

SUPPLEMENT TO “SAMPLING-BASED VERSUS DESIGN-BASED
UNCERTAINTY IN REGRESSION ANALYSIS”
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S.1. PROOFS OF THE RESULTS IN SECTION 2

S.1.1. *Preliminary Calculations*

NOTICE THAT FOR ANY INTEGER $1 \leq i \leq n$ and conditional on sample size N , such that $1 \leq N \leq n$, we obtain

$$E[R_i] = \frac{N}{n}, \quad \text{var}(R_i) = \frac{N}{n} \left(1 - \frac{N}{n}\right).$$

Also, for any integers $1 \leq j < k \leq n$,

$$\text{var}\left(\sum_{i=1}^n R_i\right) = n \text{var}(R_j) + n(n-1) \text{cov}(R_j, R_k) = 0.$$

This implies

$$\text{cov}(R_j, R_k) = -\frac{\text{var}(R_j)}{n-1} = -\frac{N}{n(n-1)} \left(1 - \frac{N}{n}\right).$$

In turn, this implies

$$E[R_i R_j] = \frac{N(N-1)}{n(n-1)}.$$

Let

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^n R_i Y_i \quad \text{and} \quad \mu = \frac{1}{n} \sum_{i=1}^n Y_i.$$

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Now,

$$E[\bar{Y}] = \frac{1}{N} \sum_{i=1}^n E[R_i] Y_i = \mu.$$

Let

$$S_Y^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \mu)^2.$$

Notice that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 &= \frac{1}{n} \sum_{i=1}^n Y_i^2 - \left(\frac{1}{n} \sum_{i=1}^n Y_i \right)^2 \\ &= \frac{n-1}{n^2} \sum_{i=1}^n Y_i^2 - \frac{2}{n^2} \sum_{i=1}^n \sum_{j=i+1}^n Y_i Y_j. \end{aligned}$$

This implies

$$nS_Y^2 = \sum_{i=1}^n Y_i^2 - \frac{2}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n Y_i Y_j.$$

Therefore,

$$\begin{aligned} \text{var}(\bar{Y}) &= \frac{1}{N^2} \sum_{i=1}^n \text{var}(R_i) Y_i^2 + \frac{2}{N^2} \sum_{i=1}^n \sum_{j=i+1}^n \text{cov}(R_i, R_j) Y_i Y_j \\ &= \frac{1}{N^2} \text{var}(R_1) \left(\sum_{i=1}^n Y_i^2 - \frac{2}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n Y_i Y_j \right) \\ &= \frac{n}{N^2} \text{var}(R_1) S_Y^2 \\ &= \frac{S_Y^2}{N} \left(1 - \frac{N}{n} \right). \end{aligned}$$

Let

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^n R_i Y_i^2 - \left(\frac{1}{N} \sum_{i=1}^n R_i Y_i \right)^2.$$

Then

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^n R_i Y_i^2 - \frac{1}{N^2} \sum_{i=1}^n R_i Y_i^2 - \frac{2}{N^2} \sum_{i=1}^n \sum_{j=i+1}^n R_i R_j Y_i Y_j.$$

Therefore,

$$E[\hat{\sigma}^2] = \frac{1}{n} \frac{N-1}{N} \sum_{i=1}^n Y_i^2 - \frac{2}{n(n-1)} \frac{N-1}{N} \sum_{i=1}^n \sum_{j=i+1}^n Y_i Y_j = \frac{N-1}{N} S_Y^2.$$

S.1.2. *Causal versus Descriptive Estimands*

Let

$$\widehat{\theta} = \frac{1}{N_1} \sum_{i=1}^n R_i X_i Y_i - \frac{1}{N_0} \sum_{i=1}^n R_i (1 - X_i) Y_i.$$

We will do all the analysis conditional on N_1, N_0 , for $N_1 > 0, N_0 > 0, n_1 > 0$, and $n_0 > 0$. To economize notation, we will leave this conditioning implicit. Notice that

$$E[\widehat{\theta}|\mathbf{X}] = \theta^{\text{desc}},$$

where

$$\begin{aligned} \theta^{\text{desc}} &= \frac{1}{n_1} \sum_{i=1}^n X_i Y_i - \frac{1}{n_0} \sum_{i=1}^n (1 - X_i) Y_i \\ &= \frac{1}{n_1} \sum_{i=1}^n X_i Y_i^*(1) - \frac{1}{n_0} \sum_{i=1}^n (1 - X_i) Y_i^*(0). \end{aligned}$$

By the law of total variance,

$$\text{var}(\widehat{\theta}) = E[\text{var}(\widehat{\theta}|\mathbf{X})] + \text{var}(\theta^{\text{desc}}).$$

The expectation of θ^{desc} over the randomization distribution is

$$\begin{aligned} E[\theta^{\text{desc}}] &= \frac{1}{n_1} \sum_{i=1}^n E[X_i] Y_i - \frac{1}{n_0} \sum_{i=1}^n (1 - E[X_i]) Y_i \\ &= \frac{1}{n_1} \sum_{i=1}^n (n_1/n) Y_i^*(1) - \frac{1}{n_0} \sum_{i=1}^n (n_0/n) Y_i^*(0) \\ &= \theta^{\text{causal}}. \end{aligned}$$

For the variance of θ^{desc} , we have to compute n square terms and $n(n - 1)$ cross-product terms. Each square term is equal to

$$\begin{aligned} &\frac{\text{var}(X_i)}{n_1^2} Y_i^*(1)^2 + \frac{\text{var}(X_i)}{n_0^2} Y_i^*(0)^2 + 2 \frac{\text{var}(X_i)}{n_1 n_0} Y_i^*(1) Y_i^*(0) \\ &= \text{var}(X_i) \left(\frac{Y_i^*(1)^2}{n_1^2} + \frac{Y_i^*(0)^2}{n_0^2} + 2 \frac{Y_i^*(1) Y_i^*(0)}{n_1 n_0} \right). \end{aligned}$$

Recall from previous calculations that

$$\text{cov}(X_i, X_j) = -\frac{\text{var}(X_i)}{n - 1}.$$

Therefore, each of the cross-product terms is equal to

$$-\frac{\text{var}(X_i)}{n - 1} \left(\frac{Y_i^*(1) Y_j^*(1)}{n_1^2} + \frac{Y_i^*(1) Y_j^*(0)}{n_1 n_0} + \frac{Y_i^*(0) Y_j^*(1)}{n_1 n_0} + \frac{Y_i^*(0) Y_j^*(0)}{n_0^2} \right).$$

Let $\theta_i = Y_i^*(1) - Y_i^*(0)$; then

$$\begin{aligned} nS_\theta^2 &= \sum_{i=1}^n (Y_i^*(1) - Y_i^*(0))^2 - \frac{2}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n (Y_i^*(1) - Y_i^*(0))(Y_j^*(1) - Y_j^*(0)) \\ &= nS_1^2 + nS_0^2 - 2 \left(\sum_{i=1}^n Y_i^*(1)Y_i^*(0) - \frac{1}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n (Y_i^*(1)Y_j^*(0) + Y_i^*(0)Y_j^*(1)) \right). \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \text{var}(\theta^{\text{desc}}) &= \text{var}(X_i) \left(\frac{nS_1^2}{n_1^2} + \frac{nS_0^2}{n_0^2} + \frac{nS_1^2}{n_1 n_0} + \frac{nS_0^2}{n_1 n_0} - \frac{nS_\theta^2}{n_1 n_0} \right) \\ &= \frac{n_1 n_0}{n^2} \left(\frac{nS_1^2}{n_1^2} + \frac{nS_0^2}{n_0^2} + \frac{nS_1^2}{n_1 n_0} + \frac{nS_0^2}{n_1 n_0} - \frac{nS_\theta^2}{n_1 n_0} \right) \\ &= \frac{n_1 n_0}{n^2} \left(\frac{n^2}{n_1^2 n_0} S_1^2 + \frac{n^2}{n_1 n_0^2} S_0^2 - \frac{nS_\theta^2}{n_1 n_0} \right) \\ &= \frac{S_1^2}{n_1} + \frac{S_0^2}{n_0} - \frac{S_\theta^2}{n}. \end{aligned}$$

Notice now that (because we condition on N_1 and N_0)

$$\begin{aligned} \text{var}(\widehat{\theta}|\mathbf{X}) &= \text{var} \left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1} - \sum_{i=1}^n R_i (1 - X_i) \frac{Y_i^*(0)}{N_0} \middle| \mathbf{X} \right) \\ &= \text{var} \left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1} \middle| \mathbf{X} \right) + \text{var} \left(\sum_{i=1}^n R_i (1 - X_i) \frac{Y_i^*(0)}{N_0} \middle| \mathbf{X} \right). \end{aligned}$$

Let us calculate the first term on the right-hand side of the last equation (the second term will be analogous):

$$\begin{aligned} &\text{var} \left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1} \middle| \mathbf{X} \right) \\ &= \text{var}(R_i | X_i = 1) \left[\sum_{i=1}^n X_i \frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1 - 1} \sum_{i=1}^n \sum_{j=i+1}^n X_i X_j \frac{Y_i^*(1)Y_j^*(1)}{N_1^2} \right]. \end{aligned}$$

Taking expectations, the right-hand side becomes

$$\begin{aligned} &\text{var}(R_i | X_i = 1) \left[\sum_{i=1}^n \frac{n_1}{n} \frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1 - 1} \sum_{i=1}^n \sum_{j=i+1}^n \frac{n_1(n_1 - 1)}{n(n_1 - 1)} \frac{Y_i^*(1)Y_j^*(1)}{N_1^2} \right] \\ &= \frac{1}{nN_1} \left(\frac{n_1 - N_1}{n_1} \right) \left[\sum_{i=1}^n Y_i^*(1)^2 - \frac{2}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n Y_i^*(1)Y_j^*(1) \right] = \frac{n_1 - N_1}{n_1 N_1} S_1^2. \end{aligned}$$

This implies

$$\text{var}(\widehat{\theta}) = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{n}.$$

Now, notice that

$$E[\widehat{\theta}|\mathbf{R}] = \theta^{\text{causal, sample}},$$

where

$$\theta^{\text{causal, sample}} = \frac{1}{N} \sum_{i=1}^n R_i (Y_i^*(1) - Y_i^*(0)).$$

Therefore, by the law of total variance,

$$\text{var}(\widehat{\theta}) = E[\text{var}(\widehat{\theta}|\mathbf{R})] + \text{var}(\theta^{\text{causal, sample}}).$$

The variance of $\theta^{\text{causal, sample}}$ is

$$\begin{aligned} & \frac{\text{var}(R_i)}{N^2} \left[\sum_{i=1}^N (Y_i^*(1) - Y_i^*(0)) - \frac{2}{n-1} \sum_{i=1}^n \sum_{j=i+1}^n (Y_i^*(1) - Y_i^*(0))(Y_j^*(1) - Y_j^*(0)) \right] \\ &= \frac{1}{N} \left(1 - \frac{N}{n} \right) S_\theta^2. \end{aligned}$$

As a result,

$$\begin{aligned} E[\text{var}(\widehat{\theta}|\mathbf{R})] &= \text{var}(\widehat{\theta}) - \text{var}(\theta^{\text{causal, sample}}) \\ &= \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{N}. \end{aligned}$$

S.1.3. EHW Variance

The EHW variance estimator for $\widehat{\theta}$ is

$$\widehat{V}^{\text{ehw}} = \frac{N_1 - 1}{N_1^2} \widehat{S}_1^2 + \frac{N_0 - 1}{N_0^2} \widehat{S}_0^2,$$

where

$$\widehat{S}_1^2 = \frac{1}{N_1 - 1} \sum_{i=1}^n R_i X_i \left(Y_i - \frac{1}{N_1} \sum_{i=1}^n R_i X_i Y_i \right)^2,$$

and \widehat{S}_0^2 is defined analogously. Using previous results, we obtain

$$\widehat{S}_1^2 = \frac{1}{N_1} \sum_{i=1}^n R_i X_i Y_i^2 - \frac{2}{N_1(N_1 - 1)} \sum_{i=1}^n \sum_{j=i+1}^n R_i R_j X_i X_j Y_i Y_j.$$

Therefore,

$$E[\widehat{S}_1^2 | \mathbf{X}] = \frac{1}{n_1} \sum_{i=1}^n X_i Y_i^2 - \frac{2}{n_1(n_1 - 1)} \sum_{i=1}^n \sum_{j=i+1}^n X_i X_j Y_i Y_j$$

and

$$\begin{aligned} E[\widehat{S}_1^2] &= \frac{1}{n} \sum_{i=1}^n Y_i^*(1)^2 - \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j=i+1}^n Y_i^*(1) Y_j^*(1) \\ &= S_1^2. \end{aligned}$$

Let

$$\widetilde{V}^{\text{chw}} = \frac{\widehat{S}_1^2}{N_1} + \frac{\widehat{S}_0^2}{N_0}.$$

Then

$$E[\widetilde{V}^{\text{chw}}] = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} \geq \text{var}(\widehat{\theta}).$$

S.1.4. Bootstrap Variance

Consider the bootstrap variance estimator that draws N_1 treated and N_0 untreated observations separately,

$$\widehat{V}_B^{\text{boot}} = \frac{1}{B-1} \sum_{b=1}^B (\widehat{\theta}^{(b)} - \bar{\theta}_B)^2,$$

where

$$\widehat{\theta}^{(b)} = \frac{1}{N_1} \sum_{i=1}^n K_{1i}^{(b)} R_i X_i Y_i - \frac{1}{N_0} \sum_{i=1}^n K_{0i}^{(b)} R_i (1 - X_i) Y_i$$

and

$$\bar{\theta}_B = \frac{1}{B} \sum_{b=1}^B \widehat{\theta}^{(b)}.$$

Conditional on \mathbf{R} , and \mathbf{X} , $K_{1i}^{(b)}$ has a multinomial distribution with parameter $\{N_1, 1/N_1, \dots, 1/N_1\}$ for units with $R_i X_i = 1$ and $K_{0i}^{(b)}$ has a multinomial distribution with parameters $\{N_0, 1/N_0, \dots, 1/N_0\}$ for units with $R_i (1 - X_i) = 1$. The variables $K_{1i}^{(b)}$ and $K_{0i}^{(b)}$ are independent of each other and independent across $b = 1, \dots, B$. As a result, for $R_i X_i = R_j X_j = 1$ with $i \neq j$, we obtain

$$E[K_{1i}^{(b)} | \mathbf{R}, \mathbf{X}] = 1,$$

$$E[(K_{1i}^{(b)})^2 | \mathbf{R}, \mathbf{X}] = (2N_1 - 1)/N_1,$$

and

$$E[K_{1i}^{(b)} K_{1i}^{(b)} | \mathbf{R}, \mathbf{X}] = (N_1 - 1)/N_1.$$

Conditional on $\mathbf{Y}(1)$, $\mathbf{Y}(0)$, \mathbf{R} , and \mathbf{X} , the mean of the bootstrap variance is

$$E[\widehat{V}_B^{\text{boot}}|\mathbf{R}, \mathbf{X}] = \left(\frac{B}{B-1}\right)E[(\widehat{\theta}^{(b)} - \bar{\theta}_B)^2|\mathbf{R}, \mathbf{X}].$$

Because of independence of the bootstrap weights between treatment samples, we obtain

$$E[\widehat{V}_B^{\text{boot}}|\mathbf{R}, \mathbf{X}] = \left(\frac{B}{B-1}\right)E[(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B})^2|\mathbf{R}, \mathbf{X}] + \left(\frac{B}{B-1}\right)E[(\widehat{\theta}_0^{(b)} - \bar{\theta}_{0B})^2|\mathbf{R}, \mathbf{X}],$$

where

$$\widehat{\theta}_1^{(b)} = \frac{1}{N_1} \sum_{i=1}^n K_{1i}^{(b)} R_i X_i Y_i$$

and

$$\bar{\theta}_{1B} = \frac{1}{B} \sum_{b=1}^B \widehat{\theta}_1^{(b)},$$

with analogous expressions for $\widehat{\theta}_0^{(b)}$ and $\bar{\theta}_{0B}$. Now, let

$$\mathbf{K}_1 = \begin{pmatrix} K_{11}^{(1)} & K_{11}^{(2)} & \cdots & K_{11}^{(B)} \\ K_{12}^{(1)} & K_{12}^{(2)} & \cdots & K_{12}^{(B)} \\ \vdots & \vdots & \cdots & \vdots \\ K_{1n}^{(1)} & K_{1n}^{(2)} & \cdots & K_{1n}^{(B)} \end{pmatrix},$$

and let $\mathbf{K}_{1\pi}$ be a random permutation of the columns of \mathbf{K}_1 . Notice that $\bar{\theta}_{1B}$ is fixed conditional on \mathbf{R} , \mathbf{X} , and $\mathbf{K}_{1\pi}$, but $\widehat{\theta}_1^{(b)}$ is not. In addition,

$$E[\widehat{\theta}_1^{(b)}|\mathbf{R}, \mathbf{X}, \mathbf{K}_{1\pi}] = \bar{\theta}_{1B}.$$

Therefore,

$$E[(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B})^2|\mathbf{R}, \mathbf{X}, \mathbf{K}_{1\pi}] = E[(\widehat{\theta}_1^{(b)})^2|\mathbf{R}, \mathbf{X}, \mathbf{K}_{1\pi}] - \bar{\theta}_{1B}^2.$$

Then

$$E[(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B})^2|\mathbf{R}, \mathbf{X}] = E[(\widehat{\theta}_1^{(b)})^2|\mathbf{R}, \mathbf{X}] - E[\bar{\theta}_{1B}^2|\mathbf{R}, \mathbf{X}].$$

Let

$$\widehat{\theta}_1 = \frac{1}{N_1} \sum_{i=1}^N R_i X_i Y_i.$$

Notice that for any b and c , such that $1 \leq b < c \leq B$, we have

$$\begin{aligned} E[\bar{\theta}_{1B}^2|\mathbf{R}, \mathbf{X}] &= \frac{1}{B}E[(\widehat{\theta}_1^{(b)})^2|\mathbf{R}, \mathbf{X}] + \frac{B-1}{B}E[\widehat{\theta}_1^{(b)}\widehat{\theta}_1^{(c)}|\mathbf{R}, \mathbf{X}] \\ &= \frac{1}{B}E[(\widehat{\theta}_1^{(b)})^2|\mathbf{R}, \mathbf{X}] + \frac{B-1}{B}\widehat{\theta}_1^2. \end{aligned}$$

Therefore,

$$E[(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B})^2 | \mathbf{R}, \mathbf{X}] = \left(\frac{B-1}{B} \right) (E[(\widehat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] - \widehat{\theta}_1^2).$$

In addition,

$$E[(\widehat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] = \frac{1}{N_1^2} \left(\sum_{i=1}^n \frac{2N_1-1}{N_1} R_i X_i Y_i^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n \frac{N_1-1}{N_1} R_i R_j X_i X_j Y_i Y_j \right).$$

Therefore,

$$\begin{aligned} E[(\widehat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] - \widehat{\theta}_1^2 &= \frac{1}{N_1^2} \left(\sum_{i=1}^n \frac{N_1-1}{N_1} R_i X_i Y_i^2 - 2 \sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{N_1} R_i R_j X_i X_j Y_i Y_j \right) \\ &= \frac{N_1-1}{N_1^2} \left(\frac{1}{N_1} \sum_{i=1}^n R_i X_i Y_i^2 - \frac{2}{N_1(N_1-1)} \sum_{i=1}^n \sum_{j=i+1}^n R_i R_j X_i X_j Y_i Y_j \right) \\ &= \frac{N_1-1}{N_1^2} \widehat{S}_1 \end{aligned}$$

and

$$E[(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B})^2 | \mathbf{R}, \mathbf{X}] = \left(\frac{B-1}{B} \right) \frac{N_1-1}{N_1^2} \widehat{S}_1^2,$$

with the analogous result holding for $E[(\widehat{\theta}_0^{(b)} - \bar{\theta}_{0B})^2 | \mathbf{R}, \mathbf{X}]$. It follows that

$$E[\widehat{V}_B^{\text{boot}} | \mathbf{R}, \mathbf{X}] = \widehat{V}^{\text{ehw}}.$$

PROOF OF LEMMA 1: Let

$$W_{n,i} = \begin{pmatrix} Y_{n,i} \\ X_{n,i} \\ Z_{n,i} \end{pmatrix} \begin{pmatrix} Y_{n,i} \\ X_{n,i} \\ Z_{n,i} \end{pmatrix}',$$

and let $W_{n,i}^{(k,l)}$ be the (k, l) element of $W_{n,i}$. Similarly, let $\widetilde{W}_{n,i}^{(k,l)}$, $W_n^{(k,l)}$, $\widetilde{W}_n^{(k,l)}$, $\widetilde{\Omega}_n^{(k,l)}$, $\Omega_n^{(k,l)}$ be the (k, l) elements of $R_{n,i} W_{n,i}$, W_n , \widetilde{W}_n , $\widetilde{\Omega}_n$, and Ω_n , respectively. In order to have $\widetilde{W}_n^{(k,l)}$ and $\widetilde{\Omega}_n^{(k,l)}$ well-defined, let $\widetilde{W}_n^{(k,l)} = \Omega_n^{(k,l)} = 0$ when $N = 0$ (this is without loss of generality). Notice that, because $n\rho_n \rightarrow \infty$, for any fixed $0 < \varepsilon < 1$, there is n_ε such that for $n > n_\varepsilon$, we have $n\rho_n > -\log(\varepsilon)$. Therefore, for $n > n_\varepsilon$, we obtain

$$\Pr(N = 0) = \left(1 - \frac{n\rho_n}{n} \right)^n < \left(1 + \frac{\log(\varepsilon)}{n} \right)^n < e^{\log(\varepsilon)} = \varepsilon.$$

As a result, $\Pr(N = 0) \rightarrow 0$ and

$$E[(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2 | N = 0] \Pr(N = 0) = (\Omega_n^{(k,l)})^2 \Pr(N = 0) \rightarrow 0$$

by Assumption 5 and Holder's inequality. Notice that for any integer, m , such that $1 \leq m \leq n$, we have

$$E\left[\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}] \mid N = m\right] = 0$$

and

$$\begin{aligned} E[(\tilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2 \mid N = m] &= E\left[\left(\frac{1}{n} \sum_{i=1}^n \left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}]\right)^2 \mid N = m\right] \\ &= \frac{1}{n^2} \sum_{i=1}^n E\left[\left(\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}]\right)^2 \mid N = m\right] \\ &\leq \frac{1}{n^2} \sum_{i=1}^n E\left[\left(\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)}\right)^2 \mid N = m\right] \\ &= \frac{1}{m} \left(\frac{1}{n} \sum_{i=1}^n E[(W_{n,i}^{(k,l)})^2]\right) \\ &\leq C/m, \end{aligned}$$

for some positive constant, C , by Assumption 5. Let

$$\tilde{\xi}_m^{(k,l)} = \begin{cases} (\tilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2 & \text{if } m > 0, \\ 0 & \text{if } m = 0, \end{cases} \quad \text{and} \quad \xi_m = \begin{cases} C/m & \text{if } m > 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Now,

$$\begin{aligned} E[(\tilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2 \mid N > 0] \Pr(N > 0) &= E[\tilde{\xi}_N^{(k,l)}] \\ &\leq E[\xi_N]. \end{aligned}$$

Applying Chernoff's bounds, for any $\varepsilon > 0$,

$$\begin{aligned} \Pr(\xi_N > \varepsilon) &\leq \Pr(0 < N < C/\varepsilon) \\ &< \Pr(N < C/\varepsilon) \\ &= \Pr\left(N < n\rho_n \left(1 - \frac{n\rho_n - C/\varepsilon}{n\rho_n}\right)\right) \\ &\leq e^{-\left(\frac{(n\rho_n - C/\varepsilon)^2}{2n\rho_n}\right)} \rightarrow 0, \end{aligned}$$

which implies that ξ_N converges in probability to zero. Because ξ_N is bounded, by the portmanteau lemma we obtain $E[\xi_N] \rightarrow 0$. As a result,

$$E[(\tilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2] \rightarrow 0.$$

For the second result, notice that

$$\begin{aligned}
 E[(\tilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)})^2 | N = m] &= E\left[\left(\sum_{i=1}^n \left(\frac{R_{n,i}}{m} - \frac{1}{n}\right) E[W_{n,i}^{(k,l)}]\right)^2 \middle| N = m\right] \\
 &= \sum_{i=1}^n \frac{1}{m^2} E\left[\left(R_{n,i} - \frac{m}{n}\right)^2 \middle| N = m\right] (E[W_{n,i}^{(k,l)}])^2 \\
 &= \sum_{i=1}^n \frac{1}{mn} \left(1 - \frac{m}{n}\right) (E[W_{n,i}^{(k,l)}])^2 \\
 &\leq \frac{1}{m} \left(\frac{1}{n} \sum_{i=1}^n (E[W_{n,i}^{(k,l)}])^2\right).
 \end{aligned}$$

Now, using the same argument as above, we obtain

$$E[(\tilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)})^2] \rightarrow 0.$$

The proof of the third result is analogous.

Q.E.D.

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