# Sasaki-Einstein Metrics on $S^{2} \times S^{3}$ 

Mitsuhiro IMADA<br>Keio University<br>(Communicated by K. Ota)


#### Abstract

In [9], Boyer and Galicki introduced a contact reduction method in the context of Sasakian manifolds, which produces 5 -dimentional Sasaki-Einstein manifolds from a 7 -sphere. In this paper, we compute very explicitly the metric obtained from the above mentioned reduction via a projection, $S^{3} \times S^{3} \rightarrow S^{2} \times S^{3}$, and show that this metric is the homogeneous Kobayashi-Tanno metric.


## 1. Introduction

Reduction techniques in symplectic geometry, such as Marsden and Ratiu [1], have natural analogues in the context of contact geometry. Depending on the geometric situation, various specializations have been considered in the literature, such as the Sasakian case by Geiges [2], Grantcharov and Ornea [3]. Later on the Sasaki-Einstein case, by Boyer and Galicki in [9]. In the latter approach (on which this paper is based) the authors constructed a 5-dimentional Sasaki-Einstein manifold by means of a $S^{1}$ reduction of the zero set of a moment map defined on $S^{7}$.

In this paper, we explicitized the above construction, and compute explicitly the reduced metric on the reduced space by means of a projection from the zero set to the reduced space, which is diffeomorphic to $S^{2} \times S^{3}$. More precisely, we consider the following moment map on $\mathbf{C}^{4}$,

$$
\mu\left(z_{1}, z_{2}, z_{3}, z_{4}\right):=\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}-\left|z_{3}\right|^{2}-\left|z_{4}\right|^{2},
$$

with the associated $U(1)$ action,

$$
\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \mapsto\left(e^{i \theta} z_{1}, e^{i \theta} z_{2}, e^{-i \theta} z_{3}, e^{-i \theta} z_{4}\right) \quad(\theta \in \mathbf{R}),
$$

and we show that $\left.\mu^{-1}(0)\right|_{S^{7}}$ is diffeomorphic to $S^{3} \times S^{3}$. Using this identification, we define a smooth projection $\pi$ from $S^{3} \times S^{3}$ to $\left(\left.\mu^{-1}(0)\right|_{S^{7}}\right) / S^{1}$ (see $\S 4$ ):

$$
\pi\left(z_{1}, z_{2}, z_{3}, z_{4}\right):=\left(2 z_{1} \bar{z}_{2},\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, z_{1} z_{3}+\bar{z}_{2} \bar{z}_{4}, z_{2} z_{3}-\bar{z}_{1} \bar{z}_{4}\right) .
$$

[^0]Later in $\S 4$, we show that this image is diffeomorphic to $S^{2} \times S^{3}$. We notice that a $S U(2) \times$ $S U(2)$ acts on $S^{3} \times S^{3}$ naturally from the left, which gives $\pi$ is an equivariant map, that is, $S^{2} \times S^{3}$ becomes a homogeneous space by this action. We then define an inner product $\langle\cdot, \cdot\rangle_{o}$ on $T_{o}\left(S^{2} \times S^{3}\right) \quad(o=(0,0,-1,1,0,0,0))$ and extend it to any point $x$ as follows

$$
\langle u, v\rangle_{x}:=\left\langle d k^{-1}(u), d k^{-1}(v)\right\rangle_{o} \quad\left(u, v \in T_{x}\left(S^{2} \times S^{3}\right)\right)
$$

where $k$ is a $(S U(2) \times S U(2)) / U(1)$ free action such that $x=k \cdot o$. This is a representation of the metric named the homogeneous Kobayashi-Tanno metric [10], [11]. Our main result (Theorem 4.1) is an explicit calculation of the metric.

## 2. Sasaki-Einstein manifolds

In this section, we recall the definition of a Sasaki-Einstein manifold [9].
DEFINITION 2.1. A Sasakian manifold is a $(2 n-1)$-dimentional Riemannian manifold $(M, g)$ whose metric cone $\left(C(M), r^{2} g+d r^{2}, J\right)$ is a Kähler manifold, where $C(M):=$ $M \times \mathbf{R}_{+}=\left\{(x, r) \mid x \in M, r \in \mathbf{R}_{+}\right\}$.

Now we check if there exists a complex structure on $C(M)$. There is a contact metric structure $(\Phi, \xi, \eta, g)$ on Sasakian $M$ where $\Phi$ is a field of endomorphisms of $T M, \xi$ is a Killing vector field and $\eta$ is a 1-form satisfying

$$
\begin{aligned}
\eta(\xi) & =1 \\
\Phi^{2} & =-I+\eta \otimes \xi
\end{aligned}
$$

We denote a vector field on $C(M)$ by $\left(X, f \frac{\partial}{\partial r}\right)$ where $X$ is tangent to $M$ and $f$ is a $C^{\infty}$ function on $C(M)$. Then we define a field of endomorphisms of $T C(M)$ by

$$
J\left(X, f \frac{\partial}{\partial r}\right):=\left(\Phi X-f \xi, \eta(X) \frac{\partial}{\partial r}\right)
$$

It is easy to check that $J^{2}=-I$. Since $J$ is integrable [8], it follows that $J$ is a complex structure on $C(M)$.

DEFINITION 2.2. A Sasaki-Einstein manifold is a $(2 n-1)$-dimentional Riemannian manifold $(M, g)$ whose metric cone $\left(C(M), r^{2} g+d r^{2}, J\right)$ is a Ricci-flat (i.e. Ricci curvature $=0)$ Kähler manifold.

We recall the definition of Ricci curvature;

$$
\operatorname{Ric}(X, Y):=\operatorname{Tr}(Z \rightarrow R(Z, X) Y)
$$

where $R$ is the curvature tensor of the metric $r^{2} g+d r^{2}$,

$$
R(Z, X) Y:=\nabla_{Z} \nabla_{X} Y-\nabla_{X} \nabla_{Z} Y-\nabla_{[Z, X]} Y
$$

Note that $R$ is a tri-linear map, and ${ }^{\prime} T r^{\prime}$ is the trace of the linear map $Z \rightarrow R(Z, X) Y$ for any given $X$ and $Y$.

EXAMPLE 2.3. An odd-dimentional sphere $S^{2 n-1}$ with induced metric $g_{0}$ from $\mathbf{C}^{n}$ is Sasaki-Einstein, as its cone $\left(C\left(S^{2 n-1}\right), r^{2} g_{0}+d r^{2}\right)$ is isometric to $\left(\mathbf{C}^{n}, g_{s t d}\right)$, where $g_{s t d}$ is the standard Ricci-flat Kähler metric on $\mathbf{C}^{n}$.

## 3. Sasakian reduction by Boyer and Galicki

In this section, we recall the special Sasakian reduction constructed by Boyer and Galicki in [9]. In particular, they focus on $n=4$ case.

DEFINITION 3.1. Let $p, q \in \mathbf{Z}_{\geq 0}$ be coprime and $p>q$, or $p=1, q=0$. We define a moment map $\mu_{p, q}: \mathbf{C}^{4} \longrightarrow \mathbf{R}$ as follows

$$
\mu_{p, q}\left(z_{1}, z_{2}, z_{3}, z_{4}\right):=p\left|z_{1}\right|^{2}+p\left|z_{2}\right|^{2}-(p-q)\left|z_{3}\right|^{2}-(p+q)\left|z_{4}\right|^{2}
$$

and $S_{p, q}^{1}$ is the associated $S^{1}$ action on $\left(\mathbf{C}^{*}\right)^{4}$,

$$
\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \mapsto\left(z_{1} e^{i p \theta}, z_{2} e^{i p \theta}, z_{3} e^{-i(p-q) \theta}, z_{4} e^{-i(p+q) \theta}\right)
$$

Theorem 3.2. We set an inclusion $\iota$ and a projection $\pi$ as

$$
\begin{aligned}
& \iota:\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}} \hookrightarrow S^{7} \\
& \pi:\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}} \rightarrow\left(\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}}\right) / S_{p, q}^{1} .
\end{aligned}
$$

Then we have the following:

1. $\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}}$ is diffeomorphic to $S^{3} \times S^{3}$.
2. $\left(\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}}\right) / S_{p, q}^{1}$ is diffeomorphic to $S^{2} \times S^{3}$.
3. There is a Sasaki-Einstein metric $g_{p, q}$ on $\left(\left.\mu_{p, q}^{-1}(0)\right|_{S^{7}}\right) / S_{p, q}^{1}$ satisfying $\iota^{*} g_{0}=\pi^{*} g_{p, q}$ where $g_{0}$ is the induced metric on $S^{7}$ from $\mathbf{C}^{4}$ (Example 2.3).
4. Computing the case of $p=1, q=0$

Let us restrict our attention for the case of $p=1$ and $q=0$, and consider the zero level set

$$
\begin{aligned}
\left.\mu_{1,0}^{-1}(0)\right|_{S^{7}} & =\left\{\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in S^{7} ;\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}=\left|z_{3}\right|^{2}+\left|z_{4}\right|^{2}=\frac{1}{2}\right\} \\
& =S^{3}\left(\frac{1}{\sqrt{2}}\right) \times S^{3}\left(\frac{1}{\sqrt{2}}\right)
\end{aligned}
$$

For any point in $\mu_{1,0}^{-1}(0) \subset S^{3} \times S^{3}$, we identify $S^{3}$ and $S U(2)$ as follows:

$$
\left(z_{1}, z_{2}\right) \in S^{3} \leftrightarrow\left(\begin{array}{cc}
z_{1} & -\bar{z}_{2} \\
z_{2} & \bar{z}_{1}
\end{array}\right) \in S U(2)
$$

The reduced space $S^{3} \times S^{3} / S^{1}$ is diffeomorphic to $S^{2} \times S^{3}$ with a projection $\pi$ defined by,

$$
\pi\left(h_{1}, h_{2}\right):=\left(\left[h_{1}\right], h_{1}{ }^{t} h_{2}\right)
$$

where $h_{1}, h_{2} \in S U(2)$ and $[\cdot]$ is the equivalence class $\sim$ given by

$$
h_{1} \sim h_{2} \Leftrightarrow h_{2}=h_{1}\left(\begin{array}{cc}
e^{i \theta} & 0 \\
0 & e^{-i \theta}
\end{array}\right) .
$$

This equivalence relation is the same as in the definition of the projective space $\mathbf{C} P^{1}$. In complex coordinates, $\pi$ is given explicitly by

$$
\pi\left(z_{1}, z_{2}, z_{3}, z_{4}\right):=\left(2 z_{1} \bar{z}_{2},\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, z_{1} z_{3}+\bar{z}_{2} \bar{z}_{4}, z_{2} z_{3}-\bar{z}_{1} \bar{z}_{4}\right)
$$

Then we have a left $S U(2) \times S U(2)$ action $\phi=\left(\phi_{1}, \phi_{2}\right)$ on $S^{3} \times S^{3}$,

$$
\phi\left(h_{1}, h_{2}\right):=\left(\phi_{1} h_{1}, \phi_{2} h_{2}\right) \quad\left(\phi_{1}, \phi_{2} \in S U(2)\right) .
$$

Let us define a $(S U(2) \times S U(2)) / U(1)$ action $\tilde{\phi}=\left(\left[\tilde{\phi}_{1}\right], \tilde{\phi}_{2}\right)$ on $S^{2} \times S^{3}$ as follows

$$
\tilde{\phi}\left(\left[h_{1}\right], h_{1}{ }^{t} h_{2}\right):=\left(\left[\tilde{\phi}_{1} h_{1}\right], \tilde{\phi}_{1} h_{1}{ }^{t} h_{2}{ }^{t} \tilde{\phi}_{2}\right) \quad\left(\tilde{\phi}_{1}, \tilde{\phi}_{2} \in S U(2)\right),
$$

such that $\phi$ induces $\tilde{\phi}$, and $\pi$ is ( $\phi, \tilde{\phi}$ )-equivariant:

$$
\begin{array}{ccc}
S^{3} \times S^{3} & \xrightarrow{\phi} & S^{3} \times S^{3} \\
\pi \downarrow & & \downarrow \pi \\
S^{2} \times S^{3} & \xrightarrow{\tilde{\phi}} & S^{2} \times S^{3} .
\end{array}
$$

Since $S^{2} \times S^{3}$ is a homogeneous space for $(S U(2) \times S U(2)) / U(1)$, we can define an inner product $\langle\cdot, \cdot\rangle_{o}$ on $T_{o}\left(S^{2} \times S^{3}\right)$, where $o$ is written with an unit matrix $I_{2}$,

$$
o:=(0,0,-1,1,0,0,0)=\left(\left[I_{2}\right], I_{2}\right)=\pi\left(I_{2}, I_{2}\right)=\pi(1,0,0,0,1,0,0,0)
$$

for the Sasaki-Einstein metric $g_{1,0}$. By Theorem 3.2, the inner product $\langle\cdot, \cdot\rangle_{o}$ satisfies a condition:

$$
\begin{aligned}
& d \pi\left(\left\{\text { an orthonormal basis of } T_{\left(I_{2}, I_{2}\right)}\left(S^{3} \times S^{3}\right)\right\}\right) \\
& \quad=\left\{\text { an orthonormal basis of } T_{o}\left(S^{2} \times S^{3}\right)\right\}
\end{aligned}
$$

By this, if we choose $\left\{\frac{\partial}{\partial s_{2}}, \frac{\partial}{\partial s_{3}}, \frac{\partial}{\partial s_{4}}, \frac{\partial}{\partial s_{6}}, \frac{\partial}{\partial s_{7}}, \frac{\partial}{\partial s_{8}}\right\}$ an orthonormal basis of $T_{\left(I_{2}, I_{2}\right)}\left(S^{2} \times\right.$ $S^{3}$ ), thus

$$
\begin{aligned}
& \left\{d \pi\left(\frac{\partial}{\partial s_{2}}\right)=d \pi\left(\frac{\partial}{\partial s_{6}}\right)=\left(\frac{\partial}{\partial x_{5}}\right)_{o}, d \pi\left(\frac{\partial}{\partial s_{3}}\right)=2\left(\frac{\partial}{\partial x_{1}}\right)_{o}+\left(\frac{\partial}{\partial x_{6}}\right)_{o}\right. \\
& \left.d \pi\left(\frac{\partial}{\partial s_{4}}\right)=2\left(\frac{\partial}{\partial x_{2}}\right)_{o}+\left(\frac{\partial}{\partial x_{7}}\right)_{o}, d \pi\left(\frac{\partial}{\partial s_{7}}\right)=-\left(\frac{\partial}{\partial x_{6}}\right)_{o}, d \pi\left(\frac{\partial}{\partial s_{8}}\right)=\left(\frac{\partial}{\partial x_{7}}\right)_{o}\right\}
\end{aligned}
$$

is an orthonormal basis of $T_{o}\left(S^{2} \times S^{3}\right)$. Then the metric $g_{o}(\cdot, \cdot)=\langle\cdot, \cdot\rangle_{o}$ defined by

$$
\left(\left\langle\left(\frac{\partial}{\partial x_{i}}\right)_{o},\left(\frac{\partial}{\partial x_{j}}\right)_{o}\right\rangle_{o}\right)_{i j}=\left(\begin{array}{ccccc}
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 1 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 1 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 1
\end{array}\right), \quad(i, j=1,2,5,6,7)
$$

Choosing the local chart $\left(U_{0}, \psi_{0}\right)$ such that

$$
\begin{aligned}
& U_{0}=\left\{\left(x_{1}, \ldots, x_{7}\right) \in S^{2} \times S^{3} ; x_{3}<0, x_{4}>0\right\} \\
& \psi_{0}:\left(x_{1}, \ldots, x_{7}\right) \mapsto\left(x_{1}, x_{2}, x_{5}, x_{6}, x_{7}\right)
\end{aligned}
$$

we extend this metric to any point $x:=\left(\left[k_{1}\right], k_{2}\right)$ by another $(S U(2) \times S U(2)) / U(1)$ action on $S^{2} \times S^{3}$ : for $k=\left(k_{1}, k_{2}\right)$,

$$
k\left(\left[h_{1}\right], h_{2}\right):=\left(\left[k_{1} h_{1}\right], k_{1} h_{2} k_{1}^{-1} k_{2}\right),
$$

noting that $x=k \cdot o$. We define the metric $g$ at $x$ by

$$
g_{x}(u, v):=g_{0}\left(d k^{-1}(u), d k^{-1}(v)\right) \quad\left(u, v \in T_{x}\left(S^{2} \times S^{3}\right)\right) .
$$

For $y=\left(y_{1}, y_{2}, y_{3}, y_{4}, y_{5}, y_{6}, y_{7}\right) \in U_{o}$, we can write $k^{-1}$ as

$$
\begin{aligned}
k^{-1}(y)= & \left(k_{1}^{-1}(y), k_{2}^{-1}(y), k_{3}^{-1}(y), k_{4}^{-1}(y), k_{5}^{-1}(y), k_{6}^{-1}(y), k_{7}^{-1}(y)\right) \\
= & \left(\frac{\left(1-x_{3}-x_{1}^{2}\right) y_{1}-x_{1} x_{2} y_{1}+x_{1}\left(1-x_{3}\right) y_{3}}{1-x_{3}}, \frac{-x_{1} x_{2} y_{1}+\left(1-x_{3}-x_{2}^{2}\right) y_{2}+x_{2}\left(1-x_{3}\right) y_{3}}{1-x_{3}},\right. \\
& -x_{1} y_{1}-x_{2} y_{2}-x_{3} y_{3}, \frac{x_{1} Y_{1}+x_{2} Y_{2}+x_{3} y_{3}+x_{4} Y_{4}}{2\left(1-x_{3}\right)}, \frac{-x_{2} Y_{1}-x_{1} y_{2}+x_{4} y_{3}-x_{3} Y_{4}}{2\left(1-x_{3}\right)}, \\
& \left.\frac{-x_{3} Y_{1}-X_{4} Y_{2}-X_{1} Y_{3}+x_{2} Y_{4}}{2\left(1-x_{3}\right)}, \frac{-x_{4} Y_{1}+x_{3} Y_{2}-x_{2} Y_{3}-x_{1} Y_{4}}{2\left(1-x_{3}\right)}\right), \text { where } \\
& X_{1}=\left(1-x_{3}\right) x_{4}+x_{1} x_{6}+x_{2} x_{7}, X_{2}=x_{2} x_{6}-x_{1} x_{7}-\left(1-x_{3}\right) x_{5}, \\
& X_{3}=x_{1} x_{4}-x_{2} x_{5}-\left(1-x_{3}\right) x_{6}, X_{4}=x_{1} x_{5}+x_{2} x_{4}-\left(1-x_{3}\right) x_{7}, \\
& Y_{1}=\left(1-x_{3}\right) y_{4}+x_{1} y_{6}+x_{2} y_{7}, Y_{2}=x_{2} y_{6}-x_{1} y_{7}-\left(1-x_{3}\right) y_{5}, \\
& Y_{3}=x_{1} y_{4}-x_{2} y_{5}-\left(1-x_{3}\right) y_{6}, Y_{4}=x_{1} y_{5}+x_{2} y_{4}-\left(1-x_{3}\right) y_{7} .
\end{aligned}
$$

Next we calculate $g_{x}\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{5}}\right)$. Let us first consider the derivation $d k^{-1}$,

$$
\begin{aligned}
d k^{-1}\left(\frac{\partial}{\partial x_{1}}\right)= & \frac{\partial k_{1}^{-1}}{\partial y_{1}}(x)\left(\frac{\partial}{\partial x_{1}}\right)_{o}+\frac{\partial k_{2}^{-1}}{\partial y_{1}}(x)\left(\frac{\partial}{\partial x_{2}}\right)_{o}+\frac{\partial k_{5}^{-1}}{\partial y_{1}}(x)\left(\frac{\partial}{\partial x_{5}}\right)_{o}+\frac{\partial k_{6}^{-1}}{\partial y_{1}}(x)\left(\frac{\partial}{\partial x_{6}}\right)_{o}+\frac{\partial k_{7}^{-1}}{\partial y_{1}}(x)\left(\frac{\partial}{\partial x_{7}}\right)_{o} \\
= & \frac{x_{2}^{2}+x_{3}-1}{x_{3}\left(1-x_{3}\right)}\left(\frac{\partial}{\partial x_{1}}\right)_{o}+\frac{-x_{1} x_{2}}{x_{3}\left(1-x_{3}\right)}\left(\frac{\partial}{\partial x_{2}}\right)_{o} \text { and } \\
d k^{-1}\left(\frac{\partial}{\partial x_{5}}\right)= & \frac{\partial k_{1}^{-1}}{\partial y_{5}}(x)\left(\frac{\partial}{\partial x_{1}}\right)_{o}+\frac{\partial k_{2}^{-1}}{\partial y_{5}}(x)\left(\frac{\partial}{\partial x_{2}}\right)_{o}+\frac{\partial k_{5}^{-1}}{\partial y_{5}}(x)\left(\frac{\partial}{\partial x_{5}}\right)_{o}+\frac{\partial k_{6}^{-1}}{\partial y_{5}}(x)\left(\frac{\partial}{\partial x_{6}}\right)_{o}+\frac{\partial k_{7}^{-1}}{\partial y_{5}}(x)\left(\frac{\partial}{\partial x_{7}}\right)_{o} \\
= & \frac{-x_{3}\left(x_{4}^{2}+x_{5}^{2}\right)+x_{1}\left(x_{4} x_{6}+x_{5} x_{7}\right)+x_{2}\left(x_{4} x_{7}-x_{5} x_{6}\right)}{x_{4}}\left(\frac{\partial}{\partial x_{5}}\right)_{o} \\
& +\frac{\left(1-x_{3}-x_{2}^{2}\right)\left(x_{5} x_{6}-x_{4} x_{7}\right)+\left(1-x_{3}\right) x_{2}\left(x_{4}^{2}+x_{5}^{2}\right)+x_{1} x_{2}\left(x_{4} x_{6}+x_{5} x_{7}\right)}{\left(1-x_{3}\right) x_{4}}\left(\frac{\partial}{\partial x_{6}}\right)_{o} \\
& +\frac{\left(1-x_{3}-x_{1}^{2}\right)\left(x_{4} x_{6}+x_{5} x_{7}\right)-\left(1-x_{3}\right) x_{1}\left(x_{4}^{2}+x_{5}^{2}\right)+x_{1} x_{2}\left(x_{5} x_{6}-x_{4} x_{7}\right)}{\left(1-x_{3}\right) x_{4}}\left(\frac{\partial}{\partial x_{7}}\right)_{o} .
\end{aligned}
$$

Then the coeffcient of $d x_{1} d x_{5}$ and $d x_{5} d x_{1}$ is given by

$$
\begin{aligned}
g_{x}\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{5}}\right) & =g_{0}\left(d k^{-1}\left(\frac{\partial}{\partial x_{1}}\right), d k^{-1}\left(\frac{\partial}{\partial x_{5}}\right)\right) \\
& =\frac{-\left(x_{1}^{2}+x_{3}^{2}\right)\left(x_{4} x_{7}-x_{5} x_{6}\right)-x_{2} x_{3}\left(x_{4}^{2}+x_{5}^{2}\right)+x_{1} x_{2}\left(x_{4} x_{6}+x_{5} x_{7}\right)}{2 x_{3} x_{4}} .
\end{aligned}
$$

Also we can find the coefficient of $d x_{i} d x_{j}$ and $d x_{j} d x_{i}$ by calculating $g_{x}\left(\frac{\partial}{\partial x_{i}}, \frac{\partial}{\partial x_{j}}\right)$. As the result, with the local coordinates $x=\left(x_{1}, x_{2}, x_{5}, x_{6}, x_{7}\right)$ on $U_{0}$, we have the formula:

$$
\begin{align*}
g_{x}= & \sum_{i=1}^{2} \frac{x_{i}^{2}+x_{3}^{2}}{2 x_{3}^{2}} d x_{i}^{2}+\frac{x_{1} x_{2}}{x_{3}^{2}} d x_{1} d x_{2}+\sum_{i=5}^{7} \frac{x_{4}^{2}+x_{i}^{2}}{x_{4}^{2}} d x_{i}^{2} \\
& +\frac{2 x_{5} x_{6}}{x_{4}^{2}} d x_{5} d x_{6}+\frac{2 x_{5} x_{7}}{x_{4}^{2}} d x_{5} d x_{7}+\frac{2 x_{6} x_{7}}{x_{4}^{2}} d x_{6} d x_{7} \\
& +\frac{-\left(x_{1}^{2}+x_{3}^{2}\right)\left(x_{4} x_{7}-x_{5} x_{6}\right)-x_{2} x_{3}\left(x_{4}^{2}+x_{5}^{2}\right)+x_{1} x_{2}\left(x_{4} x_{6}+x_{5} x_{7}\right)}{x_{3} x_{4}} d x_{1} d x_{5} \\
& +\frac{\left(x_{1}^{2}+x_{3}^{2}\right)\left(x_{4}^{2}+x_{6}^{2}\right)-x_{1} x_{2}\left(x_{4} x_{5}-x_{6} x_{7}\right)-x_{2} x_{3}\left(x_{4} x_{7}+x_{5} x_{6}\right)}{x_{3} x_{4}} d x_{1} d x_{6} \\
& +\frac{x_{1} x_{2}\left(x_{4}^{2}+x_{7}^{2}\right)+\left(x_{1}^{2}+x_{3}^{2}\right)\left(x_{4} x_{5}+x_{6} x_{7}\right)+x_{2} x_{3}\left(x_{4} x_{6}-x_{5} x_{7}\right)}{x_{3} x_{4}} d x_{1} d x_{7}  \tag{1}\\
& +\frac{x_{1} x_{3}\left(x_{4}^{2}+x_{5}^{2}\right)-x_{1} x_{2}\left(x_{4} x_{7}-x_{5} x_{6}\right)+\left(x_{2}^{2}+x_{3}^{2}\right)\left(x_{4} x_{6}+x_{5} x_{7}\right)}{x_{3}} d x_{2} d x_{5} \\
& +\frac{x_{1} x_{2}\left(x_{4}^{2}+x_{6}^{2}\right)+x_{1} x_{3}\left(x_{4} x_{7}+x_{5} x_{6}\right)-\left(x_{2}^{2}+x_{3}^{2}\right)\left(x_{4} x_{5}-x_{6} x_{7}\right)}{x_{3} x_{4}} d x_{2} d x_{6} \\
& +\frac{\left(x_{2}^{2}+x_{3}^{2}\right)\left(x_{4}^{2}+x_{7}^{2}\right)+x_{1} x_{2}\left(x_{4} x_{5}+x_{6} x_{7}\right)-x_{1} x_{3}\left(x_{4} x_{6}-x_{5} x_{7}\right)}{x_{3} x_{4}} d x_{2} d x_{7}
\end{align*}
$$

On other open sets $U_{i^{ \pm} j^{ \pm}}$of $S^{2} \times S^{3}$ defined by for $i \in\{1,2,3\}, j \in\{4,5,6,7\}$, i.e.

$$
\begin{aligned}
& U_{i^{+} j^{+}}=\left\{x_{i}>0, x_{j}>0\right\}, U_{i^{-} j^{+}}=\left\{x_{i}<0, x_{j}>0\right\} \\
& U_{i^{+} j^{-}}=\left\{x_{i}>0, x_{j}<0\right\} \text { and } U_{i^{-} j^{-}}=\left\{x_{i}<0, x_{j}<0\right\},
\end{aligned}
$$

we can calculate the metric the same way as the previous case. This is an explicit representation at $x$ of the Sasaki-Einstein metric $g_{1,0}$ called the homogeneous Kobayashi-Tanno metric by Boyer and Galicki in [9].

THEOREM 4.1. The Sasaki-Einstein metric $g_{1,0}$ on $S^{2} \times S^{3}$ at any point $x$ is given by the fomula (1).

## References

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## Present Address:

 Department of Mathematics, Keio University, Hiyoshi, KOHOKU-Ku, YoKOHAMA, 223-8532 Japan.e-mail: imaddd@a6.keio.jp


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