Tôhoku Math. Journ.
21(1969), 644-653.

SASAKIAN HYPERSURFACES IN A SPACE OF CONSTANT CURVATURE

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(Received August 1, 1969)

1. Introduction. In the previous paper [10], we defined a Sasakian manifold with pseudo-Riemannian metric, and proved that a Sasakian manifold M^{2n+1} which is properly and isometrically immersed in a pseudo-Riemannian manifold $\widetilde{M}^{2n+2}$ of constant curvature zero is of constant curvature one. It is an extension of Corollary for Theorem 2 in Tashiro-Tachibana [13]. In this paper, we prove that a Sasakian manifold M^{2n+1} (with a pseudo-Riemannian metric) which is properly and isometrically immersed in a pseudo-Riemannian manifold $\widetilde{M}^{2n+2}$ of constant curvature $\tilde{c}\neq 1$ is of constant curvature 1 (Theorem 1). In the case when $\tilde{c}=1$, we need an additional condition; namely, the Sasakian manifold to be η-Einstein, then it is of constant curvature 1 (Theorem 4). Some related results on almost contact hypersurfaces are found in Tashiro [12], Kurita [3], Tashiro-Tachibana [13] and Okumura [6], [7].

In this paper, we call a Sasakian manifold with pseudo-Riemannian metric to be a pseudo-Sasakian manifold.

I would like to express my hearty thanks to Professor S. Sasaki and Professor S. Tanno for their valuable advices.

2. Let $(\widetilde{M}^{2n+2}, \tilde{g})$ be a pseudo-Riemannian manifold of constant curvature $\tilde{c}$. Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold which is isometrically immersed in $(\widetilde{M}^{2n+2}, \tilde{g})$. Then we have the formulas of Gauss and Codazzi:

$$R(X, Y) = \tilde{c}X \wedge Y + \varepsilon AX \wedge AY\,, \tag{1}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = 0\,, \tag{2}$$

where $X \wedge Y$ denotes an endomorphism $Z \rightarrow g(Y, Z)X - g(X, Z)Y$, A is the field of the second fundamental form operators which corresponds to the field of unit normal vectors ζ to M^{2n+1} and $\varepsilon = \tilde{g}(\zeta, \zeta)$, $\varepsilon = +1$ or -1 (L. P. Eisenhart [1]). From (1), we get

$$R(X, \xi)Y = \tilde{c}\{\eta(Y)X - g(X, Y)\xi\} + \varepsilon\{\eta(AY)AX - g(AX, Y)A\xi\}\,. \tag{3}$$

On the other hand, we have (e. g. [10])

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi. \tag{4}$$

Now, suppose the immersion is proper, that is, A can be expressed as a real diagonal matrix with respect to a certain orthonormal basis at each point of M^{2n+1} (A. Fialkow [2]); the diagonal elements are called principal curvatures at the point and the vectors of the orthonormal basis are called principal directions. Let $\lambda_1, \lambda_2, \cdots, \lambda_{2n+1}$ be the principal curvatures at a point and let $e_1, e_2, \cdots, e_{2n+1}$ be the corresponding principal directions:

$$Ae_i = \lambda_i e_i, \qquad i = 1, 2, \cdots, 2n+1.$$

(3) and (4) imply

$$\tilde{c}\{\eta(e_j)e_i - g(e_i, e_j)\xi\} + \varepsilon\{\lambda_i\lambda_j\eta(e_j)e_i - \lambda_i g(e_i, e_j)A\xi\} = \eta(e_j)e_i - g(e_i, e_j)\xi\ . \tag{5}$$

In particular, we have

$$(\tilde{c} + \varepsilon\lambda_i\lambda_j - 1)\eta(e_j)e_i = 0 \quad \text{for} \quad i \neq j\ . \tag{6}$$

THEOREM 1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold which is properly and isometrically immersed in a pseudo-Riemannian manifold $(\widetilde{M}^{2n+2}, \tilde{g})$ of constant curvature $\tilde{c} \neq 1$. Then (M^{2n+1}, g) is totally umbilic in $(\widetilde{M}^{2n+2}, \tilde{g})$ and of constant curvature* 1.

PROOF. (6) implies

$$(\varepsilon\lambda_i\lambda_j - k)\eta(e_j)e_i = 0$$

for all $i \neq j$, where we have put $k = 1 - \tilde{c}$. There are two cases:

(I) $\varepsilon\lambda_i\lambda_j = k$ for all $i \neq j$,

(II) $\eta(e_j) = 0$ for some j.

In the case (I), since $k = 1 - \tilde{c} \neq 0$, there is just one principal curvature λ, and it satisfies $\varepsilon\lambda^2 = k$.

In the case (II), (5) with $e_i = e_j$ implies $\varepsilon\lambda_j A\xi = k\xi$. Hence $\lambda_j \neq 0$ and

$$A\xi = (\varepsilon k/\lambda_j)\xi\ . \tag{7}$$

Thus we may suppose $e_1 = \xi$ and hence $\eta(e_i) = 0$ for $i = 2, 3, \cdots, 2n+1$. Hence (7) holds good for $j=2, 3, \cdots, 2n+1$, which shows that $\lambda_2=\lambda_3=\cdots=\lambda_{2n+1}=\lambda\neq 0$ and $\lambda_1=\varepsilon k/\lambda$. Thus, together with the case (I), there are at most two distinct principal curvatures at each point, and if there are two distinct principal curvatures at a point, then it is in the case (II). Now, suppose $\lambda_1\neq\lambda$ at $x_0\in M^{2n+1}$. Then we may suppose that $\lambda_1\neq\lambda$ in a neighborhood U of x_0 and they are differentiable (cf. P. J. Ryan [8]). The distribution T_λ on U defined by

$$(T_\lambda)_y = \{X\in T_y(M^{2n+1});\ AX=\lambda X\}$$

is a contact distribution in the sense that for any $Z\in T_\lambda$, $\eta(Z)=0$; and dim $T_\lambda=2n$. For $Z, W\in T_\lambda$, using (2), we get

$$\begin{aligned} A[Z, W] &= A(\nabla_Z W-\nabla_W Z)\\ &= \nabla_Z(AW)-(\nabla_Z A)W-\nabla_W(AZ)+(\nabla_W A)Z\\ &= (Z\lambda)W+\lambda\nabla_Z W-(W\lambda)Z-\lambda\nabla_W Z\,. \end{aligned}$$

Hence we get

$$(A-\lambda)[Z, W]=(Z\lambda)W-(W\lambda)Z\,.$$

Since $(A-\lambda_1)(A-\lambda)[Z, W]=0$, the left hand side of the above equation is in $T_{\lambda_1}=\{\xi\}$ and the right hand side is in T_λ. Since we have $T_\lambda\cap T_{\lambda_1}=\{0\}$, $A[Z, W]=\lambda[Z, W]$; that is, the distribution T_λ is involutive, which contradicts to the fact dim $T_\lambda=2n$ by the following lemma:

LEMMA 1 (S. Sasaki [9]). *Let M^{2n+1} be a contact manifold. Then the highest dimension of integral submanifolds of the contact distribution D is equal to n.*

Thus $\lambda_1=\lambda$ at x_0, and hence $\varepsilon\lambda^2=k$, which shows that M^{2n+1} in consideration is totally umbilic.

$\varepsilon\lambda^2=k$ implies that the formula of Gauss becomes

$$R(X, Y)=X\wedge Y\,,$$

which shows that (M^{2n+1}, g) is of constant curvature 1. Q. E. D.

REMARK. In [10], we have proved the same theorem for $\tilde{c}=0$ and $n\geqq 2$. The above Theorem 1 says that we have the same conclusion without the assumption $n\geqq 2$.

We proceed to discuss in the case when $\tilde{c}=1$ for the special pseudo-Sasakian manifold in the following sections.

3. Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold. Let R and R_1 be the curvature tensor and the Ricci tensor for g, respectively. Then the following holds good as in the Sasakian case:

$$R_1(\xi, Z) = 2n\eta(Z)\,. \tag{8}$$

THEOREM 2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold which is properly and isometrically immersed in a pseudo-Riemannian manifold $(\widetilde{M}^{2n+2}, \tilde{g})$ of constant curvature* 1. *Moreover, if (M^{2n+1}, g) is Einstein, then it is totally geodesic or a developable hypersurface (i.e. the rank of the second fundamental form operator $A \leqq 1$ at each point of M^{2n+1}) in $(\widetilde{M}^{2n+2}, \tilde{g})$; in particular, it is of constant curvature* 1.

PROOF. (8) says that the Ricci curvature κ for g is $2n$. Hence the following lemma implies our theorem.

LEMMA 2 (A. Fialkow [2]). *Let (N^m, f), $m \geqq 3$, be a pseudo-Einstein manifold with the Ricci curvature κ which is properly and isometrically immersed in a pseudo-Riemannian manifold $(\widetilde{N}^{m+1}, \tilde{f})$ of constant curvature $\tilde{c}$. Then, if $\kappa = (m-1)\tilde{c}$, (N^m, f) is either totally geodesic or a developable hypersurface in $(\widetilde{N}^{m+1}, \tilde{f})$; in particular, it is of constant curvature $\tilde{c}$.*

K. Nomizu [4] has considered the following condition (*) on a hypersurface of the Euclidean space:

$$R(X, Y) \cdot R = 0 \qquad \text{for all tangent vectors } X \text{ and } Y; \tag{*}$$

and S. Tanno [11] has considered the condition (**):

$$R(X, Y) \cdot R_1 = 0 \qquad \text{for all tangent vectors } X \text{ and } Y. \tag{**}$$

On the other hand, the present author [10] has proved that if a pseudo-Sasakian manifold satisfies the condition (*), then it is of constant curvature 1. For a pseudo-Sasakian manifold with (**), we have the following theorem:

THEOREM 3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold with the condition (**). Then (M^{2n+1}, g) is Einstein with the Ricci curvature $\kappa = 2n$.*

PROOF. For any tangent vectors X and Y, we have

$$\begin{aligned}(R(X, \xi)\cdot R_1)(\xi, Y) &= -R_1(R(X, \xi)\xi, Y)-R_1(\xi, R(X, \xi)Y)\\ &= -R_1(X-\eta(X)\xi, Y)-R_1(\xi, \eta(Y)X-g(X, Y)\xi)\\ &= -R_1(X, Y)+\eta(X)R_1(\xi, Y)-\eta(Y)R_1(\xi, X)\\ &\quad +g(X, Y)R_1(\xi, \xi)\\ &= -R_1(X, Y)+2ng(X, Y),\end{aligned}$$

where we have used (4) and (8). Thus the condition (**) implies $R_1(X, Y) = 2ng(X, Y)$ for all tangent vectors X and Y, showing (M^{2n+1}, g) to be Einstein with the Ricci curvature $\kappa = 2n$. Q. E. D.

Combining Theorem 2 and Theorem 3, we get the following:

COROLLARY. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold which is properly and isometrically immersed in a pseudo-Riemannian manifold $(\tilde{M}^{2n+2}, \tilde{g})$ of constant curvature* 1. *Moreover, if (M^{2n+1}, g) satisfies the condition (**), then it is totally geodesic or a developable hypersurface in $(\tilde{M}^{2n+2}, \tilde{g})$; in particular, it is of constant curvature* 1.

4. In this section, we prove the following theorem:

THEOREM 4. *Suppose a pseudo-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqq 2$, is properly and isometrically immersed in a pseudo-Riemannian manifold $(\tilde{M}^{2n+2}, \tilde{g})$ of constant curvature* 1. *If $M^{2n+1}(\phi, \xi, \eta, g)$ is η-Einstein:*

$$R_1(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{9}$$

then (M^{2n+1}, g) is of constant curvature 1.

To prove it, we have to prepare several lemmas.

LEMMA 3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a pseudo-Sasakian manifold which is properly and isometrically immersed in a pseudo-Riemannian manifold $(\tilde{M}^{2n+2}, \tilde{g})$ of constant curvature* 1. *Let ζ be a field of unit normal vectors in a neighborhood U of a point of M^{2n+1}, and let A be the field of the second fundamental form operators. Let $\lambda_1, \lambda_2, \cdots, \lambda_{2n+1}$ be the principal curvatures and let $e_1, e_2, \cdots, e_{2n+1}$ be the corresponding principal directions.*

If $e_1=\xi$ *and* $\lambda_2=\lambda_3=\cdots=\lambda_{2n+1}=\lambda$ *on* U, *then* $\lambda=0$.

PROOF. Since the sectional curvature of 2-planes containing ξ is 1, (1) implies $\varepsilon\lambda_1\lambda=0$ on U. So, if $\lambda_1\neq 0$ at a point of U, then $\lambda_1\neq 0$ in a neighborhood $V\subset U$ of the point, and hence $\lambda=0$ on V. We can see that $\lambda_1\neq 0$ and $\lambda=0$ on V contradicts to Lemma 1 by the same method as in the proof of Theorem 1. Hence λ_1 must be equal to 0 on U.

Now suppose $\lambda\neq 0$ at a point of U, then $\lambda\neq 0$ in a neighborhood of the point which contradicts to Lemma 1 since $\lambda_1=0$ on U. Q. E. D.

LEMMA 4 (S. Sasaki). *Suppose a pseudo-Sasakian manifold* $M^{2n+1}(\phi,\xi,\eta,g)$ *is properly and isometrically immersed in a pseudo-Riemannian manifold* $(\tilde{M}^{2n+2},\tilde{g})$ *of constant curvature* 1. *If rank* $A\geqq 2$ *at* $x_0\in M^{2n+1}$, *then* $A\xi_{x_0}=0$.

PROOF. Let $\lambda_1,\lambda_2,\cdots,\lambda_{2n+1}$ be the principal curvatures at x_0 and let $e_1,e_2,\cdots,e_{2n+1}$ be the corresponding principal directions. (5) with $\tilde{c}=1$ implies

$$\varepsilon\lambda_i\lambda_j\eta(e_j)e_i=\lambda_i g(e_i,e_j)A\xi_{x_0} \tag{10}$$

for all $1\leqq i,\ j\leqq 2n+1$. Since rank $A\geqq 2$, we may suppose $\lambda_1\lambda_2\neq 0$. Then (10) with $i=1$ and $j=2$ implies $\eta(e_2)=0$. Hence (10) with $i=j=2$ implies $A\xi_{x_0}=0$. Q. E. D.

LEMMA 5. *Suppose a pseudo-Sasakian manifold* $M^{2n+1}(\phi,\xi,\eta,g)$, $n\geqq 2$, *is properly and isometrically immersed in a pseudo-Riemannian manifold* $(\tilde{M}^{2n+2},\tilde{g})$ *of constant curvature* 1. *If* $M^{2n+1}(\phi,\xi,\eta,g)$ *is* η*-Einstein, then rank* $A\leqq 1$ *on* M^{2n+1}.

PROOF. Suppose rank $A\geqq 2$ at $x_0\in M^{2n+1}$. Then, since rank $A\geqq 2$ in a neighborhood U of x_0, Lemma 4 says that $A\xi=0$ on U. We may suppose that a field of unit normals ζ is defined on U; and our argument below is just on U.

(8) and (9) imply that $a+b=2n$; moreover, since $n\geqq 2$, we can see that a and b are constant (cf. M. Okumura[5]). (1) implies

$$R_1(X,Y)=2ng(X,Y)+\varepsilon\{g(AX,Y)\mathrm{Tr}A-g(A^2X,Y)\}\,. \tag{11}$$

Let $\lambda_1,\lambda_2,\cdots,\lambda_{2n+1}$ be the principal curvatures on U and let $e_1,e_2,\cdots,e_{2n+1}$ be the corresponding principal directions. We may suppose

$$\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_{2n}\,,$$

$$e_{2n+1}=\xi,\ \lambda_{2n+1}=0\,.$$

Since $\eta(e_i)=0$ for $1 \leqq i \leqq 2n$, (9) and (11) imply

$$(12) \qquad 2n+\varepsilon\{\lambda_i \mathrm{Tr} A-\lambda_i^2\}=a$$

for $1 \leqq i \leqq 2n$. There are two cases:

(A) $\quad \lambda_1=\lambda_2 \cdots=\lambda_{2n}=\lambda$ on U,

(B) $\quad \lambda_1<\lambda_{2n}$ at some point in U.

For the case (A), Lemma 3 says that $\lambda=0$, which contradicts to the assumption rank $A \geqq 2$. Thus the case (A) does not occur. For the case (B), without loss of generality, we may suppose $\lambda_1<\lambda_{2n}$ on U. (12) implies

$$\lambda_1^2-\lambda_1 \mathrm{Tr} A=\varepsilon(2n-a),$$
$$\lambda_{2n}^2-\lambda_{2n} \mathrm{Tr} A=\varepsilon(2n-a).$$

Hence we get

$$(\lambda_1-\lambda_{2n})(\lambda_2+\lambda_3+\cdots+\lambda_{2n-1})=0,$$

which implies

$$(13) \qquad \lambda_2+\lambda_3+\cdots+\lambda_{2n-1}=0.$$

Using (13), (12) becomes

$$(14) \qquad \lambda_i^2-\lambda_i(\lambda_1+\lambda_{2n})+\varepsilon(a-2n)=0.$$

In particular, we have

$$(15) \qquad \lambda_1\lambda_{2n}=\varepsilon(a-2n).$$

Using (15), (14) becomes

$$(16) \qquad \lambda_i^2-\lambda_i(\lambda_1+\lambda_{2n})+\lambda_1\lambda_{2n}=0.$$

Thus there are just two λ's, say λ and λ':

$$\lambda=\lambda_1=\lambda_2=\cdots=\lambda_m,$$
$$\lambda'=\lambda_{m+1}=\lambda_{m+2}=\cdots=\lambda_{2n},$$

and hence (13) becomes

$$(m-1)\lambda + (2n-m-1)\lambda' = 0 . \tag{17}$$

Now, if $a \neq 2n$, then (15) says that

$$\lambda\lambda' \neq 0$$

holds good. Hence (17) and $n \geqq 2$ imply

$$1 < m < 2n-1 .$$

In this case, (15) and (17) imply

$$\lambda^2 = \frac{2n-m-1}{m-1}\, \varepsilon(2n-a) . \tag{18}$$

Since a is constant, (18) implies that λ is constant, and hence λ' is constant, too. Using the same method which we have used frequently, we can see that Lemma 1 implies

$$m = 2n - m = n .$$

Thus (17) says $\lambda + \lambda' = 0$, that is

$$A = \begin{pmatrix} \overbrace{\begin{matrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{matrix}}^{n} & & 0 \\ & \overbrace{\begin{matrix} -\lambda & & \\ & \ddots & \\ & & -\lambda \end{matrix}}^{n} & \\ 0 & & 0 \end{pmatrix}$$

In particular, we have

$$A^2 = -\lambda^2\phi^2 . \tag{19}$$

We have the following identity:

$$(\nabla_X A)AY + A(\nabla_X A)Y = \nabla_X(A^2 Y) - A^2\nabla_X Y . \tag{20}$$

Since λ is constant, (19) implies

$$\nabla_X(A^2Y) = -\lambda^2\nabla_X(-Y+\eta(Y)\xi)$$
$$= \lambda^2\nabla_X Y - \lambda^2\{[X\cdot\eta(Y)]\xi + \eta(Y)\phi X\},$$
$$A^2\nabla_X Y = -\lambda^2\{-\nabla_X Y + \eta(\nabla_X Y)\xi\}.$$

Hence (20) becomes

$$(21)\qquad (\nabla_X A)AY + A(\nabla_X A)Y = \lambda^2\{\eta(\nabla_X Y) - X\cdot\eta(Y)\}\xi - \lambda^2\eta(Y)\phi X.$$

On the other hand, we have

$$X\cdot\eta(Y) = X\cdot g(Y,\xi) = \eta(\nabla_X Y) + g(Y,\phi X).$$

Hence (21) becomes

$$(22)\qquad (\nabla_X A)AY + A(\nabla_X A)Y = \lambda^2\{g(X,\phi Y)\xi - \eta(Y)\phi X\}.$$

Interchanging X and Y in (22), we get

$$(23)\qquad (\nabla_Y A)AX + A(\nabla_Y A)X = \lambda^2\{g(Y,\phi X)\xi - \eta(X)\phi Y\}.$$

Applying (2) on (22) − (23), we get

$$(24)\qquad (\nabla_X A)AY - (\nabla_Y A)AX = \lambda^2\{2g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y\}.$$

Now, let X be a non-zero vector field in T_λ. Then, for any $Z\in T_\lambda$ and $Y\in T_{-\lambda}$, (24) implies

$$(25)\qquad -\lambda(\nabla_X A)Y - \lambda(\nabla_Y A)X = 2\lambda^2 g(X,\phi Y)\xi,$$

$$(26)\qquad \lambda(\nabla_X A)Z - \lambda(\nabla_Z A)X = 2\lambda^2 g(X,\phi Z)\xi.$$

Since $\lambda\neq 0$, (25), (26) and (2) imply that $g(X,W)=0$ for all vector fields W such that $\eta(W)=0$; this is the contradiction. Hence we must have $a=2n$; that is, (M^{2n+1},g) is Einstein. Thus Lemma 2 implies that rank $A\leqq 1$, which contradicts to the assumption rank $A\geqq 2$.

Consequently, rank $A\leqq 1$ on M^{2n+1}. Q. E. D.

PROOF OF THEOREM 4. Theorem 4 is a direct consequence of Lemma 5. Q. E. D.

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