UCRL-ID-125393

Sampling a Random Variable Distributed According to Planck's Law

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June, 1970

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-ENG-48.

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SAMPLING A RANDOM VARIABLE DISTRIBUTED ACCORDING TO PLANCK'S LAW

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In some Monte Carlo radiation transport calculations one must sample a random variable which is described by Planck's Radiation Law. That is, one must sample a random variable X where X is described by the density function

$$f_X(x) = \begin{cases} \frac{15}{\pi^4} \frac{x^3}{e^x - 1} & \text{if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases} \qquad (*)$$

We discuss a series expansion method and a rejection method in this note.

Series Expansion Method

Rewrite (*) as

$$f_X(x) = \frac{15}{\pi^4} \frac{x^3 e^{-x}}{1 - e^{-x}} \text{ if } x > 0 \qquad (**)$$

What follows is valid only for $x > 0$. Employ the expansion $\frac{1}{1-\theta} = \sum_{k=0}^{\infty} \theta^k$ if $0 < \theta < 1$ and (**) may be written as

$$f_X(x) = \frac{15}{\pi^4} x^3 e^{-x} \sum_{k=0}^{\infty} e^{-kx}$$

With a little manipulation this may be cast into the form

$$f_X(x) = \sum_{n=1}^{\infty} \left(\frac{90}{\pi^4} \cdot \frac{1}{n^4} \right) \left(\frac{n^4}{6} x^3 e^{-nx} \right)$$

If we define

$$\pi_n = \frac{90}{\pi^4} \frac{1}{n^4} \qquad 1 \leq n < \infty$$

$$f_n(x) = \frac{1}{6} n^4 x^3 e^{-nx} \qquad 1 \leq n < \infty.$$

then

$$f_X(x) = \sum_{n=1}^{\infty} \pi_n f_n(x) \qquad x > 0, \; 1 \leq n < \infty \qquad (+)$$

where $\sum_{n=1}^{\infty} \pi_n = 1$ and each f_n is a density function. Let the random variable X_n be represented by f_n. Then (+) suggests the following algorithm for sampling X:

(i) Sample a random variable which is uniformly distributed in (0,1). Call the result u_1.

(ii) If $\mu_1 \leq \pi_1$ sample X_1. Call the result x_1. Then x_1 is a sample of X.

(iii) If the test in (ii) fails, check to see if $u_1 \leq \pi_1 + \pi_2$. If the answer is yes, sample X_2. Call the result x_2. Then x_2 is a sample of X.

(iv) Continue this procedure until

$$u_1 \leq \sum_{k=1}^{n} \pi_n$$

Then sample X_n. Call the result x_n.

Then x_n is a sample of X.

To paraphrase: Sample X_n with probability π_n.

This process is potentially open ended and could go on forever. In practice, however, it is terminated after some maximum number of tests by rounding off a number in the "if" test. And in fact, except for a miracle, the sample is acquired long before this maximum number of tests is made. These assertions are supported in what follows.

Figure 1 contains a listing of the code for the series expansion method. Notice the number 1.Ø8232 in the IF TEST. This is a six place representation of $\pi^4/90$. The seventh digit is 3. By rounding down in this manner we have assured that the test will pass by the 47th cycle irrespective of what R1 is. In effect we are using a 47 term series expansion for the Planck Law density function.

We claimed that the IF TEST passes long before the 47th cycle. To show this proceed as follows:

Let U be a random variable which is uniformly distributed in (0,1). Define a random variable C by

$$C = 1 \quad \text{if } U \leq \pi_1$$

$$C = 2 \quad \text{if } \pi_1 < U \leq \pi_1 + \pi_2$$

$$\vdots$$

$$C = k \quad \text{if } \sum_{i=1}^{k-1} \pi_i < U \leq \sum_{i=1}^{k} \pi_i$$

Then C counts the number of times required for the IF TEST to pass. It is clear that C is described by the probability mass function p_C where

$$p_C(x) = \begin{cases} \frac{90}{\pi^4} \cdot \frac{1}{x^4} & \text{if } x \text{ is a counting number} \\ 0 & \text{otherwise} \end{cases}$$

The expectation of C is

$$E(C) = \sum_{k=1}^{\infty} \frac{90}{\pi^4} k \frac{1}{k^4} \cong 1.1$$

So, "on the average", 1.1 cycles are required. This makes sense when we notice that $\pi_1 \cong .92$ which implies that only one test is required 92% of the time. Figure 1 contains a flow chart of the infinite series method. When compiled on the CHIP compiler, each sample requires about 21 microseconds of 7600 time.

The algorithm for sampling a random variable which is described by a density of the form x^3e^{-nx} is discussed in UCIR # 474 .

The Rejection Method

The infinite series method described above is not theoretically exact. Practically speaking its degree of inexactness is comparable to that which results from representing real numbers by terminated decimals. But, for those who prefer theoretical exactness, we present a rejection technique.

Our density function is

$$f_X(x) = \frac{15}{\pi^4} \frac{x^3e^{-x}}{(1-e^{-x})} \qquad x \geq 0 \qquad (*)$$

Multiply and divide (*) by $1 + e^{-x}/x$ and manipulate the result somewhat to obtain

$$f(x) = \left[\frac{6}{6.25}\left(\frac{1}{6}x^3e^{-x}\right) + \frac{.25}{6.25}\left(4x^2e^{-2x}\right)\right]\frac{(6.25)\left(\frac{15}{\pi^4}\right)x}{(1-e^{-x})(x+e^{-x})} \qquad (**)$$

Identify as follows

$$\pi_1 \equiv 6/6.25 \quad , \quad \pi_2 \equiv .25/6.25 \ .$$

$$f_1(x) \equiv \frac{1}{6}x^3e^{-x} \qquad x \geq 0$$

$$f_2(x) \equiv 4x^2e^{-2x} \qquad x \geq 0$$

$$h(x) \equiv \frac{(6.25)\left(\frac{15}{\pi^4}\right)x}{(1-e^{-x})(x+e^{-x})}$$

and (**) becomes

$$f(x) = \left[\pi_1 f_1(x) + \pi_2 f_2(x)\right] h(x) \qquad x \geq 0 \qquad (+)$$

where the f_i are densities and $h(x)$ is a rejection function. $\underset{0 \leq x < \infty}{\mathrm{Max}}\, h(x) \equiv \hat{h}$ is approximately 1.16 which implies an acceptance efficiency of about 86% $(1/\hat{h})$.

Let X_1, X_2 be represented by f_1 and f_2. Then (+) suggests the following sampling procedure:

(i) Sample a random variable which is uniformly distributed in (0,1). Call the result u_1.

(ii) If $u_1 < \pi_1$, sample X_1. Call the result x_1. If $u_1 > \pi_1$ sample X_2. Call the result x_2.

(iii) Sample a random variable which is uniformly distributed in (0,1). Call the result u_2.

(iv) Let x_i, $i = 1,2$, be the result of (ii). If $u_2 \leq h(x_i)/\hat{h}$,

x_i is a sample of $\tilde{X}$. If $u_2 > h(x_i)/\hat{h}$, start again at (i).

Figure 2 shows a listing and flow chart of the code. The algorithm used for sampling f_1 and f_2 are discussed in UCIR # 474 . When compiled on the CHIP compiler, each sample requires about 30 microseconds of 7600 time.

Comparison of the Two Methods

Table 1 shows the results of a test of both the series and rejection calculations. The results are compared with each other and with a numerical integration of the density function. 100,000 samples were taken and the number which fell into 100 intervals of length .1 was determined. The columns labeled "interval number" give k where $I_k = (.1(k-1), .1k]$ is the k-th interval. The "numerical integration" column gives the number of samples which should have fallen into the k-th interval according to the density function. The other two columns give the results of the two sampling techniques.

We have not performed any statistical analysis on the data. A glance at the table indicates nothing to make us prefer one method over the other.

```
      SUBROUTINE PLANK(X)
      R1=RNFL(0)
      R2=RNFL(0)
      R3=RNFL(0)
      R4=RNFL(0)
      X=-ALOG(R1*R2*R3*R4)
      A=Y=Z=1.
      R1=RNFL(0)
    5 IF(1.08232*R1.LE.A) GO TO 10
      Y=Y+1.
      Z=1./Y
      A=A+Z*Z*Z*Z
      GO TO 5
   10 X=X*Z
      RETURN
      END
```

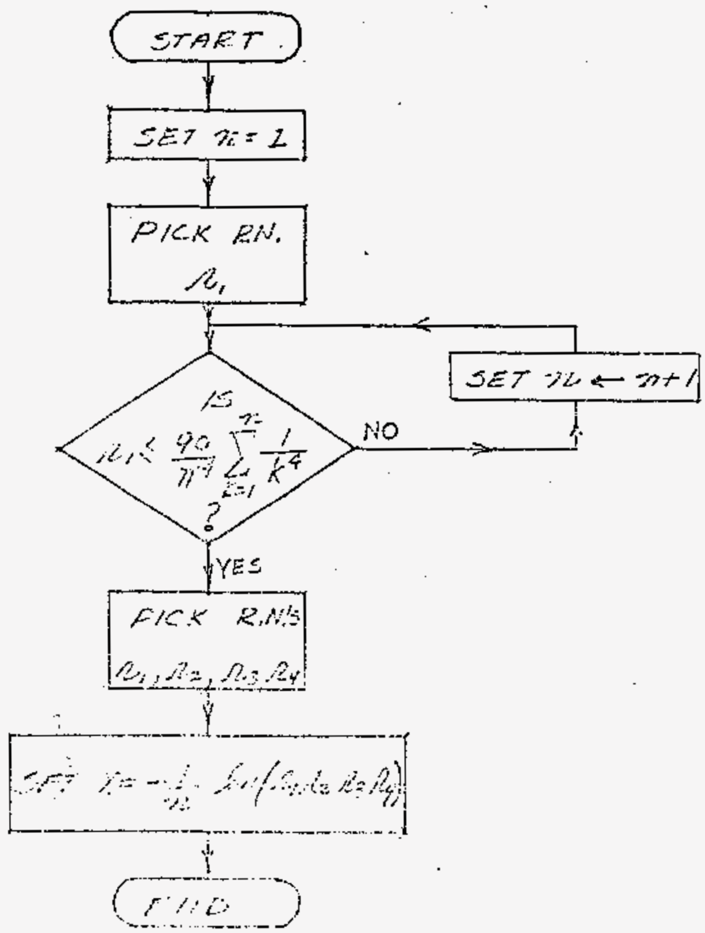

FIGURE 1. SERIES METHOD.
FORTRAN LISTING AND FLOWCHART

```
      SUBROUTINE PLANKER(X)
    4 R1=RNFL(0)
      R2=RNFL(0)
      R3=RNFL(0)
      R4=RNFL(0)
      IF(R4.LE..96) GO TO 10
      TT=R1*R2*R3
      X=-.5*ALOG(TT)
      T=SQRT(TT)
      GO TO 30
   10 R4=RNFL(0)
      T=R1*R2*R3*R4
      X=-ALOG(T)
   30 R1=RNFL(0)
      RL=1.1642*R1*(1.-T)*(X+T)
      IF(RL.GT.X) GO TO 4
      RETURN
      END
```

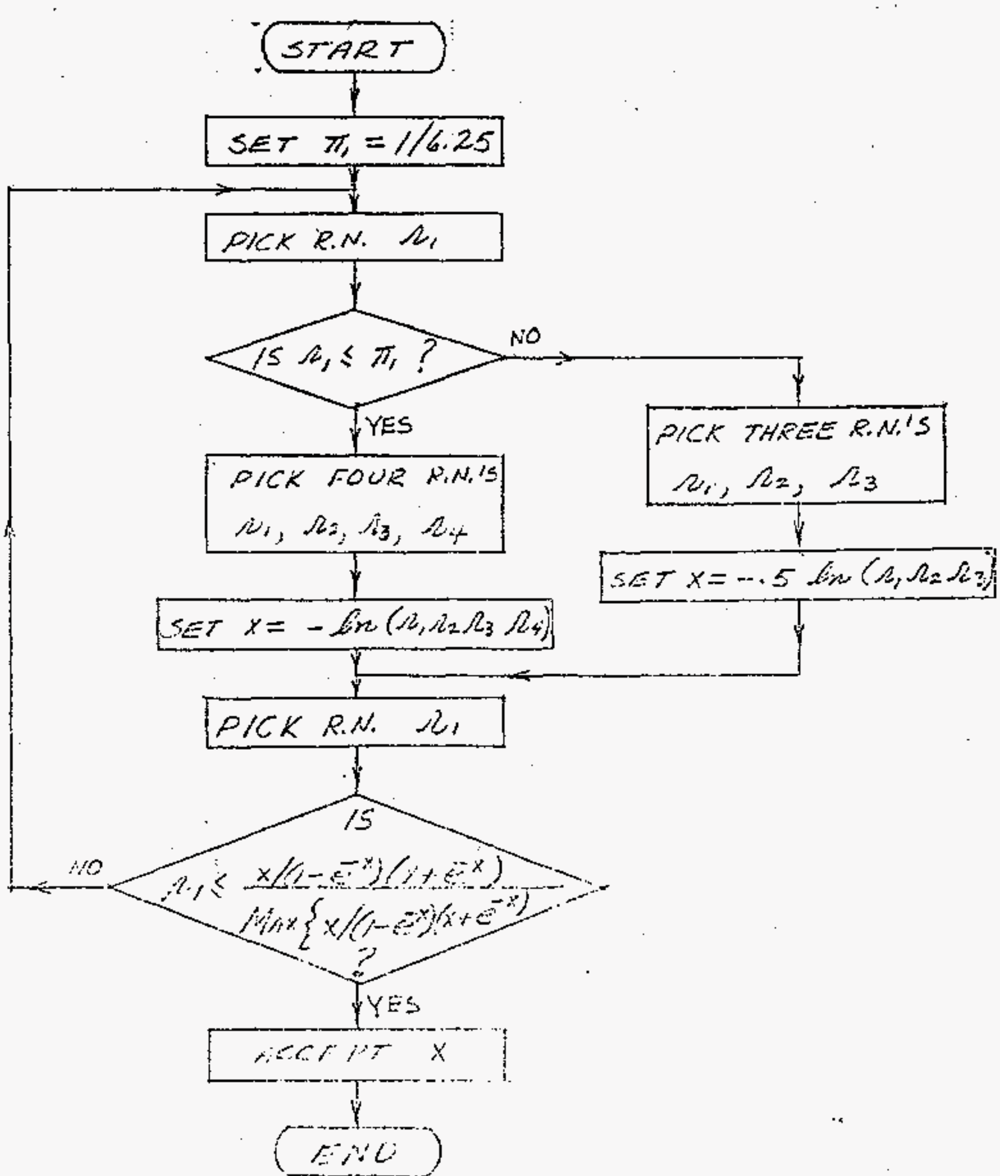

FIGURE 2. REJECTION METHOD.
FORTRAN LISTING AND FLOWCHART

INTERVAL NUMBER	NUMERICAL INTEGRATION	SERIES METHOD	REJECTION METHOD	INTERVAL NUMBER	NUMERICAL INTEGRATION	SERIES METHOD	REJECTION METHOD
1	0	1	4	51	1279	1258	1275
2	33	36	31	52	1227	1187	1202
3	86	98	80	53	1176	1195	1173
4	158	142	167	54	1125	1102	1116
5	247	257	257	55	1076	1077	1134
6	350	347	326	56	1027	1050	1006
7	462	476	460	57	983	968	953
8	582	572	561	58	935	908	925
9	706	740	707	59	890	914	892
10	833	836	820	60	848	842	851
11	960	943	951	61	806	810	826
12	1085	1114	1090	62	766	824	753
13	1208	1193	1224	63	727	730	687
14	1326	1273	1353	64	690	692	692
15	1438	1475	1488	65	654	653	638
16	1545	1521	1554	66	620	601	595
17	1644	1617	1599	67	587	576	581
18	1735	1727	1762	68	555	560	557
19	1819	1784	1748	69	525	539	523
20	1894	1948	1899	70	496	469	510
21	1960	1966	1921	71	469	444	458
22	2017	2049	2002	72	442	469	456
23	2066	2052	2127	73	417	448	438
24	2106	2101	2098	74	393	387	361
25	2138	2223	2191	75	370	363	354
26	2162	2107	2129	76	349	368	354
27	2178	2122	2212	77	328	339	334
28	2187	2148	2280	78	309	307	283
29	2188	2177	2165	79	290	265	278
30	2183	2210	2179	80	273	296	280
31	2172	2132	2212	81	256	264	238
32	2155	2120	2165	82	241	279	247
33	2132	2126	2161	83	226	229	200
34	2105	2067	2061	84	212	223	225
35	2073	2105	2068	85	199	215	218
36	2037	2055	2038	86	186	184	189
37	1998	2029	2004	87	175	188	170
38	1956	1950	2025	88	164	182	157
39	1911	1939	1919	89	153	155	139
40	1863	1845	1893	90	143	146	143
41	1814	1804	1811	91	134	131	145
42	1763	1831	1796	92	125	120	129
43	1711	1644	1733	93	117	122	112
44	1657	1679	1629	94	109	99	119
45	1603	1635	1611	95	102	93	92
46	1549	1481	1540	96	96	88	82
47	1495	1439	1472	97	89	76	82
48	1449	1484	1401	98	83	80	74
49	[illegible]	1405	1375	99	78	77	79
50	[illegible]	1333	1304	100	72	64	71

TABLE 1. A COMPARISON OF THE SERIES AND REJECTION RESULTS WITH A NUMERICAL INTEGRATION OF THE DENSITY FUNCTION.

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