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Sampling and Reconstruction of Sparse Signals in Fractional Fourier Domain

Ayush Bhandari and Pina Marziliano

Abstract—Sampling theory for continuous time signals which have a bandlimited representation in fractional Fourier transform (FrFT) domain—a transformation which generalizes the conventional Fourier transform—has blossomed in the recent past. The mechanistic principles behind Shannon’s sampling theorem for fractional bandlimited (or fractional Fourier bandlimited) signals are the same as for the Fourier domain case i.e. sampling (and reconstruction) in FrFT domain can be seen as an orthogonal projection of a signal onto a subspace of fractional bandlimited signals. As neat as this extension of Shannon’s framework is, it inherits the same fundamental limitation that is prevalent in the Fourier regime—what happens if the signals have singularities in the time domain (or the signal has a nonbandlimited spectrum)?

In this paper, we propose a uniform sampling and reconstruction scheme for a class of signals which are nonbandlimited in FrFT sense. Specifically, we assume that samples of a smoothed version of a periodic stream of Diracs (which is sparse in time-domain) are accessible. In its parametric form, this signal has a finite number of degrees of freedom per unit time. Based on the representation of this signal in FrFT domain, we derive conditions under which exact recovery of parameters of the signal is possible. Knowledge of these parameters leads to exact reconstruction of the original signal.

Index Terms—Finite-rate-of-innovation, fractional Fourier transform (FrFT), nonbandlimited signals, Shannon, sparse sampling, stream of Diracs.

I. INTRODUCTION

SHANNON’S sampling theorem [1] is at the heart of analog-to-digital conversion. Jerri [2] and Unser [3] provide an excellent survey on the state-of-the-art of the sampling theory in their respective eras. Since Almeida’s introduction of fractional Fourier transform (FrFT) [4] to the signal processing community, there has been a surge of research in this area. Since sampling theory is the theme of this paper, we would like to emphasize that at least on eight occasions including, [5]–[12], Shannon’s sampling theorem [1] was independently extended to the class of fractional Fourier bandlimited, or simply, fractional bandlimited signals. Important applications of the FrFT are mentioned in [5]–[13] and the references therein. Let $\langle x(t), y(t) \rangle = \int x(t)y^*(t)dt$ denote the L_2 -inner product operation between continuous time signals $x(t)$ and

$y(t)$ where “*” in the superscript denotes complex conjugate of $y(t)$. In context of [13], the FrFT of a signal or a function, say $x(t)$, is defined by

$$\hat{x}_\theta(\omega) = F^\theta \{x(t)\} = \langle x(t), \mathcal{K}_{-\theta}(t, \omega) \rangle \quad (1)$$

where

$$\mathcal{K}_\theta(t, \omega) \stackrel{\text{def}}{=} \begin{cases} \sqrt{\frac{1-j \cot \theta}{2\pi}} e^{j \frac{t^2 + \omega^2}{2} \cot \theta - j\omega t \csc \theta}, & \theta \neq p\pi \\ \delta(t - \omega), & \theta = 2p\pi \\ \delta(t + \omega), & \theta \pm \pi = 2p\pi \end{cases} \quad (2)$$

is the transformation kernel, parametrized by the fractional order $\theta \in \mathbb{R}$ and where p is some integer. The inverse-FrFT with respect to angle θ is the FrFT at angle $-\theta$, given by

$$x(t) = F^{-\theta} \{\hat{x}_\theta(\omega)\} = \langle \hat{x}_\theta(\omega), \mathcal{K}_\theta(t, \omega) \rangle. \quad (3)$$

Whenever $\theta = \pi/2$, (1) collapses to the classical Fourier transform definition. In sense of the FrFT, the generalized version of Shannon’s sampling theorem states,

Theorem 1 (Shannon-FrFT): Let $x(t)$ be a continuous-time signal. If the spectrum of $x(t)$, i.e. $\hat{x}_\theta(\omega)$ is fractional bandlimited to ω_m , meaning, $\hat{x}_\theta(\omega) = 0$ when $|\omega| > \omega_m$, then $x(t)$ is completely determined by giving its ordinates at a series of equidistant points spaced $T = (\pi/\omega_m) \sin \theta$ seconds apart.

The reconstruction formula for fractional bandlimited signals as given in [12] is

$$x(t) = \lambda_\theta^*(t) \sum_{n \in \mathbb{Z}} x(nT) \lambda_\theta(nT) \text{sinc} \left(\frac{t}{T} - n \right) \quad (4)$$

where $\lambda_\theta(\cdot) \stackrel{\text{def}}{=} e^{j(\cdot)^2 (\cot \theta / 2)}$ is a domain independent chirp modulation function. Let “* $_\theta$ ” denote the fractional convolution operator. Filtering $x(t)$ by a filter, say $h(t)$, in FrFT sense¹ is equivalent to [14],

$$x(t) *_\theta h(t) = \sqrt{\frac{1-j \cot \theta}{2\pi}} \lambda_\theta^*(t) \cdot ([x(t) \lambda_\theta(t)] * [h(t) \lambda_\theta(t)]) \quad (5)$$

where “*” is the conventional convolution operator. From a filtering perspective, (4) can be seen as filtering of samples $x(nT)$ with the kernel $h(t) = \lambda_\theta^*(t) \text{sinc}(t/T)$. If $\tilde{x}(t)$ is the approximation of $x(t)$, then $\|\tilde{x}(t) - x(t)\|_{L_2}^2 = 0$ whenever $\omega_m \leq \frac{\omega_s}{2} \sin \theta$ (the Nyquist rate for FrFT), where $\omega_s = 2\pi/T$ is the sampling frequency. All the aforementioned results are equivalent to Shannon’s sampling theorem with respect to Fourier domain for $\theta = \pi/2$.

¹We adhere to this modified definition of convolution operator as it inherits the fractional Fourier duality property in that $F^\theta \{x(t) *_\theta h(t)\} = \lambda_\theta^*(\omega) \cdot \hat{x}_\theta(\omega) \hat{h}_\theta(\omega)$, which does not hold for the FrFT of $x(t) * h(t)$ unless $\theta = \pi/2$.

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Although the result of Theorem 1 has notable advantages [12] over its Fourier domain based counterpart, the sampling theorem for either domain is confined to bandlimited signals only. Consider a Dirac impulse or $\delta(t)$. Using (2), we have

$$\widehat{\delta}_\theta(\omega) = F^\theta \{\delta(t)\} = \sqrt{\frac{1-j\cot\theta}{2\pi}} \lambda_\theta(\omega) \quad (6)$$

which is a nonbandlimited function (and least sparse when compared to the time-domain counterpart) and thus, Theorem 1 fails to answer the following question: If $x(t)$ is a fractional nonbandlimited signal, then, how can we sample and reconstruct such a signal?

This limitation can be quite restrictive from a practical point of view—the intense motivation behind development of interesting sampling theorems pertaining to Fourier domain by Vetterli *et al.* in [15] and their extensions in [16]–[18].

The problem of sampling nonbandlimited signals in FrFT domain has a natural/strong link with that of sparse sampling [15], [16]. The Heisenberg–Gabor uncertainty principle for the FrFT [19]—a generalization of the Fourier duality—asserts that the product of spreads of $\widehat{x}_\theta(\omega)$ and $x(t)$ has a lower bound which is proportional to $\sin^2\theta/4$ (assuming $\|x\|_{L_2} = 1$). This implies that loss of compact support/bandlimitedness in one (frequency) domain will lead to more sparse representation in canonically conjugate (time) domain.

In this paper, we propose a sampling and reconstruction scheme for a signal with sparse representation (in the time domain), whose fractional spectrum is nonbandlimited. We model the input signal as a continuous-time periodic stream of Diracs which is observed by an acquisition device which displays a sinc-based low-pass filter.

The paper is organized as follows. In Section II, we introduce our signal model and then derive an equivalent representation of our sparse/nonbandlimited signal in FrFT domain. In Section III, we present a uniform sampling theorem and describe the reconstruction process and finally, we conclude in Section IV.

II. SIGNAL MODEL AND ITS EXPANSION IN FrFT DOMAIN

A. Sparse Signal Model: Periodic Stream of Diracs

In this paper, we are interested in sampling a periodic stream of K Diracs,

$$x(t) \stackrel{\text{def}}{=} \sum_{k=0}^{K-1} c_k \sum_{n \in \mathbb{Z}} \delta(t - t_k - n\tau) \quad (7)$$

with period τ , unknown weights $\{c_k\}_{k=0}^{K-1}$ and arbitrary shifts, $\{t_k\}_{k=0}^{K-1} \subset [0, \tau)$. In sense of [15], the signal has $2K$ degrees of freedom per period and its rate of innovation is $\rho = 2K/\tau$. From now on, the signal $x(t)$ will denote the stream of Diracs.

B. Fractional Fourier Series (FrFS)

Periodic signals can be expanded in fractional Fourier domain as a fractional Fourier series or FrFS [20]. The FrFS of a periodic signal, say $x(t)$, can be written as

$$x(t) = \sum_{m \in \mathbb{Z}} \widehat{x}_\theta[m] \Phi_\theta(m, t) \quad (8)$$

where

$$\Phi_\theta^*(m, t) = \sqrt{\frac{\sin\theta - j\cos\theta}{\tau}} e^{j\frac{t^2 + (2\pi m \sin\theta/\tau)^2}{2} \cot\theta - j2\pi mt/\tau}$$

constitutes the basis for FrFS expansion for a τ -periodic $x(t)$. The FrFS coefficients are given by

$$\widehat{x}_\theta[m] = \int_{\langle\tau\rangle} x(t) \Phi_\theta^*(m, t) dt = \langle x(\cdot), \Phi_\theta(m, \cdot) \rangle \quad (9)$$

where $\langle\tau\rangle$ denotes the integral width. The well-known Fourier series (FS) is just a special case of FrFS for $\theta = \pi/2$.

C. Stream of Diracs in Fractional Fourier Domain

In Fourier analysis, the Poisson summation formula (PSF) plays an important role. It is a well-known fact that a Dirac comb (or stream of Diracs) in time-domain is another Dirac comb in Fourier domain. Generalization of the PSF for Dirac comb in FrFT domain leads to a similar result. Let $\text{III}(t) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} \delta(t - n\tau)$ be a τ -periodic Dirac comb. Then, we have

$$F^\theta \{\text{III}(t)\} = \frac{1}{\tau} \sqrt{\frac{2\pi}{1-j\cot\theta}} \times \sum_{k \in \mathbb{Z}} \widehat{\delta}_\theta[k\omega_0 \sin\theta] e^{-j\left(t^2 + \frac{(k\omega_0 \sin\theta)^2}{2}\right) \cot\theta + jk\omega_0 t} \quad (10)$$

where $\omega_0 = 2\pi/\tau$.

Proof: The proof is done by expanding $\text{III}(t)$ as FrFS:

$$\text{III}(t) = \sum_{k \in \mathbb{Z}} \underbrace{\langle \text{III}(t), \Phi_\theta(k, t) \rangle}_{\widehat{\text{III}}_\theta[k]} \Phi_\theta(k, t) \quad (11)$$

where the coefficients of expansion are given by

$$\begin{aligned} \widehat{\text{III}}_\theta[k] &\stackrel{(9)}{=} \langle \text{III}(t), \Phi_\theta(k, t) \rangle \\ &= \frac{\kappa(\theta)}{\sqrt{\tau}} \int_{t_0}^{t_0+\tau} \text{III}(t) \Phi_\theta^*(k, t) dt, \quad \forall t_0 \in \mathbb{R} \\ &= \frac{\kappa(\theta)}{\sqrt{\tau}} \int_{-\tau/2}^{\tau/2} \delta(t) e^{j\left(t^2 + (k\omega_0 \sin\theta)^2/2\right) \cot\theta - jk\omega_0 t} dt \\ &\quad (\text{since } \text{III}(t + \tau) = \text{III}(t) \text{ and} \\ &\quad \text{III}(t) = \delta(t), t \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right]) \\ &= \frac{\kappa(\theta)}{\sqrt{\tau}} e^{j\left((k\omega_0 \sin\theta)^2/2\right) \cot\theta} \\ &\stackrel{(6)}{=} \frac{\kappa(\theta)}{\sqrt{\tau}} \sqrt{\frac{2\pi}{1-j\cot\theta}} \widehat{\delta}_\theta[k\omega_0 \sin\theta] \end{aligned} \quad (12)$$

where $\kappa(\theta) = \sqrt{\sin\theta - j\cos\theta}$. Back substitution of (12) in (11) results in,

$$\text{III}(t) = \frac{\mu(\theta)}{\tau} \sum_{k \in \mathbb{Z}} \widehat{\delta}_\theta[k\omega_0 \sin\theta] e^{-j\left(t^2 + \frac{(k\omega_0 \sin\theta)^2}{2}\right) \cot\theta + jk\omega_0 t}$$

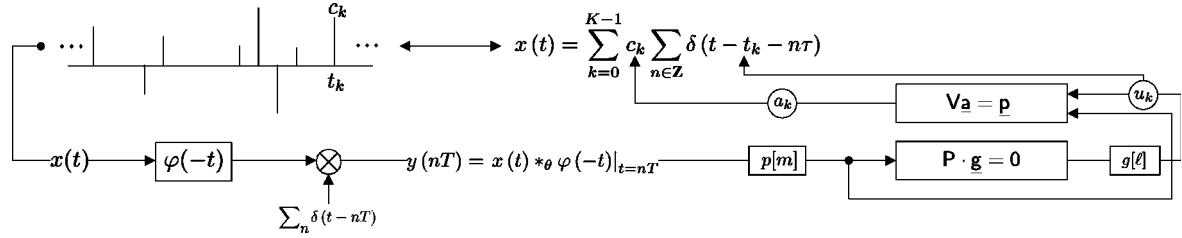


Fig. 1. Sampling and reconstruction of periodic stream of Diracs in fractional Fourier transform domain.

where $\mu(\theta) = \sqrt{2\pi/(1 - j \cot \theta)}$. This concludes the proof. ■

For sake of convenience, we will assume that the constant $\mu(\theta)$ has been absorbed in τ . Note that at $\theta = \pi/2$, $\mathbb{I}(t) = (1/\tau) \sum_{k \in \mathbb{Z}} e^{jk\omega_0 t}$ which is the result of applying the PSF on $\mathbb{I}(t)$ in Fourier domain. Our immediate goal now is to derive the FrFS equivalent of $x(t)$ in (7). Since $x(t)$ is a linear combination of some $\mathbb{I}(t)$ delayed by some time shift t_k , it will be useful to recall shift property of FrFT [4] which states that

$$F^\theta \{f(t - t_k)\} = \hat{f}_\theta(\omega - t_k \cos \theta) e^{j\frac{1}{2}t_k^2 \sin \theta \cos \theta - j\omega t_k \sin \theta}. \quad (13)$$

Therefore, call $x(t) = \sum_{k=0}^{K-1} c_k \cdot \mathbb{I}_k(t)$ where $\mathbb{I}_k(t) = \mathbb{I}(t - t_k)$ is the t_k -shifted version of $\mathbb{I}(t)$. Using (10) and the shift property of FrFT in (13), we have,

$$\begin{aligned} \mathbb{I}_k(t) &= \sum_{m \in \mathbb{Z}} F^\theta \{\delta(t - t_k)\} \Big|_{\omega = m\omega_0 \sin \theta} \Phi_\theta(m, t) \\ &= \frac{1}{\tau} \sum_{m \in \mathbb{Z}} \underbrace{e^{j\frac{\cot \theta}{2}(t_k^2 - t^2) + jm\omega_0(t - t_k)}}_{\text{PSF for Dirac Comb in FrFT}}. \end{aligned}$$

Having obtained the FrFT-version of $\mathbb{I}_k(t)$, we can write

$$\begin{aligned} x(t) &= \sum_{k=0}^{K-1} c_k \cdot \mathbb{I}_k(t) = \sum_{k=0}^{K-1} c_k \cdot \sum_{n \in \mathbb{Z}} \delta(t - t_k - n\tau) \\ &= \sum_{k=0}^{K-1} c_k \left(\frac{1}{\tau} \sum_{m \in \mathbb{Z}} e^{j\frac{\cot \theta}{2}(t_k^2 - t^2) + jm\omega_0(t - t_k)} \right) \\ &= e^{-jt^2 \frac{\cot \theta}{2}} \sum_{m \in \mathbb{Z}} \frac{1}{\tau} \left(\underbrace{\sum_{k=0}^{K-1} c_k e^{j\frac{\cot \theta}{2}(t_k^2)}}_{p[m]} \underbrace{e^{-jm\omega_0 t_k}}_{u_k^m} \right) \\ &\quad \times e^{j\omega_0 m t}. \end{aligned}$$

Note that $x(t)$ is **nonbandlimited**. However, it can be completely described by the knowledge of $p[m]$ —a linear combination of K complex exponentials.

III. MAIN RESULT: SAMPLING AND RECONSTRUCTION OF SPARSE SIGNALS IN FRACTIONAL FOURIER DOMAIN

The scheme that we will present in this section is depicted in Fig. 1. Like in most practical cases, we assume that the signal is being observed through a lowpass prefilter which is analogous to the anti-aliasing filter in Shannon's framework. In this paper, we presume that a sinc-based kernel is used to observe $x(t)$. In particular, we let the sampling kernel to be $\varphi_n(t) = \lambda_\theta^*(t) \text{sinc}(t - nT)$. We will use $\varphi(t)$ to denote $\varphi_0(t)$.

Shannon's Sampling Theorem for FrFT Domain Revisited: The family of functions $\{\varphi_n(t)\}_{n \in \mathbb{Z}}$ has two interesting

properties. Firstly, integer translates of $\varphi_n(t)$ form an orthonormal basis. Secondly, using

$$F^\theta \{\lambda_\theta^*(t) \text{sinc}(t)\} = \sqrt{\frac{1 - j \cot \theta}{2\pi}} \lambda_\theta^*(\omega) \text{rect}\left(\frac{\omega}{2\pi}\right) \quad (14)$$

it is apparent that $\{\varphi_n(t)\}_{n \in \mathbb{Z}}$ is a space of fractional bandlimited signals which we will refer to as $V_\theta(\varphi)$. Indeed, for every $f \in V_\theta(\varphi)$ it is self-evident that $f = \mathcal{P}_{V_\theta(\varphi)} f$ (where \mathcal{P} is the projection operator) and mathematically, it results in,

$$\mathcal{P}_{V_\theta(\varphi)} f = \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n \quad (\text{statement of Theorem 1})$$

which shows that Theorem 1 is simply the orthogonal projection of $f(t)$ onto the subspace of fractional bandlimited signals. We would like to highlight the fact that this procedure is optimal in least-square sense—a fact that has been overlooked in the discussions presented in [5]–[12].

In view of the filtering operation in FrFT domain which makes use of (5), $f = \mathcal{P}_{V_\theta(\varphi)} f$ can be interpreted in the following two steps.

Sampling: $f(t)$ is prefiltered with anti-aliasing filter $\varphi(-t)$ followed by sampling or $y(nT) = f(t) *_{\theta} \varphi(-t)|_{t=nT}$. Note that $f \in V_\theta(\varphi) \Rightarrow y(nT) = f(nT)$.

Reconstruction: $y(nT)$ is filtered using low-pass $\varphi(t)$ leading to approximation of $f(t)$, which is $\hat{f}(t) = y(nT) *_{\theta} \varphi(t)$.

Departing From Shannon's Framework for FrFT Domain: Theorem 1 is well applicable for the case of fractional bandlimited signals, however, it fails to recover nonbandlimited signals e.g. $x(t)$ in (7). Certainly, $x(t) \notin V_\theta(\varphi)$. We propose a remedy to this problem in the form of the following theorem.

Theorem 2: Let $x(t)$ be a τ -periodic stream of Diracs weighted by coefficients $\{c_k\}_{k=0}^{K-1}$ and locations $\{t_k\}_{k=0}^{K-1}$ with finite rate of innovation $\rho = 2K/\tau$. Let the sampling kernel/prefilter $\varphi(t)$ be an ideal low-pass filter which has fractional bandwidth $[-B\pi, B\pi]$, where B is chosen such that $B \geq \rho$. If the filtered version of $x(t)$, i.e. $y(t) = x(t) *_{\theta} \varphi(-t)$ is sampled uniformly at locations $t = nT$, $n = 0, \dots, N - 1$ then the samples,

$$y(nT) = x(t) *_{\theta} \varphi(-t)|_{t=nT}, \quad n = 0, \dots, N - 1$$

are a sufficient characterization of $x(t)$, provided that $N \geq 2M_\theta + 1$ and $M_\theta = \lfloor B\tau \csc \theta/2 \rfloor$.

Proof: We define our sampling kernel as,

$$\varphi_B(t - nT) = \lambda_\theta^*(t) \varphi(B \csc \theta(t - nT)), \quad \varphi = \text{sinc}.$$

Using (14), one can show that $\varphi_B(t)$ is compactly supported over $[-B\pi, B\pi]$. Prefiltering and sampling $x(t)$ results in,

$$\begin{aligned}
y(nT) &= x(t) *_{\theta} \varphi(-t)|_{t=nT}, \quad n = 0, \dots, N-1 \\
&= \frac{\lambda_{\theta}^*(nT)}{\tau} \sum_{m \in \mathbb{Z}} p[m] \\
&\quad \times \left\langle e^{j \frac{2\pi m}{\tau} t}, (B \csc \theta) \operatorname{sinc}((B \csc \theta)(t - nT)) \right\rangle.
\end{aligned}$$

The inner product in the above step is further simplified using the Fourier integral,

$$\begin{aligned}
&\left\langle e^{j \frac{2\pi m}{\tau} t}, (B \csc \theta) \operatorname{sinc}((B \csc \theta)(t - nT)) \right\rangle \\
&= \operatorname{rect}\left(\frac{m}{B\tau \csc \theta}\right) e^{j \frac{2\pi m}{\tau} (nT)}. \quad (15)
\end{aligned}$$

We can therefore conclude that

$$\begin{aligned}
y(nT) &= \frac{\lambda_{\theta}^*(nT)}{\tau} \sum_{m \in \mathbb{Z}} p[m] \operatorname{rect}\left(\frac{m}{B\tau \csc \theta}\right) e^{j \frac{2\pi m}{\tau} (nT)} \\
&= \frac{\lambda_{\theta}^*(nT)}{\tau} \sum_{m=-M_{\theta}}^{M_{\theta}} p[m] e^{j \frac{2\pi m}{\tau} (nT)}, \quad n = 0, \dots, N-1
\end{aligned}$$

where $M_{\theta} = \lfloor B\tau \csc \theta / 2 \rfloor$.

Signal Reconstruction From Its Samples: Call $p[m] = \sum_{k=0}^{K-1} a_k u_k^m$, $\forall m \in \mathbb{Z}$ —a linear combination of K -complex exponentials, $u_k = \lambda_{\pi/2}^*(\sqrt{2\omega_0} t_k)$ with weights $a_k = c_k \cdot \lambda_{\theta}(t_k)$. The problem of calculating $\{a_k\}_{k=0}^{K-1}$ and $\{u_k\}_{k=0}^{K-1}$ is based on finding a suitable polynomial $G(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1}) = \sum_{\ell=0}^K g[\ell] z^{-\ell}$ whose coefficients, $g[\ell]$ annihilate $p[m]$ meaning,

$$\sum_{\ell=0}^K g[\ell] p[m - \ell] = 0, \quad \forall m \in \mathbb{Z}.$$

In matrix notation, finding $g[\ell]$ is equivalent to finding a corresponding vector $\mathbf{g} = (g[0], g[1], \dots, g[K])^T$ that forms a null space of a suitable submatrix of $p[m]$ i.e., $\mathbf{P}^{(2M_{\theta}-K+1) \times (K+1)}$ which is essentially the set $\mathbf{Null}(\mathbf{P}) = \{\mathbf{g} \in \mathbb{R}^{K+1} : \mathbf{P} \cdot \mathbf{g} = \mathbf{0}\}$.

Computing u_k 's: Having computed the weights $g[\ell]$, the u_k 's are obtained by finding roots of the polynomial $G(z)$ which in turn give one set of innovative parameters i.e. t_k 's.

Computing c_k 's: Once the t_k 's are obtained, we have $c_k = a_k \cdot \lambda_{\theta}^*(t_k)$. To find the a_k 's on the other hand, we need to solve the Vandermonde system of equations $\mathbf{V}\mathbf{a} = \mathbf{p}$ where \mathbf{V} is the Vandermonde matrix with elements $V_{i,j} = u_j^{i-1}$, $i = 1, \dots, K$ and $j = 0, \dots, K-1$ and vectors $\mathbf{a} = (a_0, a_1, \dots, a_{K-1})^T$ and $\mathbf{p} = (p[0], p[1], \dots, p[K-1])^T$. This solution is unique since $u_k \neq u_{\ell}, \forall k \neq \ell$. This concludes the proof of Theorem 2. ■

Application of Theorem 2: Consider a τ -periodic chirp signal corrupted by additive impulsive noise, $f(t) = s(t) + x(t)$, where $s(t)$ is an fractional bandlimited signal ($\hat{s}_{\theta}[m] = 0, \forall m \notin [-L, L]$) and $x(t)$ as in (7) is the impulsive noise, thus the rate of innovation of $f(t)$ is $\rho_f = (2(L+K)+1)/\tau$. In view of Theorem 2, $x(t)$ can be recovered from $\hat{f}_{\theta}[m] = \hat{x}_{\theta}[m]$, $m \in [L+1, L+2K]$, which in turn will lead to the FrFS coefficients of $s(t)$, i.e. $\hat{s}_{\theta}[m] = \hat{f}_{\theta}[m] - \hat{x}_{\theta}[m]$, $m \in [-L, L]$ thus achieving the goal of denoising $f(t)$. Processing signals in FrFT domain is certainly advantageous. For a given chirp signal, its rate of innovation in FrFT domain is lower than that of Fourier domain, i.e. $\rho_{f-\text{FrFT}} \leq \rho_{f-\text{Fourier}}$.

IV. DISCUSSION & CONCLUSION

Departing from Shannon's framework, we presented a scheme for sampling and reconstruction of fractional non-bandlimited signals which have a sparse representation in time-domain. Even though these signals are nonbandlimited, in their parametric form, as we have shown, exact recovery of their parameters is possible. The constraint being, the fractional bandwidth of anti-aliasing filter exceeds the rate of innovation. Based on the recovered parameters, precise locations and amplitudes of the stream of Diracs can be obtained. Our results can be extended to Linear Canonical transform domain [13].

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