematic representation of a generalised hold function (GHF).

Lemma 3.2 *Consider the continuous-time state-space model* (2.2)–(2.3)*. If we use a GHF with impulse response $h_g(t)$ to generate the input $u(t)$, then the equivalent discrete-time model is given by:*

$$q\,x_k = x_{k+1} = A_q x_k + B_g\,u_k \tag{3.3}$$

$$y_k = C x_k \tag{3.4}$$

where $A_q = e^{A\Delta}$, and:

$$B_g = \int_0^{\Delta} e^{A(\Delta-\tau)} B h_g(\tau) d\tau \tag{3.5}$$

Proof. The proof follows similar lines as in (Feuer and Goodwin, 1996), for δ models. Specifically, from equation (2.15), we have that:

$$x_{k+1} = e^{A\Delta} x_k + \int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B u(\eta) d\eta \tag{3.6}$$

Assumption 3.1 allows one to simplify the continuous-time input (3.2). Within a single sampling period, we can write:

$$u(t) = h_g(t - k\Delta) u_k \qquad ;\, k\Delta \leq t < k\Delta + \Delta \tag{3.7}$$

Thus, we obtain:

$$\int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B u(\eta) d\eta = \left[\int_{k\Delta}^{k\Delta+\Delta} e^{A(k\Delta+\Delta-\eta)} B h_g(\eta - k\Delta) d\eta \right] u_k \tag{3.8}$$

where, changing variables or simply considering $k = 0$, the last integral is shown to be equal to B_g in (3.5).

□

Note that the previous results coincide with Lemma 2.2 on page 14, when we restrict to the ZOH case, as expected. The impulse response of the ZOH appears in Figure 2.2(a) and is defined by:

$$h_g(t) = h_{ZOH}(t) = \begin{cases} 1 & ;\, t \in [0, \Delta) \\ 0 & ;\, t \notin [0, \Delta) \end{cases} \tag{3.9}$$

Corollary 3.3 *The zeros of the discrete-time system* (3.3)–(3.4) *are given by the solutions of the equation:*

$$C\,\mathrm{adj}(zI_n - A_q) B_g = 0 \tag{3.10}$$

where $\mathrm{adj}(\cdot)$ *denotes the adjoint matrix.*

Proof. This result is a direct consequence of expressing the state-space model (3.3)–(3.4) in transfer function form:

$$G_q(z) = C(zI_n - A_q)^{-1}B_g = \frac{C\,\mathrm{adj}(zI_n - A_q)B_g}{\det(zI_n - A_q)} \tag{3.11}$$

□

Remark 3.4 *In this chapter we are interested in designing a hold function such that the asymptotic sampling zeros are arbitrarily assigned. In (Feuer and Goodwin, 1996) it is shown that (generically) the controllability of the pair (A, B) is enough to arbitrarily assign the roots of* (3.10)*.*

Remark 3.5 *Lemma 3.2 highlights the fact that sampled-data model characteristics depend both on the continuous-time system and the sampling process itself. Indeed, equation* (3.11) *shows that the poles of the system depends only on $A_q = e^{A\Delta}$, but the zeros are functions of B_g and, thus, of the GHF impulse response $h_g(t)$.*

Our focus in this chapter is to propose design methods that allow one to assign the sampled-data model zeros in (3.10), by choosing an appropriate hold device. Towards this goal, we note that a simpler expression can be obtained for the matrix B_g in (3.5) if we consider a GHF defined by the piecewise constant impulse response, shown in Figure 3.2:

$$h_g(t) = f_N(t) = \begin{cases} g_1 & ; 0 \leq t < \frac{\Delta}{N} \\ g_2 & ; \frac{\Delta}{N} \leq t < \frac{2\Delta}{N} \\ \vdots & \\ g_N & ; \frac{(N-1)\Delta}{N} \leq t < \Delta \end{cases} \tag{3.12}$$

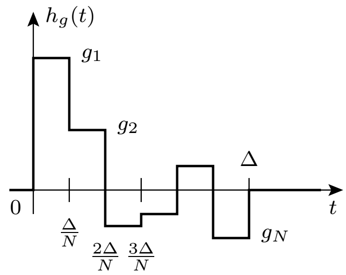


Figure 3.2: Impulse response of a piecewise constant GHF.

Substituting (3.12) in (3.5), we obtain an expression for B_g in terms of the *weights* g_ℓ, $\ell = 1, \ldots, N$. These coefficients will be used later, in Section 3.3, as design parameters to assign the sampling zeros. We note that the matrix B_g is linearly parameterised:

$$B_g = \sum_{\ell=1}^{N} g_\ell \int_{\frac{(\ell-1)\Delta}{N}}^{\frac{\ell\Delta}{N}} e^{A(\Delta-\tau)}B\,d\tau \tag{3.13}$$

Lemma 3.2 provides a state-space sampled-data model corresponding to a continuous-time system, also expressed in state-space form. However, the sampled-data model can also be obtained directly from the continuous-time transfer function $G(s)$. Specifically, the discrete-time transfer function can be obtained by computing the $\mathcal{Z}$-transform of the response of the combined GHF and continuous-time system, when the hold input is a Kronecker delta function. This line of reasoning leads us to:

$$G_q(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\{H_g(s)G(s)\}\big|_{t=k\Delta}\right\} \tag{3.14}$$

where $G(s)$ is the transfer function of the continuous-time system, and $H_g(s)$ is the Laplace transform of the GHF impulse response $h_g(t)$. This expression coincides with the result in Chapter 2, for the ZOH case. In fact, if we replace $H_g(s)$ by the ZOH *transfer function* (2.21), then (3.14) can be rewritten as in expression (2.17).

3.3 Asymptotic sampling zeros for generalised holds

In this section, we investigate the asymptotic sampling zeros that arise in sampled-data models when the continuous-time input is generated by a GHF. For simplicity, we will consider a piecewise constant GHF with impulse response as in Figure 3.2.

We first introduce the following preliminary result:

Lemma 3.6 *Using the polynomials defined in (2.44)–(2.45), and defining $B_0(z) = z^{-1}$, we have:*

$$\sum_{k=1}^{\infty} k^p z^{-k} = \frac{zB_p(z)}{(z-1)^{p+1}} \quad ; \forall p \geq 0 \tag{3.15}$$

Proof. We use induction. We first note that for $p = 0$, the result is straightforward. For $p = 1$, we have that:

$$\sum_{k=1}^{\infty} kz^{-k} = \mathcal{Z}\{k\} = -z\frac{d}{dz}\mathcal{Z}\{1\} = \frac{zB_1(z)}{(z-1)^2} \tag{3.16}$$

Assuming that (3.15) holds for p, we next prove that it also holds for $p+1$. We see that:

$$\sum_{k=1}^{\infty} k^{p+1} z^{-k} = \mathcal{Z}\left\{k^{p+1}\right\} = -z\frac{d}{dz}\mathcal{Z}\{k^p\} = \frac{z\left[z(1-z){B_p}'(z) + (pz+1)B_p(z)\right]}{(z-1)^{p+2}} \tag{3.17}$$

The result then follows from the recursion (2.48) satisfied by the polynomials $B_p(z)$:

$$z(1-z){B_p}'(z) + (pz+1)B_p(z) = B_{p+1}(z) \tag{3.18}$$

for all $p \geq 0$.

□

Using the previous result, we next extend Lemma 2.8 to the GHF case. In particular, we characterise the sampling zeros of the sampled-data model of an n-order integrator, when the input is generated by a piecewise constant GHF.

Lemma 3.7 *Consider the n-th order integrator $G(s) = s^{-n}$. If the continuous-time input $u(t)$ is generated by a piecewise constant GHF, defined as in* (3.12)*, with n different subintervals, then the corresponding discrete-time transfer function is given by:*

$$G_q(z) = \frac{\Delta^n}{n!(z-1)^n}\sum_{p=0}^{n-1} zB_p(z)(z-1)^{n-p-1}C_{n,p} \tag{3.19}$$

where the polynomials $B_p(z)$ are defined in (2.44)–(2.45), and:

$$C_{n,p} = \binom{n}{p}\left(\frac{-1}{n}\right)^{n-p}\sum_{\ell=1}^{n} g_\ell\left[(\ell-1)^{n-p} - \ell^{n-p}\right] \tag{3.20}$$

Proof. The sampled-data model will be obtained from equation (3.14). We first need to obtain the Laplace transform of the impulse response of the GHF (3.12). The latter is a piecewise constant function defined in n subintervals. Thus, we obtain:

$$f_n(t) = \sum_{\ell=1}^{n} g_\ell\left[\mu\left(t - \tfrac{(\ell-1)\Delta}{n}\right) - \mu\big(t - \tfrac{\ell\Delta}{n}\big)\right] \tag{3.21}$$

$$F_n(s) = \sum_{\ell=1}^{n} g_\ell F_{n\ell}(s) \tag{3.22}$$

where $\mu(\cdot)$ is the unitary step function, and:

$$F_{n\ell}(s) = \frac{1}{s}\left(e^{-s\frac{(\ell-1)\Delta}{n}} - e^{-s\frac{\ell\Delta}{n}}\right) \quad ; \ell = 1,\dots,n \tag{3.23}$$

We are interested in the impulse response of the combined continuous-time model:

$$G(s)F_n(s) = \sum_{\ell=1}^{n} g_\ell H_\ell(s) \tag{3.24}$$

The inverse Laplace transform of each element in the sum can readily be computed as:

$$H_\ell(s) = G(s)F_{n\ell}(s) = \frac{e^{-s\frac{(\ell-1)\Delta}{n}} - e^{-s\frac{\ell\Delta}{n}}}{s^{n+1}} \tag{3.25}$$

$$h_\ell(t) = \frac{(t-\frac{\ell-1}{n}\Delta)^n}{n!}\mu\left(t - \frac{(\ell-1)\Delta}{n}\right) - \frac{(t-\frac{\ell}{n}\Delta)^n}{n!}\mu\left(t - \frac{\ell\Delta}{n}\right) \tag{3.26}$$

We will consider this signal at the sampling instants $h_\ell[k] = h_\ell(k\Delta)$. Note that $h_\ell[0] = 0$ and, for $k \geq 1$, we can use the *binomial theorem* to obtain:

$$h_\ell[k] = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1} k^p \binom{n}{p}\left(\frac{-1}{n}\right)^{n-p}\left[(\ell-1)^{n-p} - \ell^{n-p}\right] \tag{3.27}$$

The $\mathcal{Z}$-transform of this signal is then given by:

$$H_\ell(z) = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1}\left[\binom{n}{p}\left(\frac{-1}{n}\right)^{n-p}\left[(\ell-1)^{n-p} - \ell^{n-p}\right]\sum_{k=1}^{\infty} k^p z^{-k}\right] \tag{3.28}$$

Hence, applying the result in Lemma 3.6, we have:

$$H_\ell(z) = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1}\left[\binom{n}{p}\left(\frac{-1}{n}\right)^{n-p}\left[(\ell-1)^{n-p} - \ell^{n-p}\right]\frac{z\,B_p(z)}{(z-1)^{p+1}}\right] \tag{3.29}$$

Finally, the result is obtained by substituting (3.29) into the linear combination obtained from (3.24):

$$G_q(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{G(s)F_n(s)\right\}_{t=k\Delta}\right\} = \mathcal{Z}\left\{\sum_{\ell=1}^{n} g_\ell h_l[k]\right\} = \sum_{\ell=1}^{n} g_\ell H_\ell(z) \tag{3.30}$$

.

□

Remark 3.8 *Note that (3.19) establishes that the sampled-data model of an n-th order integrator has its n poles at $z = e^{0\cdot\Delta} = 1$ (as expected). The discrete-time model also has $n-1$* sampling zeros. *In fact, the numerator of the corresponding sampled-data model can be rewritten as a polynomial of order $n-1$,* i.e.,

$$G_q(z) = \frac{\Delta^n}{n!(z-1)^n} \sum_{p=0}^{n-1} \alpha_p z^p \tag{3.31}$$

We next consider a more general system. We extend Theorem 2.18 on page 24 to the case when a piecewise constant GHF is used to generate its input.

Theorem 3.9 *Let $G(s)$ be a rational function as in (2.84), with relative degree $r = n - m$. Let $G_q(z)$ be the corresponding sampled transfer function obtained using a piecewise constant GHF with r stages.*

Assume $m < n$ (or, equivalently, $r > 0$). Then, as the sampling period Δ goes to 0, m zeros of $G_q(z)$ go to 1 as $e^{z_i\Delta}$, and the remaining $r-1$ zeros of $G_q(z)$ (the sampling zeros*) go to the roots of the polynomial:*

$$\sum_{p=0}^{r-1} zB_p(z)(z-1)^{r-p-1}C_{r,p} = \sum_{p=0}^{r-1} \alpha_p z^p \tag{3.32}$$

Proof. The proof of this result is similar to the proof of Theorem 2.18 that can be found in (Åström *et al.*, 1984). First we obtain the Laplace transform of the piecewise constant GHF with r subintervals. Proceeding as in the proof of Lemma 3.7, we have that:

$$H_g(s) = \frac{1}{s}\sum_{\ell=1}^{r} g_\ell \left(e^{-s\frac{(\ell-1)\Delta}{r}} - e^{-s\frac{\ell\Delta}{r}} \right) \tag{3.33}$$

Using the definition of the Laplace and $\mathcal{Z}$-transform (and their inverse transforms, respectively), equation (3.14) can be rewritten as:

$$\begin{aligned} G_q(z) &= \sum_{k=0}^{\infty} \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} G(s)H_g(s)e^{sk\Delta}ds\, z^{-k} = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} G(s)H_g(s)\left(\sum_{k=1}^{\infty} e^{sk\Delta}z^{-k}\right) ds \\ &= \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} G(s)H_g(s)\frac{e^{s\Delta}}{z-e^{s\Delta}}\, ds \end{aligned} \tag{3.34}$$

where γ is such that $G(s)/s$ has all its poles to the left of $\Re\{s\} = \gamma$. If we substitute the system transfer function (2.84) and the GHF (3.33), then the following expression is obtained by changing variables in the integral, using $w = s\Delta$:

$$\lim_{\Delta\to 0} \Delta^{-r} G_q(z) = \frac{K}{2\pi j}\int_{\gamma\Delta-j\infty}^{\gamma\Delta+j\infty} \frac{e^w \sum_{\ell=1}^{r} g_\ell \left(e^{-\frac{(\ell-1)w}{r}} - e^{-\frac{\ell w}{r}} \right)}{w^{r+1}(z-e^w)} dw \tag{3.35}$$

It is readily shown that this expression corresponds to replacing $G(s)$ by an r-th order integrator in (3.14). Thus, we finally obtain:

$$\lim_{\Delta\to 0} \frac{1}{\Delta^r} G_q(z) = \frac{K\sum_{p=0}^{r-1} zB_p(z)(z-1)^{r-p-1}C_{r,p}}{r!(z-1)^r} = \frac{K(z-1)^m}{r!(z-1)^n}\sum_{p=0}^{r-1}\alpha_p z^p \tag{3.36}$$

□

Based on the previous results, we next present a procedure to design a GHF such that the *sampling zeros* of the discrete-time model are asymptotically assigned to the origin, as the sampling period Δ goes to 0.

Theorem 3.10 *The coefficients g_ℓ, $\ell = 1, \ldots, r$ of the GHF in (3.33) can be chosen in such a way that the sampling zeros of the discrete-time model (3.32) converge asymptotically to $z = 0$.*

Proof. To assign the sampling zeros to the origin, it follows from (3.32) that the following condition must hold:

$$\alpha_p = 0 \quad , \forall p = 0, \ldots, r-2 \tag{3.37}$$

This is equivalent to having $r-1$ linear equations in the coefficients $C_{r,p}$, and thus for the weights g_ℓ. Moreover, the GHF must satisfy an extra condition to ensure unitary gain at zero frequency, *i.e.*,

$$\frac{1}{r}\sum_{\ell=1}^{r} g_\ell = 1 \tag{3.38}$$

Equations (3.37) and (3.38) define r conditions on the coefficients g_ℓ, $\ell = 1, \ldots, r$, which are (generically) linearly independent provided (A, B) is controllable (see Remark 3.4).

□

Remark 3.11 *A key observation is that the GHF obtained by solving (3.37)–(3.38) does not depend on the particular continuous-time system. Theorem 3.9 ensures that the sampling zeros, and, thus, the GHF design procedure, depend only on the system relative degree (see also Remark 2.11 on page 20).*

Remark 3.12 *In Theorem 3.10 we have chosen to assign the asymptotic sampling zeros to the origin. This implies that, by a continuity argument, there exists a $\Delta_\varepsilon > 0$ such that, for every sampling period $\Delta < \Delta_\varepsilon$, all the sampling zeros are stable,* i.e.*, they lie* inside *the unit circle in the complex plane z. Indeed, for Δ_ε small enough, all the sampling zeros will be inside a circle of radius $r_\varepsilon \ll 1$.*

Theorem 3.10 assigns the asymptotic sampling zeros to the origin to ensure that the sampled-data model is minimum phase. However, a different set of conditions can be imposed on the weighting coefficients if one wants to assign the sampling zeros to any other location in the complex plane.

The following example illustrates the GHF design procedure described above for a particular system.

Example 3.13 *Consider the third order system:*

$$G(s) = \frac{1}{(s+1)^3} \tag{3.39}$$

By Theorem 2.18, if we use a ZOH to generate the input, then, as the sampling period Δ tends to zero, the associated sampled-data transfer function is given by:

$$G_q(z) \xrightarrow[(ZOH)]{\Delta\approx 0} \frac{\Delta^3(z+3.732)(z+0.268)}{(z-1)^3} \tag{3.40}$$

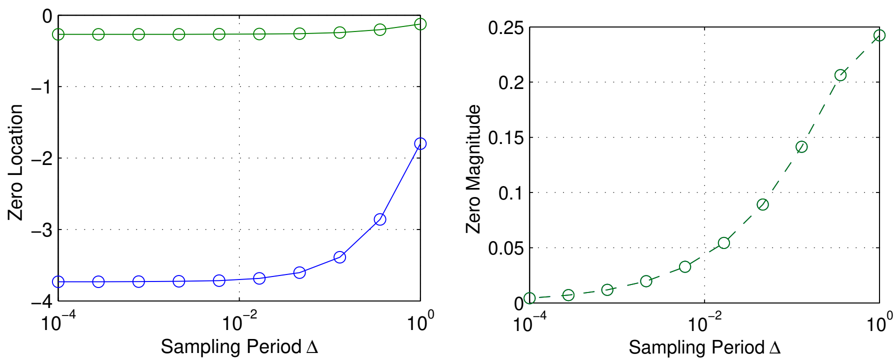
Note that the resulting discrete-time model has a non-minimum phase (NMP) zero, even though the continuous-time system has no finite zeros.

On the other hand, using (3.32), (3.37), and (3.38), we obtain a GHF given by the impulse response:

$$h_g(t) = \begin{cases} 29/2 & ;\ 0 \le t < \frac{\Delta}{3} \\ -17 & ;\ \frac{\Delta}{3} \le t < \frac{2\Delta}{3} \\ 11/2 & ;\ \frac{2\Delta}{3} \le t < \Delta \end{cases} \tag{3.41}$$

Note that this assigns the limiting sampling zeros asymptotically to the origin, i.e., *the combined hold and plant discrete-time model is, as Δ goes to 0:*

$$G_q(z) \xrightarrow[(GHF)]{\Delta\approx 0} \frac{\Delta^3 z^2}{(z-1)^3} \tag{3.42}$$



(a) Zero Order Hold (b) Generalised Hold

Figure 3.3: Sampling zeros versus sampling period in Example 3.13.

Figure 3.3(a) shows the sampling zeros as a function of Δ for the ZOH-case. Figure 3.3(b) shows the magnitude of the (complex) sampling zeros obtained when the GHF (3.41) *is used. We see that the zeros are very close to their asymptotic values if the usual* rule of thumb *is employed,* i.e., *if the sampling frequency is chosen one decade above the fastest system pole (Åström and Wittenmark, 1997).*

Furthermore, Figure 3.4 shows the zero and pole locations for the sampled version of the system using the fixed GHF (3.41), *for sampling periods from* 1 *to* 10^{-4}. *Note that all of the resulting discrete-time models are minimum phase.*

□

The proposed sampling strategy avoids the presence of unstable sampling zeros due to the discretisation process. It therefore gives a better correspondence between continuous- and discrete-time models. The sampling strategy has several potential applications. For example, it allows one to straightforwardly apply discrete-time control methods, such as model reference adaptive control (Goodwin and Sin, 1984; Åström, 1995) or Internal Model Control (Goodwin *et al.*, 2001), where the presence of NMP zeros would impose additional performance limitations. In Liang *et al.* (2003), the same goal is achieved by using fractional-order holds in a multivariable context.

The following example illustrates a possible application of the proposed GHF sampling strategy.

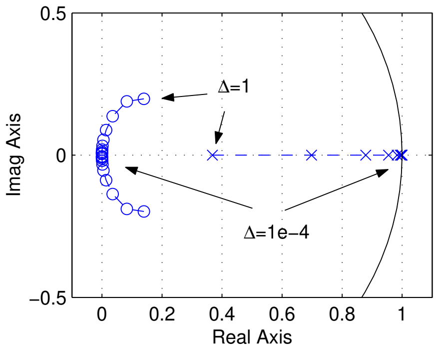


Figure 3.4: Zero ('o') and pole ('x') locations for different sampling periods Δ for GHF in Example 3.13

Example 3.14 *In this example we compare control loop performance for the system in Example 3.13, using a ZOH and the GHF in* (3.41)*. We use an internal model controller based on the Youla parameterisation (Goodwin* et al.*, 2001). This control strategy is schematically represented in Figure 3.5. The parameter $Q(z)$ is designed as an* approximate inverse *of the plant model. To ensure stability the parameter $Q(z)$ is restricted to be stable. Thus, one needs to be careful avoiding inverting NMP zeros of plant.*

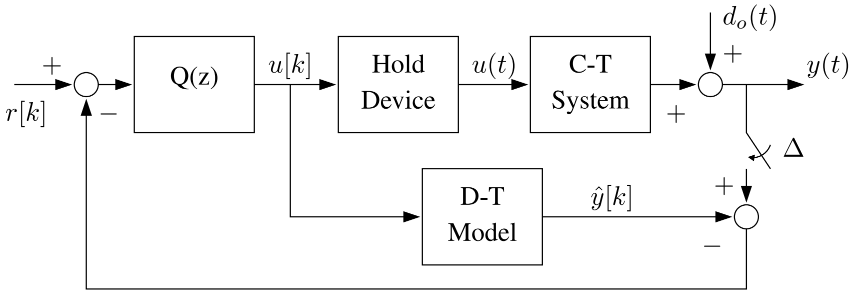


Figure 3.5: Control loop using Youla parameterisation (Example 3.14).

We consider a sampling period $\Delta = 0.1$[s]. From (3.5), the discrete-time poles are $e^{p_i \Delta} = e^{-0.1} = 0.9048$. The (sampled) zeros depend on the hold used to generate the continuous-time input. Equations (3.40) and (3.42) define the zeros for ZOH and GHF cases, respectively. Based on this, we compare the performance of two control loops, one using ZOH and the controller:

$$Q_{ZOH}(z) = \frac{(1+0.268)(z-0.9048)^3}{(1-0.9048)^3(z+0.268)z^2} \tag{3.43}$$

and the other using the GHF defined by (3.41) and the controller:

$$Q_{GHF}(z) = \frac{(z-0.9048)^3}{(1-0.9048)^3 z^3} \tag{3.44}$$

Figure 3.6 shows the control signal $u(t)$ and system output $y(t)$ obtained for a unitary step output disturbance, $d_o(t) = -\mu(t)$. The magnitude of the control signal $u(t)$ is large because the controller we have chosen tries to achieve near perfect output disturbance rejection. We can see that, even though

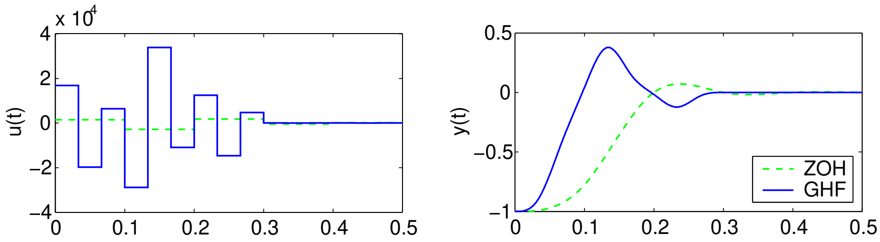


Figure 3.6: Simulation results for Example 3.14.

the response using the GHF gives overshoot, the settling time is smaller than for the ZOH case. Indeed, the discrete nominal *transfer function of the control loop for the GHF case is simply z^{-1}, whereas for the ZOH case it is $0.211z^{-1} + 0.789z^{-2}$. These discrete responses can be seen in Figure 3.6 at the sampling instants. If we consider the inter-sample response, this is also improved by the use of the GHF. The integral of the output $y(t)$ squared is reduced from 0.1088 to 0.0557, when the ZOH and $Q_{ZOH}(z)$ are replaced by the GHF (3.41) and $Q_{GHF}(z)$.*

□

3.4 Generalised sampling filters

In the previous section we have seen how the input hold device can be designed to assign the asymptotic sampling zeros of a deterministic system. A dual result holds for stochastic systems, namely, the zeros of the sampled output spectrum (and, thus, of the corresponding sampled-data model) can be assigned by choosing the generalised anti-aliasing filter used prior to instantaneous sampling.

In this section we analyse the effect of the anti-aliasing filter on the resultant stochastic sampled-data model. Later, in Section 3.5, we present a filter design procedure, such that the *sampling zeros* of the sampled output spectrum are asymptotically assigned to the origin.

We assume a sampling scheme as shown in Figure 3.7, where a generalised anti-aliasing filter is used prior to sampling the system output. This filter is chosen as a generalised sampling filter (GSF), defined by its impulse response, $h_g(t)$. Similar to Assumption 3.1 for the GHF case in Section 3.2, we will consider the class of sampling functions that have support on the interval $[0, \Delta)$. The output sequence is obtained by sampling instantaneously the output of the filter:

$$\bar{y}_k = \bar{y}(k\Delta) = \int_{k\Delta-\Delta}^{k\Delta} y(\tau) h_g(k\Delta - \tau) d\tau \tag{3.45}$$

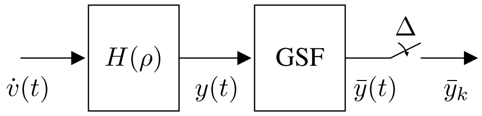


Figure 3.7: Sampling scheme using a Generalised Filter.

Remark 3.15 *The definition of a GSF as in* (3.45) *can be understood as a generalisation of the, so called,* integrating filter (2.151). *This is also called* averaging filter*, and its impulse response is given by*

(Feuer and Goodwin, 1996):

$$h_g(t) = \begin{cases} 1/\Delta & ; 0 \leq t < \Delta \\ 0 & ; \Delta > 0 \end{cases} \tag{3.46}$$

Note also that instantaneous sampling can be included in this framework by considering $h_g(t) = \delta(t)$ *in* (3.45).

The following result allows one to obtain a discrete-time description of the sampling scheme in Figure 3.7.

Lemma 3.16 *Consider the sampling scheme in Figure 3.7, where the continuous-time system* $H(\rho)$ *can be expressed in state-space form as in* (2.109)–(2.110)*, and the GSF has an impulse response* $h_g(t)$*. Then the corresponding discrete-time model is given by:*

$$\delta x_k = A_\delta x_k + v_k \tag{3.47}$$

$$\bar{y}_{k+1} = C_g x_k + w_k \tag{3.48}$$

where $\delta = \frac{q-1}{\Delta}$ *denotes the delta operator. The matrices in* (3.47)–(3.48) *are given by:*

$$A_\delta = \frac{e^{A\Delta} - I}{\Delta} \qquad\qquad C_g = \int_0^\Delta h_g(\tau) C e^{A(\Delta-\tau)} d\tau \tag{3.49}$$

and v_k *and* w_k *are white noise sequences such that:*

$$E\left\{ \begin{bmatrix} v_k \\ w_k \end{bmatrix} \begin{bmatrix} v_\ell \\ w_\ell \end{bmatrix}^T \right\} = \begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix} \frac{\delta_K[k-\ell]}{\Delta} \tag{3.50}$$

where δ_K *represents the Kronecker delta function, and where:*

$$\begin{bmatrix} \Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta \end{bmatrix} \triangleq \frac{\sigma_v^2}{\Delta} \int_0^\Delta M_g(\sigma) M_g(\sigma)^T d\sigma \tag{3.51}$$

$$M_g(\sigma) \triangleq \begin{bmatrix} e^{A\sigma} B \\ \Delta \int_0^\sigma h_g(\xi) C e^{A(\sigma-\xi)} B d\xi \end{bmatrix} \tag{3.52}$$

Proof. The proof follows the same lines as Lemma 2.33 in (Feuer and Goodwin, 1996). We first note that (3.47) can readily be obtained as for equation (2.111) in Lemma 2.25. Furthermore, the noise input v_k is given by (2.116). Equation (3.48) can be obtained on noting that, on the interval $[k\Delta, k\Delta + \Delta)$, the system output can be expressed as:

$$y(t) = C e^{A(t-k\Delta)} x(k\Delta) + C \int_{k\Delta}^t e^{A(t-\eta)} B \dot{v}(\eta) d\eta \tag{3.53}$$

Thus, the samples of the filter output can be written as:

$$\begin{aligned} \bar{y}_{k+1} &= \int_{k\Delta}^{k\Delta+\Delta} h_g(k\Delta + \Delta - \tau) y(\tau) d\tau \\ &= \left[\int_{k\Delta}^{k\Delta+\Delta} h_g(k\Delta + \Delta - \tau) C e^{A(\tau - k\Delta)} d\tau \right] x_k + w_k \end{aligned} \tag{3.54}$$

Changing variables in the integral inside the brackets, we obtain C_g as in (3.49). The noise term w_k is given by:

$$w_k = \int_{k\Delta}^{k\Delta+\Delta} h_g(k\Delta + \Delta - \tau) C \int_{k\Delta}^{\tau} e^{A(\tau-\eta)} B \dot{v}(\eta)\, d\eta\, d\tau \tag{3.55}$$

We can use Fubini's theorem (Apostol, 1974) to change the order of integration in the last integral, and then, using the expression for v_k in (2.116), we obtain:

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \frac{1}{\Delta} \int_{k\Delta}^{k\Delta+\Delta} \begin{bmatrix} e^{A(k\Delta+\Delta-\eta)} B \\ \Delta \int_{\eta}^{k\Delta+\Delta} h_g(k\Delta+\Delta-\tau) C e^{A(\tau-\eta)} B\, d\tau \end{bmatrix} \dot{v}(\eta)\, d\eta \tag{3.56}$$

Equations (3.50)–(3.52) are obtained from this last expression by proceeding as in the proof of Lemma 2.25. Indeed, the matrix Ω_δ in (3.51) is the same as in equation (2.114).

□

Remark 3.17 *The discrete-time model* (3.47)–(3.48) *can equivalently be rewritten, using the shift operator q, as:*

$$q\, x_k = x_{k+1} = A_q x_k + \tilde{v}_k \tag{3.57}$$

$$\bar{y}_{k+1} = C_g x_k + w_k \tag{3.58}$$

where $A_q = I + \Delta A_\delta = e^{A\Delta}$ and the input noise sequence is $\tilde{v}_k = \Delta v_k$. As a consequence:

$$E\left\{ \begin{bmatrix} \tilde{v}_k \\ w_k \end{bmatrix} \begin{bmatrix} \tilde{v}_\ell \\ w_\ell \end{bmatrix}^T \right\} = \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} \delta_K[k-\ell] \tag{3.59}$$

where:

$$\Omega_q = \Delta \Omega_\delta \quad ; \quad \Sigma_q = \Sigma_\delta \quad ; \quad \Gamma_q = \frac{1}{\Delta} \Gamma_\delta \tag{3.60}$$

□

Even though Lemma 3.16 provides a sampled-data model for the system in Figure 3.7, this discrete-time description depends on two noise sequences as inputs. The following lemma allows us to express $\bar{y}_k$ as the output of a system with a single white noise input, *i.e.*, the discrete-time model is expressed in *innovations form* (Anderson and Moore, 1979).

Lemma 3.18 *The state-space model* (3.57)–(3.58) *is equivalent to the following **innovations model** in the sense that their outputs share the same second order properties:*

$$z_{k+1} = A_q z_k + K_q e_k \tag{3.61}$$

$$\bar{y}_{k+1} = C_g z_k + e_k \tag{3.62}$$

where e_k is a white noise sequence with covariance matrix

$$E\{e_k^2\} = \Gamma_q + C_g P C_g^T \tag{3.63}$$

The Kalman gain K_q is given by:

$$K_q = (A_q P C_g^T + \Sigma_q)(\Gamma_q + C_g P C_g^T)^{-1} \tag{3.64}$$

and P is the state covariance matrix given by the discrete-time algebraic Riccati equation:

$$A_q P A_q^T - P - K_q(\Gamma_q + C_g P C_g^T)K_q^T + \Omega_q = 0 \tag{3.65}$$

□

Using the innovations form in Lemma 3.18, we can describe the sequence of output samples $\bar{y}_k$ by the model:

$$\bar{y}_{k+1} = H_q(q)e_k \tag{3.66}$$

where:

$$H_q(z) = C_g(zI_n - A_q)^{-1}K_q + 1 \tag{3.67}$$

and where e_k is a DTWN with variance (3.63).

Remark 3.19 *Equation* (3.67) *clearly shows that the discrete-time poles depend only on A_q and, hence, only on the continuous-time system matrix A and the sampling period Δ. However, the zeros of the model are seen to depend on C_q and K_q, and, thus, on the GSF impulse response $h_g(t)$.*

Given a GSF, Lemmas 3.16 and 3.18 provide a systematic way of obtaining a sampled-data model for the system in Figure 3.7. In particular, the zeros of $H_q(z)$ in (3.67) depend on the choice of the GSF. Indeed, the filter impulse response $h_g(t)$ determines the matrices C_g and K_q. However, to obtain K_q we need to solve the algebraic Riccati equation (3.65). Thus, the way that $h_g(t)$ appears in this matrix equation makes this approach difficult for design purposes. Hence, we explore a more direct method below based on spectral factorisation ideas. In fact, we see that the role of the impulse response $h_g(t)$ in the output spectrum can be described more easily if one avoids the form described in Lemma 3.18.

Lemma 3.20 *Given the discrete-time model* (3.57)–(3.59)*, the discrete-time output spectrum is given by:*

$$\Phi_{\bar{y}}^q(z) = \Delta \begin{bmatrix} C_g(zI_n - A_q)^{-1} & 1 \end{bmatrix} \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} \begin{bmatrix} (z^{-1}I_n - A_q^T)^{-1}C_g^T \\ 1 \end{bmatrix} \tag{3.68}$$

Proof. This result follows from the model (3.57)–(3.58). The output spectrum of this model can be obtained in the same way as in the proof of Lemma 2.35, for the integrating filter case.

□

Remark 3.21 *The result in Lemma 3.20 is closely related to Lemma 3.18, noting that the output spectrum of the innovations model* (3.61)–(3.62) *is given by:*

$$\Phi_{\bar{y}}^q(z) = H_q(z)H_q(z^{-1})\Phi_e^d \tag{3.69}$$

where the spectral factor $H_q(z)$ is given by (3.67) *and Φ_e^d is the (constant) spectral density of the innovations sequence.*

The previous remark shows that one can directly obtain a stochastic sampled-data model, with a scalar noise source as input, by spectral factorisation of the spectrum (3.68). In the next section, we follow this approach to assign the stochastic sampling zeros of $\Phi_{\bar{y}}^d(z)$ (and, thus, of the spectral factor $H_q(z)$) by choosing an appropriate GSF. Specifically, the function $h_g(t)$ will be expressed as a linear combination of more elemental functions, in such a way as to simplify the expressions for C_g, Σ_q, and Γ_q, in Lemma 3.16.

3.5 Generalised filters to assign the asymptotic sampling zeros

In this section we turn to the problem of designing a GSF such that the sampling zeros of the discrete-time spectrum (3.68) converge, as the sampling period goes to zero, to specific locations in the complex plane. In particular, we are interested in assigning them to the origin, or, equivalently, to obtain an output spectrum with no stochastic sampling zeros.

The choice of the GSF to assign the sampling zeros is not unique. Thus, we restrict ourselves, to a class of filters whose impulse response satisfies the following restriction.

Assumption 3.22 *Given a system of relative degree r, we consider a GSF such that its impulse response can be parametrised as:*

$$h_g(t) = \begin{cases} \frac{1}{\Delta}\big(h_0 + \sum_{\ell=1}^{r} h_\ell \, \phi_\ell(t)\big) & ; t \in [0, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{3.70}$$

where the weighting coefficients $h_0, \ldots, h_r \in \mathbb{R}$. The basis functions $\phi_\ell(t)$ in (3.70) *(to be specified later) are required to satisfy the following condition:*

$$\int_0^\Delta \phi_\ell(t) dt = 0 \tag{3.71}$$

□

Note that we have introduced the scaling factor $1/\Delta$ in (3.70) to resemble the *averaging* idea of the integrating filter (2.151). In fact, the averaging filter corresponds to the choice $h_0 = 1$ and $h_\ell = 0$, for $\ell = 1 \ldots r$. We see next that the condition (3.71) simplifies some of the calculations required to obtain the output spectrum (3.68).

Remark 3.23 *Note that Assumption 3.22 guarantees that, once the functions $\phi_\ell(t)$ in* (3.70) *have been chosen, then the $r+1$ coefficients $h_0, \ldots, h_r$ provide enough* degrees of freedom *to assign the r sampling zeros and the noise variance, if required.*

The design procedure presented in this section is based on the key limiting argument discussed earlier in Remark 2.11, namely, for fast sampling rates, any system of relative degree r *behaves* at high frequencies as if it were an r-order integrator. This interpretation has proven to be the key in contemporary results regarding asymptotic behaviour of sampling zeros (Åström *et al.*, 1984; Wahlberg, 1988).

We will use this idea in the following subsections. We first consider the case of first and second order integrators and show how different GSF's can be designed to assign the corresponding asymptotic sampling zeros. We then show that similar asymptotic sampling zeros are obtained when using the resultant GSF on a more general system, having relative degree 1 and 2, respectively.

The examples that follow are aimed at illustrating the general principle, *i.e.*, for a stochastic system of relative degree r, a GSF can be designed based on the r-th order integrator case (see also Remark 2.11 on page 20).

3.5.1 First order integrator

We begin by considering the first order integrator $H(\rho) = \rho^{-1}$. The matrices of the corresponding state space representation (2.109)–(2.110) are, in this case, the scalars $A = 0$, and $B = C = 1$. In the sampled-data model (3.57), this implies that $A_q = 1$ for *any* sampling period Δ.

Example 3.24 (Integrating filter) *This is one of the filters considered in (Wahlberg, 1988), where asymptotic results are obtained for fast sampling rates. The impulse response, in this case, is defined in* (3.46)*. For this choice we have:*

$$C_g = 1 \qquad \text{and} \qquad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2} \\ \frac{\Delta}{2} & \frac{\Delta}{3} \end{bmatrix} \tag{3.72}$$

Substituting into (3.68)*, yields the asymptotic result in Theorem 2.38, namely:*

$$\Phi_{\bar{y}}^{q}(z) = \frac{\Delta^2}{3!} \frac{(z + 4 + z^{-1})}{(z-1)(z^{-1}-1)} \tag{3.73}$$

A sampled-data model can be readily obtained by spectral factorisation, as in (3.69)*:*

$$H_q(z) = \frac{\Delta}{3-\sqrt{3}} \frac{(z + 2 - \sqrt{3})}{(z-1)} \tag{3.74}$$

□

Example 3.25 (Piecewise constant GSF) *This GSF has the same kind of impulse response as the generalised hold functions considered in Section 3.3. Here, however, we parameterise $h_g(t)$ in a slightly different way:*

$$h_g(t) = \begin{cases} \frac{1}{\Delta}(h_0 + h_1) & ; 0 \leq t < \frac{\Delta}{2} \\ \frac{1}{\Delta}(h_0 - h_1) & ; \frac{\Delta}{2} \leq t < \Delta \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{3.75}$$

where $h_0, h_1 \in \mathbb{R}$. For this GSF choice, we obtain:

$$C_g = h_0 \qquad \text{and} \qquad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2}\left(h_0 + \frac{1}{4}h_1\right) \\ \frac{\Delta}{2}\left(h_0 + \frac{1}{4}h_1\right) & \frac{\Delta}{3}\left(h_0^2 + \frac{3}{4}h_0 h_1 + \frac{1}{4}h_1^2\right) \end{bmatrix} \tag{3.76}$$

which, on substituting into (3.68)*, gives:*

$$\Phi_{\bar{y}}^{q}(z) = h_0^2 \frac{\Delta^2}{3!} \frac{\left(z + 4 + z^{-1}\right)}{(z-1)(z^{-1}-1)} + \frac{h_1^2}{2} \frac{\Delta^2}{3!} \frac{\left(-z + 2 - z^{-1}\right)}{(z-1)(z^{-1}-1)} \tag{3.77}$$

*If we now choose, for example, $h_0 = 1$ and $h_1 = \sqrt{2}$, we obtain a sampled spectrum with **no** zeros, or, equivalently, a stable spectral factor with zeros at the origin:*

$$\Phi_{\bar{y}}^{q}(z) = \frac{\Delta^2}{(z-1)(z^{-1}-1)} \quad \Rightarrow \quad H_q(z) = \frac{\Delta z}{(z-1)} \tag{3.78}$$

□

Example 3.26 (Sinusoidal GSF) *Another simple GSF impulse response which satisfies Assumption 3.22 is given by:*

$$h_g(t) = \begin{cases} \frac{1}{\Delta}\left(h_0 + h_1 \pi \sin\left(\frac{2\pi}{\Delta}t\right)\right) & ; 0 \leq t < \Delta \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{3.79}$$

where $h_0, h_1 \in \mathbb{R}$, and the constant π is introduced as a scaling factor. For this choice we have:

$$C_g = h_0 \qquad \textit{and} \qquad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2}(h_0 + h_1) \\ \frac{\Delta}{2}(h_0 + h_1) & \frac{\Delta}{3}\left(h_0^2 + \frac{3}{2}h_0 h_1 + \frac{9}{8}h_1^2\right) \end{bmatrix} \tag{3.80}$$

Upon substituting into (3.68)*, this gives:*

$$\Phi_{\bar{y}}^q(z) = h_0^2 \frac{\Delta^2}{3!} \frac{\left(z + 4 + z^{-1}\right)}{(z-1)(z^{-1}-1)} - \frac{9h_1^2}{4} \frac{\Delta^2}{3!} \frac{\left(z - 2 + z^{-1}\right)}{(z-1)(z^{-1}-1)} \tag{3.81}$$

If we now choose, for example, $h_0 = 1$ and $h_1 = 2/3$, we again obtain a sampled spectrum (and a stable spectral factor) as in (3.78)*. As required, this discrete-time model has sampling zeros at the origin.*

□

The GSF's obtained in Examples 3.25 and 3.26 were designed to assign the stochastic sampling zeros of a first order integrator to the origin. However, this GSF can also be used, for fast sampling rates, on *any* system of relative degree 1 to obtain sampling zeros *near* the origin. We illustrate this principle by the following example.

Example 3.27 *Consider the continuous-time system:*

$$H(\rho) = \frac{1}{\rho + 2} \tag{3.82}$$

We fix the sampling period to be $\Delta = 0.1$, which corresponds to a sampling frequency around one decade above the model bandwidth. If we use the piecewise GSF obtained in Example 3.25, we obtain the following stable spectral factor of the output spectrum:

$$H_q(z) = \frac{0.287(z - 2.489 \cdot 10^{-4})}{(z - e^{-0.2})} \tag{3.83}$$

Similarly, if we use the sinusoidal *GSF described in Example 3.26, we obtain:*

$$H_q(z) = \frac{0.287(z - 6.590 \cdot 10^{-5})}{(z - e^{-0.2})} \tag{3.84}$$

Note that, as expected, for both cases the sampling zero is very close to the origin.

□

3.5.2 Second order integrator

We next consider the GSF design problem for the second order integrator. This is a prelude to dealing with general systems of relative degree 2. The expressions that allow one to obtain the sampled-data model and, thus, to identify the stochastic sampling zeros, are more involved in this case than for the first order integrator. However, the design procedure previously outlined can be readily adapted as we show below.

Thus, consider the second order integrator $H(\rho) = 1/\rho^2$. The state space representation (2.109)–(2.110) is then given by the matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.85}$$

For any sampling period Δ, the discrete-time system matrix is given by:

$$A_q = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \tag{3.86}$$

Example 3.28 (Integrating filter) *This filter is defined in* (3.46)*. In this case we have that:*

$$C_g = \begin{bmatrix} 1 & \frac{\Delta}{2} \end{bmatrix} \qquad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \left[\begin{array}{cc|c} \frac{\Delta^3}{3} & \frac{\Delta^2}{2} & \frac{\Delta^3}{8} \\ \frac{\Delta^2}{2} & \Delta & \frac{\Delta^2}{6} \\ \hline \frac{\Delta^3}{8} & \frac{\Delta^2}{6} & \frac{\Delta^3}{20} \end{array}\right] \tag{3.87}$$

which, upon substituting into (3.68)*, gives:*

$$\Phi_{\bar{y}}^q(z) = \Phi_{\bar{y}}^{IF}(z) = \frac{\Delta^4}{5!} \frac{(z^2 + 26z + 66 + 26z^{-1} + z^{-2})}{(z-1)^2(z^{-1}-1)^2} \tag{3.88}$$

The obtained spectrum is, again, consistent with the asymptotic result in Theorem 2.38 on page 36 (see also (Wahlberg, 1988, Theorem 3.2)).

□

Example 3.29 (Piecewise constant GSF) *We consider a GSF defined by the impulse response:*

$$h_g(t) = \begin{cases} \frac{1}{\Delta}(h_0 + h_1 + h_2) & ; 0 \le t < \frac{\Delta}{4} \\ \frac{1}{\Delta}(h_0 + h_1 - h_2) & ; \frac{\Delta}{4} \le t < \frac{\Delta}{2} \\ \frac{1}{\Delta}(h_0 - h_1 + h_2) & ; \frac{\Delta}{2} \le t < \frac{3\Delta}{4} \\ \frac{1}{\Delta}(h_0 - h_1 - h_2) & ; \frac{3\Delta}{4} \le t < \Delta \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{3.89}$$

where $h_0, h_1, h_2 \in \mathbb{R}$. For this choice we have:

$$C_g = \begin{bmatrix} h_0 & \frac{\Delta}{2}(h_0 + \frac{1}{2}h_1 + \frac{1}{4}h_2) \end{bmatrix} \tag{3.90}$$

Computing the noise spectrum (3.59) *and substituting in* (3.68)*, we obtain:*

$$\Phi_{\bar{y}}^q(z) = h_0^2 \Phi_{\bar{y}}^{IF}(z) + h_1^2 \Phi_{\bar{y}}^1(z) + h_2^2 \Phi_{\bar{y}}^2(z) + h_1 h_2 \Phi_{\bar{y}}^3(z) \tag{3.91}$$

where $\Phi_{\bar{y}}^{IF}(z)$ is the spectrum (3.88) *obtained in Example 3.28, and $\Phi_{\bar{y}}^{\ell}(z)$ ($\ell = 1, 2, 3$) are other particular spectra that do not depend on the GSF parameters.*

To assign the spectrum zeros to the origin, we solve for the weighting parameters h_ℓ in (3.91)*. We see that any of the following choices:*

$$h0 = 1 \qquad h1 = \mp 9.891 \qquad h2 = \pm 23.782 \tag{3.92}$$

$$\text{or} \qquad h0 = 1 \qquad h1 = \mp 4.691 \qquad h2 = \pm 5.382 \tag{3.93}$$

lead us to a sampled spectrum with no zeros, i.e.,

$$\Phi_{\bar{y}}^{q}(z) = \frac{\Delta^4}{5!} \frac{K}{(z-1)^2(z^{-1}-1)^2} \tag{3.94}$$

□

Example 3.30 (Sinusoidal GSF) *Here we restrict the GSF impulse response to the form:*

$$h_g(t) = \begin{cases} \frac{1}{\Delta}\left(h_0 + h_1 \pi \sin\left(\frac{2\pi}{\Delta}t\right) + h_2 \pi \sin\left(\frac{4\pi}{\Delta}t\right)\right) & ; t \in [0, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{3.95}$$

where $h_0, h_1, h_2 \in \mathbb{R}$. For this choice we have:

$$C_g = \begin{bmatrix} h_0 & \frac{\Delta}{2}\left(h_0 + h_1 + \frac{1}{2}h_2\right) \end{bmatrix} \tag{3.96}$$

Computing the noise spectrum (3.59) *and substituting into* (3.68)*, gives:*

$$\Phi_{\bar{y}}^{q}(z) = h_0^2 \Phi_{\bar{y}}^{IF}(z) + h_1^2 \Phi_{\bar{y}}^{1}(z) + h_2^2 \Phi_{\bar{y}}^{2}(z) + h_1 h_2 \Phi_{\bar{y}}^{3}(z) \tag{3.97}$$

where $\Phi_{\bar{y}}^{IF}(z)$ is the spectrum (3.88) *obtained in Example 3.28, and $\Phi_{\bar{y}}^{\ell}(z)$ ($\ell = 1, 2, 3$) are other spectra that do not depend on the GSF parameters h_ℓ. Solving for these parameters to assign the zeros to the origin, we see that any of the following choices:*

$$h_0 = 1 \qquad h_1 = \pm 3.902 \qquad h_2 = \mp 9.804 \tag{3.98}$$

$$\text{or} \qquad h_0 = 1 \qquad h_1 = \pm 1.902 \qquad h_2 = \mp 1.804 \tag{3.99}$$

lead to a sampled spectrum with no zeros, i.e.,

$$\Phi_{\bar{y}}^{q}(z) = \frac{\Delta^4}{5!} \frac{K}{(z-1)^2(z^{-1}-1)^2} \tag{3.100}$$

□

The GSF's described above allow us to assign the stochastic sampling zeros of general linear models *close* to the origin, when using *fast* sampling rates. The following example illustrates the use of the GSF's obtained in Examples 3.29 and 3.30 for a general system of relative degree 2.

Example 3.31 *Consider the following second order stochastic system:*

$$H(\rho) = \frac{2}{(\rho+2)(\rho+1)} \tag{3.101}$$

We first use the piecewise GSF obtained in Example 3.29. In particular, in (3.89) *we choose* $h_0 = 1$, $h_1 = 4.691$, *and* $h_2 = -5.382$. *For a sampling period* $\Delta = 0.1$, *we obtain the following stable spectral factor:*

$$H_q(z) = \frac{5.269 \cdot 10^{-2}(z - z_1)(z - z_1^*)}{(z - e^{-0.1})(z - e^{-0.2})} \tag{3.102}$$

where $z_1 = -0.014 + j0.081$, *and where* * *denotes complex conjugation.*

We also use the sinusoidal *GSF obtained in Example 3.30. The sampling period is fixed to* $\Delta = 0.01$. *In* (3.95) *we choose* $h_0 = 1$, $h_1 = 1.902$, *and* $h_2 = -1.804$. *The sampled-data model is then given by:*

$$H_q(z) = \frac{0.22 \cdot 10^{-10}(z - z_1)(z - z_2)}{(z - e^{-0.01})(z - e^{-0.02})} \tag{3.103}$$

where $z_1 = -1.0435 \cdot 10^{-3}$ *and* $z_1 = -1.0439 \cdot 10^{-3}$.

Note that, as in Example 3.27, both GSF's assign the sampling zeros very close to the origin, as expected.

□

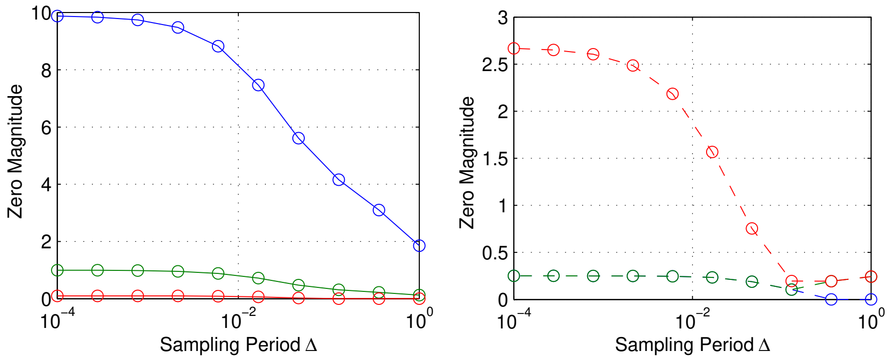
3.6 Robustness Issues

The design procedures that we have presented in the previous sections are aimed at assigning the *sampling zeros* of discrete-time models for deterministic and stochastic systems. The GHF and GSF depend on the continuous-time relative degree of the given system. Moreover, the proposed methods are asymptotic and assign the zeros to the desired location (*e.g.*, the origin $z = 0$) when the sampling period goes to zero.

In practice, one cannot sample a system infinitely fast. On the other hand, the concept of relative degree is not robustly defined for continuous-time systems, since it can be affected, for example, by high frequency unmodelled poles and/or zeros. As a consequence, when using *very fast sampling rates* the nominal model of the continuous-time system may not be appropriate. Moreover, for the stochastic system case, continuous-time white noise is a mathematical abstraction. In practice, it will correspond to a process with broad-band spectrum. This inherently implies some form of high-frequency modelling error (see Remark 2.24 on page 28). These issues may raise doubts about the practical use of the methods presented here.

We address these concerns by proposing that the generalised holds (GHF) and sampling function (GSF) be utilised in conjunction with a *bandwidth of validity* for the model relative degree. If the sampling period is chosen to be fast relative to the nominal poles (say 10 times, as per the usual *rule of thumb*) but slow relative to possible unmodelled poles (say 10 times) then one can heuristically expect that the GHF or GSF design to perform roughly as expected. On the other hand, if very fast sampling rates are used, then one can expect the relative degree to be *ill-defined*, and then the GHF or GSF design may fail to perform as desired.

These ideas are illustrated and confirmed by the following examples.



(a) Zero Order Hold (b) Generalised Hold

Figure 3.8: Magnitudes of the sampling zeros in Example 3.32: fast pole at $s = -10^2$.

Example 3.32 (Deterministic systems with GHF's) *Consider again the system in Example 3.13, but including now an* unmodelled *fast pole,* i.e.,

$$G(s) = \frac{1}{(s+1)^3(0.01s+1)} \tag{3.104}$$

For the ZOH-case, Theorem 2.18 predicts that the asymptotic sampling zeros are $\{-3.732, -0.268\}$, *based on a* nominal model *of relative degree* 3*, and* $\{-9.899, -1, -0.101\}$ *for the* true model *of relative degree* 4*. Indeed, we can see in Figure 3.8(a) that, as* Δ *decreases, the sampling zeros first approach those corresponding to the* nominal *model (of relative degree* 3*), but then move to those corresponding the* true *model (of relative degree* 4*). For this case, we see that the nominal discrete-time model (3.40) is basically reached for a sampling period* $\Delta \approx 0.2$ *but is not valid for* $\Delta < 0.1$.

Similarly, we can see in Figure 3.8(b) that the zeros obtained with the fixed GHF (3.41) are close *to the origin for* $\Delta > 0.1$. *However, when the sampling period is reduced further the unmodelled pole at* $s = -10^2$ *in (3.104) becomes significant and the zeros clearly depart from the desired values.*

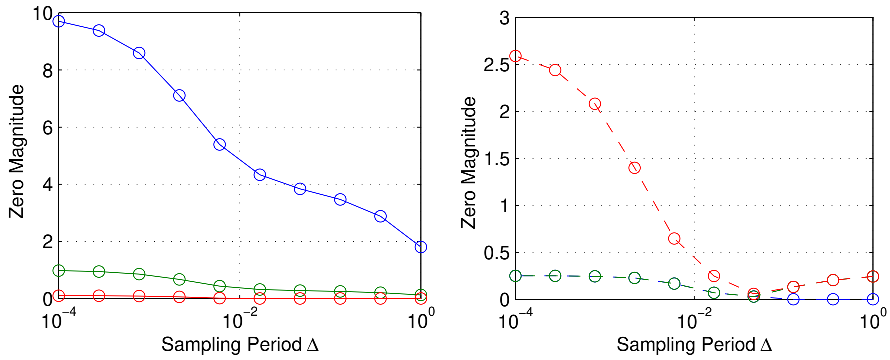
In Figure 3.9, we see even more clearly the effect of the bandwidth of validity *for the nominal model when the unmodelled fast pole is assumed to be at* $s = -10^3$. *On the other hand, the plots in Figure 3.10 correspond to an unmodelled pole at* $s = -10$, *where we can see that the GHF (3.41) is not able to assign the zeros near the origin, because the model (3.39) (used for design) is a poor representation of the true system over all sensible sampling frequencies.*

□

Example 3.33 (Stochastic systems with GSF's) *Let us consider the presence of an unmodelled fast pole in the continuous-time stochastic system defined* (3.82)*. Thus, consider the following true system:*

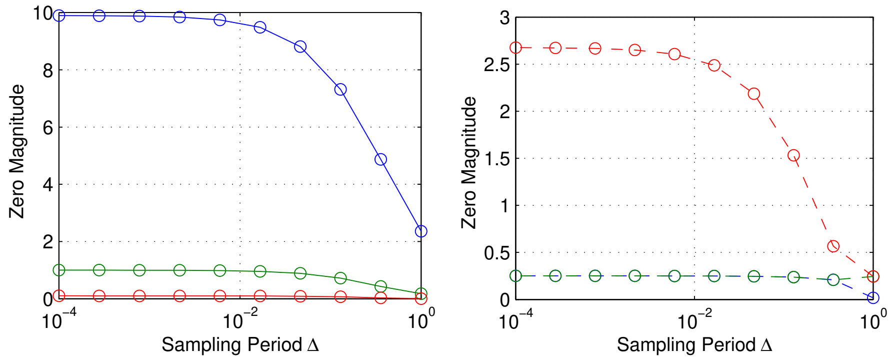
$$H(\rho) = \frac{1}{\left(\rho+2\right)\left(\frac{1}{\omega_u}\rho+1\right)} \tag{3.105}$$

We use the piecewise constant GSF obtained in Example 3.25 based on a nominal system of relative degree 1.



(a) Zero Order Hold (b) Generalised Hold

Figure 3.9: Magnitudes of the sampling zeros in Example 3.32: fast pole at $s = -10^3$.



(a) Zero Order Hold (b) Generalised Hold

Figure 3.10: Magnitudes of the sampling zeros in Example 3.32: fast pole at $s = -10$.

We assume an unmodelled fast pole located at $\omega_u = 200$*[rad/s]. We consider the following two cases for the sampling period:*

1. $\Delta = 0.1$*[s] : This corresponds to a sampling frequency* $\omega_s \approx 60$*[rad/s]. In this case, the unmodelled pole lies well beyond the sampling frequency, so we expect no significant effect on the sampled-data model. Indeed, we obtain the spectral factor:*

$$H_q(z) = \frac{1.1 \cdot 10^{-3}(z + 1.2 \cdot 10^{-2})(z - 5.3 \cdot 10^{-3})}{(z - e^{-0.2})(z - e^{-20})} \tag{3.106}$$

We can see that the sampling zeros and the fast discrete-time pole are located close the origin. Thus, the system can be roughly approximated by (3.83)*, as expected.*

In this case, we have chosen a sampling frequency inside the bandwidth *where the assumption on the relative degree is justified.*

2. $\Delta = 0.01$*[s] : We now increase the sampling frequency up to* $\omega_s \approx 600$*[rad/s]. The unmodelled pole, in this case, should ideally be considered in the GSF design. However, assuming that the presence of the high frequency pole is unknown, and then using the same GSF as above, we obtain the following sampled-data model:*

$$H_q(z) = \frac{3.8 \cdot 10^{-5}(z + 2.1 \cdot 10^{-1})(z - 3.3 \cdot 10^{-2})}{(z - e^{-0.02})(z - e^{-2})} \tag{3.107}$$

We see, in this case, that the slowest sampling zero is far from the origin. The reason for this outcome is understandable since the relative degree assumption on the nominal model is no longer valid anymore at this sampling rate.

□

The previous examples confirm the heuristic notion that the *system relative degree* and our design procedures, should be considered in terms of a *bandwidth of validity* for the nominal model of the continuous-time system.

3.7 Summary

In this chapter we have analysed the *artifacts* involved in the sampling process. Sampled-data models depend both on the continuous-time system and the details of the discretisation process. In particular, we have analysed the role of the hold device that generates the continuous-time input to the system, and the sampling device, which gives us an output sequence of samples. We have shown that these devices determine the *sampling zeros* of the corresponding discrete-time model, both in the determistic and stochastic cases.

Based on the previous analysis, we have shown that the hold device, characterised as a generalised hold function, can be designed asymptotically (as the sampling period goes to zero) to assign the sampling zeros of deterministic systems. A dual result has also been presented for stochastic systems, namely, a generalised sampling filter can be designed to assign the asymptotic zeros of the sampled output spectrum of a system.

An important observation is that the proposed procedures deal with sampling zeros **only**. Intrinsic characteristics of the system, such as non-minimum phase zeros are not *artificially* removed.

Another key point is that the proposed design procedures are independent of the particular plant. In fact, the methods rely only on the system relative degree. They are based on the key observation that, at high frequencies (*i.e.*, for fast sampling rates), any linear system of relative degree r can be *described* as an r-th order integrator.

Finally, we have made an important observation regarding the validity of *nominal* continuous-time models when using fast sampling rates. In particular, the relative degree may be an ill-defined quantity in continuous-time because of the presence of high frequency poles or zeros. Thus, in practical cases, the use of the generalised filters described here should be considered within a *bandwidth of validity* where one can rely on the relative degree assumption. This places an upper bound on the sampling rates that can be sensibly used in connection with the proposed methods.

Chapter 4

Sampling issues in continuous-time system identification

4.1 Overview

In recent years, there has been an increased interest in the problem of identifying continuous-time models (Johansson, 1994; Söderström *et al.*, 1997; Unbehauen and Rao, 1998; Johansson *et al.*, 1999; Rao and Garnier, 2002; Larsson and Söderström, 2002; Garnier *et al.*, 2003; Kristensen *et al.*, 2004; Gillberg and Ljung, 2005*a*; Gillberg and Ljung, 2005*b*). These kinds of models have several advantages compared to discrete-time models: the parameters obtained are physically meaningful, and can be related to properties of the real system; the continuous-time model obtained is independent of the sampling period; and these models may be more suitable for *fast sampling rate* applications since a continuous-time model is the (theoretical) limit when the sampling period is infitesimally small.

Even though it is theoretically possible to carry out system identification using continuous-time data (Young, 1981; Unbehauen and Rao, 1990), this will generally involve the use of analogue operations to emulate time derivatives. Thus, in practice, one is usually forced to work with sampled data (Sinha and Rao, 1991; Pintelon and Schoukens, 2001).

In this chapter we explore the issues that are associated with the use of sampled-data models in continuous-time system identification. Specifically, we use sampled-data models expressed using the δ operator, to estimate the parameters of the underlying continuous-time system. In this context, one might hope that, if one samples *quickly enough*, the difference between discrete and continuous processing would become vanishing small. Thus, say we are given a set of data $\{u_k = u(k\Delta), y_k = y(k\Delta)\}$, and we identify a sampled-data model:

$$\mathcal{M}_\Delta : \quad y_k = G_\delta(\delta, \hat{\theta})u_k + H_\delta(\delta, \hat{\theta})v_k \tag{4.1}$$

where $\hat{\theta}$ is a vector with the parameters to be estimated, then, we might hope that $\hat{\theta}$ will converge to the corresponding continuous-time parameters, as Δ goes to zero, *i.e.*,

$$\hat{\theta} \quad \xrightarrow{\Delta \to 0} \quad \theta \tag{4.2}$$

where θ represents the *true* parameter vector of the continuous-time model:

$$\mathcal{M}: \quad y(t) = G(\rho, \theta)u(t) + H(\rho, \theta)\dot{v} \tag{4.3}$$

Indeed, as we have seen in the previous chapters, there are many cases which support this hypothesis. Moreover, the delta operator has been the key tool to highlight the connections between the discrete- and the continuous-time domains (Middleton and Goodwin, 1990; Feuer and Goodwin, 1996).

The above discussion can, however, lead to a false sense of security when using sampled data. A sampled-data model asymptotically *converges* to the continuous-time representation of a given system. However, there is an inherent **loss of information** when using discrete-time models representations. In the time-domain, the use of sampled data implies that we do not know the intersample behaviour of the system. In the frequency domain, this fact translates to the well-known aliasing effect: high frequency components fold back to low frequencies, in such a way that is not possible to distinguish among them.

To fill the gap between systems evolving in continuous-time and their sampled data representations, we need to make extra assumptions on the continuous-time model and signals. This is a particularly sensitive point when trying to perform system identification using sampled-data. In Section 4.2, we pay particular attention to the impact of high frequency modelling errors. These kinds of errors may arise both in the discrete- and continuous-time domains. For discrete-time models, the *sampling zeros* go to infinity (in the γ-domain corresponding to the δ operator) as the sampling period is reduced, however, their effect cannot, in all cases, be neglected especially at high frequencies. For continuous-time systems, undermodelling errors may arise due to the presence of high frequency poles and/or zeros not included in the nominal model.

Based on the above remarks, we argue here that one should always define a *bandwidth of fidelity* of a model and ensure that the model errors outside that bandwidth do not have a major impact on the identification results. This is the identification analogue of the design restrictions discussed in Section 3.6 in the previous chapter. In Section 4.2, we propose the use of a maximum likelihood identification procedure in the frequency domain, using a **restricted bandwidth**. We show that the proposed identification method is insensitive to both high frequency undermodelling errors (in the continuous-time model), and to sampling zeros (in the sampled-data model).

A well known instance where *naive* use of sampled data can lead to erroneous results is in the identification of continuous-time stochastic systems where the noise model has relative degree greater than zero. In this case, we saw in Chapter 2 that the sampled data model will have *sampling zeros*. These are the stochastic equivalent of the well-known sampling zeros that occur in deterministic systems. We will see in Section 4.3 that these sampling zeros play a crucial role in obtaining unbiased parameter estimates in the identification of such systems from sampled data. We show that high frequency modelling errors can be equally as catastrophic as ignoring sampling zeros. These problems can be overcome by using the proposed frequency domain identification procedure, restricting the estimation to a limited bandwidth.

4.2 Limited bandwidth estimation

In this section we discuss the issues that arise when using sampled-data models to identify the underlying continuous-time system. The discrete-time description, when expressed using the δ operator, converges to the continuous-time model as the sampling period goes to zero. However, for any non-zero sampling period, there will always be a difference between the discrete- and continuous-time descriptions, due to the presence of sampling zeros. To overcome this inherent difficulty, we propose the use of maximum likelihood estimation in the frequency domain, using a restricted bandwidth.

To illustrate the differences between discrete-time models and the underlying continuous-time systems we present the following example.

Example 4.1 *Consider a second order deterministic system, described by:*

$$\frac{d^2}{dt^2}y(t) + \alpha_1 \frac{d}{dt}y(t) + \alpha_o y(t) = \beta_o u(t) \tag{4.4}$$

If we simply replace the derivatives in this continuous-time model by divided differences, we obtain the following ***approximate*** *discrete-time model described in terms of the δ operator:*

$$\delta^2 y_k + a_1 \delta y_k + a_0 y_k = b_0 u_k \tag{4.5}$$

We see that this naive *derivative replacement model has no extra zeros. However, in Section 2.2, we obtained an* ***exact*** *discrete-time model based on the use of a ZOH:*

$$\delta^2 y_k + a_1 \delta y_k + a_o y_k = b_0 u_k + b_1 \delta u_k \tag{4.6}$$

This model generically has a sampling zero*. Moreover, as the sampling period Δ goes to zero, the continuous-time coefficients are recovered, and the sampling zero can be readily characterised (see Theorem 2.19 on page 24):*

$$\delta^2 y_k + \alpha_1 \delta y_k + \alpha_o y_k = \beta_0 (1 + \tfrac{\Delta}{2}\delta) u_k \tag{4.7}$$

Figure 4.1 shows a comparison of the Bode magnitude diagrams corresponding to a second order system as (4.4) *(on the left hand side) and the exact sampled-data model* (4.6)*, obtained for different sampling frequencies (on the right):*

$$G(s) = \frac{\beta_o}{s^2 + \alpha_1 s + \alpha_o} \qquad\qquad G_\delta(\gamma) = \frac{b_1\gamma + b_o}{\gamma^2 + a_1\gamma + a_o} \tag{4.8}$$

The figure clearly illustrates the fact that, ***no matter how fast we sample****, there is always a difference (near the folding frequency) between the continuous-time model and the discretised models.*

□

The difference between discrete- and continuous-time models highlighted by the previous example, in fact, corresponds to an illustration of the *aliasing* effect predicted by the relationship in (2.20) on page 15. If we assume that the continuous-time system frequency response $G(j\omega)$ goes to zero as $|\omega| \to \infty$, then the corresponding discrete-time frequency response converges:

$$\lim_{\Delta\to 0} G_q(e^{j\omega\Delta}) = \lim_{\Delta\to 0} \sum_{\ell=-\infty}^{\infty} \left[\frac{(1-e^{-s\Delta})}{s\Delta} G(s) \right]_{s=j\omega + j\frac{2\pi}{\Delta}\ell} = G(j\omega) \tag{4.9}$$

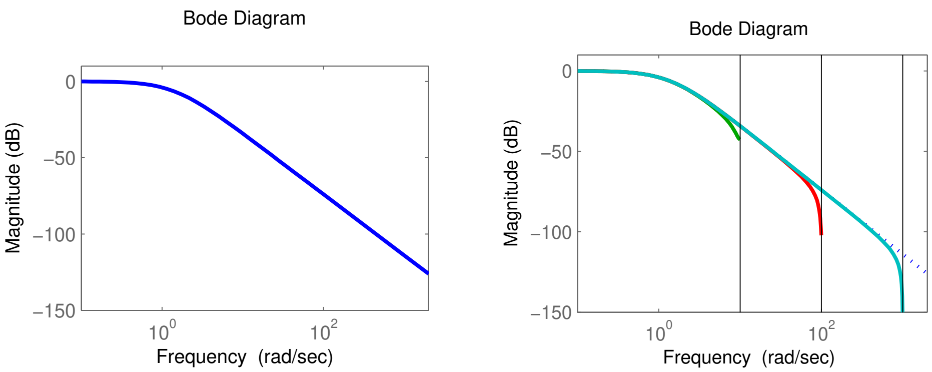


Figure 4.1: Continuous- and discrete-time frequency response magnitudes.

Remark 4.2 *Equation* (4.9) *establishes that the frequency response of sampled-data model converges to the continuous-time frequency response, as the sampling period goes to zero. However, for any finite sampling frequency, there is a difference between the continuous- and discrete-time, in particular, near the Nyquist frequency* ($\omega_N = \frac{\omega_s}{2} = \frac{\pi}{\Delta}$)*. Indeed, this is a direct consequence of the presence of the asymptotic sampling zeros.*

A different kind of problem may arise when the *true* system contains high frequency dynamics that are not included in the continuous-time model. We illustrate this by the following example.

Example 4.3 *Consider again the continuous-time system in Example 4.1. We will consider* (4.4) *as the nominal model of the system. We are interested in analysing the effect of an unmodelled fast pole. Thus, let the true system be given by:*

$$G(s) = \frac{\beta_o}{(s^2 + \alpha_1 s + \alpha_o)\left(\frac{s}{\omega_u} + 1\right)} = \frac{G_o(s)}{\left(\frac{s}{\omega_u} + 1\right)} \tag{4.10}$$

Figure 4.2 shows the comparison of nominal and true models, both for the continuous-time system and the sampled-data models. The nominal poles of the system are at $s = -1$ and $s = -2$, the sampling frequency is $\omega_s = 250$[rad/s], and the unmodelled fast pole is at $s = -50$.

Note that the true system has relative degree 3, and, thus, the corresponding discrete-time model will have 2 sampling zeros. As a consequence, while the asymptotic sampled-data model for the nominal system is given by (4.8)*, the **true** model will yield different asymptotic sampling zeros as Δ goes to zero. Thus, the nominal model satisfies:*

$$G_{o,\delta}(\gamma) \to \frac{b_o\left(1 + \frac{\Delta}{2}\gamma\right)}{\gamma^2 + a_1\gamma + a_o} \tag{4.11}$$

whereas the the true model satisfies:

$$G_\delta(\gamma) \to \frac{b_o\left(1 + \Delta\gamma + \frac{\Delta^2}{6}\gamma^2\right)}{(\gamma^2 + a_1\gamma + a_o)\left(\frac{\gamma}{\omega_u} + 1\right)} \tag{4.12}$$

□

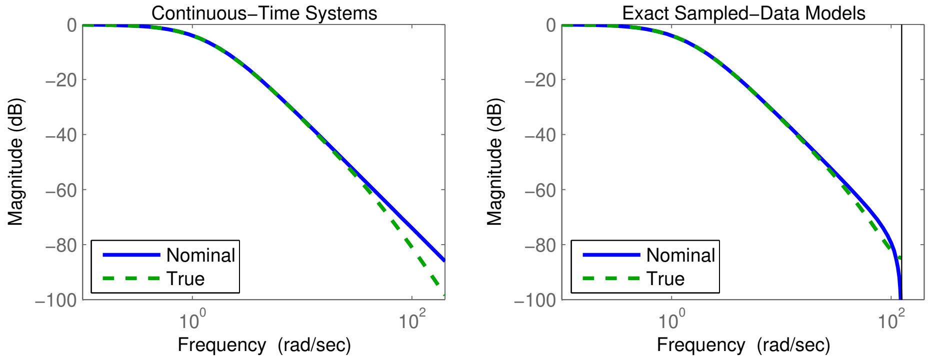


Figure 4.2: Frequency response for nominal and true models.

The previous example illustrates the problems that may arise when using *fast sampling rates*. The sampling frequency was chosen well above the nominal poles of the system, in fact, two decades. In theory, this allows one to use the asymptotic characterisation of the sampled-data model. However, we can see that, if there are any unmodelled dynamics not included in the continuous-time model (in this case, one decade above the nominal fastest pole), then there will also be undermodelling in the sampled-data description. Moreover, even though the sampling zeros go to infinity for the nominal and true models, their precise characterisation depends significantly on high frequency aspects of the model, as shown in (4.11) and (4.12).

Remark 4.4 *The above discussion highlights the issues that have to be taken into account when using sampled-data models to identify continuous-time system. Specifically:*

- *Any method that relies on high-frequency system characteristics will be inherently non robust, and, as a consequence,*
- *Models should be considered within a* bandwidth of validity*, to avoid high-frequency modelling errors — see the shaded area in Figure 4.3.*

In Section 4.3, we will see how frequency domain maximum likelihood estimation, over a *restricted bandwidth*, can be used to address these issues.

4.2.1 Frequency Domain Maximum Likelihood

In this section we describe a frequency domain maximum likelihood (FDML) estimation procedure. Specifically, if one converts the data to the frequency domain, then one can carry out the identification over a limited range of frequencies. Note, however, that one needs to carefully define the likelihood function in this case. We use the following result (for the scalar case, the result has been derived in Ljung (1993), while the multivariable case is considered in McKelvey and Ljung (1997)):

Lemma 4.5 *Assume a given set of input-output data* $\{u_k = u(k\Delta), y_k = y(k\Delta)\}$*,* $k = 0 \ldots N_d$ *, is generated by the* exact *discrete-time model:*

$$y_k = G_q(q,\theta)u_k + H_q(q,\theta)v_k \tag{4.13}$$

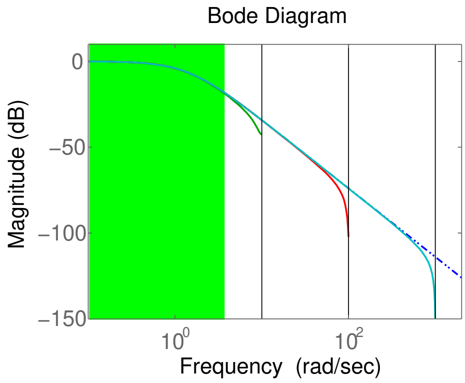


Figure 4.3: Representation of the bandwidth of validity.

where v_k is Gaussian DTWN sequence, $v_k \sim N(0, \sigma_w^2)$.

The data is transformed to the frequency domain yielding the discrete Fourier transforms U_ℓ and Y_ℓ of the input and output sequences, respectively.

Then the maximum likelihood estimate of θ, when considering frequency components up to $\omega_{\max} \leq \frac{\omega_s}{2}$, is given by:

$$\hat{\theta}_{ML} = \arg\min_{\theta} L(\theta) \tag{4.14}$$

where $L(\theta)$ is the negative logarithm of the likelihood function of the data given θ, i.e.,

$$\begin{aligned} L(\theta) &= -\log p(Y_0, \ldots, Y_{n_{\max}} | \theta) \\ &= \sum_{\ell=0}^{n_{\max}} \frac{|Y_\ell - G_q(e^{j\omega_\ell \Delta}, \theta) U_\ell|^2}{\lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2} + \log(\pi \lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2) \end{aligned} \tag{4.15}$$

where $\lambda_v^2 = \Delta N_d \sigma_v^2$, and $n_{\max}$ is the index associated with $\omega_{\max}$.

Proof. Equation (4.13) can be expressed in the frequency domain as:

$$Y_\ell = G_q(e^{j\omega_\ell \Delta}, \theta) U_\ell + H_q(e^{j\omega_\ell \Delta}, \theta) V_\ell \tag{4.16}$$

where Y_ℓ, U_ℓ, and V_ℓ are discrete Fourier transforms (DFT), *e.g.*,

$$Y_\ell = Y(e^{j\omega_\ell \Delta}) = \Delta \sum_{k=0}^{N_d - 1} y_k e^{-j\omega_\ell k \Delta} \qquad ; \omega_\ell = \tfrac{2\pi}{\Delta} \tfrac{\ell}{N_d} \tag{4.17}$$

Assuming that the DTWN sequence $v_k \sim N(0, \sigma_w^2)$, then V_ℓ are (asymptotically) independent and have a circular complex Gaussian distribution (Brillinger, 1974; Brillinger, 1981). Thus, the frequency domain noise sequence V_ℓ has zero mean and variance $\lambda_v^2 = \Delta N_d \sigma_v^2$. We therefore see that Y_ℓ is also complex Gaussian and satisfies:

$$Y_\ell \sim N(G_q(e^{j\omega_\ell \Delta}, \theta) U_\ell, \lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2) \tag{4.18}$$

The corresponding probability density function is given by:

$$p(Y_\ell) = \frac{1}{\pi \lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2} \exp\left\{ -\frac{|Y_\ell - G_q(e^{j\omega_\ell \Delta}, \theta) U_\ell|^2}{\lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2} \right\} \tag{4.19}$$

If we consider the elements Y_ℓ within a **limited-bandwidth**, *i.e.*, up to some maximum frequency $\omega_{\max}$ indexed by $n_{\max}$ with $\omega_{\max} = \omega_s \frac{n_{\max}}{N_d} \leq \frac{\omega_s}{2}$, the appropriate log-likelihood function is given by:

$$\begin{aligned} L(\theta) &= -\log p(Y_0, \ldots, Y_{n_{\max}}) = -\log \prod_{\ell=0}^{n_{\max}} p(Y_\ell) \\ &= \sum_{\ell=0}^{n_{\max}} \frac{|Y_\ell - G_q(e^{j\omega_\ell \Delta}, \theta) U_\ell|^2}{\lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2} + \log(\pi \lambda_v^2 |H_q(e^{j\omega_\ell \Delta}, \theta)|^2) \end{aligned} \tag{4.20}$$

□

Remark 4.6 *The logarithmic term in the log-likelihood function* (4.15) *plays a key role in obtaining consistent estimates of the true system. This term can be neglected* ***only*** *if (Ljung, 1993):*

- *The noise model is assumed to be known. In this case H_q does not depend on θ and, thus, plays no role in the minimisation* (4.14)*; or*

- *The frequencies ω_ℓ are equidistantly distributed over the full frequency range $[0, \frac{2\pi}{\Delta})$. This is equivalent to considering the* ***full bandwidth*** *case in* (4.15)*,* i.e.*, $n_{\max} = \frac{N}{2}$ (or N, because of periodicity). This yields:*

$$\frac{2\pi}{N_d} \sum_{\ell=0}^{N_d-1} \log |H_q(e^{j\omega_\ell \Delta}, \theta)|^2 \xrightarrow{N_d \to \infty} \int_0^{2\pi} \log |H_q(e^{j\omega}, \theta)|^2 d\omega \tag{4.21}$$

The last integral is equal to zero for any monic, stable and inversely stable transfer function $H_q(e^{j\omega}, \theta)$ (Ljung, 1993).

Remark 4.7 *In the previous lemma the discrete-time model* (4.13) *has been expressed in terms of the shift operator q. The results apply* mutatis mutandis *when the model is reparameterised using the δ-operator:*

$$G_q(e^{j\omega_\ell \Delta}) = G_\delta(\gamma_\omega) = G_\delta \left(\frac{e^{j\omega_\ell \Delta} - 1}{\Delta} \right) \tag{4.22}$$

4.3 Robust continuous-time identification

In this section we illustrate the problems that may arise when sampled-data models are used for continuous-time system identification. In particular, we illustrate the consequences of both types of undermodelling errors discussed earlier:

- Sampling zeros are not included in the sampled-data model, and

- The continuous-time system contains unmodelled high frequency dynamics.

We show that these kinds of errors can have severe consequences in the estimation results for deterministic and stochastic systems. We show that the frequency domain maximum likelihood (FDML) procedure, using restricted bandwidth, allows one to overcome these difficulties.

4.3.1 Effect of sampling zeros in deterministic systems

We first explore the consequences of neglecting the presence of sampling zeros in determistic models used for identification. Specifically, the following example considers a deterministic second order system with known input. The parameters of the system are estimated using different sampled-data model structures.

Example 4.8 *Consider again the linear system in* (4.4)*. Assume that the continuous-time parameters are* $\alpha_1 = 3$*,* $\alpha_0 = 2$*,* $\beta_0 = 2$*. We performed system identification assuming three different model structures:*

SDRM: *Simple Derivative Replacement Model. This corresponds to the structure given in* (4.5)*, where continuous-time derivatives have been replaced by divided differences.*

MIFZ: *Model Including Fixed Zero. This model considers the presence of the asymptotic zero, assuming a structure as in* (4.7)*.*

MIPZ: *Model Including Parameterised Zero. This model also includes a sampling zero, whose location has to be estimated,* i.e.*, we use the structure given by* (4.6)*.*

The three discrete-time models can be represented in terms of the δ *operator as:*

$$G_\delta(\gamma) = \frac{B_\delta(\gamma)}{\gamma^2 + \hat{\alpha}_1\gamma + \hat{\alpha}_0} \tag{4.23}$$

where:

$$B_\delta(\gamma) = \begin{cases} \hat{\beta}_0 & \textit{(SDRM)} \\ \hat{\beta}_0(1 + \frac{\Delta}{2}\gamma) & \textit{(MIFZ)} \\ \hat{\beta}_0 + \hat{\beta}_1\gamma & \textit{(MIPZ)} \end{cases} \tag{4.24}$$

We use a sampling period $\Delta = \pi/100$*[s] and choose the input* u_k *to be a random Gaussian sequence of unit variance. Note that the output sequence* $y_k = y(k\Delta)$ *can be obtained by either simulating the continuous-time system and sampling its output, or, alterntatively, by simulating the exact sampled-data model in discrete-time. Also note that the data is free of any measurement noise.*

The parameters are estimated in such a way to minimise the equation error cost function:

$$J(\hat{\theta}) = \frac{1}{N}\sum_{k=0}^{N-1} e_k(\hat{\theta})^2 = \frac{1}{N}\sum_{k=0}^{N-1}(\delta^2 y_k - \phi_k^T\theta)^2 \tag{4.25}$$

where:

$$\phi_k = \begin{cases} [-\delta y_k,\, -y_k,\, u_k]^T \\ [-\delta y_k,\, -y_k,\, (1+\frac{\Delta}{2}\delta)u_k]^T \\ [-\delta y_k,\, -y_k,\, \delta u_k,\, u_k]^T \end{cases} \quad \textit{and} \quad \hat{\theta} = \begin{cases} [\alpha_1,\, \alpha_0,\, \beta_0]^T & \textit{(SDRM)} \\ [\alpha_1,\, \alpha_0,\, \beta_0]^T & \textit{(MIFZ)} \\ [\alpha_1,\, \alpha_0,\, \beta_1, \beta_0]^T & \textit{(MIPZ)} \end{cases} \tag{4.26}$$

Table 4.1 shows the estimation results. Note that the system considered is linear, thus, the exact discrete-time parameters can be computed for the given sampling period. These are also given in Table 4.1.

	Parameters		Estimates		
	CT	Exact DT	SDRM	MIFZD	MIPZD
α_1	3	2.923	2.8804	2.9471	2.9229
α_0	2	1.908	1.9420	1.9090	1.9083
β_1	–	0.0305	–	$\frac{\beta_0\Delta}{2} = 0.03$	0.0304
β_0	2	1.908	0.9777	1.9090	1.9083

Table 4.1: Parameter estimates for a linear system

We can see that, while both models incorporating a sampling zero (MIFZ and MIPZ) are able to recover the continuous-time parameters, when using SDRM the estimate $\hat{\beta}_0$ is clearly biased.

□

The result in the previous example may be surprising since, even though the SDRM in (4.27) converge to the continuous-time system as the sampling period goes to zero, the estimate $\hat{\beta}_0$ does not converge to the underlying continuous-time parameter. This estimate is asymptotically biased. Specifically, we see that β_0 is incorrectly estimated by a factor of 2 by the SDRM. This illustrates the impact of not considering sampling effects on the sampled-data models used for continuous-time system identification.

Indeed, the following result formally establishes the asymptotic bias that was observed experimentally for the SDRM structure in the previous example. In particular, we show that β_0 is indeed underestimated by a factor of 2.

Lemma 4.9 *Consider the general second order deterministic system given in* (4.4)*. Assume that sampled data is collected from the system using a ZOH input generated from a DTWN sequence u_k, and sampling the output $y_k = y(k\Delta)$*

If an equation error identification procedure is utilised to estimate the parameters of the simple derivative replacement model:

$$\delta^2 y + \hat{\alpha}_1 \delta y + \hat{\alpha}_0 y = \hat{\beta}_0 u \tag{4.27}$$

then the parameter estimates asymptotically converge, as the sampling period Δ goes to zero, to:

$$\hat{\alpha}_1 \to \alpha_1, \qquad \hat{\alpha}_0 \to \alpha_0, \qquad \textit{and} \quad \hat{\beta}_0 \to \tfrac{1}{2}\beta_0 \tag{4.28}$$

Proof. The parameters of the approximate SDRM (4.27) model can be obtained by simple Least Squares, minimising the equation error cost function:

$$J(\hat{\theta}) = \lim_{N\to\infty} \frac{1}{N} \sum_{0}^{N-1} e_k(\hat{\theta})^2 = E\{e_k(\hat{\theta})^2\} \tag{4.29}$$

where $e_k = \delta^2 y + \hat{\alpha}_1 \delta y + \hat{\alpha}_0 y - \hat{\beta}_0 u$. The parameter estimates are given by the solution of $\frac{dJ(\hat{\theta})}{d\hat{\theta}} = 0$. Thus, differentiating the cost function with respect to each of the parameter estimates, we obtain:

$$\begin{bmatrix} E\{(\delta y)^2\} & E\{(\delta y)y\} & -E\{(\delta y)u\} \\ E\{(\delta y)y\} & E\{y^2\} & -E\{yu\} \\ -E\{y^2\} & -E\{yu\} & E\{u^2\} \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} -E\{(\delta y)(\delta^2 y)\} \\ -E\{y\delta^2 y\} \\ E\{u\delta^2 y\} \end{bmatrix} \tag{4.30}$$

This equation can be rewritten in terms of (discrete-time) correlations as:

$$\begin{bmatrix} \frac{2r_y(0)-2r_y(1)}{\Delta^2} & \frac{r_y(1)-r_y(0)}{\Delta} & \frac{r_{yu}(0)-r_{yu}(1)}{\Delta} \\ \frac{r_y(1)-r_y(0)}{\Delta} & r_y(0) & -r_{yu}(0) \\ \frac{r_{yu}(0)-r_{yu}(1)}{\Delta} & -r_{yu}(0) & r_u(0) \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} \frac{3r_y(0)-4r_y(1)+r_y(2)}{\Delta^3} \\ \frac{-r_y(0)+2r_y(1)-r_y(2)}{\Delta^2} \\ \frac{r_{yu}(0)-2r_{yu}(1)+r_{yu}(2)}{\Delta^2} \end{bmatrix} \tag{4.31}$$

To continue with the proof we need to obtain expressions for the correlations involved in the last equation. If we assume that the input sequence is a DTWN process, with unit variance then we have that:

$$r_u(k) = \delta_K[k] \quad \iff \quad \Phi_u^q(e^{j\omega\Delta}) = 1 \tag{4.32}$$

Then, the other correlation functions can be obtained from the relations:

$$r_{yu}(k) = \mathcal{F}_d^{-1}\left\{\Phi_{yu}^q(e^{j\omega\Delta})\right\} = \mathcal{F}_d^{-1}\left\{G_q(e^{j\omega\Delta})\Phi_u^q(e^{j\omega\Delta})\right\} = \mathcal{F}_d^{-1}\left\{G_q(e^{j\omega\Delta})\right\} \tag{4.33}$$

$$r_y(k) = \mathcal{F}_d^{-1}\left\{\Phi_y^q(e^{j\omega\Delta})\right\} = \mathcal{F}_d^{-1}\left\{G_q(e^{-j\omega\Delta})\Phi_{yu}^q(e^{j\omega\Delta})\right\} = \mathcal{F}_d^{-1}\left\{|G_q(e^{j\omega\Delta})|^2\right\} \tag{4.34}$$

where $G_q(e^{j\omega\Delta})$ is the exact sampled-data model corresponding to the continuous-time system (4.4). Given a sampling period Δ, the exact discrete-time model is given by:

$$G_q(z) = \frac{\beta_0(c_1 z + c_0)}{(z - e^{\lambda_1\Delta})(z - e^{\lambda_2\Delta})} \tag{4.35}$$

where:

$$c_1 = \frac{(e^{\lambda_1\Delta}-1)\lambda_2 - (e^{\lambda_2\Delta}-1)\lambda_1}{(\lambda_1-\lambda_2)\lambda_1\lambda_2} = \frac{\Delta^2}{2} + \frac{\Delta^3}{6}(\lambda_1+\lambda_2) + \ldots \tag{4.36}$$

$$c_0 = \frac{e^{\lambda_1\Delta}(e^{\lambda_2\Delta}-1)\lambda_1 - e^{\lambda_2\Delta}(e^{\lambda_1\Delta}-1)\lambda_2}{(\lambda_1-\lambda_2)\lambda_1\lambda_2} = \frac{\Delta^2}{2} + \frac{\Delta^3}{3}(\lambda_1+\lambda_2) + \ldots \tag{4.37}$$

and λ_1 and λ_2 are the continuous-time system (stable) poles of system (4.4), *i.e.*, $\alpha_1 = -(\lambda_1+\lambda_2)$ and $\alpha_0 = \lambda_1\lambda_2$.

The exact discrete-time model (4.35) can be rewritten as:

$$G_q(z) = \frac{C_1}{z - e^{\lambda_1\Delta}} + \frac{C_2}{z - e^{\lambda_2\Delta}} \tag{4.38}$$

where $C_1 = \frac{\beta_0(c_1 e^{\lambda_1\Delta}+c_0)}{(e^{\lambda_1\Delta}-e^{\lambda_2\Delta})}$ and $C_2 = \frac{\beta_0(c_1 e^{\lambda_2\Delta}+c_0)}{(e^{\lambda_2\Delta}-e^{\lambda_1\Delta})}$. Substitituting in (4.34), we obtain:

$$r_{yu}(k) = \mathcal{F}_d^{-1}\left\{G_q(e^{j\omega\Delta})\right\} = \left(C_1 e^{\lambda_1\Delta(k-1)} + C_2 e^{\lambda_2\Delta(k-1)}\right)\mu[k-1] \tag{4.39}$$

where $\mu[k]$ is the discrete-time unitary step function. From (4.35), we have that:

$$G_q(z)G_q(z^{-1}) = K_1\left(\frac{e^{\lambda_1\Delta}}{z-e^{\lambda_1\Delta}} + \frac{e^{-\lambda_1\Delta}}{z-e^{-\lambda_1\Delta}}\right) + K_2\left(\frac{e^{\lambda_2\Delta}}{z-e^{\lambda_2\Delta}} + \frac{e^{-\lambda_2\Delta}}{z-e^{-\lambda_2\Delta}}\right) \tag{4.40}$$

$$K_1 = \frac{\beta_0^2(c_1^2 e^{\lambda_1\Delta} + c_0c_1 + c_0c_1e^{2\lambda_1\Delta} + c_0^2 e^{\lambda_1\Delta})}{(e^{2\lambda_1\Delta}-1)(e^{\lambda_1\Delta}e^{\lambda_2\Delta}-1)(e^{\lambda_1\Delta}-e^{\lambda_2\Delta})} \tag{4.41}$$

$$K_2 = \frac{\beta_0^2(c_1^2 e^{\lambda_2\Delta} + c_0c_1 + c_0c_1e^{2\lambda_2\Delta} + c_0^2 e^{\lambda_2\Delta})}{(e^{2\lambda_2\Delta}-1)(e^{\lambda_2\Delta}e^{\lambda_1\Delta}-1)(e^{\lambda_2\Delta}-e^{\lambda_1\Delta})} \tag{4.42}$$

Substitituting in (4.34), we obtain:

$$r_y(k) = \mathcal{F}_d^{-1}\left\{|G_q(z=e^{j\omega\Delta})|^2\right\} = K_1 e^{\lambda_1\Delta|k|} + K_2 e^{\lambda_2\Delta|k|} \qquad ; \forall k \in \mathbb{Z} \tag{4.43}$$

The correlations (4.32), (4.39), and (4.43) can be used in the normal equation (4.30) to obtain:

$$\begin{bmatrix} \frac{-\beta_0^2}{2(\lambda_1+\lambda_2)}\Delta & 0+\mathcal{O}(\Delta^2) & -\frac{\beta_0}{2}\Delta \\ 0+\mathcal{O}(\Delta^2) & \frac{-\beta_0^2}{2(\lambda_1+\lambda_2)\lambda_1\lambda_2}\Delta & 0 \\ -\frac{\beta_0}{2}\Delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} \Delta\frac{\beta_0^2}{4} \\ \Delta\frac{-\beta_0^2}{2(\lambda_1+\lambda_2)} \\ \frac{\beta_0}{2}+\mathcal{O}(\Delta) \end{bmatrix} \tag{4.44}$$

If we consider only terms up to of order Δ we obtain:

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_0 \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} \frac{-2(\lambda_1+\lambda_2)}{2+(\lambda_1+\lambda_2)\Delta} \\ \lambda_1\lambda_2 \\ \frac{\beta_0(2-(\lambda_1+\lambda_2)\Delta)}{2(2+(\lambda_1+\lambda_2)\Delta)} \end{bmatrix} \xrightarrow{\Delta\to 0} \begin{bmatrix} -(\lambda_1+\lambda_2) \\ \lambda_1\lambda_2 \\ \beta_0/2 \end{bmatrix} \tag{4.45}$$

which corresponds to the result in (4.28).

□

The above results show that sampling zeros should be considered to obtain a sampled-data model accurate enough for estimation. Even though the sampling zero for the exact discrete-time model (4.7) goes asymptotically to infinity (in the γ-domain), if it is not considered, then the parameter estimates will be generically biased (for equation error structures).

4.3.2 Effect of sampling zeros in stochastic systems

A particular case of the above problem for stochastic systems has been studied in detail by (Söderström *et al.*, 1997; Larsson and Söderström, 2002; Larsson, 2003). These papers deal with continuous-time auto-regressive (CAR) system identification from sampled data. Such systems have relative degree n, where n is the order of the auto-regressive process. Thus, consider a system described by:

$$E(\rho)y(t) = \dot{v}(t) \tag{4.46}$$

where $\dot{v}(t)$ represents a continuous-time white noise (CTWN) process, and $E(\rho)$ is a polynomial in the differential operator $\rho = \frac{d}{dt}$, *i.e.*,

$$E(\rho) = \rho^n + a_{n-1}\rho^{n-1} + \ldots + a_0 \tag{4.47}$$

For these systems, it has been shown that one cannot ignore the presence of stochastic sampling zeros. Specifically, if derivatives are replaced by divided differences and the parameters are estimated using ordinary least squares, then the results are asymptotically biased, even when using fast sampling rates (Söderström *et al.*, 1997). Note, however that the exact discrete-time model that describes the continuous-time system (4.46) takes the following generic form:

$$E_q(q^{-1})y(k\Delta) = F_q(q^{-1})w_k \tag{4.48}$$

where w_k is a discrete-time white noise process, and E_q and F_q are polynomials in the backward shift operator q^{-1}.

As we have already seen in Chapter 2, the polynomial $E_q(q^{-1})$ in equation (4.48) is *well behaved* in the sense that it converges naturally to its continuous-time counterpart. This relationship is most readily portrayed using the delta form:

$$E_\delta(\delta) = \delta^n + \bar{a}_{n-1}\delta^{n-1} + \ldots + \bar{a}_0 \tag{4.49}$$

Using (4.49), it can be shown that, as the sampling period Δ goes to zero:

$$\lim_{\Delta\to 0} \bar{a}_i = a_i \quad ; i = n-1, \ldots, 0 \tag{4.50}$$

However, as seen in Section 2.4, the polynomial $F_q(q^{-1})$ contains the stochastic sampling zeros, with no continuous-time counterpart. Thus, to obtain the correct estimates — say via the prediction error method (Ljung, 1999) — then one needs to minimise the cost function:

$$J_{PEM} = \sum_{k=1}^{N} \left[\frac{E_q(q^{-1})y(k\Delta)}{F_q(q^{-1})} \right]^2 \tag{4.51}$$

Notice the key role played by the sampling zeros in the above expression. A simplification can be applied, when using high sampling frequencies, by replacing the polynomial $F_q(q^{-1})$ by its asymptotic expression. However, this polynomial **has to be taken into account** when estimating over full bandwidth. Hence it is not surprising that the use of ordinary least squares, *i.e.*, a cost function of the form:

$$J_{LS} = \sum_{k=1}^{N} \left[E_q(q^{-1})y(k\Delta) \right]^2 \tag{4.52}$$

leads to (asymptotically) biased results, even when using (4.49). We illustrate these ideas by the following example.

Example 4.10 *Consider the continuous-time system defined by the* nominal model*:*

$$E(\rho)y(t) = \dot{v}(t) \tag{4.53}$$

where $\dot{v}(t)$ is a CTWN process with (constant) spectral density equal to 1*, and*

$$E(\rho) = \rho^2 + 3\rho + 2 \tag{4.54}$$

From Example 2.32 on page 32, we know that the equivalent sampled-data model has the form:

$$Y(z) = \frac{F_q(z)}{E_q(z)} W(z) = \frac{K(z - z_1)}{(z - e^{-\Delta})(z - e^{-2\Delta})} W(z) \tag{4.55}$$

Moreover, from Example 2.39, we know that, as the sampling rate increases, the sampled model converges to:

$$\frac{F_q(z)}{E_q(z)} \xrightarrow{\Delta\approx 0} \frac{\Delta^2}{3!} \frac{(z - z_1^*)}{(z - e^{-\Delta})(z - e^{-2\Delta})} \tag{4.56}$$

where $z_1^* = -2 + \sqrt{3}$ is the asymptotic stochastic sampling zero, which corresponds to the stable root of the sampling zero polynomial $B_3(z) = z^2 + 4z + 1$ (see Section 2.5).

For simulation purposes we used a sampling frequency $\omega_s = 250$[rad/s]. Note that this frequency is two decades above the fastest system pole, located at $s = -2$. We performed $N_{sim} = 250$ simulations, using $N = 10000$ data points in each run.

Test 1: *If one uses ordinary least squares as in* (4.52)*, then one finds that the parameters are (asymptotically) biased, as discussed in detail in (Söderström* et al.*, 1997). The continuous-time parameters* *are extracted by converting to the delta form and then using* (4.50)*. We obtain the following (mean) parameter estimates:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 1.9834 \\ 1.9238 \end{bmatrix} \tag{4.57}$$

In particular, we observe that the estimate $\hat{a}_1$ is clearly biased with respect to the continuous-time value $a_1 = 3$.

Test 2: *We next perform least squares estimation of the parameters, but with prefiltering of the data by the asymptotic sampling zero polynomial,* i.e.*, we use the sequence of filtered output samples given by:*

$$y_F(k\Delta) = \frac{1}{1 + (2 - \sqrt{3})q^{-1}} y(k\Delta) \tag{4.58}$$

Note that this strategy is essentially as in (Larsson and Söderström, 2002; Larsson, 2003).

Again, we extract the continuous-time parameters by converting to the delta form and using (4.50)*. We obtain the following estimates for the coefficients of the polynomial* (4.54)*:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 2.9297 \\ 1.9520 \end{bmatrix} \tag{4.59}$$

The residual small bias in this case can be explained by the use of the asymptotic sampling zero in (4.56)*, while the sampling period Δ is finite.*

□

In the previous example we obtained an asymptotically biased estimation of the parameter $\hat{a}_1$ when the sampling zeros are ignored. In fact, the estimates obtained in (4.57) are predicted by the following lemma. This presents the stochastic counterpart of Lemma 4.9, namely, the asymptotic parameter estimates obtained when using the simple derivative replacement approach for second order CAR systems.

Lemma 4.11 *Consider the second order continuous-time autoregressive system:*

$$\frac{d^2}{dt^2} y(t) + \alpha_1 \frac{d}{dt} y(t) + \alpha_o y(t) = \dot{v}(t) \tag{4.60}$$

where $\dot{v}(t)$ is a continuous-time white noise process. Assume that a sequence $\{y_k = y(k\Delta)\}$ is obtained sampling instantaneously the system output. If an equation error procedure is used to estimate the parameters of (4.60) *using the model:*

$$\delta y^2 + \hat{\alpha}_1 \delta y + \hat{\alpha}_o y = e \tag{4.61}$$

Then, as the sampling period Δ goes to zero, the parameters goes to:

$$\hat{\alpha}_1 \;\rightarrow\; \tfrac{2}{3}\alpha_1 \qquad \hat{\alpha}_0 \;\rightarrow\; \alpha_0 \tag{4.62}$$

Proof. The proof follows similar lines to the proof of Lemma 4.9 on page 71. The details can be found in (Söderström *et al.*, 1992).

□

Up to this point, we have considered undermodelling errors that arise when *sampling zeros* (stochastic or deterministic) are not considered in the discrete-time model. In the next subsection, we show that high frequency modelling errors in continuous-time can be have an equally catastrophic effect on parameter estimation.

4.3.3 Continuous-time undermodelling

In this section, we illustrate the consequences of unmodelled dynamics in the continuous-time model, when using estimation procedures based on sampled data. Our focus will be on the case of stochastic systems, however, similar issues arise for deterministic system.

The input of a stochastic system is assumed to be a continuous-time white noise (CTWN) process. However, such a process is only a mathematical abstraction and does not physically exist (see Remark 2.24 on page 28). In practice, we will have *wide-band* noise processes as disturbance. This is equivalent to a form of high frequency undermodelling.

The solution of the CAR identification problem for sampled data would seem to be straightforward given the discussion in the previous subsection. Apparently, one only needs to include the *sampling zeros* to get asymptotically unbiased parameter estimates using least squares. However, this ignores the issue of fidelity of the high frequency components of the model. Indeed, as pointed out before in Section 3.6, the system relative degree cannot be robustly defined for continuous-time models due to the presence of (possibly time-varying and ill-defined) high frequency poles or zeros. If one accepts this claim, then one cannot rely upon the integrity of the extra polynomial $F_q(q^{-1})$. In particular, the error caused by ignoring this polynomial (as suggested by the cost function (4.52)) might be as catastrophic as using a sampling zero polynomial arising from some hypothetical assumption about the relative degree. Thus, this class of identification procedures are inherently non-robust. We illustrate this by continuing Example 4.10.

Example 4.12 (Example 4.10 continued) *Let us assume that the* true model *for the system* (4.53) *is given by the polynomial:*

$$E(\rho) = E^o(\rho)(0.02\rho + 1) \tag{4.63}$$

where we have renamed the polynomial (4.54) *in the original model as* $E^o(\rho)$*. The* true *system has an unmodelled pole at* $s = -50$*, which is more than one decade above the fastest nominal pole in* (4.53)–(4.54)*, but almost one decade below the sampling frequency,* $\omega_s = 250$*[rad/s].*

We repeat the estimation procedure described in Test 2*, in Example 4.10, using the filtered least squares procedure. We obtain the following estimates:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 1.4238 \\ 1.8914 \end{bmatrix} \tag{4.64}$$

These are clearly biased, even though the nominal sampling zero has been included in the model.

To analyse the effect of different types of under-modelling, we consider the true *denominator polynomial* (4.63) *to be:*

$$E(\rho) = E^o(\rho)\left(\frac{\rho}{\omega_u} + 1\right) \tag{4.65}$$

We consider different values of the parameter ω_u *in* (4.65)*, using the same simulation conditions as in the previous examples (*i.e.*,* 250 *Monte Carlo runs using* 10000 *data points each). The results are presented in Figure 4.4. The figure clearly shows the effect of the unmodelled dynamics on the parameter estimates. We see that the undermodelling has an impact even beyond the sampling frequency, which can be explained in terms of the inherent folding effect of the sampling process.*

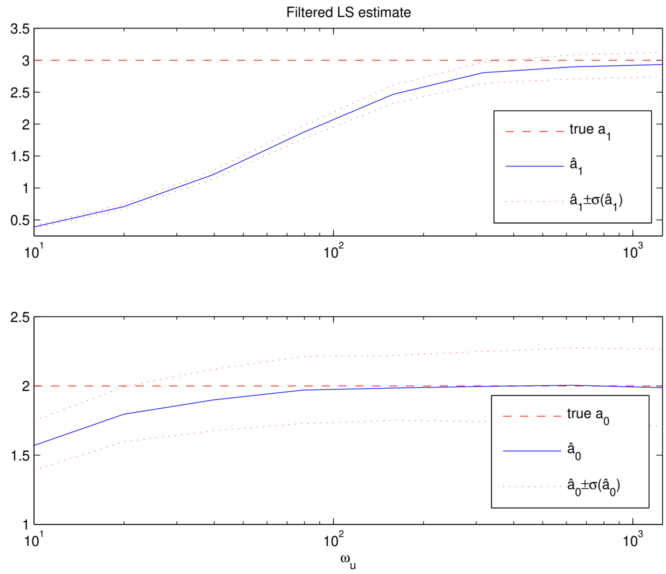


Figure 4.4: Mean of the parameter estimates as a function of the unmodelled dynamics, using filtered LS.

Figure 4.5 shows similar simulations results using an instrumental variable (IV) estimator. The IV-estimator is a basic IV method where the IV vector consists of observations of $y(t)$ *delayed one sampling period (Bigi* et al.*, 1994).*

□

The difficulties discussed above arise due to the fact that the true high frequency characteristics are not exactly as hypothesised in the algorithm. Thus, the folding that occurs is **not** governed by the anticipated sampling zero polynomial that is used to prefilter the data.

4.3.4 Restricted bandwind FDML estimation

The examples presented in the previous subsections raise the question as to how these problems might be avoided or, at least, reduced, by using an identification procedure more robust to high frequency under-modelling. Our proposal to deal with this problem is to designate a *bandwidth of validity* for the model and, then, to develop an algorithm which is insensitive to errors outside that range. This is most easily done in the frequency domain.

In the following example we will use the FDML procedure presented in Lemma 4.5 to estimate the parameters of CAR systems as (4.46).

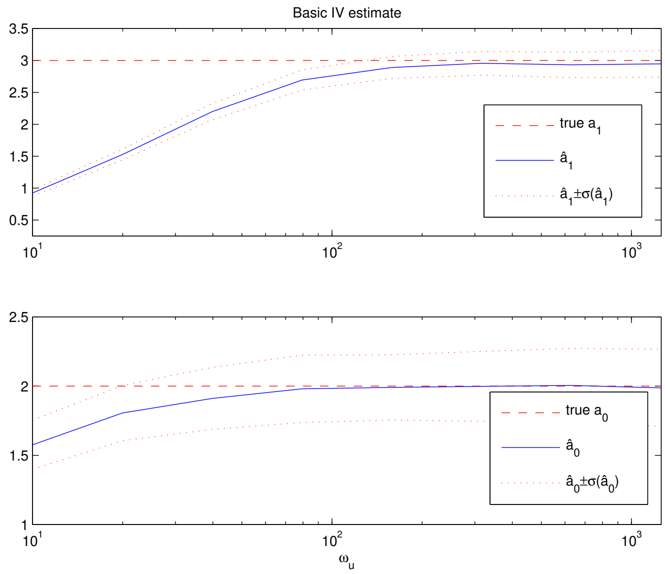


Figure 4.5: Mean of the parameter estimates as a function of the unmodelled dynamics, using simple delayed IV.

Remark 4.13 *For the a CAR system as in* (4.46)*, let us consider the (approximate) derivative replacement discrete-time model:*

$$E_q(q)y_k = w_k \tag{4.66}$$

where y_k is the sequence of instantaneous output samples of (4.46)*, and w_k is a discrete-time stationary Gaussian white-noise sequence with variance σ_w^2. Given N data points of the output sequence $y(k\Delta)$ sampled at ω_s[rad/s], the appropriate likelihood function, in the frequency domain, takes the form:*

$$L = \sum_{\ell=0}^{n_{max}} \frac{|E_q(e^{j\omega_\ell \Delta})Y(e^{j\omega_\ell \Delta})|^2}{N\sigma_w^2} - \log \frac{|E_q(e^{j\omega_\ell \Delta})|^2}{\sigma_w^2} \tag{4.67}$$

where $\omega_\ell = \frac{\omega_s \ell}{N}$ and n_{max} corresponds to the bandwidth to be considered, i.e.,*, $\omega_{max} = \frac{\omega_s n_{max}}{N}$.*

Example 4.14 *We consider again the CAR system presented in 4.10. If we use the result in Lemma 4.5, using the full bandwidth $[0, \pi/\Delta]$ (or, equivalently, up to 125[rad/s]) we obtain the following (mean) value for the parameter estimates:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 4.5584 \\ 1.9655 \end{bmatrix} \tag{4.68}$$

As expected, these parameters are clearly biased because we are not taking into account the presence of the sampling zero polynomial in the true model.

Next we consider an estimation procedure restricted to a certain ***bandwidth of validity****. For example, the usual rule of thumb is to consider up to one decade above the fastest nominal system pole, in this case, 20[rad/s]. The resultant (mean of the) parameter estimates are then given by:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 3.0143 \\ 1.9701 \end{bmatrix} \tag{4.69}$$

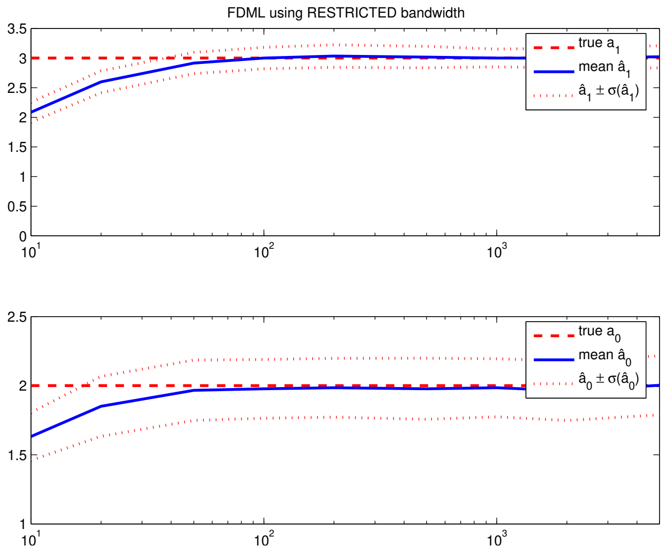


Figure 4.6: Parameter estimates using FDML as a function of unmodelled pole.

Note that these estimates are essentially equal to the (continuous-time) true values. Moreover, no prefiltering as in (4.51) *or* (4.58) *has been used! Thus, one has achieved robustness to the relative degree at high frequencies since it plays no role in the suggested procedure. Moreover, the sampling zeros can be ignored since their impact is felt only at high frequencies.*

Finally, we show that the frequency domain procedure is also robust to the presence of unmodelled fast poles. We consider again the true system to be as in (4.63)*. We restrict the estimation bandwidth up to* 20*[rad/s]. In this case, the mean of the parameter estimates is again very close to the nominal system coefficients,* i.e.*, we obtain:*

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_0 \end{bmatrix} = \begin{bmatrix} 2.9285 \\ 1.9409 \end{bmatrix} \tag{4.70}$$

A more general situation is shown in Figure 4.6. The figure shows the parameter estimates obtained using the proposed FDML procedure, with the same restricted bandwidth used before $\omega_{\max} =$ 20*[rad/s], for different locations of the unmodelled fast pole* ω_u*, as in* (4.65).

□

Remark 4.15 *Note that the likelihood function* (4.67) *is not scalable by* σ_w^2 *and hence one needs to also include this parameter in the set to be estimated. This is an important departure from the simple least squares case.*

4.4 Summary

In this chapter we have explored the robustness issues that arise in the identification of continuous-time systems from sampled data. A key observation is that the fidelity of the models at high frequencies generally plays an important role in obtaining models suitable for continuous-time system identification. In particular, we have shown that:

- Sampling zeros may have to be included in the discrete-time models to obtain accurate sampled-data descriptions.
- Unmodelled high frequency dynamics in the continuous-time model can have a critical impact on the quality of the estimation process when using sampled-data.

This implies that any result which implicitly or explicitly depends upon the folding of high frequency components down to lower frequencies will be inherently *non-robust*. As a consequence, we argue that models have to be considered within a *bandwidth of validity*.

To address these issues, we have proposed the use of frequency domain maximum likelihood estimation, using a restricted bandwidth. We have shown that this approach is robust to both the presence of sampling zeros and to high frequency modelling errors in continuous-time.

The problems discussed above have been illustrated for both, deterministic and stochastic, systems. Special attention was given to the identification of continuous-time auto-regressive stochastic models from sampled data. We have argued that traditional approaches to this problem are inherently sensitive to high frequency modelling errors. We have also argued that these difficulties can be mitigated by using the proposed FDML with restricted bandwidth.

Chapter 5

Sampled-data models in LQ problems

5.1 Introduction

In this chapter we consider a particular application of sampled-data models, namely, their use for Linear-Quadratic (LQ) optimal control problems. In particular, we examine the presence of input and/or state constraints, when using fast sampling rates. We present two main convergence results. These establish connections between continuous-time and sampled-data optimal control problems. First, we show that the constrained control problem in discrete-time has a well defined limit as the sampling rate increases. An immediate consequence of this result is the existence of a finite sampling period such that the achieved performance is arbitrarily close to the limiting performance, obtained by the hypothetical continuous-time control law. The second result considers the singular structure of sampled data LQ problems for continuous-time linear systems. We show that, as the sampling rate is increased, there is a natural convergence between the finite set of singular values of the discrete time problem, and the (infinite) countable set in continuous time.

Connections between continuous-time and sampled-data optimal control strategies have previously been addressed in the literature (Middleton and Goodwin, 1990; Feuer and Goodwin, 1996; Chen and Nett, 1995) for the *unconstrained* case. However, the constrained case has received less attention, because of the inherent difficulties involved in solving the constrained control problems in continuous-time (Berkovitz, 1974; Vinter, 2000). However, in discrete-time, different numerical procedures and algorithms can be implemented to deal with this kind of problems.

An increasingly common strategy to deal with constraints in control is Receding Horizon or Model Predictive Control (MPC) strategies (Sznaier and Damborg, 1987; Rawlings and Muske, 1993; Scokaert and Rawlings, 1998; Mayne *et al.*, 2000; De Oliveira Kothare and Morari, 2000; Cannon *et al.*, 2001; Cheng and Krogh, 2001; Goodwin *et al.*, 2004). This is a form of (discrete-time) control in which the current control action is obtained by solving *on-line*, at each sampling instant, a finite horizon optimal control problem for the open-loop plant using the current (observed) state as initial condition. The first component of the optimal control sequence is applied to the system, and the procedure is repeated again at the next sampling instant. One of the key advantages of these strategies is that constraints can be taken into account in the optimisation procedure, for example, via Quadratic Programming (QP) (Van

De Panne, 1975; Sznaier and Damborg, 1987; Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998).

Traditionally, MPC has been applied to systems having long time constants, e.g. petrochemical processes. However, increasing computational power and recent advances in this area, such as *off-line* implementations of the solution (Bemporad *et al.*, 2002; Serón *et al.*, 2003), have *opened the door* to short time constant applications including aerospace, automobile control and electro-mechanical servo problems. These applications typically use fast sampling rates and a common design goal is that the sample-hold nature of the input should have minimal impact on the achievable performance. This goal is consistent with the control of linear systems in the absence of constraints. Indeed, it is common for the sampling rate to be chosen so that the response of the sampled-data control system is *practically indistinguishable* from the corresponding continuous-time solution. The goal of constrained control will often be similar, *i.e.*, it is desirable to choose a sampling period such that the artifacts of sampling are *practically unobservable*.

We begin in Section 5.2 by presenting the (constrained) LQ optimal control problem, both in continuous- and discrete-time. In particular, the latter problem is formulated in such a way that it corresponds to the sampled-data version of the underlying continuous-time problem. A particular feature of the discrete-time problem statement is that the input and/or state constraints are tightened by a scaling factor $\alpha(\Delta)$, whose choice is explained later.

The analysis presented in Section 5.3 shows that there exists a finite sampling period such that the sampled-data response of a finite horizon constrained linear controller is arbitrarily close to the response which would be achieved by a continuous-time constrained linear controller. Previous work having a connection with our work has been reported in (Kojima and Morari, 2004), where the finite horizon constrained LQ problem is solved in continuous-time, using spectral properties. In their approach, however, the constraints are satisfied only at a finite set of points over the control horizon. By way of contrast, we show that the continuous-time solution, with constraints imposed for *all* time, can be arbitrarily approximated using a standard discrete-time approach provided one chooses a suitably fast sampling rate and, possibly, tighter constraints at the sampling instants.

In Section 5.4 we study the spectral properties of the optimal control problem, both for the continuous-time formulation and its corresponding sampled-data version. Specifically, we obtain the singular structure in continuous- and discrete-time domains. We show that, as the sampling rate is increased, there is a natural convergence between the finite set of singular values of the discrete time problem, and the (infinite) countable set in continuous time.

The motivation for this work was the results reported in (Kojima and Morari, 2001), where a singular value decomposition of linear operators is used to approximate the solution of constrained LQ problems in continuous time. Related work regarding singular value structures has also been reported in the context of cross directional control (Rojas *et al.*, 2002) and constrained receding horizon control (Rojas *et al.*, 2003; Rojas and Goodwin, 2004) for discrete time systems. This body of work raises the more general system theoretic question regarding the connection between the singular value structure of discrete-time LQ problems and the associated continuous-time case. A deeper understanding of this connection could, for example, lead to approximate algorithms for the continuous time problem, which are solved using standard discrete time methods. Moreover, the existence of a well defined limit as the sampling rate increases could be exploited in high speed applications, using ad-hoc algorithms for constrained systems in MPC strategies. Furthermore, connections with intrinsic properties of the continuous-time system, such as its frequency response, have also been established (Rojas and Goodwin, 2004; Rojas *et al.*, 2004; Rojas, 2004).

5.2 Linear-Quadratic optimal control problems

In this section we present the LQ optimal control problem formulation in continuous-time and its corresponding sampled-data version. The latter problem is usually solved in *open loop* using discrete-time MPC strategies at each sampling instant. Then a *closed loop* implementation is obtained by exploiting a receding horizon strategy. We will initially focus our analysis on the fixed horizon case. The moving horizon problem will be discussed later in Section 5.3.2.

In the following subsections we consider two related problems, one defined in continuous-time and an associated discrete-time sampled-data problem.

5.2.1 Continuous-time problem

As a benchmark problem, we consider a fixed horizon constrained control problem $\mathcal{P}$, defined in the continuous-time domain as follows:

(i) A continuous-time model in state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; x(0) = x_o \tag{5.1}$$
$$y(t) = Cx(t) \tag{5.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$.

(ii) A fixed time horizon $T_f < \infty$.

(iii) A quadratic cost function:

$$J(u) = J_1 + J_\infty \tag{5.3}$$

where:

$$J_1 = \int_0^{T_f} \left(x(t)^T Qx(t) + u(t)^T Ru(t) \right) dt \tag{5.4}$$
$$J_\infty = x(T_f)^T P\, x(T_f) \tag{5.5}$$

with $Q \geq 0$, $R > 0$, and where the final state weighting matrix, P, gives rise to the infinite horizon optimal *unconstrained* cost associated with the cost J_1 when the initial state is $x(T_f)$. Thus P satisfies the *continuous-time* algebraic Riccati equation:

$$0 = Q + A^T P + PA - PBR^{-1}B^T P \tag{5.6}$$

(iv) And a set of (continuous-time) state and/or input constraints, written in the following general form:

$$\begin{aligned} L_u\, u(t) &\leq M_u \\ L_x\, x(t) &\leq M_x \end{aligned} \tag{5.7}$$

for all $t \in [0, T_f]$. Note that L_u will typically have rank m whilst L_x will have rank less than or equal to n.

Let $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{X} \subset \mathbb{R}^n$ be the sets of all possible values of $u(t)$ and $x(t)$, $t \in [0, T_f]$, such that (5.1) and (5.7) are satisfied. Then, the only requirements on L_u, L_x, M_u, and M_x are that the sets $\mathcal{U}$ and $\mathcal{X}$ are bounded (possibly as a function of x_o in the case of $\mathcal{X}$) and contain the origin of the respective space — see (Chmielewski and Manousiouthakis, 1996). Note that this implies that all entries in both M_u and M_x are positive.

Given (i)–(iv), the continuous-time problem $\mathcal{P}$ is defined to be: find the optimal control signal $u^* = u^*(t)$ such that the cost function (5.3) is minimised, *i.e.*,

$$u^*(t) = \arg \min_{u(t) \in \mathcal{U}} J(u) \tag{5.8}$$

Remark 5.1 *There has been substantial work, spanning three centuries, but particularly since the 1950's, on the general conditions under which optimal control problems such as $\mathcal{P}$ have a solution. These results give necessary and sufficient conditions under which a solution is truly a minimiser. Moreover, existence theorems for this minimiser assume that the problem is* ***feasible****,* i.e.*, that the class of admissible pairs $\{u(t), x(t)\}$ which satisfy the system dynamic equation (5.1) and constraints (5.7) is non-empty (Berkovitz, 1974; Vinter, 2000).*

We will thus assume the existence of a solution of the continuous-time constrained optimal control problem $\mathcal{P}$, even though an explicit expression will not be obtained. Instead, we show that, subject to existence, the solution can be approximated to any desired degree of accuracy by solving an associated sampled-data constrained optimal control problem $\mathcal{P}_\Delta$, described in the next subsection.

Remark 5.2 *Note that it is common in Receding Horizon Control to utilise a final state weighting matrix as in* (5.5) *(see also equation* (5.14) *below). This choice ensures that, if the constrains are not* active *at the end of the fixed horizon T_f, the cost* (5.3) *represents in fact the infinite horizon cost (Chmielewski and Manousiouthakis, 1996; Mayne* et al.*, 2000; Goodwin* et al.*, 2004).*

5.2.2 Sampled-data problem

A natural way to approximate the continuous-time problem $\mathcal{P}$ is to use a (small) sampling period Δ together with a Zero Order Hold (ZOH) approximation to the input signal. In this framework, the optimal solution can be found using standard numerical algorithms such as QP.

For a given sampling interval Δ, we define $u_\Delta(t)$ as the piece-wise constant continuous-time input to the system, generated by a Zero Order Hold (ZOH) as in (2.10) on page 13, *i.e.*,

$$u(t) = u_\Delta(t) = u_k \quad ; k\Delta \leq t < k\Delta + \Delta \tag{5.9}$$

where $k \in \mathbb{Z}$ is the discrete time domain index. Furthermore, we assume that the sampling interval is an integer fraction of the fixed time horizon T_f, *i.e.*,

$$\Delta = \frac{T_f}{N} \iff N\Delta = T_f \tag{5.10}$$

for some $N \in \mathbb{N}$.

Given the continuous-time problem $\mathcal{P}$, we next define an associated discrete-time constrained optimal control problem $\mathcal{P}_\Delta$. This, in fact, corresponds to a sampled-data version of problem $\mathcal{P}$, but where the input and state constraints have been slightly modified. The choice of the scaling factor in the discrete-time constraints will be clarified later in Section 5.3.

We consider:

(i) A discrete-time model expressed in state-space form:

$$x_{k+1} = A_q x_k + B_q u_k \quad ; x_o \text{ as in (5.1)} \tag{5.11}$$

$$y_k = Cx_k \tag{5.12}$$

As discussed earlier in Section 2.2, if the system matrices are given by:

$$A_q = e^{A\Delta} \quad ; \quad B_q = \int_0^\Delta e^{A\eta} B \, d\eta \tag{5.13}$$

the model (5.11)–(5.12) corresponds, in fact, to the sampled version of the continuous-time system (5.1)–(5.2), *i.e.*, $x_k = x(k\Delta)$ and $y_k = y(k\Delta)$.

(ii) A fixed discrete-time horizon $N = \frac{T_f}{\Delta}$.

(iii) A quadratic cost function:

$$J_\Delta(u_k) = \sum_{k=0}^{N-1} \begin{bmatrix} x_k^T & u_k^T \end{bmatrix} \begin{bmatrix} Q_q & S_q \\ S_q^T & R_q \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_\Delta \, x_N \tag{5.14}$$

where:

$$Q_q = \int_0^\Delta e^{A^T t} Q e^{At} \, dt \tag{5.15}$$

$$S_q = \int_0^\Delta e^{A^T t} Q h(t) \, dt \tag{5.16}$$

$$R_q = \int_0^\Delta h(t)^T Q h(t) \, dt + R\Delta \tag{5.17}$$

$$h(t) = \int_0^t e^{A\tau} B \, d\tau \tag{5.18}$$

and where P_Δ satisfies the following *discrete-time* algebraic Riccati equation:

$$P_\Delta = Q_q + A_q^T P_\Delta A_q - \left(A_q^T P_\Delta B_q + S_q\right)\left(R_q + B_q^T P_\Delta B_q\right)^{-1}\left(B_q^T P_\Delta A_q + S_q^T\right) \tag{5.19}$$

(iv) And a set of **discrete-time** constraints for the state and/or the input signal, written in the following form:

$$\begin{aligned} L_u\, u_k \; &\leq \alpha(\Delta) M_u \\ L_x\, x_k \; &\leq \alpha(\Delta) M_x \end{aligned} \tag{5.20}$$

for all $k \in \{0, \ldots, N-1\}$ and where $\alpha(\Delta) \in (0, 1]$ is a scaling factor, whose choice will be discussed in the next section.

Given (i)–(iv), the sampled-data problem $\mathcal{P}_\Delta$ is to find the optimal control sequence $u^* = u^*_{\Delta,k}$ such that the cost function (5.14) is minimised, *i.e.*,

$$u^*_{\Delta,k} = \arg\min_{u_k} J_\Delta(u_k) \tag{5.21}$$

The previous formulation of the discrete-time problem $\mathcal{P}_\Delta$ is clearly justified by the following two remarks.

Remark 5.3 *It can be readily shown (Middleton and Goodwin, 1990; Åström and Wittenmark, 1997) that the choices (5.15) to (5.19) ensure that:*

$$J(u_\Delta) = J_\Delta(u_k) \tag{5.22}$$

where $J_\Delta(u_k)$ denotes the discrete-time cost value, as defined in (5.14), when the control sequence u_k is applied to the system (5.11)–(5.12), and $J(u_\Delta)$ denotes the continuous-time cost value in (5.3)–(5.6) when $u_\Delta(t)$, as defined in (5.9), is applied to the continuous-time system (5.1)–(5.2).

Remark 5.4 *It has been shown earlier in this thesis that delta operator models provide a natural connection between discrete- and continuous-time. In fact, as the sampling period goes to zero, we have that (Feuer and Goodwin, 1996; Middleton and Goodwin, 1990):*

$$\frac{A_q - I}{\Delta} \to A, \quad \frac{B_q}{\Delta} \to B, \quad \frac{Q_q}{\Delta} \to Q, \quad \frac{S_q}{\Delta} \to 0, \quad \frac{R_q}{\Delta} \to R, \quad P_\Delta \to P \tag{5.23}$$

*This means that the **unconstrained** sampled-data problem given by (i)–(iii) converges to the description of the underlying **unconstrained** continuous-time problem given by (i)–(iii) in Section 5.2.1.*

In the next section we consider the corresponding constrained case. We show that, under some additional requirements on the scaling factor $\alpha(\Delta)$, the formulation of the sampled-data problem $\mathcal{P}_\Delta$ converges, as $\Delta \to 0$, to the formulation of problem $\mathcal{P}$, defined in continuous-time.

5.3 Constrained control using fast sampling rates

While the explicit solution of the continuous time problem $\mathcal{P}$ is very difficult to obtain, solving $\mathcal{P}_\Delta$ for a given sampling period Δ is relatively straightforward using standard numerical procedures, *e.g.*, quadratic programming (QP) (Goodwin *et al.*, 2004).

In this section, we present results that will help us to relate the solution of $\mathcal{P}_\Delta$ to the solution of $\mathcal{P}$. We first discuss the scalar $\alpha(\Delta)$ introduced earlier, in equation (5.20), as a scaling factor associated with the discrete-time constraints (5.20). We then present two results that will be used later, in Theorem 5.11 in the next section, to establish the convergence of the solution of $\mathcal{P}_\Delta$ to the solution of $\mathcal{P}$.

We first introduce some definitions and notation that will be used in the sequel. Given the continuous-time system (with initial condition x_o) in (5.1), and the constraints (5.7), we define the following sets:

$$S = \{u \in \mathcal{L}_2[0, T_f] : L_u\, u(t) \leq M_u \text{ and } L_x\, x(t) \leq M_x\,,\ \forall t \in [0, T_f]\} \tag{5.24}$$

and:

$$\begin{aligned} S_\Delta = \big\{u \in \mathcal{L}_2[0, T_f] : L_u\, u(t) \leq \alpha(\Delta) M_u\,,\ \forall t \in [0, T_f] \\ \text{and} \quad L_x\, x(k\Delta) \leq \alpha(\Delta) M_x\,,\ \forall k \in \{0, \ldots, N-1\}\big\} \end{aligned} \tag{5.25}$$

Remark 5.5 *Note that the set S contains all possible signals $u(t)$ among which we need to find the one that minimises the cost function $J(u)$ in* (5.3)–(5.6), *i.e.,* $u^*(t) \in S \subset \mathcal{L}_2[0, T_f]$.

On the other hand, every sequence u_k (including u_k^*, the solution of $\mathcal{P}_\Delta$) satisfying the difference equation (5.11) *and the constraints* (5.20)*, will generate a piece-wise signal $u_\Delta(t)$, as in* (5.9)*, that belongs to S_Δ. However, even if we choose a scaling factor $\alpha(\Delta) < 1$, assuming $u_\Delta(t) \in S_\Delta$ is not* sufficient *to ensure that $u_\Delta(t) \in S$, because of the inter-sample state trajectory.*

The previous remark highlights the fact that further conditions are required on the scalar $\alpha(\Delta)$ to ensure that, when using a ZOH to implement in continuous-time the solution of the discrete-time problem $\mathcal{P}_\Delta$, the conditions of the continuous-time problem $\mathcal{P}$ are also satisfied.

Given a sequence of sampling periods $\{\Delta_i > 0\}$, such that:

$$\Delta_i > \Delta_{i+1} \quad \text{and} \quad \lim_{i\to\infty} \Delta_i = 0 \tag{5.26}$$

we require the following properties for the corresponding scaling factors $\alpha(\Delta_i)$:

$$\lim_{i\to\infty} \alpha(\Delta_i) = \lim_{\Delta_i \to 0} \alpha(\Delta_i) = 1 \tag{5.27}$$

and for the resulting sequence of sets $\{S_{\Delta_i}\}$:

$$S_{\Delta_i} \subseteq S_{\Delta_{i+1}} \tag{5.28}$$

Remark 5.6 *Condition* (5.27) *ensures that, as the sampling rate increases, the sequence of discrete-time constraints* (5.20) *approaches the continuous-time constraints in* (5.7)*. Moreover, from definitions in* (5.24) *and* (5.25)*, we have that:*

$$\lim_{i\to\infty} S_{\Delta_i} = \lim_{\Delta_i \to 0} S_{\Delta_i} = S \tag{5.29}$$

Furthermore, the requirement (5.28) *ensures that the sequence of sets $\{S_{\Delta_i}\}$ approach the set S* from the interior*, i.e.,*

$$S_{\Delta_0} \subseteq S_{\Delta_1} \subseteq \ldots \subseteq S \tag{5.30}$$

□

A particular choice for the sequences $\{\Delta_i\}$ and $\{\alpha(\Delta_i)\}$ ensuring that the above requirements are satisfied, is described in the following result:

Lemma 5.7 *Consider the sequence of sampling intervals* $\{\Delta_i\}$*, defined by:*

$$\Delta_i = \frac{2^{2-i}}{\|A\|} \ln \left(\frac{\sqrt{1+4\gamma^2}}{2\gamma} \right) \tag{5.31}$$

Then, one particular choice for the scaling factor $\alpha(\Delta)$ *that satisfies* (5.27)–(5.28) *is given by:*

$$\alpha(\Delta) = \frac{1}{1+(2\gamma+1)(e^{\|A\|\Delta}-1)} \tag{5.32}$$

where γ *is a constant given by:*

$$\gamma = \left(\max_i \frac{\|L_x^i\|}{M_x^i} \right) \left(X + \frac{\|B\|}{\|A\|} U \right) \tag{5.33}$$

and where L_x^i *and* M_x^i *are the* i*-th row and* i*-th entry of* L_x *and* M_x *respectively, and where* X *and* U *denote upper bounds on the norms of* $x(t)$ *and* $u(t)$ *in the bounded sets* $\mathcal{X}$ *and* $\mathcal{U}$*, respectively, defined in Section 5.2.1.*

Proof. We show that the choice of the sequence of sampling periods $\{\Delta_i\}$ and the scaling factor $\alpha(\Delta)$ satisfies the given requirements (5.26)–(5.28).

If the sampling period Δ_i is chosen as in (5.31), we clearly have the strictly decreasing condition in (5.26). As a consequence, the scaling factor $\alpha(\Delta)$ defined as in (5.32) satisfies condition (5.27).

To show that (5.28) holds, let us take any $u \in S_{\Delta_i}$, which by definition (5.25) implies that:

$$L_u\, u(t) \le \alpha(\Delta_i) M_u \qquad \text{and} \qquad L_x\, x(k\Delta_i) \le \alpha(\Delta_i) M_x \tag{5.34}$$

Since $\alpha(\Delta_i) < \alpha(\Delta_{i+1})$ we have that $L_u\, u(t) \le \alpha(\Delta_{i+1}) M_u$. (Recall that the entries of M_u are positive). Furthermore, solving the differential equation (5.1) for $t = \sigma + k\Delta_i$, $0 \le \sigma < \Delta_i$, we have:

$$\begin{aligned}
L_x\, x(t) &= L_x \left(e^{A\sigma} x(k\Delta_i) + \int_0^\sigma e^{A(\sigma-\tau)} B u(\tau + k\Delta_i)\, d\tau \right) \\
&= L_x x(k\Delta_i) + L_x \left((e^{A\sigma} - I) x(k\Delta_i) + \int_0^\sigma e^{A(\sigma-\tau)} B u(\tau + k\Delta_i)\, d\tau \right) \\
&\le \alpha(\Delta_i) M_x + \begin{bmatrix} \|L_x^1\| \\ \|L_x^2\| \\ \vdots \end{bmatrix} \left(\left\| e^{A\sigma} - I \right\| \|x(k\Delta_i)\| + \left\| \int_0^\sigma e^{A(\sigma-\tau)} B u(\tau + k\Delta_i)\, d\tau \right\| \right) \\
&\le \alpha(\Delta_i) M_x + M_x \left(\max_r \frac{\|L_x^r\|}{M_x^r} \right) \\
&\qquad \left((e^{\|A\|\sigma} - 1) \|x(k\Delta_i)\| + \int_0^\sigma e^{\|A\|(\sigma-\tau)} \|B\| \, \|u(\tau + k\Delta_i)\| d\tau \right)
\end{aligned} \tag{5.35}$$

From the definitions of the bounds X and U and equation (5.34) we have:

$$\|x(k\Delta_i)\| \le \alpha(\Delta_i) X \quad \text{and} \quad \|u(\tau + k\Delta_i)\| \le \alpha(\Delta_i) U \tag{5.36}$$

Then:

$$\begin{aligned}L_x\, x(t) &\le \alpha(\Delta_i)M_x + \alpha(\Delta_i)M_x \left(\max_r \frac{\|L_x^r\|}{M_x^r}\right)\left(e^{\|A\|\sigma}-1\right)\left(X+\frac{\|B\|}{\|A\|}U\right)\\ &\le \alpha(\Delta_i)M_x\left(1+(e^{\|A\|\Delta_i}-1)\gamma\right) = M_x\frac{1+\gamma(e^{\|A\|\Delta_i}-1)}{1+(2\gamma+1)(e^{\|A\|\Delta_i}-1)}\\ &\le M_x\frac{1+\gamma(e^{2\|A\|\Delta_{i+1}}-1)}{1+(2\gamma+1)(e^{2\|A\|\Delta_{i+1}}-1)}\end{aligned} \tag{5.37}$$

We next show that:

$$\frac{1+\gamma(e^{2\|A\|\Delta_{i+1}}-1)}{1+(2\gamma+1)(e^{2\|A\|\Delta_{i+1}}-1)} \le \frac{1}{1+(2\gamma+1)(e^{\|A\|\Delta_{i+1}}-1)} = \alpha(\Delta_{i+1}) \tag{5.38}$$

for:

$$0<\Delta_{i+1}\le \frac{1}{\|A\|}\ln\left(\frac{\sqrt{1+4\gamma^2}}{2\gamma}\right) \iff 1< e^{\|A\|\Delta_{i+1}} \le \frac{\sqrt{1+4\gamma^2}}{2\gamma} \tag{5.39}$$

After some manipulation, the inequality (5.38) is easily seen to be equivalent to:

$$\gamma(2\gamma+1)e^{2\|A\|\Delta_{i+1}} - (\gamma+1)e^{\|A\|\Delta_{i+1}} - 2\gamma^2 \le 0 \tag{5.40}$$

The left hand side is negative for both $e^{\|A\|\Delta_{i+1}}=1$ and $e^{\|A\|\Delta_{i+1}}=\frac{\sqrt{1+4\gamma^2}}{2\gamma}$, hence it is negative in the whole range which verifies (5.39). In (5.37) we then have that, for all $t\in[0,T_f]$:

$$L_x\, x(t) \le \alpha(\Delta_{i+1})M_x \;\text{ for }\; u\in S_{\Delta_i} \tag{5.41}$$

This implies, by definition, that $u \in S_{\Delta_{i+1}}$. Hence (5.28) follows.

□

We next present two additional technical results, which will be utilised in the proof of the main convergence result in Section 5.3.1.

Lemma 5.8 *Let $u(t)\in S$. Then, for any $\delta>0$ there exists a $\Delta_\delta>0$ such that, for all $\Delta\le\Delta_\delta$ and $u_\Delta(t)$ such that:*

$$u_\Delta(t) = u(k\Delta) \qquad ;\ k\Delta \le t < (k+1)\Delta \tag{5.42}$$

we have:

$$\|u-u_\Delta\|_2 < \delta \tag{5.43}$$

Proof. We consider the sequence of decreasing sampling periods $\{\Delta_i>0\}$ in (5.26). For every Δ_i, the piecewise constant signal $u_{\Delta_i}(t)$ belongs to $\mathcal{L}_2[0,T_f]$, because it is obtained by sampling and holding the signal $u(t)$, which belongs to $S\subseteq\mathcal{L}_2[0,T_f]$.

Thus, the sequence of functions $\{u_{\Delta_i}(t)\}$ converges point-wise almost everywhere (*i.e.*, except on a set of zero measure) to $u(t)$. Using (Lang, 1993, Theorem 1.6), this implies that $\{u_{\Delta_n}(t)\}$ converges to $u(t)$ in an $\mathcal{L}_2$ sense, *i.e.*,

$$\lim_{i\to\infty}\|u-u_{\Delta_i}\|_2 = \lim_{\Delta_i\to 0}\|u-u_{\Delta_i}\|_2 = 0 \tag{5.44}$$

□

Lemma 5.9 *For any* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *such that if:*

$$\|u - u^*\|_2 < \delta_\varepsilon \tag{5.45}$$

then:

$$J(u) - J(u^*) < \varepsilon \tag{5.46}$$

Proof. This follows from the operator factorisation approach that we will study in more detail later in Section 5.4.1 on page 94. In particular, if we use the definitions and relations detailed later in (5.67)–(5.74), the continuous-time cost function (5.3) — see also (5.71) — can be rewritten as:

$$J = J(\tilde{u}) = J_{opt} + \langle \tilde{u}\,,\, (R + \mathcal{G}^*\mathcal{G})\tilde{u}\rangle \tag{5.47}$$

where:

$$\tilde{u} = u - u_{opt} \tag{5.48}$$

$$u_{opt} = -(R + \mathcal{G}^*\mathcal{G})^{-1}\mathcal{G}^*\mathcal{F}x_o \tag{5.49}$$

$$J_{opt} = x_o^T\mathcal{F}^*(I + \mathcal{G}R^{-1}\mathcal{G}^*)^{-1}\mathcal{F}x_o \tag{5.50}$$

We note that u_{opt} and J_{opt} are the **unconstrained** optimal control signal and cost.

The operator $R + \mathcal{G}^*\mathcal{G}$ in (5.47) is a compact bounded self-adjoint operator. It can thus be expressed as:

$$R + \mathcal{G}^*\mathcal{G} = \mathcal{S}^*\mathcal{S} \tag{5.51}$$

We then have that, for $u_1, u_2 \in \mathcal{L}_2[0, T_f]$:

$$\begin{aligned} |J(u_1) - J(u_2)| &= |\langle\, \tilde{u}_1\,,\, \mathcal{S}^*\mathcal{S}\tilde{u}_1 \,\rangle - \langle\, \tilde{u}_2\,,\, \mathcal{S}^*\mathcal{S}\tilde{u}_2 \,\rangle| = |\langle\, \mathcal{S}\tilde{u}_1\,,\, \mathcal{S}\tilde{u}_1 \,\rangle - \langle\, \mathcal{S}\tilde{u}_2\,,\, \mathcal{S}\tilde{u}_2 \,\rangle| \\ &= \big|\|\, \mathcal{S}\tilde{u}_1 \,\|^2 - \|\, \mathcal{S}\tilde{u}_2 \,\|^2\big| \leq \|\, \mathcal{S}\tilde{u}_1 - \mathcal{S}\tilde{u}_2 \,\|^2 \\ &\leq \|\mathcal{S}\|^2 \,\|\, \tilde{u}_1 - \tilde{u}_2 \,\|^2 \leq \sigma_{max}\|u_1 - u_2\|^2 \end{aligned} \tag{5.52}$$

where $\sigma_{max} > 0$ is the largest singular value of the operator $\mathcal{S}$. The result then follows by taking $u_2 = u^*$, $\delta_\varepsilon = \sqrt{\varepsilon/\sigma_{max}}$, and on recalling that u^* is the optimal control signal, so $J(u) \geq J(u^*)$ for all u.

□

Remark 5.10 *Lemma 5.8 establishes that any input signal* $u(t) \in S$ *can be arbitrarily approximated, in an* $\mathcal{L}_2$ *sense, by the sample-and-hold signal* $u_\Delta(t)$*. On the other hand, Lemma 5.9 establishes that the optimal continuous-time performance* $J(u^*)$ *can be arbitrarily approximated by choosing any signal* $u(t)$ *sufficiently close, in an* $\mathcal{L}_2$ *sense, to* $u^*(t)$*. These two facts will be used in the proof of the convergence of the solution of problem* $\mathcal{P}_\Delta$ *to the solution of* $\mathcal{P}$*, in the next section.*

5.3.1 Sampled-data LQ problem convergence

In this section we present one of the main results of this chapter, namely, that the optimal performance obtained by solving the continuous-time problem $\mathcal{P}$ can be arbitrarily approximated by solving the discrete-time problem $\mathcal{P}_\Delta$.

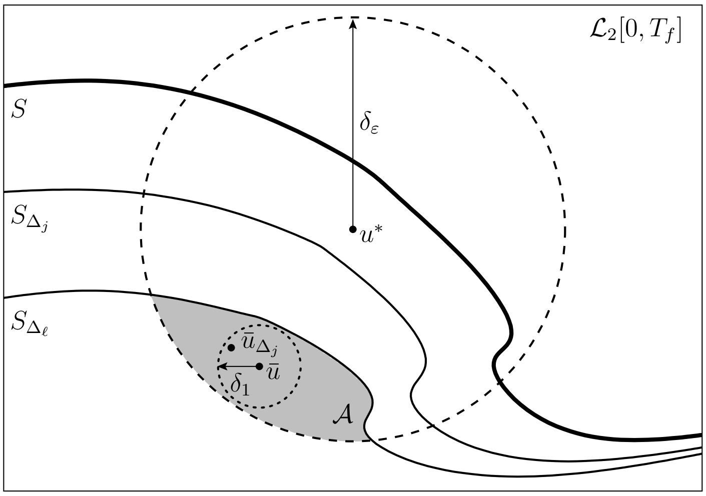


Figure 5.1: Schematic representation of proof of Theorem 5.11.

We note first that, for any chosen Δ, the corresponding problem $\mathcal{P}_\Delta$ is well defined and can be solved. We denote the resulting optimal control sequence by $\{u^*_{\Delta,k}\}_{k=0}^{N-1}$.

We can now state our main result.

Theorem 5.11 *Provided $\mathcal{P}$ has a solution $u^*(t) \in S$, then, for every $\varepsilon > 0$ there exists a $\Delta_\varepsilon > 0$, such that:*

$$J(u^*_{\Delta_\varepsilon}) - J(u^*) < \varepsilon \tag{5.53}$$

where $u^*_{\Delta_\varepsilon}(t)$ is generated, using the ZOH (5.9)*, by the sequence $u^*_{\Delta_\varepsilon,k}$, the solution of problem $\mathcal{P}_{\Delta_\varepsilon}$.*

Proof. Given $\varepsilon > 0$ we know by Lemma 5.9 that there exists a $\delta_\varepsilon > 0$ such that for every $u \in \mathcal{L}_2[0, T_f]$ for which (5.45) holds, (5.46) also holds.

Since $u^* \in S$ and $\lim_{i\to\infty} S_{\Delta_i} = S$, there exists $\Delta_\ell > 0$ for which the set

$$\mathcal{A} = \{u \in \mathcal{L}_2[0, T_f] : \|u - u^*\|_2 < \delta_\varepsilon\} \cap S_{\Delta_\ell} \tag{5.54}$$

is non-empty. Let $\bar{u}$ be in this set. The set $\{u \in \mathcal{L}_2[0, T_f] : \|u - u^*\|_2 < \delta_\varepsilon\}$ is open. Hence, there exists $\delta_1 > 0$ such that:

$$\{u \in \mathcal{L}_2[0, T_f] : \|u - \bar{u}\|_2 < \delta_1\} \subset \mathcal{A} \tag{5.55}$$

Now, using Lemma 5.8, we know that $\bar{u}$ can be arbitrarily approximated in an $\mathcal{L}_2$ sense by a piecewise constant function. This means that there exists a sampling period $\Delta_j < \Delta_\ell$ such that:

$$\|\bar{u} - \bar{u}_{\Delta_j}\|_2 < \delta_1 \tag{5.56}$$

Hence, $\bar{u}_{\Delta_j} \in S_{\Delta_\ell} \subset S_{\Delta_j} \subset S$ and $\|\bar{u}_{\Delta_j} - u^*\|_2 < \delta_\varepsilon$. This is represented schematically in Figure 5.1. Using Lemma 5.9, the latter implies that:

$$J(\bar{u}_{\Delta_j}) - J(u^*) < \varepsilon \tag{5.57}$$

If we now consider $u^*_{\Delta_j,k}$, the solution of the problem $\mathcal{P}_{\Delta_j}$, we know that:

$$J_\Delta(\bar{u}_{\Delta_j,k}) \geq J_\Delta(u^*_{\Delta_j,k}) \tag{5.58}$$

where $\bar{u}_{\Delta_j,k}$ is the sequence with the values taken by $\bar{u}_{\Delta_j}(t)$. Using (5.22), we then have that:

$$J(\bar{u}_{\Delta_j}) \geq J(u^*_{\Delta_j}) \tag{5.59}$$

and equation (5.53) follows by choosing $\Delta_\varepsilon = \Delta_j$. This completes the proof.

□

The above theorem establishes that there exists a finite sampling period Δ, such that the performance achieved with the optimal fixed horizon discrete-time constrained controller is arbitrarily close to the performance achievable by the optimal fixed horizon continuous-time constrained controller.

We illustrate the results in the previous sections by a simple example:

Example 5.12 *Consider the second order system:*

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u \qquad ; x_o = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{5.60}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \tag{5.61}$$

with continuous cost function (5.3)*, where* $T_f = 5$*,* $Q = C^TC$*,* $R = 0.1$*, and where* P *is the solution of the algebraic Riccati equation* (5.6)*. We impose constraints on the input and on one of the states as follows:*

$$|u(t)| \leq 1 \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.62}$$

$$|x_1(t)| \leq 1 \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x(t) \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.63}$$

The matrices of the discrete-time sampled model (5.11)–(5.12) *are obtained from equation* (5.13). *Similarly, the matrices for the sampled-data cost function* (5.14)*, are obtained from equations* (5.15)–(5.19)

Figure 5.2 shows the evolution of the cost function as N *increases (*i.e.*,* Δ *decreases), for the constrained and unconstrained cases. Note that a logarithmic scale has been used. For the unconstrained case, it can be seen that the value of the cost function approaches the continuous time optimal result. Similarly, for the constrained case, we can see that beyond* $N = 16$*, the minimum achievable for the cost function is almost constant. This is also confirmed in Figure 5.3, which shows the convergence of the (ZOH or piece-wise constant) control and state signals for the constrained case.*

□

5.3.2 Receding horizon control problem

To conclude this section, we consider the moving horizon control problem. An issue here is that usually, in discrete receding horizon strategies, only the first element of the fixed horizon control solution is

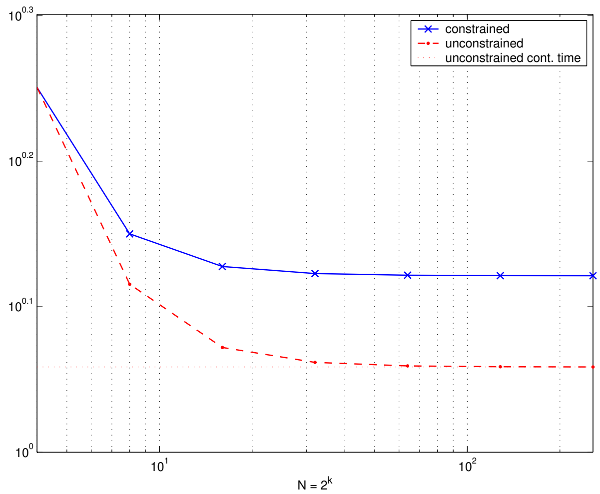


Figure 5.2: Cost function values v/s N

applied to the system at each step. The problem is then solved again at the next sampling instant (Goodwin *et al.*, 2004).

If we take the limit of the receding horizon procedure as Δ tends to zero, we end up with an ill-defined control law because the optimal continuous-time solution is only unique up to an $\mathcal{L}_2$ equivalence.

To address this issue, we will adopt a form of discrete MPC which mirrors the common strategy suggested for continuous-time predictive control (Cannon and Kouvaritakis, 2000). The control input applied to the plant is changed every Δ (seconds), but the fixed horizon optimisation is done only every $\bar{\Delta}$ (seconds), where $\bar{\Delta} > \Delta$. Let ℓ be an integer, and define:

$$u^*_{FH}(\ell\bar{\Delta}, \tau) = u^*(\tau) \qquad \forall \tau \in [0, \bar{\Delta}) \tag{5.64}$$

where $u^*(\tau)$ is the solution to the fixed horizon constrained continuous-time problem $\mathcal{P}$ on the interval $\tau \in [0, T_f]$. We then define the continuous time **moving** horizon optimal solution in terms of (5.64) as:

$$u^*_{MH}(t) = u^*_{FH}(\ell\bar{\Delta}, t - \ell\bar{\Delta}) \qquad \forall t \in [\ell\bar{\Delta}, (\ell+1)\bar{\Delta}) \tag{5.65}$$

Theorem 5.13 . *The discrete-time approximation of the receding horizon strategy defined above, where the control is restricted to be piece-wise constant over every interval* $[k\Delta, (k+1)\Delta)$*, converges to the continuous-time result,* i.e.,

$$\lim_{\Delta \to 0} \| u^*_{MH\,\Delta}(t) - u^*_{MH}(t) \|_2 = 0 \tag{5.66}$$

Proof. The result readily follows from the definitions of u^*_{MH} in (5.65) and (5.64), and the previous convergence result in Theorem 5.11

□

5.4 Spectral properties of LQ problems

In this section we will explore a related aspect of the link between continuous- and discrete-time LQ optimal control problems. Our goal here is to examine spectral properties of the dicrete and continuous

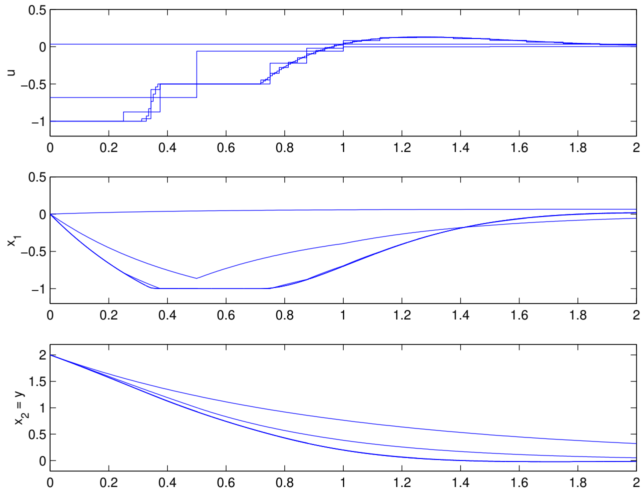


Figure 5.3: Control and state signals for $N = 1, 4, \ldots, 256$

problems via a functional analysis approach.

Functional analysis has proven to be a powerful tool in the development of control and estimation theory, since their early years. In fact, minimisation of quadratic functionals in $\mathcal{L}^2$-spaces is still the paradigm behind optimal control and optimal estimation problems (Kwakernaak and Sivan, 1972; Leigh, 1980; Grimble and Johnson, 1988). The use of linear operator theory provides a concise conceptual framework and a wide range of useful theorems, both for linear control and estimation (Kailath, 1969; Hagander, 1973; De Doná *et al.*, 2000; Kojima and Morari, 2004). A brief review on linear operators in Hilbert spaces is presented in Appendix B.

In this section we study the singular structure of the operators involved in LQ control problems $\mathcal{P}$, in continuous-time, and its sampled-data version $\mathcal{P}_\Delta$, defined in discrete-time. In particular, we are interested in the relationship between the singular values and functions associated with the continuous-time system operator, to its sampled-data counterpart. We will show that the (finite set of) singular values of the discrete-time problem, converge to a subset of the (infinite, but countable) set in continuous-time.

In the following sections we first obtain the singular structure characterisation in the continuous- and discrete-time domain, to then show the convergence result previously mentioned.

5.4.1 Continuous-time singular structure

Our results will build on earlier work on the singular value structure of problem $\mathcal{P}$ as presented in (Kojima and Morari, 2004). We summarise these results below.

We consider the Hilbert spaces $\mathcal{V} = \mathcal{L}_2(0, T_f; \mathbb{R}^m)$ and $\mathcal{Z} = \mathbb{R}^n \times \mathcal{L}_2(0, T_f; \mathbb{R}^n)$, with inner products:

$$\langle f_1, f_2 \rangle_{\mathcal{V}} = \int_0^{T_f} f_1(t)^T f_2(t)\, dt \qquad ;\ f_1, f_2 \in \mathcal{V} \tag{5.67}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} = \left(g_1^0\right)^T g_2^0 + \int_0^{T_f} g_1^1(t)^T g_2^1(t)\, dt \qquad ;\ g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{5.68}$$

We rewrite the input and response signal as:

$$v = v(t) = R^{\frac{1}{2}} u(t) \in \mathcal{V} \tag{5.69}$$

$$z = \begin{bmatrix} z^0 \\ z^1(t) \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}\, x(T_f) \\ Q^{\frac{1}{2}}\, x(t) \end{bmatrix} \in \mathcal{Z} \tag{5.70}$$

The cost function (5.3) and the system dynamics can then be expressed as:

$$J = \|v\|_{\mathcal{V}}^2 + \|z\|_{\mathcal{Z}}^2 \tag{5.71}$$

$$z = \mathcal{F} x_o + \mathcal{G} v \tag{5.72}$$

where $x_o \in \mathbb{R}^n$, and $\mathcal{F}$ and $\mathcal{G}$ are *linear operators*:

$$\mathcal{F} x_o = \begin{bmatrix} (\mathcal{F} x_o)^0 \\ (\mathcal{F} x_o)^1(t) \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}} e^{AT_f} x_o \\ Q^{\frac{1}{2}} e^{At} x_o \end{bmatrix} \tag{5.73}$$

$$\mathcal{G} v = \begin{bmatrix} (\mathcal{G} v)^0 \\ (\mathcal{G} v)^1(t) \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}} \int_0^{T_f} e^{A(T_f - \xi)} B R^{-\frac{1}{2}} v(\xi)\, d\xi \\ Q^{\frac{1}{2}} \int_0^{t} e^{A(t - \xi)} B R^{-\frac{1}{2}} v(\xi)\, d\xi \end{bmatrix} \tag{5.74}$$

The following theorem establishes the singular values of the operator $\mathcal{G}$, which satisfy:

$$\sigma > 0 : \quad \mathcal{G} f = \sigma g \quad , \quad \mathcal{G}^* g = \sigma f \tag{5.75}$$

where $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$ (see Example B.11 on page 149).

Theorem 5.14 *The set of singular values $\{\sigma_i\}$ of the linear operator $\mathcal{G}$ in (5.74) are given by the roots of the equation:*

$$\det\left(\begin{bmatrix} -\sigma^{-1} P & I_n \end{bmatrix} e^{M(\sigma) T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \right) = 0 \tag{5.76}$$

where $\sigma > 0$, and:

$$M(\sigma) = \begin{bmatrix} A & \sigma^{-1} B R^{-1} B^T \\ -\sigma^{-1} Q & -A^T \end{bmatrix} \tag{5.77}$$

The corresponding singular vectors $f_i \in \mathcal{V}$ and $g_i \in \mathcal{Z}$, are given by the following functions:

$$f_i(\xi) = R^{-\frac{1}{2}} B^T \begin{bmatrix} 0 & I_n \end{bmatrix} e^{M(\sigma_i)\xi} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{5.78}$$

$$g_i^0 = P^{\frac{1}{2}} \begin{bmatrix} I_n & 0 \end{bmatrix} e^{M(\sigma_i) T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{5.79}$$

$$g_i^1(\xi) = Q^{\frac{1}{2}} \begin{bmatrix} I_n & 0 \end{bmatrix} e^{M(\sigma_i)\xi} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{5.80}$$

where $d_i \neq 0$ is a vector such that:

$$\begin{bmatrix} -\sigma^{-1}P & I_n \end{bmatrix} e^{M(\sigma_i)T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i = 0 \tag{5.81}$$

Proof. See Kojima and Morari (2004).

□

5.4.2 Discrete-time singular structure

We next explore the singular value structure of the associated sampled-data problem. In Section 5.4.3 we will show that it naturally converges to that of the underlying continuous problem.

We consider the Hilbert spaces $\mathcal{V} = l_2(0, N-1; \mathbb{R}^m)$ and $\mathcal{Z} = \mathbb{R}^n \times l_2(0, N-1; \mathbb{R}^n)$, with inner products:

$$\langle f_1, f_2 \rangle_{\mathcal{V}} = \sum_{0}^{N-1} f_1^T f_2 \qquad ; f_1, f_2 \in \mathcal{V} \tag{5.82}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} = \left(g_1^0\right)^T g_2^0 + \sum_{0}^{N-1} \left(g_1^1\right)^T g_2^1 \qquad ; g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{5.83}$$

We rewrite the input and response signals as:

$$v = v_k = R_q^{\frac{1}{2}}\, u_k \in \mathcal{V} \tag{5.84}$$

$$z = \begin{bmatrix} z^0 \\ z_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{\frac{1}{2}}\, x_N \\ Q_q^{\frac{1}{2}}\, x_k \end{bmatrix} \in \mathcal{Z} \tag{5.85}$$

In the cost function (5.14) we can neglect the coupled term depending on S_q based on its convergence properties given earlier in (5.23). In this case, the cost and the system dynamics can then be expressed as:

$$J_\Delta = \|v\|_{\mathcal{V}}^2 + \|z\|_{\mathcal{Z}}^2 \tag{5.86}$$

$$z = \mathcal{F}_\Delta x_o + \mathcal{G}_\Delta v \tag{5.87}$$

where $x_o \in \mathbb{R}^n$, and $\mathcal{F}_\Delta$ and $\mathcal{G}_\Delta$ are *linear operators* defined by:

$$\mathcal{F}_\Delta x_o = \begin{bmatrix} (\mathcal{F}_\Delta x_o)^0 \\ (\mathcal{F}_\Delta x_o)_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{\frac{1}{2}} A_q^N x_o \\ Q_q^{\frac{1}{2}} A_q^k x_o \end{bmatrix} \tag{5.88}$$

$$\mathcal{G}_\Delta v = \begin{bmatrix} (\mathcal{G}_\Delta v)^0 \\ (\mathcal{G}_\Delta v)_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{\frac{1}{2}} \sum_{l=0}^{N-1} A_q^{N-1-l} B_q R_q^{-\frac{1}{2}} v_l \\ Q_q^{\frac{1}{2}} \sum_{l=0}^{k-1} A_q^{k-1-l} B_q R_q^{-\frac{1}{2}} v_l \end{bmatrix} \tag{5.89}$$

where $0 \leq k < N$.

Theorem 5.15 *For a given Δ, the set of singular values $\{\sigma_i\}$ of the linear operator $\mathcal{G}_\Delta$ in (5.89) are given by the roots of the equation:*

$$\det\left(\begin{bmatrix}-\sigma^{-1}P_\Delta & A_q^T\end{bmatrix} M_\Delta(\sigma)^N \begin{bmatrix}0\\ I_n\end{bmatrix}\right) = 0 \tag{5.90}$$

where $\sigma > 0$, and:

$$M_\Delta(\sigma) = \begin{bmatrix} A_q & \frac{1}{\sigma}B_q R_q^{-1}B_q^T \\ -\frac{1}{\sigma}(A_q^T)^{-1}Q_q A_q & (A_q^T)^{-1}\left(I - \frac{1}{\sigma^2}Q_q B_q R_q^{-1} B_q^T\right)\end{bmatrix} \tag{5.91}$$

The corresponding singular vectors $f_i \in \mathcal{V}$ and $g_i \in \mathcal{Z}$, are given by the following functions:

$$f_i[k\Delta] = R_q^{-\frac{1}{2}}B_q^T \begin{bmatrix}0 & I_n\end{bmatrix} M_\Delta(\sigma_i)^k \begin{bmatrix}0\\ I_n\end{bmatrix} d_i \tag{5.92}$$

$$g_i^0 = P_\Delta^{\frac{1}{2}} \begin{bmatrix}I_n & 0\end{bmatrix} M_\Delta(\sigma_i)^N \begin{bmatrix}0\\ I_n\end{bmatrix} d_i \tag{5.93}$$

$$g_i^1[k\Delta] = Q_q^{\frac{1}{2}} \begin{bmatrix}I_n & 0\end{bmatrix} M_\Delta(\sigma_i)^k \begin{bmatrix}0\\ I_n\end{bmatrix} d_i \tag{5.94}$$

where $d_i \neq 0$ is a vector such that:

$$\begin{bmatrix}-\sigma^{-1}P_\Delta & A_q^T\end{bmatrix} M_\Delta(\sigma_i)^N \begin{bmatrix}0\\ I_n\end{bmatrix} d_i = 0 \tag{5.95}$$

Proof. The singular values and singular vectors of the linear operator $\mathcal{G}_\Delta$ are defined by:

$$\sigma > 0: \quad \mathcal{G}_\Delta f = \sigma g \quad , \quad \mathcal{G}_\Delta^* g = \sigma f \tag{5.96}$$

where $f \in \mathcal{V}$, $g \in \mathcal{Z}$, and $\mathcal{G}_\Delta^*$ is the *adjoint* operator of $\mathcal{G}_\Delta$ given by (see Example B.12 on page 150):

$$\mathcal{G}_\Delta^* g = \mathcal{G}_\Delta^* \begin{bmatrix}g^0\\ g_k^1\end{bmatrix} = R_q^{-\frac{1}{2}}B_q^T \left[(A_q^T)^{N-1-l}P_\Delta^{\frac{1}{2}}g^0 + \sum_{k=l+1}^{N-1}(A_q^T)^{k-1-l}Q_q^{\frac{1}{2}}g_k^1\right] \tag{5.97}$$

We define the variables:

$$p_k = \sum_{l=0}^{k-1} A_q^{k-1-l}B_q R_q^{-\frac{1}{2}} f_l \tag{5.98}$$

$$q_l = (A_q^T)^{N-1-l}P_\Delta^{\frac{1}{2}}g^0 + \sum_{k=l+1}^{N-1}(A_q^T)^{k-1-l}Q_q^{\frac{1}{2}}g_k^1 \tag{5.99}$$

Using (5.96) it is readily seen that these functions satisfy:

$$\begin{bmatrix}p_{j+1}\\ q_{j+1}\end{bmatrix} = M_\Delta(\sigma)\begin{bmatrix}p_j\\ q_j\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}p_j\\ q_j\end{bmatrix} = M_\Delta(\sigma)^j \begin{bmatrix}p_0\\ q_0\end{bmatrix} \tag{5.100}$$

Also the following relations hold:

$$\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} \begin{bmatrix} p_N \\ q_N \end{bmatrix} = 0 \quad , \quad \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 \tag{5.101}$$

Using (5.101) in the solution of (5.100), with $j = N$, we have:

$$\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} M_\Delta(\sigma)^N \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 = 0 \tag{5.102}$$

However, a necessary condition to have a non trivial solution, different from $(f, g) = 0$, is $q_0 \neq 0$. Therefore condition (5.90) is a necessary condition. Similarly, sufficiency can be easily verified.

The singular vectors can be obtained using the definitions of $\mathcal{G}_\Delta$ in (5.89), $\mathcal{G}_\Delta^*$ in (5.97), (5.98) and (5.99):

$$f_k = R_q^{-\frac{1}{2}} B_q^T\, q_k \, ; \quad g^0 = P_\Delta^{\frac{1}{2}}\, p_N \, ; \quad g_k^1 = Q_q^{\frac{1}{2}}\, p_k \tag{5.103}$$

Denoting the vector $q_0 = d \neq 0$, equations (5.92)–(5.94), are obtained from:

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix} = M_\Delta(\sigma)^k \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = M_\Delta(\sigma)^k \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 \tag{5.104}$$

□

The singular values of the operator $\mathcal{G}_\Delta$ correspond, in fact, to the (squared) eigenvalues of the compact self-adjoint operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. They satisfy also the following important properties (Kreyszig, 1978):

1. The singular vectors $\{f_i\}$ are orthogonal, and

2. The singular values $\{\sigma_i\}$ are real and positive, being $\sigma = 0$ the only possible point of accumulation:

$$\sigma_1 \geq \ldots \geq \sigma_N > 0 \tag{5.105}$$

Theorem 5.16 *Given a sampling period $\Delta = T_f/N$ the operator $\mathcal{G}_\Delta$ has exactly Nm singular values.*

Proof. Following the same lines as in (Kojima and Morari, 2004), for the continuous-time case, it can be proved that $\sigma = 0$ is not an eigenvalue of the operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. Then the set of (orthogonal) eigenvectors form a complete basis of $\mathcal{V} = l_2(0, N-1; \mathbb{R}^m)$. This space has dimension Nm and, hence, the operator $\mathcal{G}_\Delta$ has exactly Nm singular vectors and singular values.

□

5.4.3 Singular structure convergence

We next explore the relations and connections between the continuous and discrete singular structure characterised in the previous sections. In particular, we show that, as the sampling period Δ goes to zero, the singular values in discrete-time approach the singular values in continuous-time.

We start by presenting a limiting result for matrix $M_\Delta(\sigma)$, defined in equation (5.90).

Lemma 5.17 *The matrix $M_\Delta(\sigma)^N$ has a well defined limit when Δ goes to zero. Specifically:*

$$\lim_{\Delta \to 0} M_\Delta(\sigma)^N = e^{M(\sigma)T_f} \tag{5.106}$$

where:

$$M(\sigma) = \begin{bmatrix} A & \sigma^{-1}BR^{-1}B^T \\ -\sigma^{-1}Q & -A^T \end{bmatrix} \tag{5.107}$$

Proof. We write $M_\Delta(\sigma) = M_\Delta$, and then we consider the following expression:

$$\log M_\Delta^N = N \cdot \log\left(I + (M_\Delta - I)\right) = \frac{T_f}{\Delta}\left[(M_\Delta - I) - \frac{(M_\Delta - I)^2}{2} + \dots\right] \tag{5.108}$$

Using the delta domain state space matrices $A_\delta = (A_q - I)/\Delta$ and $B_\delta = B_q/\Delta$ (Middleton and Goodwin, 1990), we have:

$$M_\Delta - I = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \tag{5.109}$$

where:

$$L_{11} = A_q - I = A_\delta\,\Delta \tag{5.110}$$

$$L_{12} = \frac{1}{\sigma}B_q R_q^{-1} B_q^T = \frac{1}{\sigma}B_\delta \left(\frac{R_q}{\Delta}\right)^{-1} B_\delta^T\,\Delta \tag{5.111}$$

$$L_{21} = -\frac{1}{\sigma}(A_q^T)^{-1}Q_q A_q = -\frac{1}{\sigma}(I + \Delta A_\delta^T)^{-1}\frac{Q_q}{\Delta}(I + \Delta A_\delta)\,\Delta \tag{5.112}$$

$$\begin{aligned} L_{22} &= (A_q^T)^{-1}\left(I - \frac{1}{\sigma^2}Q_q B_q R_q^{-1} B_q^T - A_q^T\right) \\ &= -(I + \Delta A_\delta^T)^{-1}\left(A_\delta^T + \frac{\Delta}{\sigma^2}\frac{Q_q}{\Delta}B_\delta\left(\frac{R_q}{\Delta}\right)^{-1}B_\delta^T\right)\Delta \end{aligned} \tag{5.113}$$

Using the convergence properties in (5.23), we can thus see that:

$$\lim_{\Delta \to 0}\frac{1}{\Delta}\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} A & \sigma^{-1}BR^{-1}B^T \\ -\sigma^{-1}Q & -A^T \end{bmatrix} \tag{5.114}$$

which corresponds exactly to $M(\sigma)$ in (5.107). Therefore we finally have:

$$\lim_{\Delta \to 0}\log\left(M_\Delta(\sigma)^N\right) = T_f \cdot M(\sigma) \tag{5.115}$$

and (5.106) is obtained by applying the exponential function, which is continuous, on both sides of the equation. □

Remark 5.18 *Based on the previous result, and the convergence properties of matrices P_Δ and A_q when the sampling rate grows to infinity, we can notice that equation (5.90) is transformed to (5.76), as $\Delta \to 0$.*

Finally, we show that the finite set of singular values of the discrete problem converge, in a well defined fashion, to a countable subset of singular values for the continuous problem.

Theorem 5.19 *When the sampling period Δ goes to zero, the singular values of the operator $\mathcal{G}_\Delta$, for the discrete time problem, converge to a subset of the singular values of the operator $\mathcal{G}$, for the continuous time problem.*

Proof. The singular values of the operator $\mathcal{G}_\Delta$ are the square root of the eigenvalues of the self-adjoint operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. The last statement is true for every operator, so the same holds for the continuous time operators $\mathcal{G}$ defined in (5.74) and $\mathcal{G}^*\mathcal{G}$. The adjoint operators $\mathcal{G}^*$ and $\mathcal{G}_\Delta^*$ are explicitly found in Appendix B.

The proof uses the fact that the discrete self-adjoint operator:

$$\mathcal{G}_\Delta^*\mathcal{G}_\Delta v = R_q^{-\frac{1}{2}}B_q^T\left[(A_q^T)^{N-1-l}P_\Delta\sum_{k=0}^{N-1}A_q^{N-1-k}B_qR_q^{-\frac{1}{2}}v_k\right.$$
$$\left.+\sum_{j=l+1}^{N-1}(A_q^T)^{j-1-l}Q_q\sum_{k=0}^{j-1}A_q^{j-1-k}B_qR_q^{-\frac{1}{2}}v_k\right] \tag{5.116}$$

can be rewritten as:

$$\mathcal{G}_\Delta^*\mathcal{G}_\Delta v = \left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}\frac{B_q^T}{\Delta}\left[e^{A^T\Delta(N-1-l)}P_\Delta\sum_{k=0}^{N-1}e^{A\Delta(N-1-k)}\frac{B_q}{\Delta}\left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}v_k\Delta\right.$$
$$\left.+\sum_{j=l+1}^{N-1}e^{A^T\Delta(j-1-l)}\frac{Q_q}{\Delta}\Delta\sum_{k=0}^{j-1}e^{A\Delta(j-1-k)}\frac{B_q}{\Delta}\left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}v_k\Delta\right] \tag{5.117}$$

This operator converges *point-wise* exactly to its continuous time counterpart:

$$\mathcal{G}^*\mathcal{G}v = R^{-\frac{1}{2}}B^T\left[e^{A^T(T_f-\beta)}P\int_0^{T_f}e^{A(T_f-\xi)}BR^{-\frac{1}{2}}v(\xi)\,d\xi\right.$$
$$\left.+\int_\beta^{T_f}e^{A^T(\tau-\beta)}Q\int_0^{\tau}e^{A(\tau-\xi)}BR^{-\frac{1}{2}}v(\xi)\,d\xi\,d\tau\right] \tag{5.118}$$

as the sampling period Δ goes to zero, where we have used the convergence properties in Remark 5.4 on page 86, and:

$$N\Delta = T_f\,,\quad (l+1)\Delta = \beta\,,\quad j\Delta = \tau\,,\quad k\Delta = \xi \tag{5.119}$$

According to Theorem 1.6 in (Lang, 1993), given a sequence of bounded functions $\{\mathcal{G}_{\Delta_n}^*\mathcal{G}_{\Delta_n}v\}$ in $\mathcal{L}_2$ which converges point-wise to $\mathcal{G}^*\mathcal{G}v$, then $\mathcal{G}^*\mathcal{G}v$ is in $\mathcal{L}_2$ and $\{\mathcal{G}_{\Delta_n}^*\mathcal{G}_{\Delta_n}v\}$ is $\mathcal{L}_2$-convergent to $\mathcal{G}^*\mathcal{G}v$, *i.e.*, the discrete time operator converges in norm to the continuous time one.

Then, following the same arguments used in (De Doná *et al.*, 2000), we prove that every limiting point of the set of eigenvalues of $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$ (singular values of $\mathcal{G}_\Delta$) converges to an eigenvalue of $\mathcal{G}^*\mathcal{G}$. Let us suppose that $\lambda \neq 0$ is a limit point of a sequence of eigenvalues λ_Δ of $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$, hence, $(\lambda_\Delta I - \mathcal{G}_\Delta^*\mathcal{G}_\Delta)$ converges in norm to $(\lambda I - \mathcal{G}^*\mathcal{G})$. Since the set of invertible operators is open, if $(\lambda_\Delta I - \mathcal{G}_\Delta^*\mathcal{G}_\Delta)$ is not invertible (*i.e.*, λ_Δ belongs to the spectrum of the self-adjoint operator) for all Δ then $(\lambda I - \mathcal{G}^*\mathcal{G})$ is not invertible. This establishes that λ belongs to the spectrum of the compact self-adjoint operator $\mathcal{G}^*\mathcal{G}$, different from zero, so it is one of its eigenvalues.

□

Finally, we illustrate the previous convergence results for a simple example:

Example 5.20 *Consider the double integrator system:*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5.120}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{5.121}$$

with continuous cost function (5.3), where $T_f = 5$, $Q = C^T C$, $R = 0.1$, and P is the solution of the algebraic Riccati equation associated to J_1 in (5.3).

The discrete time model matrices, from (5.13), are:

$$A_q = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \qquad B_q = \Delta \begin{bmatrix} \Delta/2 \\ 1 \end{bmatrix} \tag{5.122}$$

The matrices for the sampled data cost function (5.14), are:

$$Q_q = \Delta \begin{bmatrix} 1 & \Delta/2 \\ \Delta/2 & \Delta^2/3 \end{bmatrix} \qquad R_q = \Delta \left(\Delta^4/20 + 0.1\right) \qquad S_q = \Delta \begin{bmatrix} \Delta^2/6 \\ \Delta^3/8 \end{bmatrix} \tag{5.123}$$

and P_Δ is obtained from the discrete time algebra Riccati equation associated with (5.14).

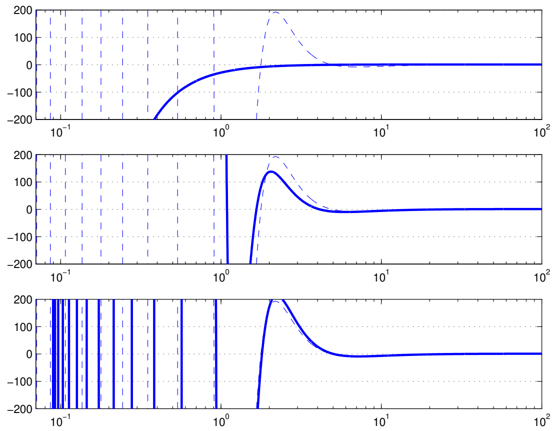


Figure 5.4: Plot of $f_\Delta(\sigma)$ (solid) and $f(\sigma)$ (dashed) for $N = 1, 4$ and 16.

We define $f(\sigma)$ to be the function on the left hand side of equation (5.76), whose positive roots are the singular values of $\mathcal{G}$, and analogously, we use $f_\Delta(\sigma)$ to define the function on the left hand side of (5.90), whose positive roots are the singular values of $\mathcal{G}_\Delta$. Figure 5.4 shows the functions $f(\sigma)$ and $f_\Delta(\sigma)$ for three different values of $\Delta = T_f/N$. We can see that the finite set of roots (singular values) obtained in discrete time approach the countable set of roots in the continuous time set. Note that the zero crossings by $f(\sigma)$ and $f_\Delta(\sigma)$ correspond to the singular values.

To illustrate the above results, Figure 5.5 shows the evolution of the first 10 singular values for the discrete problem, $\mathcal{P}_\Delta$, compared with the singular values obtained in (Kojima and Morari, 2004) for the continuous-time problem. Note that given any $N \in \mathbb{N}$, and the corresponding sampling period $\Delta = T_f/N$, the operator $\mathcal{G}_\Delta$ has only N singular values given by equation (5.90), as stated in Theorem

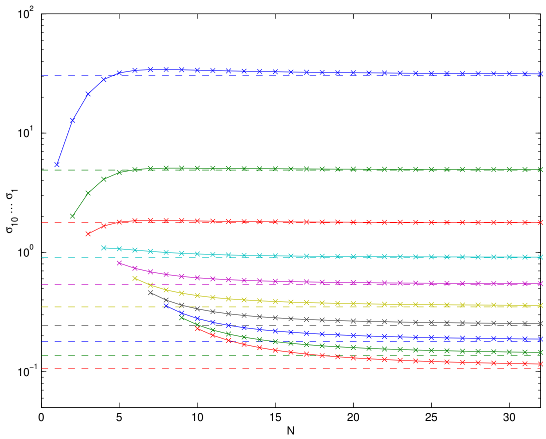


Figure 5.5: Convergence of the (first 10) singular values of $\mathcal{G}_\Delta$

5.16. In the figure we can see that the discrete-time singular values quickly approach their continuous-time counterpart. For example, with $N = 4$ *the four singular values obtained are within* 10% *of the continuous time ones.*

□

5.5 Summary

This chapter has explored the use sampled-data models in (constrained) optimal control problems, when using fast sampling rates. We have established two natural convergence results as the sampling period goes to zero:

First, we have shown that a constrained optimal control problem, defined in continuous-time, can be approximated arbitrarily closely, by considering an associated sampled-data problem, with (possibly) tighter constraints. The existence of a well defined limit ensures that there exists a finite sampling period Δ, for which the performance achieved by the discrete-time constrained controller is arbitrarily close to the achievable performance by the hypothetical continuous-time constrained control law.

Secondly, we have explored the connections between the singular structure of LQ problems in continuous and discrete-time. In particular, we have shown that there is a natural convergence between the finite set of singular values of the sampled-data problem, and the (infinite) countable set in continuous time. This result can be applied in suboptimal continuous-time control strategies by exploiting the singular structure of the problem, which is implemented in discrete-time.

Part II

Nonlinear Systems

Chapter 6

Sampled-data models for deterministic nonlinear systems

6.1 Introduction

Models for continuous-time systems typically take the form of (nonlinear) differential equations. However, in practice we need alternative models that describe the relationship between the discrete-time actions taken on the system and the samples taken from its signals. In this second part of the thesis, our interest is on sampled-data models for nonlinear systems. A key departure point from to the linear sampled-data models presented in Chapter 2, is that, for nonlinear systems, only **approximate** sampled-data models can be obtained. However, the accuracy of these models can be characterised in a precise way, as we will see later.

In this chapter we consider the sampling of deterministic nonlinear systems. Later, in Chapter 7, we will consider the stochastic case. A similar separate analysis was presented for linear systems in Part I. Even though *superposition* does not generally apply for nonlinear systems, we have kept this separation both for the sake of simplicity and to better reveal the relations to the linear case.

The use of sampled-data models raises the question of the relationship between the discrete-time description of the samples and the original continuous-time model. It is tempting to simply sample quickly and then to replace derivatives in the continuous-time model by divided differences (*i.e.*, simply replacing the differential operator ρ by the δ operator) in the sampled-data model. This certainly leads to an approximate model. However, one can obtain more accurate models. For the linear case studied in Part I, we saw that exact sampled-data models can be generated by including extra zeros due to the sampling process (Åström *et al.*, 1984; Wahlberg, 1988). One would reasonably expect similar results to hold in the more general nonlinear framework. However, the situation, in this case, is more complex than for linear systems. Indeed, to the best of our knowledge, an explicit characterisation of the *sampling zero dynamics* has previously remained unresolved, although, an implicit characterisation of the (nonlinear) sampling zeros was given in Monaco and Normand-Cyrot (1988). On the other hand, in (Kazantzis and Kravaris, 1997), system-theoretic properties of sampled-data models for nonlinear system are studied.

Any sampled-data model for a nonlinear system will, in general, be an approximate description of the combination of two elements: the continuous-time system itself, together with the sample and hold devices. An exact discrete-time description of such a hybrid nonlinear system is, in most cases, not known or impossible to compute (Nešić *et al.*, 1999). Thus, one needs to be clear about the potential accuracy achieved by any model. In fact, the accuracy of the approximate sampled-data plant model has proven to be a key issue in the context of control design, where a controller designed to stabilise an approximate model may fail to stabilise the exact discrete-time model, no matter how small the sampling period Δ is chosen (Nešić and Teel, 2004).

In this chapter, we present an approximate sampled-data model for deterministic nonlinear systems. We show how a particular strategy can be used to approximate the system output and its derivatives in such a way as to obtain a *local truncation error*, between the output of the resulting sampled-data model and the true continuous-time system output, of order Δ^{r+1}, where Δ is the sampling period and r is the (nonlinear) relative degree.

An insightful interpretation of the sampled-data model described here can be made in terms of additional zero dynamics. As in the linear case, these extra zero dynamics, due to the sampling process, have no continuous-time counterpart. We give an explicit characterisation of these *sampling zero dynamics* and show that they are a function only of the (nonlinear) system relative degree r. Moreover, the sampling zero dynamics turn out to be identical to those found in the linear case.

The occurrence of nonlinear sampling zero dynamics is also relevant to the problem of sampled-data control of nonlinear continuous-time systems. In this context, topics such as relative degree, normal form, and zero dynamics of nonlinear systems have been extensively studied. In particular, these elements play a key role in feedback linearisation techniques (Isidori, 1995; Isidori, 1999; Khalil, 2002; Byrnes and Isidori, 1988; Marino, 1986). Some of these results have also been extended to discrete-time and sampled nonlinear systems (Grizzle, 1986; Monaco *et al.*, 1986; Lee *et al.*, 1987; Jakubczyk, 1987; Glad, 1988; Arapostathis *et al.*, 1989; Jakubczyk and Sontag, 1990; Barbot *et al.*, 1992; Barbot *et al.*, 1993; Castillo *et al.*, 1997; Teel *et al.*, 1998; Dabroom and Khalil, 2001; Hamzi and Tall, 2003; Chen and Narendra, 2004; Monaco and Normand-Cyrot, 2005). However, the theory for the discrete-time case is less well developed than for the continuous-time case (Monaco and Norman-Cyrot, 1997) and the absence of good models for sampled-data nonlinear plants is still recognised as an important issue for control design (Nešić and Teel, 2001).

The approximate sampled-data models presented in this chapter are believed to give important insights into nonlinear systems theory. By way of illustration, in Section 6.4, we examine their implications in the system identification context.

6.2 Background on nonlinear systems

In this section we review some concepts and results from nonlinear system theory that will be used later in Section 6.3. The results presented here are based on (Isidori, 1995), for continuous-time systems, and partially based on (Monaco and Normand-Cyrot, 1988; Barbot *et al.*, 1992; Hamzi and Tall, 2003), for the discrete-time case.

6.2.1 Continuous-Time Systems

Much of the work regarding control of (continuous-time) nonlinear systems is based on a model consisting of a set of ordinary differential equations affine in the control signals (Isidori, 1995):

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{6.1}$$

$$y(t) = h(x(t)) \tag{6.2}$$

where $x(t)$ is the state evolving in an open subset $\mathcal{M} \subset \mathbb{R}^n$, and where the vector fields $f(\cdot)$ and $g(\cdot)$, and the output function $h(\cdot)$ are analytic.

Definition 6.1 (Relative degree) *The nonlinear system* (6.1)–(6.2) *is said to have relative degree r at a point x_o if:*

(i) $L_g L_f^k h(x) = 0$ *for x in a neighbourhood of x_o and for $k = 0, \ldots, r-2$, and*

(ii) $L_g L_f^{r-1} h(x_o) \neq 0$.

where L_g and L_f correspond to Lie derivatives. For example, $L_g h(x) = \frac{\partial h}{\partial x} g(x)$.

□

Intuitively, the relative degree, as defined above, corresponds to the number of times that we need to differentiate the output $y(t)$ to make the input $u(t)$ appear explicitly. For example:

$$\frac{dy}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u = L_f h(x) + L_g h(x)u \tag{6.3}$$

We next show that there is a local coordinate transformation that allows one to rewrite the nonlinear system (6.1)–(6.2) in the, so called, *normal form*.

Lemma 6.2 (Local coordinate transformation) *Suppose that the system has relative degree r at x_o. Consider the new coordinate defined as:*

$$z_1 = \phi_1(x) = h(x) \tag{6.4}$$

$$z_2 = \phi_2(x) = L_f h(x) \tag{6.5}$$

$$\vdots$$

$$z_r = \phi_r(x) = L_f^{r-1} h(x) \tag{6.6}$$

Furthermore, if $r < n$ it is always possible to define $z_{r+1} = \phi_{r+1}(x), \ldots, z_n = \phi_n(x)$ such that:

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \Phi(x) \tag{6.7}$$

has a nonsingular Jacobian at x_o. Then, $\Phi(\cdot)$ is a local coordinate transformation in a neighbourhood of x_o. Moreover, it is always possible to define $z_{r+1} = \phi_{r+1}(x), \ldots, z_n = \phi_n(x)$ in such a way that:

$$L_g \phi_i(x) = 0 \tag{6.8}$$

in a neighbourhood of x_o, for all $i = r+1, \ldots, n$.

Proof. See (Isidori, 1995).

□

Lemma 6.3 (Normal form) *The state space description of the nonlinear system* (6.1)–(6.2) *in the new coordinate defined by Lemma 6.2 is given by the, so called, normal form:*

$$\dot{\zeta} = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \\ \hline 0 & 0 & \dots & 0 \end{array}\right] \zeta + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \big(b(\zeta,\eta) + a(\zeta,\eta)u(t)\big) \tag{6.9}$$

$$\dot{\eta} = c(\zeta,\eta) \tag{6.10}$$

where the output is $z_1 = h(x) = y$, the state vector is:

$$z(t) = \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} \quad \begin{cases} \zeta(t) = [z_1(t), z_2(t), \dots, z_r(t)]^T \\ \eta(t) = [z_{r+1}(t), z_{r+2}(t), \dots, z_n(t)]^T \end{cases} \tag{6.11}$$

and:

$$b(\zeta,\eta) = b(z) = L_f^r h(\Phi^{-1}(z)) \tag{6.12}$$

$$a(\zeta,\eta) = a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \tag{6.13}$$

$$c(\zeta,\eta) = c(z) = \begin{bmatrix} L_f\phi_{r+1}(\Phi^{-1}(z)) \\ \vdots \\ L_f\phi_n(\Phi^{-1}(z)) \end{bmatrix} \tag{6.14}$$

Proof. See (Isidori, 1995).

□

Remark 6.4 *Note that the state variables contained in $\zeta(t)$, defined in* (6.4)–(6.6)*, correspond to the output $y(t)$ and its first $r-1$ derivatives:*

$$z_\ell(t) = z_1^{(\ell-1)}(t) = y^{(\ell-1)}(t) \quad ; \ell = 1, \dots, r \tag{6.15}$$

This fact will be used later, in Section 6.3, where a sampled-data model for a nonlinear system is obtained based on its normal form.

Definition 6.5 (Zero dynamics) *The zero dynamics of the nonlinear system* (6.1)–(6.2) *are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero,* i.e.*, $y(t) \equiv 0$, for all $t > 0$.*

Using the coordinate transformation, and, thus, the system expressed in the normal form (6.9)–(6.10), we can see that the zero dynamics satisfy:

$$\dot{\eta} = c(0,\eta) \tag{6.16}$$

for any initial condition $z(0) = [0, \eta(0)^T]^T$, and, from (6.9), for an input:

$$u(t) = u_{zd}(t) = -\frac{b(0,\eta)}{a(0,\eta)} \tag{6.17}$$

The concepts presented above, of course, apply *mutatis mutandis* for linear systems. This is formally stated below.

Remark 6.6 *For linear systems, the zero dynamics correspond to the system zeros. In this case, equation* (6.16) *reduces to a linear differential equation $\dot{\eta} = S\eta$, where the eigenvalues of the matrix S are the roots of the polynomial $F(s)$ in* (2.6) *on page 13 — see also (Isidori, 1995).*

6.2.2 Discrete-Time Systems

In this section, we consider the case of nonlinear systems defined in discrete-time. We summarise, in a similar fashion to the continuous-time case in the previous section, several concepts and results partially based on (Monaco and Normand-Cyrot, 1988; Barbot *et al.*, 1992; Califano *et al.*, 1998).

We consider the class of nonlinear discrete-time systems expressed as:

$$\delta x_k = F(x_k) + G(x_k)u_k \tag{6.18}$$

$$y_k = H(x_k) \tag{6.19}$$

where $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are assumed analytic. Note that the state equation (6.18) can also be easily rewritten using the shift operator:

$$qx_k = x_{k+1} = F_q(x_k) + G_q(x_k)u_k \tag{6.20}$$

where, using the delta operator definition (2.30), the functions can be readily obtained as:

$$F_q(x_k) = x_k + \Delta F(x_k) \quad \text{and} \quad G_q(x_k) = \Delta G(x_k) \tag{6.21}$$

Definition 6.7 (Discrete-time relative degree) *The discrete-time system* (6.18)–(6.19) *has relative degree r if (Barbot* et al.*, 1992):*

(i) $\left.\dfrac{\partial y_{k+\ell}}{\partial u_k}\right|_{(x_k,u_k)} = 0$, *for all* $\ell = 0, \ldots, r-1$

(ii) $\left.\dfrac{\partial y_{k+r}}{\partial u_k}\right|_{(x_k,u_k)} \neq 0.$

Intuitively, the discrete-time relative degree corresponds to the number of time shifts before an element u_k of the input sequence u appears explicitly in the output sequence y. The relative degree r can be also characterised in terms of divided differences of y_k, as follows:

Lemma 6.8 *The conditions in Definition 6.7 are equivalent to:*

(a) $\left.\dfrac{\partial(\delta^\ell y_k)}{\partial u_k}\right|_{(x_k,u_k)} = 0$, *for all* $\ell = 0, \ldots, r-1$

(b) $\left.\dfrac{\partial(\delta^r y_k)}{\partial u_k}\right|_{(x_k,u_k)} \neq 0.$

Proof. We next prove that (i)–$(ii) \iff (a)$–(b).

(i)–$(ii) \Rightarrow (a)$–(b) : Using the delta operator definition (2.30), we have that:

$$\frac{\partial(\delta^\ell y_k)}{\partial u_k} = \frac{\partial}{\partial u_k}\left(\left(\tfrac{q-1}{\Delta}\right)^\ell y_k\right) = \frac{1}{\Delta^\ell}\frac{\partial}{\partial u_k}\sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^{\ell-i}(q^i y_k)$$
$$= \frac{1}{\Delta^\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}(-1)^{\ell-i}\frac{\partial y_{k+i}}{\partial u_k} \tag{6.22}$$

where clearly if $\frac{\partial y_{k+i}}{\partial u_k} = 0$, for all $i = 0, \ldots, r-1$, then $\frac{\partial(\delta^\ell y_k)}{\partial u_k} = 0$, for all $\ell = 0, \ldots, r-1$. Furthermore, $\frac{\partial y_{k+r}}{\partial u_k} \neq 0$ implies $\frac{\partial(\delta^r y_k)}{\partial u_k} \neq 0$.

(a)–$(b) \Rightarrow (i)$–(ii) : This follows from similar arguments, on noting that $q = 1 + \Delta\delta$. Then we have that:

$$\frac{\partial y_{k+\ell}}{\partial u_k} = \frac{\partial(q^\ell y_k)}{\partial u_k} = \frac{\partial}{\partial u_k}\left((1+\Delta\delta)^\ell y_k\right) = \frac{\partial}{\partial u_k}\left(\sum_{i=0}^{\ell}\binom{\ell}{i}\Delta^i\delta^i y_k\right)$$
$$= \sum_{i=0}^{\ell}\binom{\ell}{i}\Delta^i\frac{\partial(\delta^i y_k)}{\partial u_k} \tag{6.23}$$

where clearly if $\frac{\partial(\delta^i y_k)}{\partial u_k} = 0$, for all $i = 0, \ldots, r-1$, then $\frac{\partial y_{k+\ell}}{\partial u_k} = 0$ for all $\ell = 0, \ldots, r-1$. Furthermore, $\frac{\partial(\delta^r y_k)}{\partial u_k} \neq 0$ implies $\frac{\partial y_{k+r}}{\partial u_k} \neq 0$.

□

Definition 6.9 (Discrete-time normal form) *Consider the nonlinear discrete-time system* (6.18)–(6.19) *and assume that it has relative degree r. We say that the system is expressed in its discrete-time normal form when it is rewritten as:*

$$\delta\zeta_k = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{r-1} & \\ 0 & & & \\ \hline 0 & 0 & \ldots & 0 \end{array}\right]\zeta_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\big(B(z_k) + A(z_k)u_k\big) \tag{6.24}$$

$$\delta\eta_k = C(z_k) \tag{6.25}$$

where the state vector is:

$$z_k = \begin{bmatrix}\zeta_k \\ \eta_k\end{bmatrix} \quad \begin{cases}\zeta_k = [z_{1,k}, z_{2,k}, \ldots, z_{r,k}]^T \\ \eta_k = [z_{r+1,k}, z_{r+2,k}, \ldots, z_{n,k}]^T\end{cases} \tag{6.26}$$

and the output is $z_{1,k} = H(x_k) = y_k$.

Remark 6.10 *The state variables contained in ζ_k, defined in* (6.26)*, correspond, in fact, to y_k and its first $r-1$ divided differences, i.e.:*

$$z_{\ell,k} = \delta^{\ell-1}z_{1,k} = \delta^{\ell-1}y_k \quad \forall \ell = 1, \ldots, r \tag{6.27}$$

Califano *et al.* (1998) discuss necessary and sufficient for the existence of a *discrete-time normal form* defined in a different way. They consider nonlinear discrete-time models expressed in terms of the shift operator, and, thus, their result cannot be directly applied to the normal form considered in Definition 6.9.

Definition 6.11 (Discrete-time zero dynamics) *The discrete-time zero dynamics of the nonlinear system* (6.18)–(6.19) *are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero,* i.e.*, $y_k \equiv 0$, for all $k \geq 1$.*

If the system is expressed in the normal form (6.24)–(6.25), we can see that the zero dynamics satisfy:

$$\delta\eta_k = C(0, \eta_k) \tag{6.28}$$

for any initial condition $z_0 = [0, \eta_0^T]^T$, and, from (6.24), for an input:

$$u_k = u_k^{zd} = -\frac{B(0, \eta_k)}{A(0, \eta_k)} \tag{6.29}$$

Remark 6.12 *Similarly to the continuous-time case in Remark 6.6, when restricting ourselves to linear systems, the discrete-time zero dynamics* (6.28) *reduce to a linear difference equation $\delta\eta = S\eta$, where the eigenvalues of the matrix S correspond to the zeros of the discrete-time transfer function.*

The following result re-establishes Lemma 2.12 regarding the sampled model for an n-th order integrator. Here, we restate the result in a novel form. In particular, we show, via use of the normal form, that the eigenvalues of the zero dynamics in this case correspond to the *sampling zeros* of the discrete-time transfer function (2.54). This result will be used for the nonlinear case in Section 6.3, specifically, as a key building block in the proof of Theorem 6.22.

Lemma 6.13 (Sampled n-th order integrator in normal form) *Given a sampling period Δ, the discrete-time sampled-data model corresponding to the n-th order integrator $G(s) = s^{-n}$, $n \geq 1$, for a ZOH input, can be written in the normal form:*

$$\delta z_1 = q_{11} z_1 + Q_{12}\eta + \tfrac{\Delta^{n-1}}{n!} u_k \tag{6.30}$$

$$\delta\eta = Q_{21} z_1 + Q_{22}\eta \tag{6.31}$$

with output $y = z_1$. The scalar q_{11} and the matrices Q_{12}, Q_{21}, and Q_{22} take specific forms as given below in (6.35)*. Furthermore, the sampling zeros in* (2.54) *appear as eigenvalues of the matrix Q_{22},* i.e.*, the following equation holds:*

$$p_n(\Delta\gamma) = \det M_n = \tfrac{\Delta^{n-1}}{n!} \det\left(\gamma I_{n-1} - Q_{22}\right) \tag{6.32}$$

Proof. An n-th order integrator can be represented in state-space form (2.2)–(2.3), on page 12, where the matrices are given by (2.57). We consider the corresponding sampled-data model, in delta form, given by (2.31)–(2.32), where the matrices are given by (2.59), on page 21.

If we apply the state space similarity transformation $z = Tx$, where the nonsingular matrix T is given by:

$$T = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline T_{21} & I_{n-1} \end{array}\right] \quad \Longleftrightarrow \quad T^{-1} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline -T_{21} & I_{n-1} \end{array}\right] \tag{6.33}$$

where:

$$T_{21} = \begin{bmatrix} -\frac{n}{\Delta} & \cdots & -\frac{n!}{\Delta^{n-1}} \end{bmatrix}^T \tag{6.34}$$

Then, the new state space representation is given by the following matrices:

$$\begin{aligned} \bar{A}_\delta = TA_\delta T^{-1} &= \left[\begin{array}{c|c} q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} 1 & 0 \\ \hline T_{21} & I_{n-1} \end{array}\right] \left[\begin{array}{c|c} 0 & A_{12} \\ \hline \mathbf{0} & A_{22} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline -T_{21} & I_{n-1} \end{array}\right] \\ &= \left[\begin{array}{c|c} -A_{12}T_{21} & A_{12} \\ \hline -\left(T_{21}A_{12} + A_{22}\right)T_{21} & T_{21}A_{12} + A_{22} \end{array}\right] \end{aligned} \tag{6.35}$$

where, from (2.59):

$$A_{12} = \begin{bmatrix} 1 & \frac{\Delta}{2} & \cdots & \frac{\Delta^{n-2}}{(n-1)!} \end{bmatrix} \tag{6.36}$$

$$A_{22} = \begin{bmatrix} 0 & 1 & \cdots & \frac{\Delta^{n-3}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{6.37}$$

and:

$$\bar{B}_\delta = TB_\delta = \begin{bmatrix} \frac{\Delta^{n-1}}{n!} & 0 & \ldots & 0 \end{bmatrix}^T \tag{6.38}$$

These state space matrices give the *normal form* that appears in (6.30)–(6.31).

To prove (6.32), we first note that:

$$M_n \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} \frac{\Delta^{n-1}}{n!} & A_{12} \\ \hline -\frac{\Delta^{n-1}}{n!}T_{21} & A_{22} - \gamma I_{n-1} \end{array}\right] \tag{6.39}$$

Computing the determinant of the matrices on both sides of the above equation (using matrix results in Appendix A), we have that:

$$(\det M_n)(-1)^{n-1} = \tfrac{\Delta^{n-1}}{n!} \det\left(A_{22} - \gamma I_{n-1} + T_{21}A_{12}\right) \tag{6.40}$$

where, from definition of Q_{22} in (6.35), we finally have that:

$$\begin{aligned} \det M_n &= \tfrac{\Delta^{n-1}}{n!}(-1)^{n-1} \det\left(-\gamma I_{n-1} + (A_{22} + T_{21}A_{12})\right) \\ &= \tfrac{\Delta^{n-1}}{n!} \det\left(\gamma I_{n-1} - Q_{22}\right) \end{aligned} \tag{6.41}$$

□

6.3 A sampled data model for deterministic nonlinear systems

In this section we present the main result of this chapter, namely, a sampled-data model that approximates the input-output mapping of a given nonlinear system. We also show that this discrete-time model contains extra *zero dynamics* which correspond to the asymptotic sampling zeros of the linear case.

We are interested in obtaining a discrete-time model that closely approximates the nonlinear input-output mapping given by (6.1)–(6.2), when the input $u(t)$ is generated by a digital device using a ZOH. This will result in a model of the form:

$$\delta x^S = f^S(x^S) + g^S(x^S)u \tag{6.42}$$

$$y^S = h^S(x^S) \tag{6.43}$$

where $x^S = x_k^S \in \mathbb{R}^n$ is the discrete-time state sequence, $u = u_k$ is the input sequence, $y^S = y_k^S$ is the output sequence, and $k \in \mathbb{Z}$ is the discrete-time index.

Our goal is to define the discrete-time model (6.42)–(6.43), such that y^S is *close* (in a well defined sense) to the continuous-time output $y(t)$ in (6.2) at the sampling instants $t = k\Delta$, when the input $u(t)$ is generated from u_k with the ZOH (2.10). Theorem 6.15 (below) explicitly defines the vector fields $f^S(\cdot)$, $g^S(\cdot)$, and $h^S(\cdot)$ in (6.42)–(6.43) in terms of the sampling period Δ and the vector fields $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ in Lemma 6.3, which are function of $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ in the original continuous-time nonlinear model (6.1)–(6.2).

We first introduce the following assumption:

Assumption 6.14 *The continuous-time nonlinear system* (6.1)–(6.2) *has uniform relative degree $r \leq n$ in the open subset $\mathcal{M} \subset \mathbb{R}^n$, where the state $x(t)$ evolves.*

This assumption ensures that there is a coordinate transformation, as in Lemma 6.2, that allows us to express the system in its normal form.

We then have the following key result:

Theorem 6.15 *Consider the continuous-time nonlinear system* (6.1)–(6.2) *subject to Assumption 6.14. Then the local truncation error between the output $y^S = z_1^S$ of the following discrete-time nonlinear model and the true system output $y(t)$ is of order Δ^{r+1}:*

$$\delta\zeta^S = \begin{bmatrix} 0 & 1 & \frac{\Delta}{2} & \cdots & \frac{\Delta^{r-2}}{(r-1)!} \\ 0 & 0 & 1 & \cdots & \frac{\Delta^{r-3}}{(r-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & 1 \\ 0 & 0 & & \cdots & 0 \end{bmatrix} \zeta^S + \begin{bmatrix} \frac{\Delta^{r-1}}{r!} \\ \frac{\Delta^{r-2}}{(r-1)!} \\ \vdots \\ \frac{\Delta}{2} \\ 1 \end{bmatrix} \big(b + a\,u\big) \tag{6.44}$$

$$\delta\eta^S = c(\zeta^S, \eta^S) \tag{6.45}$$

where $a = a(\zeta^S, \eta^S)$, $b = b(\zeta^S, \eta^S)$, and $c(\zeta^S, \eta^S)$ are defined in Lemma 6.3, u is the discrete-time input to the ZOH, and the discrete-time state vector is:

$$z^S = \begin{bmatrix} \zeta^S \\ \eta^S \end{bmatrix} \qquad \begin{cases} \zeta^S = [z_1^S, z_2^S, \ldots, z_r^S]^T \\ \eta^S = [z_{r+1}^S, z_{r+2}^S, \ldots, z_n^S]^T \end{cases} \tag{6.46}$$

Proof. Assumption 6.14 ensures the existence of the normal form for the nonlinear model (6.1)–(6.2). In Lemma 6.3, the vector fields $b(\cdot)$, $a(\cdot)$, and $c(\cdot)$ are continuous and, thus, the state variables $z_1(t), \ldots, z_r(t)$ are continuous functions of t. This implies (see Remark 6.4) that the output signal $y(t)$ and its first $r-1$ derivatives are continuous. However, when the input signal $u(t)$ is generated by a ZOH, the r-th derivative, $y^{(r)}(t) = \dot{z}_r(t) = b(z) + a(z)u(t)$, is well defined but is, in general, discontinuous at the sampling instants $t = k\Delta$, when the ZOH control signal (2.10) is updated. This allows us to apply the *Taylor's formula with remainder* (Apostol, 1974, Theorem 5.19) to $y(t)$ and to each one of its $r-1$ derivatives at any point t_o as:

$$y(t_o + \tau) = y(t_o) + y^{(1)}(t_o)\tau + \ldots + \frac{y^{(r)}(\xi_1)}{r!}\tau^r \tag{6.47}$$

$$y^{(1)}(t_o + \tau) = y^{(1)}(t_o) + \ldots + \frac{y^{(r)}(\xi_2)}{(r-1)!}\tau^{r-1} \tag{6.48}$$

$$\vdots$$

$$y^{(r-1)}(t_o + \tau) = y^{(r-1)}(t_o) + y^{(r)}(\xi_r)\tau \tag{6.49}$$

for some $t_o < \xi_\ell < t_o + \tau$, for all $\ell = 1, \ldots, r$.

In turn, this implies that, taking $t_o = k\Delta$ and $\tau = \Delta$, the state variables z_ℓ at $t = k\Delta + \Delta$ can be **exactly** expressed by:

$$z_1(k\Delta + \Delta) = z_1(k\Delta) + \Delta\, z_2(k\Delta) + \ldots + \frac{\Delta^r}{r!}[b + a\,u]_{t=\xi_1} \tag{6.50}$$

$$z_2(k\Delta + \Delta) = z_2(k\Delta) + \ldots + \frac{\Delta^{r-1}}{(r-1)!}[b + a\,u]_{t=\xi_2} \tag{6.51}$$

$$\vdots$$

$$z_r(k\Delta + \Delta) = z_r(k\Delta) + \Delta\,[b + a\,u]_{t=\xi_r} \tag{6.52}$$

and

$$\eta(k\Delta + \Delta) = \eta(k\Delta) + \Delta[q]_{t=\xi_{r+1}} \tag{6.53}$$

for some time instants $k\Delta < \xi_\ell < k\Delta + \Delta$, $\ell = 1, \ldots, r+1$.

Next we rewrite (6.50)–(6.53) using the δ-operator. We also replace the signals at sampling instants by their sampled counterparts, using the superscript ^{S}:

$$\delta z_1^S = z_2^S + \frac{\Delta}{2} z_3^S + \ldots + \frac{\Delta^{r-1}}{r!}[b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_1} \tag{6.54}$$

$$\delta z_2^S = z_3^S + \ldots + \frac{\Delta^{r-2}}{(r-1)!}[b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_2} \tag{6.55}$$

$$\vdots$$

$$\delta z_r^S = [b(\zeta,\eta) + a(\zeta,\eta)u]_{t=\xi_r} \tag{6.56}$$

$$\delta \eta^S = [c(\zeta,\eta)]_{t=\xi_{r+1}} \tag{6.57}$$

Note that this is an exact discrete-time description of the nonlinear system together with a ZOH input, for some (undetermined) time instants ξ_ℓ, $\ell = 1, \ldots, r+1$. Replacing these unknown time instants by $k\Delta$ we obtain the approximate discrete-time model in (6.44)–(6.45).

We next analyse the *local truncation error* (Butcher, 1987) between the true system output and the output of the obtained sampled data model, assuming that, at $t = k\Delta$, the state z^S is equal to the true system state $z(k\Delta)$. We compare the true system output at the end of the sampling interval, $y(k\Delta + \Delta) = z_1(k\Delta + \Delta)$ in (6.50), with the first (shifted) state of the approximate sampled-data model in (6.44), *i.e.*, with:

$$qz_1^S = (1 + \Delta\delta)\, z_1^S = z_1^S + \Delta z_2^S + \ldots + \tfrac{\Delta^r}{r!}[b(\zeta^S, \eta^S) + a(\zeta^S, \eta^S)u] \tag{6.58}$$

This yields the following local truncation output error:

$$\begin{aligned} e &= \left| y(k\Delta + \Delta) - q\, y^S \right| \\ &= \tfrac{\Delta^r}{r!} \Big| [b(\zeta, \eta) + a(\zeta, \eta)u_k]_{t=\xi_1} - [b(\zeta, \eta) + a(\zeta, \eta)u_k]_{t=k\Delta} \Big| \\ &\leq \tfrac{\Delta^r}{r!} \cdot L \big\| (\zeta, \eta)_{t=\xi_1} - (\zeta, \eta)_{t=k\Delta} \big\| \\ &= \tfrac{\Delta^r}{r!} \cdot L \big\| z(\xi_1) - z(k\Delta) \big\| \end{aligned} \tag{6.59}$$

where the existence of the Lipschitz constant $L > 0$ is guaranteed by the analyticity of $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ in (6.1)–(6.2) and, as a consequence, of $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$. Indeed, any $\mathcal{C}^1$ map satisfies locally at each point a Lipschitz condition (Lang, 1997).

Furthermore, according to (Butcher, 1987, Theorem 112E), the Lipschitz condition guarantees that the variation of the state trajectory $z(t)$ can be bounded as:

$$\big\| z(\xi_1) - z(k\Delta) \big\| \leq C \cdot \frac{e^{L|\xi_1 - k\Delta|} - 1}{L} < C \cdot \frac{e^{L\Delta} - 1}{L} = \mathcal{O}(\Delta) \tag{6.60}$$

The result then follows from equation (6.59). □

Remark 6.16 *The Taylor series truncation used in the proof of Theorem 6.15 is closely related to Runge-Kutta methods (Butcher, 1987), commonly used to simulate nonlinear systems. In fact, the model in Theorem 6.15 describes an approximate model for the output $y(t)$ and its derivatives to* solve *the nonlinear differential equation in one sampling interval. Furthermore, we will see in Theorem 6.22 that this improved numerical integration technique can be interpreted as incorporating sampling zero dynamics into the discrete-time model.*

Remark 6.17 *Theorem 6.15 shows that the accuracy of the approximate sampled-data model improves with the continuous-time system relative degree r. Thus, in general, one obtains a more accurate model than would result from simple derivative replacement using an Euler approximation.*

Remark 6.18 *Note that the sampled-data model described in Theorem 6.15 can be obtained for any equivalent representation of the nonlinear system of the form* (6.1)–(6.2)*. Specifically, the approximate sampled-data model* (6.44)–(6.45) *is described in terms of $b(\cdot)$, $a(\cdot)$, and $c(\cdot)$ which are functions of $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ — see Lemma 6.3.*

Remark 6.19 *In (Barbot* et al.*, 1992), a* sampled normal form *is obtained by a Taylor series expansion of all the elements of the state vector* (6.11) *to the same order in the sampling period Δ. By way of contrast, we have considered the smoothness of the input $u(t)$, and, thus, of $y(t)$ and its derivatives, to obtain the exact representation given in* (6.54)–(6.57) *and, from there, the approximate discrete-time model* (6.44)–(6.45).

Remark 6.20 *Note that the result in Theorem 6.15 can be equally applied to the nonuniform sampling case. In the latter case, the local truncation output error will be of order in Δ_k^{r+1}, where Δ_k is the length of the sampling interval $[t_k, t_{k+1})$.*

Remark 6.21 *An interesting observation can be made when obtaining the proposed sampled-data model for a linear n-th order integrator. The normal form for this particular system is readily obtained by substituting $a(x) \equiv 1$ and $b(x) \equiv 0$ in Lemma 6.3. Thus, the model given by Theorem 6.15 corresponds to the **exact** discrete-time description of the integrator obtained in Lemma 2.12 on page 20 — see, in particular, the matrices in* (2.59).

Next we present a result which shows that the discrete-time zero dynamics of the sampled-data model presented in Theorem 6.15 are given by the sampled counterpart of the continuous-time zero dynamics, together with extra zero dynamics produced by the sampling process. Perhaps surprisingly, these sampling zero dynamics turn out to be the same as those which appear asymptotically for the linear case.

Theorem 6.22 *The sampled-data model* (6.44)–(6.45) *generically has relative degree* 1*, with respect to the output $z_1^S = y^S$. Furthermore, the discrete-time zero dynamics are given by two subsystems:*

(i) The sampled counterpart of the continuous-time zero dynamics:

$$\delta\eta^S = c(0, \tilde{z}_{2:r}^S, \eta^S) \tag{6.61}$$

where $\tilde{z}_{2:r}^S \triangleq [\tilde{z}_2^S, \ldots, \tilde{z}_r^S]^T$, and

(ii) A linear subsystem of dimension $r-1$:

$$\delta\, \tilde{z}_{2:r}^S = Q_{22}\, \tilde{z}_{2:r}^S \tag{6.62}$$

where the eigenvalues of matrix Q_{22} are the same sampling zeros as in the asymptotic linear case, namely, the roots of $p_r(\Delta\gamma)$ defined in (2.55).

Proof. Using the definition of discrete-time relative degree given in Lemma 6.8, we have that:

$$\frac{\partial y^S}{\partial u} = \frac{\partial z_1^S}{\partial u} = 0 \tag{6.63}$$

$$\frac{\partial(\delta y^S)}{\partial u} = \frac{\partial(\delta z_1^S)}{\partial u} = \frac{\partial}{\partial u}\left(z_2^S + \ldots + \frac{\Delta^{r-1}}{r!}[b+au]\right) \neq 0 \tag{6.64}$$

which shows that (6.44)–(6.45) generically has relative degree 1. This result is consistent with (Barbot *et al.*, 1992, Lemma 2.2).

Now, in order to extract the zero dynamics of the discrete-time nonlinear system (6.44)–(6.45), we rewrite it in its normal form. To do so, we proceed as in the proof of Lemma 6.13 on page 111 for the n-th order integrator. We first define the following linear state transformation:

$$\tilde{\zeta}^S = \begin{bmatrix} \tilde{z}_1^S \\ \vdots \\ \tilde{z}_r^S \end{bmatrix} = T \begin{bmatrix} z_1^S \\ \vdots \\ z_r^S \end{bmatrix} = T\zeta^S \tag{6.65}$$

where matrix T is defined analogously to (6.33):

$$T = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline T_{21} & I_{r-1} \end{array}\right] \iff T^{-1} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline -T_{21} & I_{r-1} \end{array}\right] \tag{6.66}$$

where:

$$T_{21} = -\begin{bmatrix} \frac{r}{\Delta} & \cdots & \frac{r!}{\Delta^{r-1}} \end{bmatrix} \tag{6.67}$$

Substituting (6.65)–(6.66) in (6.44), we obtain a discrete-time normal form:

$$\begin{aligned} \delta\tilde{\zeta}^S &= \delta \begin{bmatrix} \tilde{z}_1^S \\ \tilde{z}_{2:r}^S \end{bmatrix} = \left[\begin{array}{c|c} q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \begin{bmatrix} \tilde{z}_1^S \\ \tilde{z}_{2:r}^S \end{bmatrix} \\ &\qquad + \begin{bmatrix} \frac{\Delta^{n-1}}{n!} \left(b(\tilde{\zeta}^S, \eta^S) + a(\tilde{\zeta}^S, \eta^S) u \right) \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{6.68}$$

$$\delta\eta^S = c(\tilde{\zeta}^S, \eta^S) = q\left(\tilde{z}_1^S, \tilde{z}_{2:r}^S, \eta^S\right) \tag{6.69}$$

where the sub-matrices in (6.68) are given by expressions analogous to (6.35)–(6.34).

Taking the output $y^S = z_1^S = \tilde{z}_1^S = 0$, for all (discrete) time instants $k \in \mathbb{Z}$, we now see that the discrete-time zero dynamics are described by two subsystems:

$$\delta\, \tilde{z}_{2:r}^S = Q_{22}\, \tilde{z}_{2:r}^S \tag{6.70}$$

$$\delta\, \eta^S = q\left(0, \tilde{z}_{2:r}^S, \eta^S\right) \tag{6.71}$$

and the eigenvalues of Q_{22} are clearly the same as the roots of $p_r(\Delta\gamma)$ as in Lemma 6.13.

□

Remark 6.23 *If the continuous-time input $u(t)$ is generated by a different hold device, for example, a First Order Hold as in* (2.11) *on page 13, this information can be used to include more terms in the Taylor's expansion* (6.50)–(6.52). *This, of course, would lead us to a different approximate discrete-time model in Theorem 6.15, with different sampling zeros in Theorem 6.22.*

Indeed, this fact corresponds to the results for linear systems in Chapter 3, where it was shown that the asymptotic sampling zeros depend on the system relative degree and also *on the nature of the hold device used to generate the continuous-time system input.*

6.4 Implications in nonlinear system identification

We believe that the results in the previous sections give additional insight into many problems in nonlinear system theory. As a specific illustration, we next consider the problem of nonlinear system identification based on sampled output observations. Note that we do not explicitly consider noise in this section since our focus is on the deterministic (bias) errors resulting from under-modelling in sampled-data models.

The results in Section 6.3 describe an approximate sampled-data discrete-time model for a nonlinear system. This model shows that the accuracy of the sampled data model can be improved by using a better approximation than simple Euler integration (The latter procedure is equivalent to replace $\frac{d}{dt}$, in the continuous model, by the δ operator in the approximate sampled-data model). The more accurate discrete-time model can be interpreted as including *sampling zero dynamics*, which are the same as in the linear system case.

In this section we illustrate the use of the approximate sampled-data model (6.44)–(6.45) for parameter estimation of a particular nonlinear system. This model, which includes sampling zero dynamics, gives better results than those achieved by simply replacing time derivatives by divided differences, even when fast sampling rates are utilised.

Example 6.24 *Consider the nonlinear system defined by the differential equation:*

$$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t)(1 + \varepsilon_1 y^2(t)) = \beta_0(1 + \varepsilon_2 y(t))u(t) \tag{6.72}$$

This model can be expressed in state-space form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ f(x_1, x_2, u) \end{bmatrix} \tag{6.73}$$

$$y = x_1 \tag{6.74}$$

where we have defined the function:

$$f(x_1, x_2, u) = -\alpha_1 x_2 - \alpha_0 x_1(1 + \varepsilon_1 x_1^2) + \beta_0(1 + \varepsilon_2 x_1)u \tag{6.75}$$

This system has relative degree $r = 2$ for all $x_o \in \mathbb{R}^2$, and is already in normal form (6.9)–(6.10). *The nonlinear function* (6.75) *can be linearly reparameterised as $f(x_1, x_2, u) = \phi(t)^T\theta$, where:*

$$\phi(t) = \begin{bmatrix} -x_2(t) \\ -x_1(t) \\ -x_1(t)^3 \\ u(t) \\ x_1(t)u(t) \end{bmatrix} \quad \text{and} \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \varepsilon_1\alpha_0 \\ \beta_0 \\ \varepsilon_1\beta_0 \end{bmatrix} \tag{6.76}$$

We next perform system identification by applying an equation error procedure on three different model structures (compare with the linear case treated in Section 4.3.1):

SDRM: *A Simple Derivative Replacement Model. This model is obtained by simply replacing the time derivatives by divided differences in the state-space model* (6.73)–(6.74). *This leads to the approximate model:*

$$\textit{SDRM:} \quad \delta^2 y = -\theta_1 \delta y - \theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 uy \tag{6.77}$$

where the parameters θ_i are given in (6.76).

MIFZD: *A Model Incorporating Fixed Zero Dynamics. This is based on our proposed discrete-time nonlinear model in Theorem 6.15. The corresponding state space representation is given by:*

$$\delta x_1 = x_2 + \tfrac{\Delta}{2} f(x_1, x_2, u) \tag{6.78}$$

$$\delta x_2 = f(x_1, x_2, u) \tag{6.79}$$

where $f(x_1, x_2, u)$ *is defined in* (6.75). This particular system *can be rewritten as a divided difference equation as follows:*

$$\textit{MIFZD:} \quad \delta^2 y = -\theta_1 \delta y + (1 + \tfrac{\Delta}{2}\delta)(-\theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 y u) \tag{6.80}$$

where the parameters θ_i *are given in* (6.76)*.*

MIPZD: *A Model Incorporating Parameterised Zero Dynamics. This is also based on our proposed discrete-time nonlinear model* (6.78)–(6.79)*, with the difference being that here we expand* (6.80) *and relax the existing relation between the parameters of the different terms. This yields:*

$$\textit{MIPZD:} \quad \delta^2 y = -\theta_1 \delta y - \theta_2 y - \theta_3 y^3 + \theta_4 u + \theta_5 y u - \theta_6 \delta(y^3) + \theta_7 \delta u + \theta_8 \delta(yu) \tag{6.81}$$

where $\theta_1 = \alpha_1 + \frac{\Delta}{2}\alpha_0$, $\{\theta_2, \ldots, \theta_5\}$ *are given in* (6.76), $\theta_6 = \frac{\Delta}{2}\alpha_0\varepsilon_1$, $\theta_7 = \frac{\Delta}{2}\beta_0$, *and* $\theta_8 = \frac{\Delta}{2}\beta_0\varepsilon_2$.

Note that the MIPZD in (6.81) *can be rewritten in state-space form as:*

$$\delta x_1 = x_2 - \theta_1 x_1 - \theta_6 x_1^3 + \theta_7 u + \theta_8 u x_1 \tag{6.82}$$

$$\delta x_2 = -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 u + \theta_5 u x_1 \tag{6.83}$$

with output $y = x_1$.

The parameters for the three models, SDRM in (6.77)*, MIFZD in* (6.80)*, and MIPZD in* (6.81)*, can be estimated using the ordinary least squares method by minimising the* Equation Error *cost function:*

$$J_{ee}(\theta) = \frac{1}{N}\sum_{k=0}^{N-1} e_k(\theta)^2 = \frac{1}{N}\sum_{k=0}^{N-1} (\delta^2 y - {\phi_k}^T \theta)^2 \tag{6.84}$$

where:

$$\phi_k = \begin{cases} [-\delta y,\ -y,\ -y^3,\ u,\ uy]^T & \textit{(SDRM)} \\ [-\delta y,\ -(1+\frac{\Delta}{2}\delta)y,\ -(1+\frac{\Delta}{2}\delta)(y^3),\ (1+\frac{\Delta}{2}\delta)u,\ (1+\frac{\Delta}{2}\delta)(uy)]^T & \textit{(MIFZD)} \\ [-\delta y,\ -y,\ -y^3,\ u,\ uy,\ -\delta(y^3),\ \delta u,\ \delta(uy)]^T & \textit{(MIPZD)} \end{cases} \tag{6.85}$$

The parameters for each model were estimated by performing 50 Monte Carlo simulations, using different realisations of a Gaussian random input sequence u_k *(zero mean, unit variance). The sampling period was* $\Delta = \pi/20$*[s]. The results are summarised in Table 6.1. We can see that both MIFZD and MIPZD give good estimates for the continuous-time parameters, whereas SDRM is not able to find the right values, especially for the parameters* $\{\theta_3, \theta_4, \theta_5\}$*. Of course, small discrepancies from the continuous-time parameters are explained by the non infinitesimal sampling period.*

To explore the convergence of the parameter estimates to the continuous-time values, we repeat the simulations for different sampling period. Table 6.2 shows the root mean square error between the average parameters obtained by running 50 Monte Carlo simulations for each sampling period. Note that we are able to compare only the first five parameters of the MIPZD. In fact, we can see that, as the

	CT	SDRM		MIFZD		MIPZD	
		avg	std	avg	std	avg	std
θ_1	3	2.6987	0.4622	2.6479	0.0241	2.6414	0.0141
θ_2	2	1.5080	1.2832	1.5999	0.0487	1.5876	0.0330
θ_3	1	5.8089	30.2745	0.6442	1.0831	0.7299	0.3986
θ_4	2	0.7431	0.1467	1.6054	0.0081	1.5882	0.0052
θ_5	1	0.1597	0.9703	0.7752	0.0557	0.7770	0.0317
θ_6	–	–	–	–	–	0.1152	0.0959
θ_7	–	–	–	–	–	0.1345	0.0004
θ_8	–	–	–	–	–	0.0665	0.0022
	$J_{ee}(\theta)$	0.6594	0.1493	0.0069	0.0021	0.0001	0.0001
Validation		0.7203		0.0076		0.0003	

Table 6.1: Parameter estimates using equation error procedures.

	$\sqrt{\frac{1}{5}\sum_{i=1}^{5}\left(\bar{\theta}_i - {\theta_i}^{ct}\right)^2}$		
Δ	SDRM	MIFZD	MIPZD
$\pi/20$	2.2691	0.3513	0.3438
$\pi/100$	9.6156	0.0744	0.0714
$\pi/200$	53.6027	0.0508	0.0366
$\pi/500$	109.5187	0.0167	0.0146

Table 6.2: Convergence of parameter estimates.

sampling period is reduced this is the model that gives the best estimation of the true parameter vector. On the other hand, the estimate corresponding to the SDRM is clearly asymptotically biased.

We also tested the three models, SDRM, MIFZD, and MIPZD, with the average estimated parameters *that appear in Table 6.1, using a longer validation data set of length* 100*[s] and the same sampling period* $\Delta = \pi/20$*[s]. Part of the output of the nonlinear continuous-time system and the discrete-time models, when using the validation input, are shown in Figure 6.1. We see that both models based on our proposed state-space model as described in Section 6.3 replicate the continuous-time output very accurately. On the other hand, the SDRM has a clear bias.*

The value of the Equation Error cost function (6.84) *for each one of the three discrete-time models, when considering the sampled input and output validation data, appears in the last row of Table 6.1.*

□

The results obtained for the nonlinear models in the previous example highlight that the inclusion of zero dynamics (as in MIFZD and MIPZD) allows one to obtain better results than a simple derivative replacement approach (as in SDRM). Actually, the results presented here are a nonlinear extension of the results presented earlier in Chapter 4 for linear systems. In particular, if we consider $\varepsilon_1 = \varepsilon_2 = 0$ in (6.72), we obtain the same second order linear system considered earlier in Example 4.1 on page 65.

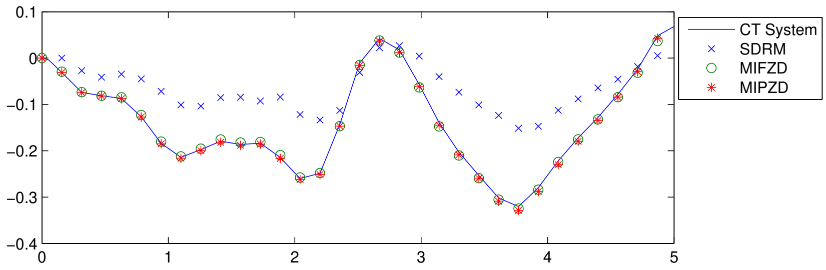


Figure 6.1: Simulated output sequences for the validation input.

We recall that in the linear case presented in Section 4.3.1 on page 70, we were able to show analytically that the system gain was underestimated by a factor of 2 using the SDRM approach. On the other hand, all the continuous-time parameters were recovered when using MIFZD and MIPZD. Thus, the results presented in Example 6.24 are clearly consistent with the simpler linear results presented in Part I.

Remark 6.25 *The asymptotic bias in the SDRM estimates obtained in Example 6.24 can be explained by reviewing Lemma 4.9 on page 71. In that result, we show asymptotic bias is also obtained in the linear case (see Example 4.8). This bias can be mitigated, for example, if we use output error system identification instead of least squares estimation, but at the expense of using non-convex optimisation.*

6.5 Summary

In this chapter we have developed an approximate discrete-time model for deterministic nonlinear systems. The sampled-data model described here has several interesting features:

- It is simple to obtain, in particular, by expressing the continuous-time system in its *normal form*.
- It provides a *local truncation error* between the output of the approximate discrete-time model and the output of the underlying continuous-time system of order Δ^{r+1}, where r is the (nonlinear) system relative degree and Δ is the sampling period.
- It is obtained through a more sophisticated derivative approximation than the simple Euler approach.
- An insightful interpretation is given in terms of explicit characterisation of the nonlinear *sampling zero dynamics* of the obtained discrete-time model.

These results extend well-known results for models of sampled linear systems to the nonlinear case. Of particular interest is the occurence of sampling zero dynamics, with no counterpart in the underlying continuous-time nonlinear system. This mirrors the linear case.

The results are believed to give important insights in different problems in nonlinear systems theory. By way of illustration, we have shown that models obtained using equation error system identification methods have higher fidelity when nonlinear *sampling zero dynamics* are included in the model.

Chapter 7

Sampled-data models for stochastic nonlinear systems

7.1 Introduction

In this chapter we consider sampled-data models for stochastic nonlinear systems. These systems are usually expressed, in continuous-time, using **stochastic differential equations** (SDEs).

Stochastic differential equations are of interest in many different areas of science and engineering. They have been successfully applied to problems such as population growth models, signal estimation, optimal control, boundary value problems, and mathematical finance (Øksendal, 2003). SDE models have also been used for nonlinear system identification (Bohlin and Graebe, 1995; Kristensen *et al.*, 2003; Kristensen *et al.*, 2004) and control of stochastic systems (Deng and Krstic, 1997*a*; Deng and Krstic, 1997*b*; Pan, 2001; Pan, 2002).

In this chapter we are interested in the use of SDEs to model continuous-time stochastic systems (also called *noise models*), and from those models, to obtain sampled-data descriptions which are accurate in a well defined sense.

The mathematical treatment of SDEs has similarities, but also slight differences, to the usual theory of deterministic differential equations. One needs to be aware of these similarities and differences when considering numerical methods to solve them. This will turn out to be important in the context of the current chapter, where we study how one can obtain discrete-time models for nonlinear systems based on numerical solutions of stochastic differential equations.

Explicitly solvable SDEs are rare in practical applications (Kloeden and Platen, 1992). Thus, numerical solutions play a key role in filling the gap between a well developed theory and applications. In this framework, sampled-data models can be understood as numerical algorithms for solving (approximately) a given SDE.

The basic theory of SDEs presented in Section 7.2 is mainly based on (Øksendal, 2003), whereas topics related to numerical methods for SDEs can be found in (Kloeden and Platen, 1992). There also exist many other references in the area of stochastic calculus applied to control and estimation (Bucy and Joseph, 1968; Åström, 1970; Jazwinski, 1970; Kallianpur, 1980). More advanced mathematical details can be found in (Protter, 1990; Kushner and Dupuis, 1992; Klebaner, 1998).

In Section 7.3 we present a sampled-data model to represent stochastic nonlinear systems. To find the exact solution of a (stochastic) nonlinear differential equation will usually be impossible, as was found for the deterministic case in the previous chapter. Thus, only **approximate** sampled-data models will be obtained. However, the accuracy of the proposed model will be characterised in terms of a very specific order of convergence. Connections to the linear case are also established.

7.2 Background on stochastic differential equations

In this section we review some concepts and results from stochastic calculus and stochastic differential equations. We deal with these topics in a general framework, including linear systems as a particular case of the nonlinear theory.

We will consider stochastic nonlinear systems expressed as a set of differential equations:

$$\frac{dx(t)}{dt} = a(t,x) + b(t,x)\dot{v}(t) \tag{7.1}$$

$$y(t) = c(t,x) \tag{7.2}$$

where the input $\dot{v}(t)$ is a *continuous-time white noise* (CTWN) process having constant spectral density $\sigma_v^2 = 1$. The functions $a(\cdot)$ and $b(\cdot)$ are assumed analytical, *i.e.*, $\mathcal{C}^\infty$. This latter assumption can sometimes be relaxed to *smooth enough* functions, ensuring that the required derivatives are well defined.

Note that the model structure in (7.1) is similar to the deterministic description in (6.1), *i.e.*, the system equation is affine in the input signal. Conditions for existence of diffeomorphisms that transform stochastic linear systems to different canonical forms can be found in (Pan, 2002).

The model (7.1)–(7.2) depends on the CTWN process $\dot{v}(t)$. However, as previously discussed in Section 2.4, white noise processes in continuous-time do not exist in any meaningful sense (see Remark 2.24 on page 28). In fact, for a proper mathematical treatment, equation (7.1) should be understood as a stochastic differential equation (SDE):

$$dx_t = a(t,x_t)\,dt + b(t,x_t)\,dv_t \tag{7.3}$$

where the driving input to the system are the **increments** of $v_t = v(t)$, a Wiener process of unitary incremental variance. Equation (7.3) is, in fact, usually rewritten as the integral equation:

$$x_t = x_o + \int_0^t a(\tau,x_\tau)d\tau + \int_0^t b(\tau,x_\tau)dv_\tau \tag{7.4}$$

which consists of an initial condition x_o (possibly, random), a slowly varying continuous component called the **drift** term, and a rapidly varying continuous random component called the **diffusion** term.

Remark 7.1 *The last integral involved in expression* (7.4) *cannot be interpreted in the* usual *Riemann-Stieltjes sense (Øksendal, 2003). In the literature, two constructions of this integral are usually considered, leading to different calculi:*

- *The* ***Ito integral*** *construction:*

$$\int f(t)dv_t = \lim \sum_\ell f(t_\ell)[v_{t_{\ell+1}} - v_{t_\ell}] \tag{7.5}$$

- *The **Stratonovich integral** construction:*

$$\int f(t) \circ dv_t = \lim \sum_{\ell} f\left(\tfrac{t_{\ell+1}-t_{\ell}}{2}\right)[v_{t_{\ell+1}} - v_{t_\ell}] \tag{7.6}$$

Each one of these definitions presents some advantages and drawbacks. For example, by using the Stratonovich construction (7.6)*, the usual* chain rule *for transformations can be applied, whereas when using* (7.5) *we need to apply the* Ito rule *(see Section 7.2.1). On the other hand, the Ito integral* (7.5) *has the specific feature of* not looking into the future*. In fact, the Ito integral is a **martingale**, whereas the Stratonovich integral is not, and, conversely, every martingale can be represented as a Ito integral (Øksendal, 2003)* .

Here we will consider the Ito construction (7.5) only, but equivalent results can be obtained by using the Stratonovich definition (7.6). We will refer to (7.4) as an Ito integral, and x_t as an Ito process described either by this integral equation or by the SDE in (7.3).

7.2.1 The Ito rule

The Ito construction of a stochastic integral in (7.5) implies an important departure point from the *usual* calculus. Specifically, the usual *chain rule* for transformations has to be modified. The key point that leads to this result is given by the properties of the Wiener process $v(t)$ (see Section 2.4.2 on page 27). In particular, its incremental variance can be obtained as:

$$E\left\{(v(t) - v(s))^2\right\} = |t - s| \quad ; \forall t \neq s \tag{7.7}$$

$$\Rightarrow \quad E\left\{dv^2\right\} = E\left\{(v(t + dt) - v(t))^2\right\} = dt \tag{7.8}$$

Lemma 7.2 (Ito rule for scalar processes) *Let us consider a scalar Ito process x_t as in* (7.3)*, and a transformation of this process:*

$$y = g(t, x) \tag{7.9}$$

where $g(t, x) \in \mathcal{C}^2$, i.e., *g has at least its second order continuous derivatives. Then $y = y_t$ is also an Ito process, and:*

$$dy = \frac{\partial g}{\partial t}(t, x)dt + \frac{\partial g}{\partial x}(t, x)dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, x)(dx)^2 \tag{7.10}$$

The differential in the last term, $(dx)^2 = (dx(t))(dx(t))$, is computed according to:

$$dt \cdot dt = dt \cdot dv = dv \cdot dt = 0 \tag{7.11}$$

$$dv \cdot dv = dt \tag{7.12}$$

Proof. See, for example, (Øksendal, 2003).

□

Note that the Ito rule arises from (7.7), where we can see that the variance of the increments dv is of order dt. As a consequence, the last term in (7.10) has to be considered.

Lemma 7.2 presents the derivative rule for transformations of a scalar process $x(t)$. The next result considers the general case for a vector process X_t.

Lemma 7.3 (Multidimensional Ito rule) *Let us consider an Ito process X_t defined by the set of SDE:*

$$dX = A(t, X)dt + B(t, X)dV \tag{7.13}$$

where:

$$X = X_t = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad A(t, X) = \begin{bmatrix} a_1(t, X) \\ \vdots \\ a_n(t, X) \end{bmatrix} \quad B(t, X) = \begin{bmatrix} b_{11}(t, X) & \cdots & b_{1m}(t, X) \\ \vdots & \ddots & \vdots \\ b_{n1}(t, X) & \cdots & b_{nm}(t, X) \end{bmatrix} \tag{7.14}$$

and dV are increments of a multidimensional Wiener (vector) process:

$$V = V_t = \begin{bmatrix} v_1(t) & \cdots & v_m(t) \end{bmatrix}^T \tag{7.15}$$

We consider the transformation given by a $\mathcal{C}^2$ vector map:

$$Y = G(t, X) = \begin{bmatrix} g_1(t, X) & \cdots & g_p(t, X) \end{bmatrix}^T \tag{7.16}$$

Then $Y = Y_t$ is again an Ito process, given (component-wise) by:

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_{\ell=1}^{n} \left(\frac{\partial g_k}{\partial x_\ell}(t, X)dx_\ell \right) + \frac{1}{2} \sum_{\ell,m=1}^{n} \frac{\partial^2 g_k}{\partial x_\ell \partial x_m}(t, X)(dx_\ell)(dx_m) \tag{7.17}$$

for all $k = 1, \ldots, p$; and where the terms in the last sum are computed according to:

$$dv_\ell \cdot dt = dt \cdot dv_m = dt \cdot dt = 0 \tag{7.18}$$

$$dv_\ell \cdot dv_m = \delta_K[\ell - m]\, dt \tag{7.19}$$

Proof. See (Øksendal, 2003).

□

Remark 7.4 *A **linear** stochastic system as considered in Section 2.4 on page 25 can be expressed as the (vector) SDE:*

$$dX_t = AX_t\, dt + Bdv_t \tag{7.20}$$

where the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$.

Note that, in this case, the driving input comprises increments of a single scalar Wiener process $v_t = v(t)$. The solution to this SDE can be obtained by applying the result in Lemma 7.3 for the transformation:

$$Y = e^{-At}X \quad \Rightarrow \quad d(e^{-At}X) = (-A)e^{-At}Xdt + e^{-At}dX \tag{7.21}$$

Note that, in this case, all the second order derivatives in (7.17) *vanish. Thus, reordering terms in* (7.20) *and multiplying by the* integrating factor e^{-At}*, we can see that:*

$$e^{-At}dX_t - Ae^{-At}X_tdt = e^{-At}Bdv_t \tag{7.22}$$

$$d(e^{-At}X_t) = e^{-At}Bdv_t \tag{7.23}$$

$$e^{-At}X_t = X_0 + \int_0^t e^{-A\tau}Bdv_\tau \tag{7.24}$$

where we finally obtain the solution:

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-\tau)}Bdv_\tau \tag{7.25}$$

Note that this solution corresponds, in fact, to the same state transition equation as in the deterministic linear case (see, for example, (2.15) *on page 14) where the deterministic input is replaced by the CTWN process,* i.e., $u(\tau)d\tau = \dot{v}(\tau)d\tau = dv_\tau$.

□

7.2.2 Ito-Taylor expansions

In this section we review stochastic Taylor expansions. These expansions generalise the deterministic Taylor formula as well as the Ito stochastic rule. They allow one to obtain higher order approximations to functions of stochastic processes and, thus, will prove useful in the context of numerical solutions of SDEs in the next section.

We first review the usual Taylor formula we used to obtain the deterministic sampled-data model in Chapter 6. However, in this case we will reexpress it in integral form. We thus consider the following nonlinear differential equation and its *implicit* solution in integral form:

$$\frac{dx}{dt} = a(x) \qquad \Longleftrightarrow \qquad x(t) = x(0) + \int_0^t a(x)d\tau \tag{7.26}$$

If we now consider a general (continuously differentiable) function $f(x)$, then, by using the *usual* chain rule, we have that:

$$\frac{df(x)}{dt} = a(x)\frac{\partial f}{\partial x} \qquad \Longleftrightarrow \qquad f(x) = f(x_0) + \int_0^t Lf(x)d\tau \tag{7.27}$$

where we have used the notation $L = a(x)\frac{\partial}{\partial x}$, and $x_0 = x(0)$.

Note that the integral relation in the last equation is valid, in particular, for $f = a$. Thus, it can be used to substitute $a(x)$ into the integral on the right hand side of (7.26), *i.e.*,

$$\begin{aligned} x(t) &= x_0 + \int_0^t \left(a(x_0) + \int_0^{\tau_1} La(x)d\tau_2 \right) d\tau_1 \\ &= x_0 + a(x_0)\,t + R_2 \end{aligned} \tag{7.28}$$

where we have a *residual* term:

$$R_2 = \int_0^t \int_0^{\tau_1} La(x)d\tau_2 d\tau_1 \tag{7.29}$$

We can use again relation in (7.27), with $f = a$, to replace $a(x)$ in R_2, obtaining:

$$x(t) = x_0 + a(x_0)\,t + (La)(x_0)\,\frac{t^2}{2} + R_3 \tag{7.30}$$

$$R_3 = \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} L^2 a(x)d\tau_3 d\tau_2 d\tau_1 \tag{7.31}$$

We thus have the following general result.

Lemma 7.5 *Given a function* $f \in \mathcal{C}^{r+1}$, i.e., $r+1$ *continuously differentiable, we can express it using the* Taylor formula in integral form*:*

$$f(x(t)) = f(x_0) + \sum_{\ell=1}^{r} (L^\ell f)(x_0) \frac{t^\ell}{\ell!} + R_{r+1} \tag{7.32}$$

where the residual term is given by:

$$R_{r+1} = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_r} L^{r+1} f(x) d\tau_{r+1} \dots d\tau_2 d\tau_1 \tag{7.33}$$

□

Note that (7.28) and (7.30) are particular cases of (7.32), considering $f(x) = x$ and, thus, $Lf(x) = a(x)$.

For the stochastic case we can follow a similar line of reasoning. Thus, if we consider an Ito process:

$$x_t = x_0 + \int_0^t a(x_\tau) d\tau + \int_0^t b(x_\tau) dv_\tau \tag{7.34}$$

and a transformation $f(x) \in \mathcal{C}^2$, we can apply the Ito rule in (7.10) to obtain:

$$\begin{aligned} f(x_t) &= f(x_0) + \int_0^t \left(a(x_\tau) \frac{\partial f(x_\tau)}{\partial x} + \frac{1}{2} b^2(x_\tau) \frac{\partial^2 f(x_\tau)}{\partial x^2} \right) d\tau + \int_0^t b(x_\tau) \frac{\partial f(x_\tau)}{\partial x} dv_\tau \\ &= f(x_0) + \int_0^t L^0 f(x_\tau) d\tau + \int_0^t L^1 f(x_\tau) dv_\tau \end{aligned} \tag{7.35}$$

where we have defined the operators:

$$L^0 = a \frac{\partial}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2} \qquad\qquad L^1 = b \frac{\partial}{\partial x} \tag{7.36}$$

Analogously to the deterministic case, if we now apply the Ito formula (7.35) to $f = a$ and $f = b$ in (7.34), we obtain:

$$x_t = x_0 + a(x_0) \int_0^t d\tau + b(x_0) \int_0^t dv_\tau + R_2^s \tag{7.37}$$

$$\begin{aligned} R_2^s = &\int_0^t \int_0^{\tau_1} L^0 a(x_{\tau_2})\, d\tau_2\, d\tau_1 + \int_0^t \int_0^{\tau_1} L^1 a(x_{\tau_2})\, dv_{\tau_2}\, d\tau_1 \\ &+ \int_0^t \int_0^{\tau_1} L^0 b(x_{\tau_2})\, d\tau_2\, dv_{\tau_1} + \int_0^t \int_0^{\tau_1} L^1 b(x_{\tau_2})\, dv_{\tau_2}\, dv_{\tau_1} \end{aligned} \tag{7.38}$$

which is the stochastic analogue for the Taylor formula of second order. Indeed, if the diffusion term is $b(x_t) \equiv 0$, then equations (7.37)–(7.38) reduce to the deterministic expressions in (7.28)–(7.29)

It is possible to go further to obtain Ito-Taylor expansions where we use again (7.35) to substitute $f = a$ and $f = b$ in (7.38). However, the expressions become increasingly involved, including multiple stochastic integrals. Kloeden and Platen (1992) give a systematic notation to manipulate the required multiple integrals and, thus, to obtain higher order Ito-Taylor expansions for a general SDE, by using *multi-indices* and *hierarchical sets*. For example, in (Kloeden and Platen, 1992, p.182) the following expression is obtained for the stochastic analogue to the expansion obtained in (7.30):

$$\begin{aligned} x_t = x_0 &+ aI_{(0)} + bI_{(1)} + \left(aa' + \tfrac{1}{2} b^2 a''\right) I_{(0,0)} \\ &+ \left(ab' + \tfrac{1}{2} b^2 b''\right) I_{(0,1)} + ba' I_{(1,0)} + bb' I_{(1,1)} + R_3^s \end{aligned} \tag{7.39}$$

where:

$$a = a(x_0) \qquad a' = \frac{\partial a}{\partial x}(x_0) \qquad a'' = \frac{\partial^2 a}{\partial x^2}(x_0) \tag{7.40}$$

$$b = b(x_0) \qquad b' = \frac{\partial b}{\partial x}(x_0) \tag{7.41}$$

and:

$$I_{(0)} = \int_0^t d\tau_1 \qquad I_{(1)} = \int_0^t dv_{\tau_1} \tag{7.42}$$

$$I_{(0,0)} = \int_0^t \int_0^{\tau_1} d\tau_2 d\tau_1 \qquad I_{(0,1)} = \int_0^t \int_0^{\tau_1} d\tau_2 dv_{\tau_1} \tag{7.43}$$

$$I_{(1,0)} = \int_0^t \int_0^{\tau_1} dv_{\tau_2} d\tau_1 \qquad I_{(1,1)} = \int_0^t \int_0^{\tau_1} dv_{\tau_2} dv_{\tau_1} \tag{7.44}$$

In the next section we will discuss numerical solutions for SDEs that can be derived from Ito-Taylor expansions of the type described above. We first present the following result which establishes the convergence of truncated Ito-Taylor expansions.

Lemma 7.6 *Consider an Ito process x_t as in* (7.34) *and its corresponding k-th order truncated Ito-Taylor expansion $x_k(t)$, around $t = t_o$. Then we have that:*

$$E\left\{|x_t - x_k(t)|^2\right\} \leq C_k(t - t_o)^{k+1} \tag{7.45}$$

where C_k is a constant that depends only on the truncation order k.

Proof. The details of the proof and can be found in (Kloeden and Platen, 1992, Section 5.9).

□

The previous lemma establishes that an Ito-Taylor expansion converges to the original Ito process in the mean square sense, as k goes to infinity. Under additional assumptions, the previous result can be strengthened to convergence with probability one, uniformly on the interval $[t_o, t]$.

7.2.3 Numerical solution of SDEs

In the previous section we presented Ito-Taylor expansions that allow higher order approximations of an Ito process defined by an SDE. Analogously to the deterministic case in the previous chapter, Ito-Taylor expansions can be used to derive discrete-time approximations to solve SDEs. The simplest of these approximations can be obtained by truncating the expansion in (7.37):

$$x_t = x_0 + a(x_0)t + b(x_0)\int_0^t dv_\tau \tag{7.46}$$

This is the stochastic equivalent of the Euler approximation for ordinary differential equations, and is sometimes called the **Euler-Maruyama approximation**. Note that from this approximation a simple sampled-data model can readily be obtained as:

$$\bar{x}((k+1)\Delta) = \bar{x}(k\Delta) + a(\bar{x}(k\Delta))\Delta + b(\bar{x}(k\Delta))\Delta v_k \tag{7.47}$$

where $\Delta v_k = v_{(k+1)\Delta} - v_{k\Delta}$ are increments of the Wiener process v_τ:

$$E\left\{\Delta v_k\right\} = 0 \qquad\qquad E\left\{(\Delta v_k)^2\right\} = \Delta \tag{7.48}$$

Naturally, other algorithms can be derived by considering more terms in the Ito-Taylor expansions. To analyse the quality of different algorithms we will employ two different criteria to measure their accuracy, namely, strong and weak convergence. These concepts are formally defined as follows:

Definition 7.7 *We say that a sampled-data approximation $\bar{x}(k\Delta)$, obtained using a sampling period Δ,* ***converges strongly*** *to the continuous-time process x_t at time $t = k\Delta$ if*

$$\lim_{\Delta\to 0} E\{|\bar{x}(k\Delta) - x_{k\Delta}|\} = 0 \tag{7.49}$$

Furthermore, we will say that it converges strongly with order $\gamma > 0$, *if there exists a positive constant C and a sampling period $\Delta_o > 0$ such that*

$$E\left\{\left|\bar{x}(k\Delta) - x_{k\Delta}\right|\right\} \leq C\Delta^\gamma \tag{7.50}$$

for all $\Delta < \Delta_o$.

This type of convergence is also called *path-wise* convergence: the sampled data model is required to replicate the continuous-time system output when the same realisation of noise process is used as input.

The error between the discrete-time model and the continuous-time process can also be bounded using the Lyapunov inequality:

$$E\left\{\left|\bar{x}(k\Delta) - x_{k\Delta}\right|\right\} \leq \sqrt{E\left\{\left|\bar{x}(k\Delta) - x_{k\Delta}\right|^2\right\}} \tag{7.51}$$

Example 7.8 *If we consider again the Euler-Maruyama scheme introduced previously in* (7.46)*, we see that it corresponds to the truncated Ito-Taylor expansion containing only the time and Wiener integrals of multiplicity one. Thus, it can be interpreted as an order* 0.5 *strong Ito-Taylor approximation (Kloeden and Platen, 1992, p. 341)*

□

A weaker definition of convergence can be obtained by not considering each *path* of the process involved, but instead focusing on the associated statistical properties.

Definition 7.9 *We say that a sampled-data approximation $\bar{x}(k\Delta)$, obtained using a sampling period Δ,* ***converges weakly*** *to the continuous-time process x_t at time $t = k\Delta$, for a class $\mathcal{T}$ of test functions if*

$$\lim_{\Delta\to 0} \left|E\left\{g(x_t)\right\} - E\left\{g(\bar{x}(k\Delta))\right\}\right| = 0 \tag{7.52}$$

Note that if $\mathcal{T}$ contains all polynomials this definition implies the convergence of all moments. Furthermore, we will say that it converges weakly with order $\beta > 0$, *if there exists a positive constant C and a sampling period $\Delta_o > 0$ such that*

$$\left|E\left\{g(x_k)\right\} - E\left\{g(\bar{x}(k\Delta))\right\}\right| \leq C\Delta^\beta \tag{7.53}$$

for all $\Delta < \Delta_o$.

7.3 A sampled data model for stochastic systems

In this chapter we will restrict ourselves to the class of nonlinear stochastic systems described in (7.1)–(7.2). Note, in particular, that this continuous-time model has a single scalar noise source as driving input. Before presenting a sampled-data model for this class nonlinear stochastic system, we consider the following preliminary case.

Example 7.10 *Consider the stochastic n-th order integrator:*

$$\frac{d^n y(t)}{dt^n} = \dot{v}(t) \tag{7.54}$$

This system can be described by the linear SDE:

$$dX = A\,X\,dt + B\,dv_t \tag{7.55}$$

where:

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad A = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{7.56}$$

and the output is $y(t) = x_1(t)$.

If we consider the SDE equation corresponding to the last state x_n, we can see that the expansion in (7.37) *gives the **exact** solution, because $R_2^s \equiv 0$,* i.e.,

$$dx_n = dv_t \quad \Rightarrow \quad x_n(\Delta) = x_n(0) + \int_0^\Delta dv_{\tau_n} \tag{7.57}$$

If we now use this expression in the equation corresponding to x_{n-1}, we have that:

$$\begin{aligned} dx_{n-1} = x_n\,dt \quad \Rightarrow \quad x_{n-1}(\Delta) &= x_{n-1}(0) + \int_0^\Delta x_n(\tau_{n-1})d\tau_{n-1} \\ &= x_{n-1}(0) + x_n(0)\,\Delta + \int_0^\Delta \int_0^{\tau_{n-1}} dv_{\tau_n} d\tau_{n-1} \end{aligned} \tag{7.58}$$

Proceeding in this way for the other state components of the SDE (7.55)–(7.56) *we obtain:*

$$X_\Delta = \begin{bmatrix} 1 & \Delta & \dots & \frac{\Delta^{n-1}}{(n-1)!} \\ 0 & 1 & \dots & \frac{\Delta^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} X_0 + \begin{bmatrix} I_{(1,0,\dots,0,0)} \\ I_{(1,0,\dots,0)} \\ \vdots \\ I_{(1)} \end{bmatrix} \tag{7.59}$$

where we have used multi-indices as in (7.42), i.e.,

$$I_{(1,\underbrace{0,\dots,0}_{m\ zeros})} = \int_0^\Delta \int_0^{\tau_1} \cdots \int_0^{\tau_m} dv_{\tau_{m+1}} d\tau_m \dots d\tau_1 \tag{7.60}$$

Note that since this system is linear, the same discrete-time model can be obtained using our earlier results in Chapter 2. Specifically, we can use Lemma 2.25 on page 29. Substituting the matrices A *and* B *given in* (7.56) *into equation* (2.115) *we obtain:*

$$X_\Delta = \underbrace{\begin{bmatrix} 1 & \Delta & \dots & \frac{\Delta^{n-1}}{(n-1)!} \\ 0 & 1 & \dots & \frac{\Delta^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{A_q} X_0 + \underbrace{\int_0^\Delta e^{A(\Delta-\tau)} B dv_\tau}_{\tilde{V}} \tag{7.61}$$

where $A_q = e^{A\Delta}$*, and the elements of* $\tilde{V}$ *can be obtained from* (2.116) *on page 29,* i.e.,

$$\tilde{V} = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \cdots & \tilde{v}_n \end{bmatrix}^T \tag{7.62}$$

$$\Rightarrow \quad \tilde{v}_\ell = \int_0^\Delta \frac{(\Delta-\tau)^{(n-\ell)}}{(n-\ell)!} dv_\tau \qquad ; \ell = 1, \dots, n \tag{7.63}$$

The first two moments of vector $\tilde{V}$ *in* (7.61) *can be obtained as in Lemma 2.25,* i.e.,

$$E\left\{\tilde{V}\right\} = 0 \tag{7.64}$$

$$\begin{aligned} E\left\{\tilde{V}\tilde{V}^T\right\} &= \int_0^\Delta e^{A\eta} B B^T e^{A^T\eta} d\eta = \int_0^\Delta \begin{bmatrix} \frac{\eta^{n-1}}{(n-1)!} \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \frac{\eta^{n-1}}{(n-1)!} \\ \vdots \\ 1 \end{bmatrix}^T d\eta \\ &= \int_0^\Delta \left[\frac{\eta^{n-i}}{(n-i)!} \frac{\eta^{n-j}}{(n-j)!} \right]_{i,j=1,\dots,n} d\eta \\ &= \left[\frac{\Delta^{2n-i-j+1}}{(n-i)!(n-j)!(2n-i-j+1)} \right]_{i,j=1,\dots,n} \end{aligned} \tag{7.65}$$

where $\left[m_{ij}\right]_{i,j=1,\dots,n}$ *represents an* $n \times n$ *matrix whose entries are* m_{ij}*.*

For example, if we consider the case of a the second order stochastic integrator, we have that:

$$E\left\{ \begin{bmatrix} \tilde{v}_1^2 & \tilde{v}_1\tilde{v}_2 \\ \tilde{v}_2\tilde{v}_1 & \tilde{v}_2^2 \end{bmatrix} \right\} = \begin{bmatrix} \frac{\Delta^3}{3} & \frac{\Delta^2}{2} \\ \frac{\Delta^2}{2} & \Delta \end{bmatrix} \tag{7.66}$$

□

Remark 7.11 *The results in the previous example are consistent with (Kloeden and Platen, 1992, Section 5.7) where moments of multiple stochastic integrals such as the ones given in* (7.59) *are obtained.*

Remark 7.12 *Note that the discretised model* (7.61) *was obtained by expanding each of the states of the continuous-time description* (7.55) *to different orders. However, for this particular system, each one of the Ito-Taylor expansions involved is exact and, thus, the sampled-data model for the stochastic* n*-th order integrator* ***converges strongly*** *to the true solution with order* $\gamma = \infty$*.*

Note that we have found two alternative ways to define the noise vector $\tilde{V}$. First, its elements were defined in terms of multiple stochastic integrals in (7.59) and (7.60). Secondly, using the (simpler) approach in Lemma 2.25 in Chapter 2, we obtained (7.61)–(7.63). We next prove that both expressions are equivalent.

Lemma 7.13 *Consider a standard Wiener process v_t and a time period Δ. Then the following* reduction rule *for multiple integrals holds:*

$$I_{(1,\underbrace{0,\ldots,0}_{m\ zeros})} = \int_0^{\Delta}\int_0^{\tau_m}\cdots\int_0^{\tau_1} dv_\tau\, d\tau_1 \ldots d\tau_m = \int_0^{\Delta} \frac{(\Delta-\tau)^m}{m!}\, dv_\tau \quad ; m \geq 1 \tag{7.67}$$

Proof. Relation (7.67) can be proven using induction. For $m = 0$, we have, by definition:

$$I_{(1)} = \int_0^{\Delta} dv_\tau \tag{7.68}$$

For $m = 1$ we have:

$$I_{(1,0)} = \int_0^{\Delta}\int_0^{\tau_1} dv_\tau d\tau_1 = \int_0^{\Delta}\int_\tau^{\Delta} d\tau_1 dv_\tau = \int_0^{\Delta}(\tau-\Delta)dv_\tau \tag{7.69}$$

where we changed the order of integration in the double integral.

Finally, we assume that the result holds for $m = k$, and we will prove it for $m = k+1$:

$$\begin{aligned} I_{(1,\underbrace{0,\ldots,0}_{k+1\ zeros})} &= \int_0^{\Delta}\left(\int_0^{\tau_{k+1}}\cdots\int_0^{\tau_1} dv_\tau\, d\tau_1 \ldots d\tau_k\right) d\tau_{k+1} \\ &= \int_0^{\Delta}\left(\int_0^{\tau_{k+1}} \frac{(\tau_{k+1}-\tau)^k}{k!} dv_\tau\right) d\tau_{k+1} \end{aligned} \tag{7.70}$$

where we have used the result (7.67) for $m = k$ setting $\Delta = \tau_{k+1}$. The result is obtained by changing the order of integration, *i.e.*,

$$\begin{aligned} I_{(1,\underbrace{0,\ldots,0}_{k+1\ zeros})} &= \int_0^{\Delta}\int_0^{\tau_{k+1}} \frac{(\tau_{k+1}-\tau)^k}{k!} dv_\tau d\tau_{k+1} = \int_0^{\Delta}\int_\tau^{\Delta} \frac{(\tau_{k+1}-\tau)^k}{k!} d\tau_{k+1} dv_\tau \\ &= \int_0^{\Delta} \frac{(\Delta-\tau)^{k+1}}{(k+1)!} dv_\tau \end{aligned} \tag{7.71}$$

□

Corollary 7.14 *As a consequence of Lemma 7.13 we have the following relation between* (7.60) *and* (7.63)*:*

$$\tilde{v}_\ell = I_{(1,\underbrace{0,\ldots,0}_{n-\ell\ zeros})} \tag{7.72}$$

□

We next turn to a general class of nonlinear stochastic systems. We will obtain a sampled-data model for this class of systems based on a similar expansion on the different state variables. We will consider that these systems can be expressed in a particular state-space description which mimics the, so called, *normal form* for deterministic nonlinear systems. Normal forms for stochastic differential equations have been studied in (Arnold and Kedai, 1995; Arnold and Imkeller, 1998).

Assumption 7.15 *Given the stochastic nonlinear system* (7.1)–(7.2)*, we assume that there exists a transformation* $Z = \Phi(X)$*, such that, in the new coordinates, the system is expressed as:*

$$\frac{dZ(t)}{dt} = A\,Z + B\left(a(t,Z) + b(t,Z)\dot{v}(t)\right) \tag{7.73}$$

$$y(t) = z_1 \tag{7.74}$$

where the matrices A *and* B *are as in* (7.56).

Renaming the state variables, the previous assumption implies that we will restrict ourselves to stochastic nonlinear systems that can be expressed as the following set of SDEs:

$$dx_1 = x_2 dt \tag{7.75}$$

$$\vdots$$

$$dx_{n-1} = x_n dt \tag{7.76}$$

$$dx_n = a(x)dt + b(x)dv_t \tag{7.77}$$

where the output is $y = x_1$. The functions $a(\cdot)$ and $b(\cdot)$ are assumed analytic or $\mathcal{C}^\infty$.

Note that, as in Example 7.10, if we expand the last state equation (7.77), we can use the first Ito-Taylor expansion as:

$$x_n(\Delta) = x_n(0) + a(X_0)\,\Delta + b(X_0)\tilde{v}_n \tag{7.78}$$

where $X_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^T$ and $\tilde{v}_n$ is defined in (7.63).

If we proceed in the same manner with the second last state SDE in (7.76), we obtain:

$$\begin{aligned} x_{n-1}(\Delta) &= x_{n-1}(0) + \int_0^\tau x_n(\tau)d\tau \\ &= x_{n-1}(0) + x_n(0)\,\Delta + a(X_0)\,\frac{\Delta^2}{2} + b(X_0)\tilde{v}_{n-1} \end{aligned} \tag{7.79}$$

where we have used Corollary 7.14 to substitute $I_{(1,0)} = \tilde{v}_{n-1}$. Proceeding in a similar way with the rest state components, we can obtain truncated Ito-Taylor expansions up to the first state in (7.75), *i.e.*,

$$x_1(\Delta) = x_1(0) + x_2(0)\Delta + \ldots + x_n(0)\frac{\Delta^{n-1}}{(n-1)!} + a(X_0)\,\frac{\Delta^n}{n!} + b(X_0)\tilde{v}_1 \tag{7.80}$$

Note that the last equation corresponds, in fact, to the Ito-Taylor expansion for the system output $y = x_1$ of order n.

Note that the truncated Ito-Taylor expansions obtained above for each of the state components can be readily rewritten and summarised in terms of the δ operator:

$$\frac{X_\Delta - X_0}{\Delta} = \delta X_\Delta = A_\delta\, X_\Delta + B_\delta a(X_0) + b(X_0)V \tag{7.81}$$

where:

$$X_\Delta = \begin{bmatrix} x_1(\Delta) \\ x_2(\Delta) \\ \vdots \\ x_n(\Delta) \end{bmatrix} \quad A_\delta = \begin{bmatrix} 0 & 1 & \cdots & \frac{\Delta^{n-2}}{(n-1)!} \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad B_\delta = \begin{bmatrix} \frac{\Delta^{n-1}}{n!} \\ \frac{\Delta^{n-2}}{(n-1)!} \\ \vdots \\ 1 \end{bmatrix} \quad V = \frac{1}{\Delta}\tilde{V} \tag{7.82}$$

The following theorem presents a sampled-data model based on the expansions in (7.78)–(7.80), and precisely characterises its accuracy.

Theorem 7.16 *Consider the continuous-time stochastic system* (7.75)–(7.77)*, with output $y = x_1$, and the following sampled-data model:*

$$\delta \bar{X}(k\Delta) = A_\delta \bar{X}(k\Delta) + B_\delta a(\bar{X}(k\Delta)) + b(\bar{X}(k\Delta))V_k \tag{7.83}$$

and A_δ and B_δ are defined as in (7.82)*, and $V_k = \frac{1}{\Delta}\tilde{V}_k$ corresponds to* (7.62)–(7.63) *where the integration interval has to be changed to $[k\Delta, k\Delta + \Delta)$ (By stationarity of v_t, vectors V_k and V have the same covariance).*

Then, the output $\bar{y}(k\Delta) = \bar{x}_1(k\Delta)$ of this model ***converges strongly*** *to the* true *output with order $\gamma = n/2$, provided that:*

$$E\left\{\left|y_0 - \bar{y}(0)\right|^2\right\} \leq C_0\,\Delta^n \tag{7.84}$$

for some constant C_0.

Proof. The proof follows from the fact that the discrete-time expression obtained for the output $\bar{y} = \bar{x}_1$ corresponds to the truncated Ito-Taylor expansion of order n, and thus from (Kloeden and Platen, 1992, Section 10.6) we have that:

$$E\left\{\left|y_{t=k\Delta+\Delta} - \bar{y}(k\Delta + \Delta)\right|^2\right\} \leq C_{k+1}\,\Delta^n \tag{7.85}$$

provided that $E\left\{\left|y_{t=k\Delta} - \bar{y}(k\Delta)\right|^2\right\} \leq C_k\,\Delta^n$.

The result then follows by applying the Lyapunov inequality (7.51) and the Definition 7.7.

□

Remark 7.17 *From* (7.65) *we can see that the covariance structure of V_k will be given by:*

$$E\left\{V_k V_\ell^T\right\} = \frac{1}{\Delta^2} E\left\{\tilde{V}_k \tilde{V}_\ell^T\right\} = \left[\frac{\Delta^{2n-i-j}}{(n-i)!(n-j)!(2n-i-j)}\right]_{i,j=1,\ldots,n} \frac{\delta_K[k-\ell]}{\Delta} \tag{7.86}$$

As the sampling period goes to zero, this covariance approaches the continuous-time covariance of $B\dot{v}(t)$ (see equation (7.73)*). This results parallels the linear case discussed earlier in Remark 2.27 on page 30.*

Remark 7.18 *We can see that the sampled-data model presented in* (7.83) *closely resembles the discrete-time model obtained for the n-th order stochastic integrator in Example 7.10. In particular, the vector $\tilde{V}_k$ plays a key role as the driving input in both cases. If we substitute $a(X) \equiv 0$ and $b(X) \equiv 1$ in* (7.83)*, we can see that this model reduces to the* ***exact*** *sampled-data model obtained* (7.61)*. Thus, for this particular case, the order of strong convergence of the sampled-data model is $\gamma = \infty$.*

The previous remarks show strong parallels between the stochastic nonlinear case discussed here and the corresponding results for linear stochastic sampled-data models. The analysis reinforces the notion that the n-th integrator case gives important insights into obtaining accurate sampled-data models for more general systems. The role of the integrator was first highlighted in the context of linear systems in Part I to characterise asymptotic sampled-data models for continuous-time system. Furthermore, in the previous chapter we noticed that the sampled-data model for deterministic nonlinear systems happens to be exact when considering the case of linear integrator (see Remark 6.21 on page 116).

7.4 Summary

In this chapter we have presented a sampled-data model for stochastic nonlinear systems described by stochastic differential equations (SDEs).

We have briefly reviewed basic SDE theory and associated numerical solution schemes. These schemes are generally based on Ito-Taylor expansions of stochastic processes. These expansion are the stochastic counterpart of the *usual* deterministic Taylor expansion in integral form.

We have considered a particular class of stochastic nonlinear systems that can be expressed in a form that resembles the, so called, *normal form* for deterministic nonlinear models. The discrete-time description obtained for this class of systems was shown to be accurate in a well defined sense.

Moreover, a connection to the linear case was established. Specifically, the associated input of the proposed stochastic sampled-data model has the same covariance structure as the discrete-time input that arises in the sampling of an n-th order stochastic linear integrator.

Chapter 8

Summary and conclusions

8.1 General Overview

In this thesis we have studied sampled-data models for linear and nonlinear systems. We have reviewed existing results and presented novel contributions. In this final chapter we summarise the main results presented throughout the thesis. We also discuss the implications and inherent difficulties of using sampled-data models, defined at discrete-time instants, to represent real systems evolving in continuous-time. There are still many open and new problems in this area and, thus, we also present some future research directions based on the issues raised along the chapters of this thesis.

8.1.1 Sampling of continuous-time systems

We have studied the sampling process of continuous-time systems: linear and nonlinear, deterministic and stochastic. The sampled-data models obtained were shown to depend not only on the underlying continuous-time system, but also on the details of the sampling process itself. Specifically, the hold device, used to generate a continuous-time input, and the sampling device, that gives us the output sequence of samples, both play an important role in the sampling process. The effect of these *artifacts* of sampling becomes negligible when the sampling period goes to zero. However, for any finite sampling rate, their role has to be considered to obtain accurate sampled-data models.

8.1.2 Sampling zeros

We have seen that sampled-data models have, in general, more zeros than the underlying continuous-time system. These extra zeros, called *sampling zeros*, have no continuous-time counterpart. For the linear case, their presence can be interpreted as a consequence of the *aliasing* effect of the system frequency response (or spectrum), where high frequency components are folded back to low frequencies due to the sampling process. We have seen that sampling zeros arise in both deterministic and stochastic systems. Exact expressions for these zeros are not easy to obtain. However, they can be asymptotically characterised in terms of the Euler-Fröbenius polynomials.

The presence of these *sampling zeros* in discrete-time models is an illustration of the inherent difference between continuous- and discrete-time system descriptions. When using δ-operator models these zeros go to infinity as the sampling period goes to zero, nonetheless they generally have to be taken into account to obtain accurate discrete-time descriptions.

We have seen that the above ideas can be also extended to the nonlinear case. In fact, the sampled-data model obtained for nonlinear systems contains extra zero dynamics with no counterpart in continuous time. These *sampling zero dynamics* are a consequence of using a more accurate discretisation procedure than simple Euler integration. Surprisingly, the extra zero dynamics turn out to be the same as the dynamics associated with the asymptotic sampling zeros in the linear case.

8.1.3 Use of sampled-data models

We have seen that sampled-data can be successfully applied in estimation and control. In particular, when expressed in terms of the δ-operator, these models provide a natural framework to deal with continuous-time problems in real applications, where data and control actions are defined at specific time instants only.

The use of sampled data taken from continuous-time systems inherently implies a **loss of information**. Even though it is possible to obtain accurate models, there will always exist a gap between the discrete- and continuous-time representations. As a consequence, one needs to rely on **assumptions** on the inter-sample behaviour of signals or, equivalently, on the characteristics of the system response beyond the sampling frequency.

We have seen that frequency *aliasing* and the presence of sampling zeros are strongly connected. The characterisation of the asymptotic sampling zeros relies on the continuous-time system relative degree. However, relative degree is an *ill-defined* quantity in continuous-time because that can be affected by high frequency under-modelling.

In a similar fashion, the unknown input to stochastic systems is assumed to be a continuous-time white noise process. However, this is a mathematical abstraction which has no physical counterpart. In practice it is only used to approximate (coloured) broad-band noise processes. This implies the presence of potential modelling errors in the continuous-time stochastic system description.

Based on these issues we have repeatedly stressed the concept of **bandwidth of validity** for continuous-time models, within which assumptions, such as relative degree, can be trusted. We have emphasised the importance of this concept, in particular, when utilising asymptotic results for fast sampling rates. We introduced this concept for sample and hold designs in Chapter 3, and we showed its importance also for continuous-time system identification from sampled data in Chapter 4.

8.2 Summary of chapter contributions

There has been significant ongoing research regarding the use of sampled-data models to represent continuous-time systems. However, there remain many open problems and associated research opportunities. In this context, we believe that the current thesis has presented novel contributions and new insights into the sampling process and into the use of sampled-data models for control and estimation.

In **Chapter 2** we presented the basic framework associated with the sampling process of linear systems, both deterministic and stochastic. The results presented have been expressed in two equivalent forms: using the shift operator q, and the divided difference operator δ. In particular, we have presented a novel characterisation of the sampling zeros that arise in sampled-data models in the δ-domain. Even though this result is well known in the z-domain, the alternative formulation in the δ-domain turns out to be one of the key enabling tools for the nonlinear case in Chapter 6. We have also presented a novel recursive relation between the polynomials that define the sampling zeros in the δ-domain.

In **Chapter 3** the role of sample and hold devices in obtaining sampled-data models was studied in detail. It is well known that the zeros of sampled-data models for deterministic and stochastic systems depend on the particular choice of the hold used to generate the continuous-time system input, and the pre-filter used prior to obtain the output sequence of samples, respectively. The contribution in this chapter has been to show that these devices can be designed in such a way as to asymptotically assign the sampling zeros of the discrete-time model. This result clearly illustrates that the *artifacts* of the sampling process do play an important role in obtaining accurate discrete-time descriptions for continuous-time systems. We have also stressed that, even though the design procedures depend only on the continuous-time system relative degree, this may be *ill-defined*. In fact, we have shown that (continuous-time) modelling errors beyond the sampling frequency can have a significant impact on the discrete-time results. Thus, as a second contribution in the chapter, we have introduced the concept of *bandwidth of validity* for continuous-time models, within which one can rely on characteristics such as relative degree.

The issue of validity of continuous-time models was also one of the key motivations for the work presented in **Chapter 4**, in the context of continuous-time system identification from sampled data. We studied the role of the sampling zeros and the effect of high frequency (continuous-time) under-modelling on sampled-data models used in parameter estimation. Our contribution in this chapter has been to show that these issues can be addressed by using a *restricted bandwidth* maximum likelihood estimation in the frequency domain. Indeed, the proposed procedure was successfully applied to CAR systems and shown to be robust to (continuous- and discrete-time) modelling errors beyond the considered bandwidth of validity.

In **Chapter 5** we have shown how sampled-data models can be successfully applied in LQ optimal control problems. The presence of input and/or state constraints can render the continuous-time solution of the problem impossible to find explicitly. Thus, one is usually forced to use optimisation algorithms in discrete-time. As a first contribution, we have shown that the solution of an associated sampled-data problem (with possibly tighter constraints) converges to the hypothetical solution of the original problem, satisfying the continuous-time constraints. An immediate consequence of this result is the existence of a finite sampling period such that the achieved performance is arbitrarily close to the limiting continuous-time performance. Thus, sampled-data models are a useful tool to deal with problems defined in continuous-time that may be more difficult or even impossible to solve.

A second novel result in Chapter 5 was the convergence established between the singular structures of the discrete- and the continuous-time LQ problems. In particular, the singular values of a linear operator associated with sampled-data LQ problem were shown to converge, as the sampling period goes to zero, to (a subset of) the singular values of the operator corresponding to the continuous-time problem. The existence of a well defined limit, as the sampling rate increases, could be exploited in approximate algorithms for the continuous-time problem in high speed applications, which are solved using standard discrete-time methods for constrained systems.

In **Chapter 6** we have presented an approximate sampled-data model for deterministic nonlinear systems. The proposed model has several interesting features: it is simple to obtain, it is accurate in a well defined sense (as a function of the nonlinear relative degree and the sampling period), and it has *sampling zero dynamics*, with no counterpart in continuous-time. The model was obtained by a *Runge-Kutta*-like discretisation procedure more accurate than simple Euler integration, where derivatives are replaced by divided differences. As a way of illustration, the model was used for nonlinear system identification and shown to lead to better results than when using simple Euler integration models.

The presence of sampling zero dynamics in sampled-data models for nonlinear systems has been previously been established. However, a contribution in Chapter 6 has been to show that the sampling zero dynamics of the proposed model are exactly the same as the dynamics corresponding to the asymptotic sampling zeros in the linear case.

Finally, in **Chapter 7**, we have considered sampled-data models for stochastic nonlinear systems. An approximate model was obtained by similar considerations as in the deterministic case, but making use of the stochastic Ito-Taylor expansions. The resultant discrete-time description is, in fact, closely related to existing approaches for numerical solution of stochastic differential equations.

8.3 Future research

The work reported in this thesis raises several interesting lines of research for the future. We next list some of these possible research topics.

Robust continuous-time system identification. We have seen in Chapter 4 that issues associated with the use of sampled data can have a key impact on continuous-time system identification results. We have shown that procedures such as frequency domain maximum likelihood estimation can be modified, *e.g.*, using a restricted bandwidth, in such a way as to reduce the sensitivity to the inherent loss of information due to sampling. In this regard, other identification approaches can be analysed and modified accordingly to achieve robustness to the issues associated with the use of sampled data and discrete-time models.

Singular structure for the infinite horizon case. In Section 5.4 we established the convergence, from discrete- to continuous-time, of the singular structure of linear operators associated with **finite horizon** LQ optimal control problems. On the other hand, in (Rojas and Goodwin, 2004; Rojas *et al.*, 2004; Rojas, 2004) connections are established between the singular values of the associated Hessian of discrete-time LQ problems and the frequency response of the system, when the discrete-time horizon tends to infinity. These results should be understood in a common framework represented schematically by the following diagram:

$$\begin{array}{ccc} \text{Discrete-time LQ problem} & \xrightarrow{\Delta\to 0} & \text{Continuous-time LQ problem} \\ \text{with finite horizon } T = N\Delta & & \text{with finite horizon } T \\ {\scriptstyle N\to\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle T\to\infty} \\ \text{Discrete-time LQ problem} & \xrightarrow[\Delta\to 0]{} & \text{Continuous-time LQ problem} \\ \text{with infinite horizon} & & \text{with infinite horizon} \end{array} \tag{8.1}$$

The result presented in Section 5.4 corresponds to the convergence described accross the top of the diagram: the singular structure of a sampled-data problem converges, as the sampling period goes to zero, to the singular structure of the underlying continuous-time problem, both specified for a fixed finite horizon.

On the other hand, the results in (Rojas and Goodwin, 2004; Rojas *et al.*, 2004; Rojas, 2004) correspond to the left column of the diagram. They showed that when considering a (pure) discrete-time problem, the singular values of the Hessian matrix associated with the LQ problem converges to the frequency response of an associated normalised system, as the discrete-time horizon grows to infinity.

The two aforementioned results are certainly strongly connected. In particular, singular values of matrices are known to be equivalent to singular values of linear operators defined in Hilbert spaces.

To complete the relations in the above diagram would unveil deeper mathematical connections between finite and infinite horizon LQ problems in discrete- and continuous-time. In particular, we believe that once the continuous-time infinite-horizon problem (bottom left corner of the previous diagram) is fully understood, the remaining three cases in the previous scheme can be obtained by truncating the time horizon and/or introducing sampled-data models. This certainly constitutes an interesting and challenging research topic.

Applications of nonlinear sampled-data models. The sampled-data model presented in Chapter 6 is believed to give insights into nonlinear systems theory. In particular, Section 6.4 has explored the use of this model for nonlinear system identification. However, the latter is only one example used here to illustrate the advantages of using a sampled-data model that is simple but more accurate than simple Euler integration. The sampled-data model presented here may give similar advantages in other areas such as control of nonlinear systems. In this framework, it could be used either to have an accurate discrete-time description of the plant, or to digitally implement a continuous-time controller. These issues remain an interesting research area.

Stochastic sampling zero dynamics. An insightful interpretation of the sampled-data model for *deterministic* nonlinear systems in Chapter 6 was given in terms of the presence of sampling zero dynamics. Moreover, these extra zero dynamics of the proposed (discrete-time) model were shown to be the same as the dynamics associated with the asymptotic sampling zeros in the linear case. An immediate question that arises is how this relation between linear and nonlinear sampling zeros can also be established for the stochastic case. We envisage two possible lines of research to unveil this connection:

- In the linear case, stochastic sampled-data models (and, thus, stochastic sampling zeros) are obtained by spectral factorisation of the output sampled spectrum. Thus, extensions of spectrum and spectral factorisation of sampled-data models are required for the nonlinear case. For continuous-time systems, nonlinear spectral factorisation has been considered in (Ball and Petersen, 2003; Ball *et al.*, 2004).
- Pan (2002) and Arnold and Imkeller (1998) have studied canonical and normal forms for (continuous-time) stochastic systems. If such analysis is extended to sampled-data (or purely discrete-time) models then, writing these models in a *sampled normal form*, one would be able to recognise its (sampling) zero dynamics.

Appendix A

Matrix results

Exponential Matrix

The **exponential matrix**, for a given matrix $M \in \mathbb{R}^{n\times n}$, is given by the formal power series:

$$e^M = I_n + M + \frac{1}{2!}M^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}M^n \tag{A.1}$$

Continuous-time Lyapunov equation

The continuous-time Lyapunov equation is given by:

$$AP + PA^H + Q = 0 \tag{A.2}$$

where Q is hermitian, *i.e.*, $Q = Q^H = (Q^*)^T$.

- There is a unique solution for P if, and only if, no eigenvalue of A has a zero real part and no two eigenvalues are negative complex conjugates of each other. If this condition is satisfied then the unique P is hermitian.
- If A is stable then P is unique, hermitian, and:

$$P = \int_0^\infty e^{A\tau} Q e^{A^H \tau} d\tau \tag{A.3}$$

- If A is stable and Q is positive definite (or semi-definite) then P is unique, hermitian and positive definite (or semi-definite).

Discrete-time Lyapunov equation

The discrete-time Lyapunov equation is given by:

$$A_q P A_q^H - P + Q = 0 \tag{A.4}$$

where Q is hermitian, *i.e.*, $Q = Q^H = (Q^*)^T$.

- There is a unique solution P if, and only if, no eigenvalue of A_q is the reciprocal of an eigenvalue of A_q^H. If this condition is satisfied, the unique P is hermitian.
- If A_q is stable (eigenvalues inside the unit circle) then P is unique, hermitian, and:

$$P = \sum_{k=0}^{\infty} A_q^k Q (A_q^H)^k \tag{A.5}$$

- If A_q is stable and Q is positive definite (or semi-definite) then P is unique, hermitian, and positive definite (or semi-definite).

Computation of covariance matrix

Lemma A.1 *Let us consider a matrix Ω_c hermitian, positive semi-definite, and the following matrix:*

$$\Omega_\delta = \frac{1}{\Delta} \int_0^\Delta e^{A\tau} \Omega_c e^{A^H \tau} d\tau \tag{A.6}$$

Then the following relation holds:

$$\Omega_\delta = \frac{1}{\Delta} \left(P - e^{A\Delta} P e^{A^H \Delta} \right) \tag{A.7}$$

where P is the solution of the Lyapunov equation (A.2) *where $Q = \Omega_c$.*

Proof. It follows from the derivative of a matrix product:

$$\frac{d}{d\tau}\left(e^{A\tau} P e^{A^H \tau} \right) = \frac{d}{d\tau}\left(e^{A\tau} \right) P e^{A^H \tau} + e^{A\tau} P \frac{d}{d\tau}\left(e^{A^H \tau} \right) \tag{A.8}$$

But noting, from (A.1), that $\frac{d}{d\tau} e^{M\tau} = M e^{M\tau} = e^{M\tau} M$, we obtain:

$$\frac{d}{d\tau}\left(e^{A\tau} P e^{A^H \tau} \right) = e^{A\tau} \left(AP + PA^H \right) e^{A^H \tau} = -e^{A\tau} \Omega_c e^{A^H \tau} \tag{A.9}$$

if, and only if, $AP + PA^H + \Omega_c = 0$. If now integrate on both sides of the last equation we obtain the result:

$$\int_0^\Delta e^{A\tau} \Omega_c e^{A^H \tau} d\tau = - \left[e^{A\tau} P e^{A^H \tau} \right]_0^\Delta = P - e^{A\Delta} P e^{A^H \Delta} \tag{A.10}$$

□

Block Matrices

Theorem A.2 *Consider the block matrix:*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{A.11}$$

where A is an $n \times n$ matrix, B and C^T are an $n \times m$ matrices, and D is an $m \times m$ matrix.

The schur complements *of A and D are defined (if they exist), respectively, as:*

$$P = D - CA^{-1}B \tag{A.12}$$

$$Q = A - BD^{-1}C \tag{A.13}$$

Then the following results hold:

(Block decomposition)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & I_m \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix} \tag{A.14}$$

$$= \begin{bmatrix} I_n & B \\ 0 & D \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I_m \end{bmatrix} \tag{A.15}$$

(Determinant)

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \det(A)\det(D - CA^{-1}B) \tag{A.16}$$

$$= \det(D)\det(A - BD^{-1}C) \tag{A.17}$$

(Inverse)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} & -Q^{-1}BD^{-1} \\ -D^{-1}CQ^{-1} & D^{-1}(I_m + CQ^{-1}BD^{-1}) \end{bmatrix} \tag{A.18}$$

$$= \begin{bmatrix} A^{-1}(I_n + BP^{-1}CA^{-1}) & -A^{-1}BP^{-1} \\ -P^{-1}CA^{-1} & P^{-1} \end{bmatrix} \tag{A.19}$$

Matrix Inversion Lemma

A very important consequence of the previous results regarding block matrices is the matrix inversion lemma, expressed in the following general form:

Lemma A.3 *Provided that the matrices have the appropriate dimensions and the inverses exist, the following equation hold:*

$$(M_1 + M_2M_3M_4)^{-1} = M_1^{-1}(I - M_2(M_3^{-1} + M_4M_1^{-1}M_2)^{-1}M_4M_1^{-1}) \tag{A.20}$$

Proof. The result can be obtained, for example, by comparing the top left blocks of matrices (A.18) and (A.19), where we replace $A = M_1$, $B = -M_2$, $C = M_4$, and $D = M_3^{-1}$.

□

Appendix B

Linear operators in Hilbert spaces

Hilbert spaces

In this appendix we present some concepts and definitions related to Hilbert spaces and spectral properties of linear operators (Helmberg, 1969; Balakrishnan, 1976; Kreyszig, 1978; Leigh, 1980; Lang, 1993).

This brief review is included as supporting material for the topics and results in Chapter 5, related to singular structure of LQ optimal control problems.

Definition B.1 *A **linear space** is a nonvoid set $\mathcal{S}$ for which two operations are defined: addition and scalar multiplication. Addition is commutative and associative. Multiplication by scalars (either from the real or complex field) is associative, and disctributive with respect to addition of elements of $\mathcal{S}$ as well as addition of scalars.*

Definition B.2 *An **inner product** in a linear space $\mathcal{S}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ satisfying the following properties:*

(i) $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$ *; for all scalars α_1, α_2 and $x_1, x_2, y \in \mathcal{S}$.*

(ii) $\langle x, y \rangle = \langle y, x \rangle^*$ *, where * denotes complex conjugation.*

(iii) $\langle x, x \rangle \geq 0$ *, and the equality holds only if x is zero.*

Definition B.3 *A linear space $\mathcal{S}$ endowed with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space** (or pre-Hilbert space). It is a normed linear space, where the norm is induced by the inner product:*

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{B.1}$$

Definition B.4 *A sequence x_n in a normed linear space $\mathcal{S}$ is a **Cauchy sequence** if, and only if:*

$$\forall \varepsilon > 0 \,, \, \exists N_\varepsilon > 0 \quad \textit{such that} \quad n, m > N_\varepsilon \; \Rightarrow \|x_n - x_m\| < \varepsilon \tag{B.2}$$

Definition B.5 *An inner product space (normed linear) $\mathcal{S}$ is called **complete** if all Cauchy sequences $\{x_n\}$ converge to an element in the space,* i.e.,

$$\lim_{n \to \infty} x_n = x \in \mathcal{S} \tag{B.3}$$

Definition B.6 *A complete inner product space is called a **Hilbert space***

Linear Operators

Functions can be defined for the elements of a Hilbert space. A given function may defined only on a subset of the Hilbert space, called the **domain** of definition of the function, D. The **range** R of the function is the set into which the function maps the domain. In particular, the term **functional** is generally used for functions whose range is scalar valued ($\mathbb{R}$ or $\mathbb{C}$). It is common to refer to the function as an **operator** when the domain is a dense subspace (and hence, as a special case, the whole Hilbert space), and the range is contained into a Hilbert space.

Definition B.7 *Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, an operator $L : D \subseteq \mathcal{H}_1 \to R \subseteq \mathcal{H}_2$ is **linear** if*

$$L(\alpha x + \beta y) = \alpha L x + \beta L y \tag{B.4}$$

for all $x, y \in \mathcal{H}_1$ and all scalars α, β.

Definition B.8 *A linear operator L is **bounded** if $D = \mathcal{H}_1$ and:*

$$\sup_{x \in \mathcal{H}_1} \frac{\|Lx\|}{\|x\|} = M < \infty \tag{B.5}$$

*If the above supremum exists, M is called the **norm** of the operator L.*

Lemma B.9 *Given a linear operator $L : \mathcal{H}_1 \to \mathcal{H}_2$, the following statements are equivalent:*

(i) The linear operator L is bounded.

(ii) The linear operator L is continuous at the origin.

(iii) The linear operator L is continuous at every point $x \in \mathcal{H}_1$.

Proof. See any of the references, for example, (Kreyszig, 1978).

□

Adjoints operators

Definition B.10 *Consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and a linear bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$. Then a linear operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, is called the **adjoint** operator of T, if and only if:*

$$\langle y, T(x) \rangle_{\mathcal{H}_2} = \langle T^*(y), x \rangle_{\mathcal{H}_1} \qquad ; \forall x \in \mathcal{H}_1 \, , \, \forall y \in \mathcal{H}_2 \tag{B.6}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ denote the inner products in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

In the following examples we show how to obtain the adjoint operators considered in Section 5.4. These operators are associated with LQ optimal control problem in continuous- and discrete-time, respectively.

Example B.11 *Consider the Hilbert spaces* $\mathcal{V} = \mathcal{L}_2(0, T_f; \mathbb{R}^m)$ *and* $\mathcal{Z} = \mathbb{R}^n \times \mathcal{L}_2(0, T_f; \mathbb{R}^n)$*, with inner products:*

$$\langle f_1, f_2 \rangle_{\mathcal{V}} = \int_0^{T_f} f_1(t)^T f_2(t)\, dt \qquad ;\ f_1, f_2 \in \mathcal{V} \tag{B.7}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} = \left(g_1^0\right)^T g_2^0 + \int_0^{T_f} g_1^1(t)^T g_2^1(t)\, dt \qquad ;\ g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{B.8}$$

Let us define the linear operator $\mathcal{G} : \mathcal{V} \to \mathcal{Z}$ *(as in* (5.74)*), such that:*

$$\mathcal{G} f = \begin{bmatrix} (\mathcal{G} f)^0 \\ (\mathcal{G} f)^1(t) \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}} \int_0^{T_f} e^{A(T_f - \xi)} B R^{-\frac{1}{2}} f(\xi)\, d\xi \\ Q^{\frac{1}{2}} \int_0^{t} e^{A(t - \xi)} B R^{-\frac{1}{2}} f(\xi)\, d\xi \end{bmatrix} \tag{B.9}$$

where the matrices P*,* Q*, and* R *are symmetric (see Chapter 5).*

The adjoint operator can be obtained by manipulation of the inner product expressions. Let us consider $f \in \mathcal{V}$ *and* $g \in \mathcal{Z}$*, then we have that:*

$$\begin{aligned} \langle g, \mathcal{G} f \rangle_{\mathcal{Z}} &= (g^0)^T (\mathcal{G} f)^0 + \int_0^{T_f} (g^1(t))^T (\mathcal{G} f)^1(t) dt \\ &= (g^0)^T P^{\frac{1}{2}} \int_0^{T_f} e^{A(T_f - \xi)} B R^{-\frac{1}{2}} f(\xi)\, d\xi \\ &\quad + \int_0^{T_f} (g^1(t))^T Q^{\frac{1}{2}} \int_0^{t} e^{A(t - \xi)} B R^{-\frac{1}{2}} f(\xi)\, d\xi dt \end{aligned} \tag{B.10}$$

Note that in the last equation we can use Fubini's theorem to change the order of integration in the second integral as:

$$\begin{aligned} &\int_0^{T_f} \int_0^{t} (g^1(t))^T Q^{\frac{1}{2}} e^{A(t - \xi)} B R^{-\frac{1}{2}} f(\xi)\, d\xi dt \\ &\qquad = \int_0^{T_f} \int_\xi^{T_f} (g^1(t))^T Q^{\frac{1}{2}} e^{A(t - \xi)} B R^{-\frac{1}{2}} f(\xi)\, dt d\xi \\ &\qquad = \int_0^{T_f} \left[\int_\xi^{T_f} R^{-\frac{1}{2}} B^T e^{-A^T(\xi - t)} Q^{\frac{1}{2}} g^1(t)\, dt \right]^T f(\xi) d\xi \end{aligned} \tag{B.11}$$

If we now substitute (B.11) *in* (B.10)*, and rearrange terms into the integral, we obtain:*

$$\begin{aligned} \langle g, \mathcal{G} f \rangle_{\mathcal{Z}} &= \int_0^{T_f} \Big[R^{-\frac{1}{2}} B^T e^{A^T(T_f - t)} P^{\frac{1}{2}} (g^0) \\ &\qquad + \int_\xi^{T_f} R^{-\frac{1}{2}} B^T e^{-A^T(\xi - t)} Q^{\frac{1}{2}} g^1(t)\, dt \Big]^T f(\xi)\, d\xi \\ &= \int_0^{T_f} (G^*(g))(\xi)^T f(\xi)\, d\xi = \langle G^*(g), f \rangle_{\mathcal{V}} \end{aligned} \tag{B.12}$$

where we have obtained the adjoint operator corresponding to $\mathcal{G}$*,* i.e.,

$$\begin{aligned} G^*(g) &= G^* \begin{bmatrix} g^0 \\ g^1(t) \end{bmatrix} \\ &= R^{-\frac{1}{2}} B^T e^{A^T(T_f - t)} P^{\frac{1}{2}} (g^0) + \int_\xi^{T_f} R^{-\frac{1}{2}} B^T e^{-A^T(\xi - t)} Q^{\frac{1}{2}} g^1(t)\, dt \end{aligned} \tag{B.13}$$

□

Example B.12 *Consider the Hilbert spaces* $\mathcal{V} = l_2(0, N-1; \mathbb{R}^m)$ *and* $\mathcal{Z} = \mathbb{R}^n \times l_2(0, N-1; \mathbb{R}^n)$, *with inner products:*

$$\langle f_1, f_2 \rangle_{\mathcal{V}} = \sum_{0}^{N-1} f_1^T f_2 \qquad ; f_1, f_2 \in \mathcal{V} \tag{B.14}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} = \left(g_1^0\right)^T g_2^0 + \sum_{0}^{N-1} \left(g_1^1\right)^T g_2^1 \qquad ; g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{B.15}$$

Let us define the linear operator $\mathcal{G} : \mathcal{V} \to \mathcal{Z}$ *(as in* (5.89)*), such that:*

$$\mathcal{G}_\Delta f = \begin{bmatrix} (\mathcal{G}_\Delta f)^0 \\ (\mathcal{G}_\Delta f)_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{\frac{1}{2}} \sum_{l=0}^{N-1} A_q^{N-1-l} B_q R_q^{-\frac{1}{2}} f_l \\ Q_q^{\frac{1}{2}} \sum_{l=0}^{k-1} A_q^{k-1-l} B_q R_q^{-\frac{1}{2}} f_l \end{bmatrix} \tag{B.16}$$

where the matrices P_Δ, Q_q, *and* R_q *are symmetric (see Chapter 5).*

The adjoint operator can be obtained by manipulation of the inner product expressions. Let us consider $f \in \mathcal{V}$ *and* $g \in \mathcal{Z}$, *then we have that:*

$$\begin{aligned} \langle g, \mathcal{G}_\Delta f \rangle_{\mathcal{Z}} &= \left(g^0\right)^T (\mathcal{G}_\Delta f)^0 + \sum_{k=0}^{N-1} \left(g^1\right)_k^T (\mathcal{G}_\Delta f)_k^1 \\ &= \left(g^0\right)^T P_\Delta^{\frac{1}{2}} \sum_{l=0}^{N-1} A_q^{N-1-l} B_q R_q^{-\frac{1}{2}} f_l + \sum_{k=0}^{N-1} \left(g^1\right)_k^T Q_q^{\frac{1}{2}} \sum_{l=0}^{k-1} A_q^{k-1-l} B_q R_q^{-\frac{1}{2}} f_l \end{aligned} \tag{B.17}$$

Note that in the last equation we can interchange the order of the sums in the second term as:

$$\begin{aligned} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \left(g^1\right)_k^T Q_q^{\frac{1}{2}} A_q^{k-1-l} B_q R_q^{-\frac{1}{2}} f_l &= \sum_{l=0}^{N-1} \sum_{k=l+1}^{N-1} \left(g^1\right)_k^T Q_q^{\frac{1}{2}} A_q^{k-1-l} B_q R_q^{-\frac{1}{2}} f_l \\ &= \sum_{l=0}^{N-1} \left[\sum_{k=l+1}^{N-1} R_q^{-\frac{1}{2}} B_q^T \left(A_q^{k-1-l}\right)^T Q_q^{\frac{1}{2}} \left(g^1\right)_k \right]^T f_l \end{aligned} \tag{B.18}$$

If we now substitute (B.18) *in* (B.17)*, and rearrange terms into the sum, we obtain:*

$$\begin{aligned} \langle g, \mathcal{G}_\Delta f \rangle_{\mathcal{Z}} &= \sum_{l=0}^{N-1} \left[R_q^{-\frac{1}{2}} B_q^T \left(A_q^{N-1-l}\right)^T P_\Delta^{\frac{1}{2}} g^0 + \sum_{k=l+1}^{N-1} R_q^{-\frac{1}{2}} B_q^T \left(A_q^{k-1-l}\right)^T Q_q^{\frac{1}{2}} \left(g^1\right)_k \right]^T f_l \\ &= \sum_{l=0}^{N-1} \mathcal{G}_\Delta^*(g)_l^T f_l = \langle G_\Delta^*(g), f \rangle_{\mathcal{V}} \end{aligned} \tag{B.19}$$

where we have obtained the adjoint operator corresponding to $\mathcal{G}_\Delta$, i.e.,

$$\begin{aligned} G_\Delta^*(g) &= G_\Delta^* \begin{bmatrix} g^0 \\ (g^1)_k \end{bmatrix} \\ &= R_q^{-\frac{1}{2}} B_q^T \left(A_q^{N-1-l}\right)^T P_\Delta^{\frac{1}{2}} g^0 + \sum_{k=l+1}^{N-1} R_q^{-\frac{1}{2}} B_q^T \left(A_q^{k-1-l}\right)^T Q_q^{\frac{1}{2}} \left(g^1\right)_k \end{aligned} \tag{B.20}$$

□

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