Korean J. Math. **29** (2021), No. 3, pp. 547–554
http://dx.doi.org/10.11568/kjm.2021.29.3.547

SASAKIAN 3-MANIFOLDS ADMITTING A GRADIENT RICCI-YAMABE SOLITON

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Abstract. The object of the present paper is to characterize Sasakian 3-manifolds admitting a gradient Ricci-Yamabe soliton. It is shown that a Sasakian 3-manifold M with constant scalar curvature admitting a proper gradient Ricci-Yamabe soliton is Einstein and locally isometric to a unit sphere. Also, the potential vector field is an infinitesimal automorphism of the contact metric structure. In addition, if M is complete, then it is compact.

1. Introduction

In 1982, the concept of Ricci flow was introduced by Hamilton [11]. The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M^n, g) given by

$$\frac{\partial g}{\partial t} = -2S,$$

where g is the Riemannian metric and S denotes the $(0,2)$-symmetric Ricci tensor.

The notion of Yamabe flow was proposed by Hamilton [13] in 1989, which is defined on a Riemannian manifold (M^n, g) as

$$\frac{\partial g}{\partial t} = -rg,$$

where r is the scalar curvature of the manifold.

In 2019, Güler and Crasmareanu [10] consider a scalar combination of the Ricci flow and the Yamabe flow and introduced the notion of the Ricci-Yamabe flow on a Riemannian manifold (M^n, g) as

$$\frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t) g(t) = 0,$$

where g is the Riemannian metric, S is the $(0,2)$-symmetric Ricci tensor, r is the scalar curvature and α, β are two constants. Since α and β are arbitrary constants, we can choose the signs of α and β according to our choice. This freedom of choice of the signs of α and β is very useful in differential geometry and theory of relativity. Recently

Received December 15, 2020. Revised August 22, 2021. Accepted August 25, 2021.
2010 Mathematics Subject Classification: 53C25, 35Q51.
Key words and phrases: Sasakian 3-manifold, Ricci-Yamabe soliton, Gradient Ricci-Yamabe soliton, Infinitesimal automorphism.

in [1] and [4], the authors used a bi-metric approach of the space-time geometry. Recently, the notion of η-Ricci-Yamabe soliton [21], Ricci-Yamabe soliton and gradient Ricci-Yamabe soliton [8] were introduced from the Ricci-Yamabe flow. The Ricci-Yamabe soliton is defined on a Riemannian manifold as follows:

DEFINITION 1.1. A Riemannian manifold (M^n, g), $n > 2$ is said to admit a Ricci-Yamabe soliton (in short, RYS) $(g, V, \lambda, \alpha, \beta)$ if

$$\pounds_V g + 2\alpha S = (2\lambda - \beta r)g, \tag{1.1}$$

where $\lambda,\ \alpha,\ \beta \in \mathbb{R}$.

If V is gradient of some smooth function f on M, then the above notion is called a gradient Ricci-Yamabe solition (in short, GRYS) and then (1.1) reduces to

$$\nabla^2 f + \alpha S = (\lambda - \frac{1}{2}\beta r)g, \tag{1.2}$$

where $\nabla^2 f$ is the Hessian of f.

The GRYS is said to be expanding, steady or shrinking according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. The above notion generalizes a large class of soliton like equations. A GRYS is said to be a

- gradient Ricci soliton (see [12]) if $\alpha = 1$, $\beta = 0$.
- gradient Yamabe soliton (see [13] if $\alpha = 0$, $\beta = 2$.
- gradient Einstein soliton (see [6]) if $\alpha = 1$, $\beta = -1$.
- gradient ρ-Einstein soliton (see [7]) if $\alpha = 1$, $\beta = -2\rho$.

The GRYS is said to be proper if $\alpha \neq 0,\ 1$.

Sasakian geometry is an odd dimensional analogue of the Kaehler geometry. The notion of Sasakian manifolds were firstly studied by Sasaki [20]. The Kaehler cone over a Sasakian Einstein manifolds has application in superstring theory (see [5], [16]). Since then, Sasakian geometry has been widely studied as it perceived relevance in string theory. In [18], Sharma showed that a K-contact metric satisfying a gradient Ricci soliton is Einstein. Further in [19], the author studied a 3-dimensional Sasakian metric as Yamabe soliton and proved that either the manifold has constant curvature 1 or the potential vector field is an infinitesimal automorphism of the contact metric structure . In 2019, Venkatesha and Naik [23] studied the notion of the Yamabe soliton on 3-dimensional contact metric manifolds under certain condition. In [9], Ghosh and Sharma studied Sasakian 3-metric as a Ricci soliton and identify the Sasakian metric on the Heisenberg group as a non-trivial solution. Motivated by the above studies, we consider a proper GRYS in the framework of three dimensional Sasakian manifolds with constant scalar curvature and proved some related results.

2. Preliminaries

An odd dimensional differentiable manifold M is said to be an almost contact metric manifold if it admits a structure (ϕ, ξ, η, g) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y on M, where ϕ is a $(1,1)$-tensor field, ξ is a unit vector field called the Reeb vector field, η is a 1-form defined by $\eta(X) = g(X, \xi)$ and g is the Riemannian metric. Using (2.2), we can easily see that

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M. An almost contact metric manifold with $d\eta = \Phi$ is called a contact metric manifold. If the Reeb vector field ξ is Killing type, then a contact metric manifold is called a K-contact manifold and if the structure (ϕ, ξ, η, g) is normal, then a contact metric manifold is called Sasakian. Also, an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi Y) = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

for any vector fields X, Y on M. A Sasakian manifold is K-contact but the converse holds only in dimension 3. It may not be true for higher dimension (see [14]). On a 3-dimensional Sasakian manifold, the following relations are well known:

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$S(X, \xi) = 2\eta(X), \quad Q\xi = 2\xi, \tag{2.9}$$

where R, Q and S denotes the Riemann curvature tensor, the Ricci operator and the Ricci tensor respectively which is defined as $S(X, Y) = g(QX, Y)$. Since a 3-dimensional Riemannian manifold is conformally flat, it's curvature tensor can be expressed as

$$\begin{aligned} R(X, Y)Z &= [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.10}$$

where r is the scalar curvature defined by $r = \sum S(e_i, e_i) = \sum g(Qe_i, e_i)$ for any orthonormal basis $\{e_i\}$ of the tangent space at any point of M. Now, the Ricci tensor for a Sasakian 3-manifold can be obtained from here as

$$S(X, Y) = \frac{1}{2}[(r - 2)g(X, Y) + (6 - r)\eta(X)\eta(Y)]. \tag{2.11}$$

For further details on Sasakian geometry, we refer the reader to go through the references ([2], [3], [19]).

3. Gradient Ricci-Yamabe Solitons

We now consider the notion of a proper GRYS in the framework of Sasakian 3-manifolds with constant scalar curvature. For existence of Sasakian 3-manifolds with constant scalar curvature, see example in [15]. To prove our first theorem regarding a GRYS, we need the followings:

Definition 3.1. ([22]) A vector field X is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensors ϕ, ξ, η, g invariant.

Lemma 3.2. *On a Sasakian 3-manifold M with constant scalar curvature, the following relation holds*

$$(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ = (6-r)[\eta(Y)g(\phi X,Z) + \eta(X)g(\phi Y,Z)].$$

Proof. Differentiating (2.11) covariantly along any vector field Z, we obtain

$$(\nabla_Z S)(X,Y) = \frac{1}{2}(6-r)[\eta(Y)(\nabla_Z\eta)X + \eta(X)(\nabla_Z\eta)Y].$$

Using (2.6) in the foregoing equation yields

$$(\nabla_Z S)(X,Y) = \frac{1}{2}(6-r)[\eta(Y)g(Z,\phi X) + \eta(X)g(Z,\phi Y)]. \tag{3.1}$$

In a similar manner, we obtain

$$(\nabla_X S)(Y,Z) = \frac{1}{2}(6-r)[\eta(Z)g(X,\phi Y) + \eta(Y)g(X,\phi Z)]. \tag{3.2}$$

$$(\nabla_Y S)(X,Z) = \frac{1}{2}(6-r)[\eta(Z)g(Y,\phi X) + \eta(X)g(Y,\phi Z)]. \tag{3.3}$$

Using (3.1)-(3.3) and (2.3), we compute

$$(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ = (6-r)[\eta(Y)g(\phi X,Z) + \eta(X)g(\phi Y,Z)]. \tag{3.4}$$

This comoletes the proof. □

Theorem 3.3. *Let $(g,V,\lambda,\alpha,\beta)$ be a proper GRYS on a Sasakian 3-manifold M with constant scalar curvature. Then*

(1) *M is Einstein.*
(2) *M is locally isometric to a unit sphere.*
(3) *the potential vector field V is an infinitesimal automorphism of the contact metric structure.*
(4) *if M is complete, then it is compact.*

Proof. Let V be the gradient of a non-zero smooth function $f: M \to \mathbb{R}$, that is, $V = Df$, where D is the gradient operator. Then from (1.2), we can write

$$\nabla_X Df = (\lambda - \frac{1}{2}\beta r)X - \alpha QX \tag{3.5}$$

for any vector field X on M. With the help of (3.5), we can easily obtain

$$R(X,Y)Df = \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y]. \tag{3.6}$$

Substituting $X = \xi$ in (3.6) and then taking inner product with ξ yields

$$g(R(\xi,Y)Df,\xi) = \alpha[(\nabla_Y S)(\xi,\xi) - (\nabla_\xi S)(Y,\xi)].$$

With the help of (3.2) and (3.3), we can easily see that

$$g(R(\xi,Y)Df,\xi) = 0. \tag{3.7}$$

Since $g(R(\xi,Y)Df,\xi) = -g(R(\xi,Y)\xi,Df)$, then using (2.7), we obtain

$$g(R(\xi,Y)Df,\xi) = (Yf) - (\xi f)\eta(Y). \tag{3.8}$$

Equating (3.7) and (3.8), we have

$$(Yf) - (\xi f)\eta(Y) = 0,$$

which implies

$$V = Df = (\xi f)\xi. \tag{3.9}$$

This shows that V is pointwise collinear with ξ. For simplicity, we write $V = b\xi$, where $b = (\xi f)$ is some smooth function. Now using (2.3) and (2.5), we obtain

$$(\pounds_V g)(X,Y) = (\pounds_{b\xi} g)(X,Y) = (Xb)\eta(Y) + (Yb)\eta(X). \tag{3.10}$$

Using (3.10), we get from (1.1)

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2\alpha S(X,Y) = (2\lambda - \beta r)g(X,Y). \tag{3.11}$$

Substituting $X = Y = \xi$ in (3.11) and using (2.9), we obtain

$$2(\xi b) = 2\lambda - \beta r - 4\alpha. \tag{3.12}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at any point of M. Now, substituting $X = Y = e_i$ in (3.11) and then summing over i yields

$$2(\xi b) = 3(2\lambda - \beta r) - 2\alpha r. \tag{3.13}$$

Equating (3.12) and (3.13), we get

$$(2\lambda - \beta r - 4\alpha) + \alpha(6 - r) = 0. \tag{3.14}$$

Now from (1.1), we have

$$(\pounds_V g)(X,Y) + 2\alpha S(X,Y) = [2\lambda - \beta r]g(X,Y). \tag{3.15}$$

Differentiating the previous equation covariantly along any vector field Z, we obtain

$$(\nabla_Z \pounds_V g)(X,Y) = -2\alpha(\nabla_Z S)(X,Y). \tag{3.16}$$

Due to Yano [24], the following commutation formula

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).$$

leads to

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) \;=\; & \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ & -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned}$$

Using (3.16) in the forgoing formula and then applying lemma 3.2, we obtain

$$g((\pounds_V \nabla)(X,Y),Z) = \alpha(6 - r)[\eta(Y)g(\phi X, Z) + \eta(X)g(\phi Y, Z)],$$

which implies

$$(\pounds_V \nabla)(X,Y) = \alpha(6 - r)[\eta(Y)\phi X + \eta(X)\phi Y]. \tag{3.17}$$

Putting $Y = \xi$ in (3.17), we get

$$(\pounds_V \nabla)(X,\xi) = \alpha(6 - r)\phi X. \tag{3.18}$$

Differentiating (3.18) covariantly along any vector field Z, then using (3.17)-(3.18) and (2.1)-(2.5) in (3.12), we obtain

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = \alpha(6 - r)[g(X,Y)\xi - 2\eta(X)Y + \eta(X)\eta(Y)\xi]. \tag{3.19}$$

Now, it is well known that (see [24])

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

We now use (3.19) in the foregoing equation to obtain

$$(\pounds_V R)(X,\xi)\xi = -2\alpha(6-r)(X-\eta(X)\xi). \tag{3.20}$$

Now, substituting $Y=\xi$ in (3.15) and using (2.9), we get

$$(\pounds_V g)(X,\xi) = [2\lambda - \beta r - 4\alpha]\eta(X),$$

which implies

$$(\pounds_V \eta)X - g(X, \pounds_V \xi) = [2\lambda - \beta r - 4\alpha]\eta(X). \tag{3.21}$$

Setting $X=\xi$ in (3.21), we obtain

$$\eta(\pounds_V \xi) = -\frac{1}{2}[2\lambda - \beta r - 4\alpha]. \tag{3.22}$$

From (2.7), we write

$$R(X,\xi)\xi = X - \eta(X)\xi.$$

Lie differentiating the above equation and using (3.21)-(3.22)) and (2.7)-(2.8), we obtain

$$(\pounds_V R)(X,\xi)\xi = [2\lambda - \beta r - 4\alpha](X - \eta(X)\xi). \tag{3.23}$$

Equating (3.20) and (3.23), we infer that

$$(2\lambda - \beta r - 4\alpha) + 2\alpha(6-r) = 0. \tag{3.24}$$

Using (3.14) in (3.24) yields

$$\alpha(6-r) = 0. \tag{3.25}$$

Since the GRYS is proper, then $\alpha \neq 0$ and hence $r=6$. Therefore, from (2.11), we get

$$S(X,Y) = 2g(X,Y), \tag{3.26}$$

which implies that the manifold M is Einstein. This proves (1).

Now using (3.26) in (2.10), we can easily obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{3.27}$$

This shows that the manifold is of constant curvature 1, that is, locally isometric to a unit sphere. This proves (2).

Using (3.25) in (3.24), we get

$$2\lambda - \beta r - 4\alpha = 0. \tag{3.28}$$

Using (3.26) and (3.28) in (3.15), we obtain $(\pounds_V g)(X,Y) = 0$, for any vector fields X, Y on M. This proves that V is a Killing vector field or V leaves the metric tensor invariant. Applying (3.28) in (3.12) yields $(\xi b) = 0$. Using $\pounds_V g = 0$ and $(\xi b) = 0$ in (3.10), we obtain $(Xb) = 0$, for any vector field X on M, which implies b is a constant. Therefore, V is a constant multiple of ξ. Now, it can be easily calculated that $\pounds_V \xi = 0$, that is, V leaves the Reeb vector field invariant. Applying (3.28) and $\pounds_V \xi = 0$ in (3.21) yields $(\pounds_V \eta)X = 0$ for any vector field X on M. This shows that

V leaves η invariant or V is a strict infinitesimal contact transformation. Also, using (2.5), we can easily obtain $\pounds_V \phi = 0$, that is, V leaves ϕ invariant. Hence, V leaves the structure (ϕ, ξ, η, g) invariant. This proves (3).

Since the Ricci curvature $r = 6 > 0$, then by Myers theorem [17], if M is complete, then it is necessarily compact. This proves (4). $\square$

REMARK 3.4. We have obtained $r = 6$ and $2\lambda - \beta r - 4\alpha = 0$. These two together implies $\lambda = 2\alpha + 3\beta$. Therefore, the GRYS is expanding, steady or shrinking according as $(2\alpha + 3\beta)$ is negative, zero or positive respectively.

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