

## SASAKIAN MANIFOLDS WITH QUASI-CONFORMAL CURVATURE TENSOR

UDAY CHAND DE, JAE BOK JUN, AND ABUL KALAM GAZI

ABSTRACT. The object of the paper is to study a Sasakian manifold with quasi-conformal curvature tensor.

### 1. Introduction

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [11]. According to them a quasi-conformal curvature tensor  $\tilde{C}$  is defined by

$$(1.1) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] - \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where  $a$  and  $b$  are constants and  $R, S, Q$  and  $r$  are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by  $g(QX, Y) = S(X, Y)$  and the scalar curvature of the manifold respectively. If  $a = 1$  and  $b = -\frac{1}{n-2}$ , then (1.1) takes the form

$$\begin{aligned} \tilde{C}(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] + \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] \\ &= C(X, Y)Z, \end{aligned}$$

where  $C$  is the conformal curvature tensor [4]. Thus the conformal curvature tensor  $C$  is a particular case of the tensor  $\tilde{C}$ . For this reason  $\tilde{C}$  is called the quasi-conformal curvature tensor. A manifold  $(M^n, g)(n > 3)$  shall be called quasi-conformally flat if the quasi-conformal curvature tensor  $\tilde{C} = 0$ . It is known [1] that the quasi-conformally flat manifold is either conformally flat if  $a \neq 0$  or, Einstein if  $a = 0$  and  $b \neq 0$ . Since, they give no restrictions for manifolds if  $a = 0$  and  $b = 0$ , it is essential for us to consider the case of  $a \neq 0$  or  $b \neq 0$ .

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An almost contact metric manifold is said to be an  $\eta$ -Einstein manifold if the Ricci tensor  $S$  satisfies the condition

$$(1.2) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where  $a, b$  are certain scalars. It is known [10] that in a Sasakian manifold  $a, b$  are constants. A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric [8] if  $R(X, Y).R = 0$ , where  $R$  is the Riemannian curvature tensor and  $R(X, Y)$  is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors  $X, Y$ . If a Riemannian manifold satisfies  $R(X, Y) \cdot \tilde{C} = 0$ , where  $\tilde{C}$  is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

It is known [5] that a conformally flat Sasakian manifold is of constant curvature and a Weyl semi-symmetric Sasakian manifold is locally isometric with the unit sphere  $S^n(1)$  [3]. In the present paper we have studied quasi-conformally flat and quasi-conformally semi-symmetric Sasakian manifolds. At first we prove that a Sasakian manifold is quasi-conformally flat if and only if it is locally isometric with the unit sphere  $S^n(1)$ . Also it is proved that a compact orientable quasi-conformally flat Sasakian manifold can not admit a non-isometric conformal transformation. Finally, we have shown that a Sasakian manifold is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.

## 2. Preliminaries

Let  $S$  and  $r$  denote respectively the Ricci tensor of type (0,2) and the scalar curvature in a Sasakian manifold  $(M^n, g)$ . It is known that in a Sasakian manifold  $M^n$ , the following relations hold [6], [2], [7]:

$$(2.1) \quad \phi(\xi) = 0$$

$$(2.2) \quad \eta(\xi) = 1$$

$$(2.3) \quad \phi^2 X = -X + \eta(X)\xi$$

$$(2.4) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

$$(2.5) \quad g(\xi, X) = \eta(X)$$

$$(2.6) \quad \nabla_X \xi = -\phi X$$

$$(2.7) \quad S(X, \xi) = (n-1)\eta(X)$$

$$(2.8) \quad g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y)$$

$$(2.9) \quad R(\xi, X)\xi = -X + \eta(X)\xi$$

$$(2.10) \quad g(R(X, Y)\xi, Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X)$$

$$(2.11) \quad \text{and } (\nabla_X \phi)(Y) = R(\xi, X)Y$$

for any vector fields  $X, Y$ .

The above results will be used in the next section.

### 3. $\eta$ -Einstein Sasakian manifold

Let  $l^2$  be the square of the length of the Ricci tensor, then

$$(3.1) \quad l^2 = \sum_{i=1}^n S(Qe_i, e_i),$$

where  $Q$  is the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor  $S$  and  $\{e_i\}$ ,  $i = 1, 2, \dots, n$ , is an orthonormal basis of the tangent space at any point. Now putting  $X = Y = \{e_i\}$  in (1.2), and taking summation over  $i, 1 \leq i \leq n$ , we get

$$(3.2) \quad r = na + b,$$

where  $r$  is the scalar curvature. Again from (1.2) we obtain

$$(3.3) \quad S(\xi, \xi) = a + b.$$

Now we get from (1.2) with the help of (3.1), (3.2) and (3.3)

$$(3.4) \quad l^2 = (n - 1)a^2 + (a + b)^2.$$

Since the scalars  $a$  and  $b$  are constants of an  $\eta$ -Einstein Sasakian manifold, it follows from (3.2) that  $r$  is constant and so also is the length of the Ricci tensor. Next we suppose that the manifold under consideration admits a non-isometric conformal motion generated by a vector field  $X$ . Since  $l^2$  is constant, it follows that

$$(3.5) \quad L_X l^2 = 0,$$

where  $L_X$  denotes Lie-differentiation with respect to  $X$ . Now it is known [9] that if a compact Riemannian manifold  $M^n (n > 2)$  with constant scalar curvature admits an infinitesimal nonisometric conformal transformation  $X$  such that  $L_X l^2 = 0$ , then  $M$  is isometric to a sphere. But a sphere is an Einstein manifold. Hence we can state the following:

**Theorem 3.1.** *A compact orientable  $\eta$ -Einstein Sasakian manifold does not admit a nonisometric conformal transformation.*

### 4. Quasi-conformally flat Sasakian manifold

If the manifold under consideration is quasi-conformally flat, then we have from (1.1)

$$(4.1) \quad \begin{aligned} {}'R(X, Y, Z, W) = & \frac{b}{a} [S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ & + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\ & + \frac{r}{na} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \end{aligned}$$

where  $a$  and  $b$  are constants and  $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$ .

Now putting  $Z = \xi$  in (4.1) and using (2.2), (2.5), (2.7) and (2.10) we get

$$\begin{aligned}
 & g(X, W)\eta(Y) - g(Y, W)\eta(X) \\
 (4.2) \quad &= \frac{b}{a}[(n-1)g(Y, W)\eta(X) - (n-1)g(X, W)\eta(Y) \\
 & \quad + S(Y, W)\eta(X) - S(X, W)\eta(Y)] + \frac{r}{na}\left[\frac{a}{n-1}\right. \\
 & \quad \left. + 2b\right][g(X, W)\eta(Y) - g(Y, W)\eta(X)].
 \end{aligned}$$

Again putting  $X = \xi$  in (4.2) and using (2.2), (2.5) and (2.7) it follows that

$$(4.3) \quad S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W),$$

where

$$(4.4) \quad A = [-(n-1) + \frac{r}{nb}\left(\frac{a}{n-1} + 2b\right) - \frac{a}{b}]$$

and

$$(4.5) \quad B = [2(n-1) - \frac{r}{nb}\left(\frac{a}{n-1} + 2b\right) + \frac{a}{b}].$$

Here  $A + B = (n-1)$ . This leads to the following theorem:

**Theorem 4.1.** *A quasi-conformally flat Sasakian manifold is an  $\eta$ -Einstein manifold.*

Now from Theorem 3.1 we can state the following:

**Corollary 4.1.** *A compact orientable quasi-conformally flat Sasakian manifold can not admit a nonisometric conformal transformation.*

Putting  $Y = W = \{e_i\}$  in (4.3) and taking summation over  $i$ ,  $1 \leq i \leq n$ , we get

$$(4.6) \quad r = nA + B.$$

Now with the help of (4.4) and (4.5) the equation (4.6) gives

$$(4.7) \quad [(n-2) + \frac{a}{b}]\left[\frac{r}{n} + (1-n)\right] = 0.$$

Hence either

$$(4.8) \quad b = \frac{a}{2-n},$$

or,

$$(4.9) \quad r = n(n-1).$$

If  $b = \frac{a}{(2-n)}$  then putting it into (1.1) we get

$$(4.10) \quad \tilde{C}(X, Y)Z = aC(X, Y)Z,$$

where  $C(X, Y)Z$  denotes the Weyl conformal curvature tensor. So the quasi-conformally flatness and conformally flatness are equivalent in this case. A conformally flat Sasakian manifold  $(M^n, g)$  ( $n \geq 5$ ) is of constant curvature. But a manifold of constant curvature is conformally flat. Hence a Sasakian

manifold is conformally flat if and only if it is locally isometric with a unit sphere  $S^n(1)$ . So in this case  $M^n$  is locally isometric to the unit sphere.

If  $r = n(n - 1)$ , then from (4.3), (4.4) and (4.5) we obtain

$$(4.11) \quad S(Y, W) = (n - 1)g(Y, W).$$

This implies that  $M^n$  is an Einstein manifold. So putting (4.8), (4.9) and (4.11) into (4.1) we obtain

$$R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W).$$

Then  $M^n$  is of constant curvature +1. Hence it is locally isometric with the unit sphere  $S^n(1)$ . If  $M^n$  is locally isometric to the unit sphere  $S^n(1)$  then it is easy to see that  $M^n$  is quasi-conformally flat. This leads to the following theorem:

**Theorem 4.2.** *Let  $(M^n, g)(n \geq 5)$  be a Sasakian manifold. Then  $M^n$  is quasi-conformally flat if and only if  $M^n$  is locally isometric to the unit sphere  $S^n(1)$ .*

### 5. Sasakian manifolds satisfying $R(X, Y) \cdot \tilde{C} = 0$

In this section we consider a Sasakian manifold  $M^n$  satisfying the condition

$$(5.1) \quad R(X, Y) \cdot \tilde{C} = 0.$$

Then we obtain from (1.1) by using (2.5), (2.7) and (2.10)

$$(5.2) \quad \begin{aligned} \eta(\tilde{C}(X, Y)Z) &= [a + b(n - 1) - \frac{r}{n}(\frac{a}{n - 1} + 2b)][g(Y, Z)\eta(X) \\ &\quad - g(X, Z)\eta(Y)] + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \end{aligned}$$

For  $Z = \xi$ , we get from (5.2)

$$(5.3) \quad \eta(\tilde{C}(X, Y)\xi) = 0.$$

Again putting  $X = \xi$  in (5.2) we get

$$(5.4) \quad \begin{aligned} \eta(\tilde{C}(\xi, Y)Z) &= [a + b(n - 1) - \frac{r}{n}(\frac{a}{n - 1} + 2b)][g(Y, Z) \\ &\quad - \eta(Y)\eta(Z)] + b[S(Y, Z) - (n - 1)\eta(Y)\eta(Z)]. \end{aligned}$$

In virtue of (5.1) we get

$$(5.5) \quad \begin{aligned} R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W \\ - \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W = 0, \end{aligned}$$

which implies that

$$(5.6) \quad \begin{aligned} ' \tilde{C}(U, V, W, Y) - \eta(Y)\eta(\tilde{C}(U, V)W) + \eta(U)\eta(\tilde{C}(Y, V)W) \\ + \eta(V)\eta(\tilde{C}(U, Y)W) + \eta(W)\eta(\tilde{C}(U, V)Y) - g(Y, U)\eta(\tilde{C}(\xi, V)W) \\ - g(Y, V)\eta(\tilde{C}(U, \xi)W) - g(Y, W)\eta(\tilde{C}(U, V)\xi) = 0, \end{aligned}$$

where  $' \tilde{C}(U, V, W, Y) = g(\tilde{C}(U, V)W, Y)$ .

Putting  $U = Y$  in (5.6) and with the help of (5.2) and (5.3) we get

$$(5.7) \quad \begin{aligned} & \tilde{C}(U, V, W, U) + \eta(W)\eta(\tilde{C}(U, V)U) \\ & - g(U, U)\eta(\tilde{C}(\xi, V)W) - g(U, V)\eta(\tilde{C}(U, \xi)W) = 0. \end{aligned}$$

Now putting  $U = \{e_i\}$ , where  $\{e_i\}, i = 1, 2, \dots, n$ , be an orthonormal basis of the tangent space at each point of the manifold, in (5.7) and taking the summation over  $i, 1 \leq i \leq n$ , and using (5.2), (5.4) we get

$$(5.8) \quad S(V, W) = \lambda g(V, W) + \mu \eta(V)\eta(W),$$

where

$$(5.9) \quad \lambda = \frac{-br + (n-1)^2b + (n-1)a}{a-b}$$

and

$$(5.10) \quad \mu = \frac{b[r - n(n-1)]}{a-b}.$$

Hence (5.8) leads to the following theorem:

**Theorem 5.1.** *A quasi-conformally semi-symmetric Sasakian manifold is an  $\eta$ -Einstein manifold.*

Now contracting (5.8) we get

$$(5.11) \quad r = n\lambda + \mu.$$

By (5.9) and (5.10) the equation (5.11) gives

$$[a + (n-2)b][r - n(n-1)] = 0.$$

Therefore, either

$$(5.12) \quad \begin{aligned} b &= \frac{a}{2-n} \quad \text{or} \\ r &= n(n-1). \end{aligned}$$

From (5.9) and (5.12) we obtain

$$(5.13) \quad \lambda = (n-1).$$

By (5.10) and (5.12) we get

$$(5.14) \quad \mu = 0.$$

So, from (5.8), (5.13) and (5.14) we have

$$(5.15) \quad S(V, W) = (n-1)g(V, W).$$

Therefore,  $M^n$  is an Einstein manifold. Now with the help of (5.12) and (5.15) the equations (5.2) and (5.4) imply that

$$(5.16) \quad \eta(\tilde{C}(U, V)W) = 0$$

and

$$(5.17) \quad \eta(\tilde{C}(\xi, U)V) = 0,$$

respectively. So using (5.16), (5.17) and (5.3) into the equation (5.6) we get

$$(5.18) \quad {}^1\tilde{C}(U, V, W, Y) = 0.$$

Therefore,  $M^n$  is quasi-conformally flat. Then it is trivially quasi-conformally semi-symmetric. So we have the following result:

**Theorem 5.2.** *Let  $(M^n, g)(n > 3)$  be a Sasakian manifold. Then  $M^n$  is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.*

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UDAY CHAND DE  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF KALYANI  
KALYANI 741235  
WEST BENGAL, INDIA  
*E-mail address:* uc\_de@yahoo.com

JAE BOK JUN  
DEPARTMENT OF MATHEMATICS  
COLLEGE OF NATURAL SCIENCE  
KOOK-MIN UNIVERSITY  
SEOUL 136-702, KOREA  
*E-mail address:* jbjun@kookmin.ac.kr

ABUL KALAM GAZI  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF KALYANI  
KALYANI 741235  
WEST BENGAL, INDIA