Proceedings of the 42nd IEEE Conference on Decision and Control
Maui, Hawaii USA, December 2003

Sampling Random Transfer Functions*

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Abstract— **Recently, considerable attention has been paid to the use of probabilistic algorithms for analysis and design of robust control systems. However, since these algorithms require the generation of random samples of the uncertain parameters, their application has been mostly limited to the case of parametric uncertainty. Notable exceptions to this limitation are the algorithm for generating FIR transfer functions in Lagoa et al. and the algorithm for generating random fixed order state space representations in Calafiore et al. In this paper, we provide the means for further extending the use of probabilistic algorithms for the case of dynamic causal uncertain parameters. More precisely, we exploit both time and frequency domain characterizations to develop efficient algorithms for generation of random samples of causal, linear time-invariant uncertain transfer functions. The usefulness of these tools will be illustrated by developing an algorithm for solving some multi-disk problems arising in the context of synthesizing robust controllers for systems subject to structured dynamic uncertainty.**

I. Introduction

A large number of control problems of importance can be reduced to the robust performance analysis framework illustrated in Figure 1. The family of systems under consideration consists of the interconnection of a known stable LTI plant with a bounded uncertainty $\Delta \subset \mathbf{\Delta}$. The goal is to compute the worst–case, with respect to $\mathbf{\Delta}$, of the norm of the output to some class of exogenous disturbances.

Depending on the choice of models for the input signals and on the criteria used to assess performance, this prototype problem leads to different mathematical formulations such as $\mathcal{H}_\infty$, ℓ^1, $\mathcal{H}_2$ and ℓ^∞ control. A common feature to all these problems is that, with the notable exception of the $\mathcal{H}_\infty$ case, no tight performance bounds are available for systems with uncertainty Δ being a causal bounded LTI operator[1]. Moreover, even in the $\mathcal{H}_\infty$ case, the problem of computing a tight performance bound is known to be NP–hard in the case of structured uncertainty, with more than two uncertainty blocks [4].

Given the difficulty of computing these bounds, over the past few years, considerable attention has been devoted to the use of probabilistic methods. This approach furnishes, rather than worst case bounds, risk–adjusted bounds; i.e., bounds for which the probability of performance violation is no larger than a prescribed risk level ε. An appealing feature

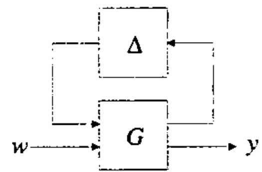

Fig. 1. The Robust Performance Analysis Problem

of this approach is that, contrary to the worst–case approach case, here, the computational burden grows moderately with the size of the problem. Moreover, in many cases, worst–case bounds can be too conservative, in the sense that performance can be substantially improved by allowing for a small level of performance violation. The application of Monte Carlo methods to the analysis of control systems was recently in the work by Stengel, Ray and Marrison in [11], [13], [17] and was followed, among others, by [2], [6], [7], [8], [18], [19], [20].

At the present time the domain of applicability of Monte Carlo techniques is largely restricted to the finite–dimensional parametric uncertainty case. The main reason for this limitation resides in the fact that up to now, the problem of sampling causal bounded operators (rather than vectors or matrices) has not appeared in the systems literature. Notable exceptions to this limitation are the algorithm for generating FIR transfer functions in [10] and the algorithm for generating random fixed order state space representations in [5]. In this paper, we provide two algorithms aimed at removing this limitation when the set $\mathbf{\Delta}$ consists of balls in $\mathcal{H}_\infty$. We use results on Nevanlinna-Pick and boundary Nevanlinna-Pick interpolation theory to develop two new procedures for random transfer function generation. The first one generates random transfer functions having the property that, for a given frequency, the frequency response is uniformly distributed over the interior of the unit circle. This algorithm is useful for problems such as model (in)validation, where the uncertainty that validates the model description is not necessarily on the boundary of the uncertainty set $\mathbf{\Delta}$. The second algorithm provides samples uniformly distributed over the unit circle, and is useful for cases such as some robust performance analysis/synthesis problems where the worst–case uncertainty is known to be on the boundary of $\mathbf{\Delta}$.

The usefulness of these tools is illustrated by developing algorithms for solving some multi–disk problems arising in the context of synthesizing robust controllers for systems subject to structured dynamic uncertainty. More precisely,

*Funding for this research was provided by the National Science Foundation under grants ECS-9984260 and ECS-0115946 and the Air Force Office of Scientific Research under grant AFSOR-F49620-00-1-0020.

[1] *Recently some tight bounds have been proposed for the $\mathcal{H}_2$ case, but these bounds do not take causality into account; see [12].*

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we provide a modification of the algorithm in [9] that when used together with the sampling schemes mentioned above, enables one to solve the problem of designing a controller that robustly stabilizes the system for a "large" set of uncertainties while guaranteeing a given performance level on a "smaller" uncertainty subset.

II. Preliminaries

Given a matrix M, let M^T and M^* denote the transpose and Hermitian conjugate respectively. As usual $M > 0$ ($M \geq 0$) indicates that M is positive definite (positive semi-definite), and $M < 0$ that M is negative definite. Furthermore, let $\mathrm{Re}(M)$ denote the real part of M and $\mathrm{Trace}(M)$ its trace.

By $\mathcal{L}_\infty$, we denote the Lebesgue space of complex-valued matrix functions essentially bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|=1} \overline{\sigma}(G(z))$, where $\overline{\sigma}$ represents the largest singular value. By $\mathcal{H}_\infty$, we denote the subspace of functions in $\mathcal{L}_\infty$ with bounded analytic continuation outside the unit disk, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|>1} \overline{\sigma}(G(z))$. Finally, we use $\mathcal{B}$ and $\mathcal{R}$ to denote unit balls and subspaces composed of real rational transfer matrices, respectively. The $\mathcal{H}_\infty$ ball of radius r is denoted by $\mathcal{BH}_\infty(r)$.

Also, let $\mathcal{H}_2$ denote the Hilbert space of complex matrix valued functions analytic in the set $\{z \in \mathbf{C} : |z| \geq 1\}$, equipped with the inner product

$$\langle H, T\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\{\mathrm{Trace}[H(e^{j\theta})^* T(e^{j\theta})]\} d\theta.$$

Also, let $\mathcal{RH}_2$ denote the subspace of all rational functions in $\mathcal{H}_2$ analytic in $\{z \in \mathbf{C} : |z| \geq 1\}$. Moreover, define the space $\mathcal{G}$ as the space of rational functions $G : \mathbf{C} \rightarrow \mathbf{C}^{n\times m}$ that can be represented as

$$G(z) = G_s(z) + G_u(z).$$

where $G_s(z) \in \mathcal{RH}_2$ and $G_u(z)$ is strictly proper and analytic in the set $\{z \in \mathbf{C} : |z| < 1\}$. Now, given two functions $G, H \in \mathcal{G}$ and $0 < \gamma < 1$ define the distance function d as

$$d(G,H) \doteq \left(\|G_s(z) - H_s(z)\|_2^2 + \|G_u(\gamma/z) - H_u(\gamma/z)\|_2^2\right)^{\frac{1}{2}}.$$

The results later in the paper that make use of this distance function are similar for any value of γ. Finally, define the projection $\pi_s : \mathcal{G} \rightarrow \mathcal{H}_2$ as $\pi_s(G) \doteq G_s$.

Now, consider a convex function $g : H_2 \rightarrow \mathbf{R}$. Given any $G_0 \in H_2$, there exists a $\partial_G g(G_0) \in H_2$ such that

$$g(G) - g(G_0) \geq \langle \partial g_G(G_0), G - G_0\rangle. \qquad (1)$$

for all $G \in H_2$. The quantity $\partial_G g(G_0)$ is said to be a subgradient of g at the point G_0.

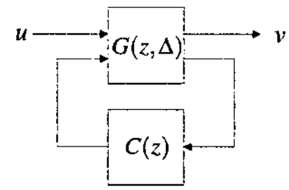

Fig. 2. Closed Loop System

A. Closed Loop Transfer Function Parametrization

Central to the results presented in this paper is the parametrization of all closed loop transfer functions. Consider the closed loop plant in Figure 2 with uncertain parameters $\Delta \in \mathbf{\Delta}$. The uncertainty Δ can include static uncertainty, uncertain transfer function matrices or a combination of both. The Youla parametrization (e.g., see [16]) indicates that, given $\Delta \in \mathbf{\Delta}$ and a stabilizing controller $C \in \mathcal{G}$, the closed loop transfer function can be represented as

$$T_{CL}(z,\Delta,C) = T_\Delta^1(z) + T_\Delta^2(z) Q_{\Delta,C}(z) T_\Delta^3(z), \qquad (2)$$

where $T_\Delta^1, T_\Delta^2, T_\Delta^3 \in RH_2$ are determined by the plant $G(z,\Delta)$ (and, hence, they also depend on the uncertainty Δ) and $Q_{\Delta,C} \in RH_2$ depends on both the open loop plant $G(z,\Delta)$ and the controller $C(z)$. Also, given any $Q_{\Delta,C}(s) \in RH_2$, there exists a controller $C \in \mathcal{G}$ such that the equality above is satisfied. This parametrization also holds for all closed loop transfer functions, stable and unstable. Using a frequency scaling reasoning, one can prove the following result: Given $\Delta \in \mathbf{\Delta}$ and a controller $C \in \mathcal{G}$, the closed loop transfer function can be represented as (2) Note that the mapping from Δ to $T_\Delta^1, T_\Delta^2, T_\Delta^3$ is not unique. In what follows, we assume that a unique mapping has been selected. Results to follow do not depend on how this mapping is chosen.

III. Sampling $\mathcal{BH}_\infty$

We now present two algorithms for generating random transfer functions in $\mathcal{BH}_\infty$.

A. Sampling the "Inner" $\mathcal{BH}_\infty$

The first one, based on "ordinary" Nevanlinna-Pick interpolation, provides transfer functions with $\mathcal{H}_\infty$ norm less or equal than 1 and whose frequency response, at given frequency grid points, is uniformly distributed over the complex plane unit circle.

Algorithm 1: 1) Given an integer N, pick N frequencies λ_i such that $|\lambda_i| = 1$, $i = 1, 2, \ldots, N$.

2) Generate N independent samples w_i uniformly distributed over the set $\{w \in \mathbf{C} : |w| < 1\}$.
3) Find $0 < r < 1$ such that the matrix Λ with entries

$$\Lambda_{i,j} = \frac{1 - w_i w_j^*}{1 - r^2 \lambda_i \lambda_j^*}$$

is positive definite.

4) Find a rational function $h_r(\lambda)$ analytic inside the unit circle satisfying

$$\|h_r\|_\infty \leq 1 \quad \text{and} \quad h_r(r\lambda_i) = w_i; i = 1,2,\ldots,N$$

by solving a "traditional" Nevanlinna-Pick interpolation problem.

5) The random transfer function is given by

$$h(z) = h_r(rz^{-1}).$$

We refer the reader to the Appendix for a brief review of results on Nevanlinna-Pick interpolation and state space descriptions of the interpolating transfer function $h(z)$.

Remark: Note that, there always exists an $0 < r < 1$ that will make the matrix Λ positive definite. This is a consequence of the fact that the diagonal entries are positive real numbers and that, as one increases $r < 1$, the matrix will eventually be diagonally dominant.

B. Sampling the Boundary of $\mathcal{BH}_\infty$

We now present a second algorithm for random generation of rational functions. The algorithm below generates random transfer functions whose frequency response, at given frequency grid points, is uniformly distributed over the boundary of the unit circle. Recall that the rational for generating these samples is that in many problems it is known that the worst case uncertainty is located in the boundary of the uncertainty set , and thus there is no point in generating and testing elements with $\|\Delta\|_\infty < 1$.

Algorithm 2: 1) Given an integer N, pick N frequencies λ_i such that $|\lambda_i| = 1$, $i = 1,2,\ldots,N$.

2) Generate N independent samples w_i uniformly distributed over the set $\{w \in \mathbf{C} : |w| = 1\}$.

3) Find the smallest possible $\rho \geq 0$ such that the matrix Λ with entries

$$\Lambda_{i,j} = \begin{cases} \frac{1-w_i^* w_j}{1-\lambda_i^* \lambda_j} & i \neq j \\ \rho & i = j \end{cases}$$

is positive definite.

4) Let

$$\theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix}$$

be a 2×2 transfer function matrix given by

$$\theta(\lambda) = I + (\lambda - \lambda_0) C_0 (\lambda I - A_0)^{-1} \Lambda^{-1} (\lambda I - A_0^*)^{-1} C_0^* J$$

where

$$C_0 = \begin{bmatrix} w_1 & \ldots & w_N \\ 1 & \ldots & 1 \end{bmatrix}; \quad A_0 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N);$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and λ_0 is a complex number of magnitude 1 and not equal to any of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$.

5) The random transfer function is given by

$$h(z) = \frac{\theta_{12}(z^{-1})}{\theta_{22}(z^{-1})}.$$

The algorithm above provides a solution of the boundary Nevanlinna-Pick interpolation problem

$$\|h\|_\infty = 1; \quad h(\lambda_i) = w_i; \quad h'(\lambda_i) = \rho \lambda_i^* w_i,$$

for $i = 1,2,\ldots,N$. A proof of this result can be found in [1]. A more complete description of the results on boundary Nevanlinna-Pick interpolation used here is given in the Appendix.

Remark: The search for the lowest ρ that results in a positive definite matrix Λ is equivalent to finding the interpolant with the lowest derivative.

IV. MULTI–DISK DESIGN PROBLEM

In this section we use discuss an application of the sampling algorithms developed in this paper. More precisely, we introduce an stochastic gradient based algorithm to solve the so–called multi–disk design problem. We aim at solving the problem of design a robustly stabilizing controller that results in guaranteed performance in a subset of the uncertainty support set. The algorithm presented is an extension of the algorithms developed in [9]. Before providing the controller design algorithm, we first provide a precise definition of the problem to be solved and the assumptions that are made.

A. Problem Statement

Consider the closed-loop system in Figure 2 and a convex objective function $g_1 : \mathcal{H}_2 \to \mathbf{R}$. Given a performance value γ_1 and uncertainty radii $r_2 > r_1 > 0$, we aim at designing a controller $C^*(s)$ such that the closed loop system $T_{CL}(z, \Delta, C^*)$ is stable for all $\|\Delta\|_\infty \leq r_2$ and satisfies

$$g\left[T_{CL}(z, \Delta, C^*)\right] \leq \gamma_1$$

for all $\|\Delta\|_\infty \leq r_1$. Throughout this paper, we will assume that the problem above is feasible. More precisely, the following assumption is made:

Assumption 1: There exists a controller C^* and an $\varepsilon > 0$ such that

$$d(Q_{\Delta,C^*}, Q) < \varepsilon \Rightarrow g_1\left[T_\Delta^1(z) + T_\Delta^2(z) Q(z) T_\Delta^3(z)\right] \leq \gamma_1$$

for all $\|\Delta\|_\infty \leq r_1$ and there exists a γ_2 (sufficiently large) such that

$$d(Q_{\Delta,C^*}, Q) < \varepsilon \Rightarrow g_2\left[T_\Delta^1(z) + T_\Delta^2(z) Q(z) T_\Delta^3(z)\right] \leq \gamma_2$$

for all $\|\Delta\|_\infty \leq r_2$, where

$$g_2\left[T_\Delta^1(z) + T_\Delta^2(z) Q(z) T_\Delta^3(z)\right] \doteq \left\|T_\Delta^1(z) + T_\Delta^2(z) Q(z) T_\Delta^3(z)\right\|_2.$$

Remark: Even though it is a slightly stronger requirement than robust stability, the existence of a large constant γ_2 satisfying the second condition above can be considered to be, from a practical point of view, equivalent to robust stability.

B. Controller Design Algorithm

We now state the proposed robust controller design algorithm. This algorithm has a free parameter η that has to be specified. This parameter can be arbitrarily chosen from the interval $(0,2)$.

Algorithm 3: 1) Let $k=0$. Pick a controller $C_0(z)$.

2) Generate sample i^k with equal probability or being 1 or 2.
3) Draw sample Δ^k over $\mathcal{BH}_\infty(r_{i^k})$. Given $G(z,\Delta^K)$, compute $T^1_{\Delta^k}(z)$, $T^2_{\Delta^k}(z)$, $T^3_{\Delta^k}(z)$ as described in [16].
4) Let $Q_k(z)$ be such that the closed loop transfer function using controller $C_k(s)$ is

$$T_{CL}(z,\Delta^k,\ C_k) = T^1_{\Delta^k}(z) + T^2_{\Delta^k}(z)Q_k(z)T^3_{\Delta^k}(z)$$

5) Do the stabilizing projection $Q_{k,s}(z) = \pi_s(Q_k(z))$.
6) Perform update

$$Q_{k+1}(z) = Q_{k,s}(z) - \alpha_k(Q_{k,z},\Delta^k)(z)\partial_Q g_{i^k}(T_{CL}(z,\Delta^k,Q))|_{Q_{k,s}}$$

where

$$\alpha_k(Q_k,\Delta) = \begin{cases} \eta \frac{g_{i^k}(T_{CL}(z,\Delta,Q_k)) - \gamma_{i^k} + \varepsilon \|\partial_Q g_{i^k}(T_{CL}(z,\Delta,Q))|_{Q_k}\|_2}{\|\partial_Q g_{i^k}(T_{CL}(z,\Delta,Q))|_{Q_k}\|_2^2} & \text{if } g_{i^k}(T_{CL}(z,\Delta,Q_k)) > \gamma_{i^k} \\ 0 & \text{otherwise} \end{cases}. \quad (3)$$

7) Determine the controller C_{k+1} so that $Q_{\Delta^k,C_{k+1}} = Q_{k+1}$.
8) Let $k=k+1$. Go to Step 2.

It can be proven that the algorithm described above indeed converges to a controller that robustly satisfies the performance specifications. The exact statement is given below. The proof is follows the same line of reasoning as in [9] and it is omitted due to space constraints.

Theorem 1: Let $g_1 : \mathcal{H}_2 \to \mathbf{R}$ be a convex function with subgradient $\partial g_1 \in \mathcal{RH}_2$ and let $\gamma_1 > 0$ be given. Also let $g_2(H) = \|H\|_2$. Define

$$P_{k,1} \doteq Prob\{g_1(T_{CL}(z,\Delta,C_k)) > \gamma_1\}$$

with Δ having the distribution over $\mathcal{BH}_\infty(r_1)$ used in the algorithm. Similarly take

$$P_{k,2} \doteq Prob\{g_2(T_{CL}(z,\Delta,C_k)) > \gamma_2\}$$

with Δ having the distribution over $\mathcal{BH}_\infty(r_2)$ used in the algorithm. Given this, define $P_k \doteq (P_{k,1}+P_{k,2})/2$. Then, if Assumption 1 holds, the algorithm described above generates a sequence of controllers C_k for which the risk of performance violation satisfies $\lim_{k\to\infty} P_k = 0$.

V. Numerical Example

Consider the uncertain system $P(z,\Delta) = P_0(z) + \Delta(z)$, with nominal plant

$$P_0(z) = \frac{0.006135z^2 + 0.01227z + 0.006135}{z^2 - 1.497z + 0.5706}$$

and stable causal dynamic uncertainty Δ. The objective is to find a controller $C(z)$ such that, for all $\|\Delta\|_\infty \le r_1 = 1$,

$$\|W(z)(1+C(z)P(z,\Delta))^{-1}\|_2 \le \gamma_1 = 0.089$$

where

$$W(z) = \frac{0.0582z^2 + 0.06349z + 0.005291}{z^2 + 0.2381z - 0.6032}.$$

and the closed loop system is stable for all $\|\Delta\|_\infty \le r_2 = 2$. Since the plant $P(z,\Delta)$ is stable in spite of the uncertainty, according to the Youla parametrization, all stabilizing controllers are of the form

$$C = \frac{Q(z)}{1 - Q(z)P(z,\Delta)}$$

where $Q(z)$ is a stable rational transfer function. To solve this problem using the algorithm presented in the previous section, we take $\gamma_2 = 10^9$ (which is in practice equivalent to requiring robust stability for $\|\Delta\| \le r_2$) and generate the random uncertainty samples using Algorithm 2 by taking $z_i = e^{j2\pi i/11}$, $i = 1,2,\ldots,10$. We first consider a design using only the nominal plant. Using `Matlab`'s function `dh2lqg()`, we obtain the nominal $\mathcal{H}_2$ optimal controller

$$C_{nom}(z) = \frac{138.2z^3 - 93.78z^2 - 90.4z + 64.5}{z^4 + 2.238z^3 + 0.8729z^2 - 0.9682z - 0.6031}$$

and a nominal performance $\|T_{cl}(z)\|_2 = 0.0583$. However, this controller does not robustly stabilize the closed loop plant for $\|\Delta\|_\infty \le 2$. We next apply Algorithm 3 to design a risk–adjusted controller and, after 1,500 iterations, we obtain

$$C_1(z) = \frac{-0.003808z^{14} - 0.01977z^{13}}{z^{14} - 0.1778z^{13} + 0.6376z^{12} + 0.09269z^{11} + 0.2469z^{10}} \frac{-0.002939z^{12}}{+0.06291z^9 + 0.08426z^8 + 0.0433z^7 + 0.07403z^6 + 0.0004z^5} \frac{+0.04627z^{11}}{-0.1107z^4 - 0.07454z^3 - 0.08156z^2 - 0.05994z + 0.01213}.$$

Monte Carlo simulations were performed to estimate $P_{k,1}$ and $P_{k,2}$ for each controller $C_k(z)$ and the estimate of $(P_{k,1} + P_{k,2})/2$ is shown in Figure 3. One can see that both the probability of performance violation for $\|\Delta\|_\infty \le 1$ and the probability of instability for $\|\Delta\|_\infty \le 2$ quickly converge to zero, being negligible after iteration 200.

VI. Concluding Remarks

In this paper, we provide efficient algorithms for generation of random samples of causal, linear time-invariant uncertain transfer functions. Results on Nevanlinna-Pick and boundary Nevanlinna-Pick interpolation are exploited to develop two algorithms. The first one generates samples inside the unit $\mathcal{H}_\infty$ ball and the second one generates random transfer function on the boundary of the unit $\mathcal{H}_\infty$ ball. The usefulness of these tools is illustrated by developing an algorithm for solving some multi–disk problems arising in

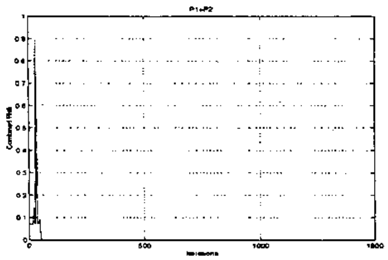

Fig. 3. Estimated $(P_{k,1}+P_{k,2})/2$.

the context of synthesizing robust controllers for systems subject to structured dynamic uncertainty.

The results presented suggest several directions for further research. First, we believe that effort should be put in the development of efficient numerical implementations of the algorithms put forth in this paper. Another possible direction for further research is the development of stochastic gradient algorithms for controller design which would guarantee that one would obtain a robustly stabilizing controller after a finite number of steps.

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APPENDIX

A. Generalized Interpolation Framework

We start by focusing our attention in a more general result in interpolation theory. Let $\mathcal{T}$ and $\mathcal{BT}$ denote the space of complex valued rational functions continuous in $|\lambda| = 1$ and analytic in $|\lambda| < 1$, equipped with the $\|.\|_{\mathcal{L}_\infty}$ norm, and the (open) unit ball in this space, respectively (i.e. $f(\lambda) \in \mathcal{BT} \iff f(\frac{1}{z}) \in \mathcal{BH}_\infty$). We now present a fundamental result whose proof can be found in [1], [14].

Theorem 2: There exists a transfer function $f(\lambda) \in \mathcal{BT}$ ($\overline{\mathcal{BT}}$) such that:

$$\sum_{\lambda_o \in \mathcal{D}} \mathrm{Res}_{\lambda=\lambda_o} f(\lambda) C_-(\lambda I - A)^{-1} = C_+ \tag{4}$$

if and only if the following discrete time Lyapunov equation has a unique positive (semi) definite solution.

$$M = A^* M A + C_-^* C_- - C_+^* C_+ \tag{5}$$

where A, C_- and C_+ are constant complex matrices of appropriate dimensions and $\mathcal{D}$ denotes the open unit circle. If $M > 0$ then the solution $f(\lambda)$ is non-unique and the set of solutions can be parameterized in terms of $q(\lambda)$, an arbitrary element of $\overline{\mathcal{BT}}$, as follows:

$$f(\lambda) = \frac{T_{11}(\lambda) q(\lambda) + T_{12}(\lambda)}{T_{21}(\lambda) q(\lambda) + T_{22}(\lambda)}; \quad T(\lambda) = \begin{bmatrix} T_{11}(\lambda) & T_{12}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) \end{bmatrix} \tag{6}$$

where $T(\lambda)$ is the J-lossless[2] matrix:

$$T(\lambda) \equiv \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]; \quad A_T = A$$

$$B_T = M^{-1}(A^* - I)^{-1} \begin{bmatrix} -C_+^* & C_-^* \end{bmatrix}; C_T = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (A - I)$$

$$D_T = I + \begin{bmatrix} C_+ \\ C_- \end{bmatrix} M^{-1}(A^* - I)^{-1} \begin{bmatrix} -C_+^* & C_-^* \end{bmatrix}$$

Note that the matrices A and C_- provide the structure of the interpolation problem while C_+ provides the interpolation values. The following corollary shows that Nevanlinna-Pick problem is a special case of this theorem, corresponding to an appropriate choice of the matrices A and C_-.

Corollary 1 (Nevanlinna-Pick): Let $\Gamma = \mathrm{diag}\{\lambda_i\} \in \mathbf{C}^{r \times r}$ and take

$$A = \Gamma \tag{7}$$

$$C_- = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \in \mathcal{R}^r \tag{8}$$

$$C_+ = \begin{bmatrix} w_1 & w_2 & \dots & w_r \end{bmatrix} \tag{9}$$

then (4) is equivalent to $f(\lambda_i) = w_i$, $i = 1, \dots, r$ and the solution to (5) is the standard Pick matrix:

$$P = \left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j} \right]_{ij} \tag{10}$$

B. Using this results for boundary interpolation

In the case of boundary interpolation $|\lambda_i| = 1$, $|w_i| < 1$, these results can be used as follows:

1) Find a scalar $r < 1$ such that the equation:

$$M = r^2 A^* M A + C_-^* C_- - C_+^* C_+ \tag{11}$$

has a positive definite solution $M > 0$.

2) Find the modified interpolant using the formulas (6) with $A = r\Gamma = r\mathrm{diag}\{\lambda_i\}$
3) The desired interpolant is given by $G(\lambda) = G_r(r\lambda)$.

[2] A transfer function $H(\lambda)$ is said to be J-lossless if $H^T(1/\lambda) J H(\lambda) = J$ when $|\lambda| = 1$, and $H^T(1/\lambda) J H(\lambda) < J$ when $|\lambda| < 1$. Here $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

C. Boundary Nevanlinna-Pick Interpolation

We now elaborate on the results on boundary Nevanlinna-Pick interpolation used in this paper. For a extensive treatment of this problem see [1]. Let $\mathcal{D}$ denote the unit circle in the complex plane with boundary $\partial\mathcal{D}$ and consider the following interpolation problem:

Problem 1: Given N distinct points $\lambda_1, \lambda_2, \dots, \lambda_N$ in $\partial\mathcal{D}$, N complex numbers $w_1, w_2, \dots, w_N$ of unit magnitude and N positive real numbers $\rho_1, \rho_2, \dots, \rho_N$, find all rational functions $f(\lambda)$ mapping $\mathcal{D}$ into $\mathcal{D}$ such that, for all $i = 1, 2, \dots, N$,

$$f(\lambda_i) = w_i; \quad f'(\lambda_i) = \lambda_i^* w_i \rho_i.$$

The following theorem provides a solution for the problem above. The proof of this result can be found in [1].

Theorem 3: Let $\lambda_1, \lambda_2, \dots, \lambda_N$, $w_1, w_2, \dots, w_N$ and $\rho_1, \rho_2, \dots, \rho_N$ be as in the statement of Problem 1 and define the matrix $\Lambda = [\Lambda_{ij}]_{1 \le i,j \le N}$ by

$$\Lambda_{i,j} = \begin{cases} \frac{1 - w_i^* w_j}{1 - \lambda_i^* \lambda_j} & i \neq j \\ \rho_i & i = j \end{cases}.$$

Then a necessary condition for Problem 1 to have a solution is that Λ be positive semidefinite and a sufficient condition is that Λ be positive definite. In the latter case, the set of all solutions is given by

$$f(\lambda) = \frac{\theta_{11}(\lambda) g(\lambda) + \theta_{12}(\lambda)}{\theta_{21}(\lambda) g(\lambda) + \theta_{22}(\lambda)}$$

where $g(\lambda)$ is an arbitrary scalar rational function analytic on $\mathcal{D}$ with $\sup\{|g(\lambda)| : z \in \mathcal{D}\} \le 1$ such that $\theta_{21}(\lambda) g(\lambda) + \theta_{22}(\lambda)$ has a simple pole at the points $\lambda_1, \lambda_2, \dots, \lambda_N$. Here

$$\theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix}$$

is given by

$$\theta(\lambda) = I + (\lambda - \lambda_0) C_0 (\lambda I - A_0)^{-1} \Lambda^{-1} (\lambda I - A_0^*)^{-1} C_0^* J$$

where

$$C_0 = \begin{bmatrix} w_1 & \dots & w_N \\ 1 & \dots & 1 \end{bmatrix}; A_0 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}; J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and λ_0 is a complex number of magnitude 1 and not equal to any of the numbers $\lambda_1, \lambda_2, \dots, \lambda_N$.

Note that if only the values $w_1, w_2, \dots, w_N$ of magnitude one are specified at the boundary points $\lambda_1, \lambda_2, \dots, \lambda_N$, then the matrix Λ in the theorem above can always be made positive definite by choosing the unspecified quantities $\rho_1, \rho_2, \dots, \rho_N$ sufficiently large. This leads to the following corollary.

Corollary 2: Let 2N complex numbers of magnitude one $\lambda_1, \lambda_2, \dots, \lambda_N$ and $w_1, w_2, \dots, w_N$ be given, where $\lambda_1, \lambda_2, \dots, \lambda_N$ are distinct. Then, there always exist scalar rational functions $f(\lambda)$ analytic in $\mathcal{D}$ with $\sup\{|f(\lambda)| : \lambda \in \mathcal{D}\} \le 1$ which satisfy the set of interpolation conditions $f(\lambda_i) = w_i; \quad i = 1, 2, \dots, N$.