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SASAKIAN MANIFOLDS WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR AND CONSTANT SCALAR CURVATURE

BY

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As a complex analogue to the Weyl conformal curvature tensor, Bochner [1] (see also Yano and Bochner [18]) introduced the so-called Bochner curvature tensor using a complex local coordinate system. The Bochner curvature tensor with respect to a real coordinate system has been given by Tachibana [13]. In [19], Yano and Ishihara proved the following

THEOREM A. *Let M be a Kählerian manifold of real dimension n with constant scalar curvature whose Bochner curvature tensor vanishes and whose Ricci tensor is positive semi-definite. If M is compact, then the universal covering manifold is a complex projective space $CP^{n/2}$ or a complex space $C^{n/2}$.*

For a Kähler manifold having the vanishing Bochner curvature tensor and constant scalar curvature, Matsumoto and Tanno [10] proved important theorems (see also Matsumoto [8]).

In Sasakian manifolds, Matsumoto and Chūman [9] defined the contact Bochner curvature tensor, which is constructed from the Bochner curvature tensor by the fibering of Boothby and Wang [2] (see also Yano [16]). Recently, the contact Bochner curvature tensor was studied by Ikawa [4] and Yano [16], [17] in the theory of submanifolds.

The purpose of this paper is to study a Sasakian manifold with vanishing contact Bochner curvature tensor and constant scalar curvature.

1. Sasakian manifolds. In this section we would like to recall definition and some fundamental properties of a Sasakian manifold.

Let M be a $(2n+1)$-dimensional differentiable manifold of class C^∞, and let φ, ξ and η be a tensor field of type $(1, 1)$, a vector field and a 1-form on M, respectively, such that

$$(1.1)\qquad \varphi^2 X = -X+\eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X on M. Then M is said to have an *almost contact structure* (φ, ξ, η) and is called an *almost contact manifold.* The almost contact structure is said to be *normal* if $N+d\eta\otimes\xi = 0$, where N denotes the Nijenhuis tensor formed with φ, and $d\eta$ is the differential of the 1-form η. If a Riemannian metric tensor field $\langle,\rangle$ is given on M and satisfies

$$(1.2) \qquad \langle\varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \qquad \eta(X) = \langle X, \xi\rangle$$

for any vector fields X and Y, then a $(\varphi, \xi, \eta, \langle,\rangle)$-structure is called an *almost contact metric structure*, and M is called an *almost contact metric manifold.* If $d\eta(X, Y) = \langle\varphi X, Y\rangle$, then an almost contact metric structure is called a *contact metric structure.* If, moreover, the structure is normal, then a contact metric structure is called a *Sasakian structure*, and a manifold with Sasakian structure is called a *Sasakian manifold.* It is well known that in a Sasakian manifold with structure $(\varphi, \xi, \eta, \langle,\rangle)$ we have

$$(1.3) \qquad \nabla_X\xi = \varphi X, \qquad (\nabla_X\varphi)Y = \eta(Y)X - \langle X, Y\rangle\xi = R(X, \xi)Y,$$

where ∇ denotes the covariant differentiation in M, and R denotes the Riemannian curvature tensor of M.

In the following, let M be a Sasakian manifold with structure tensors $(\varphi, \xi, \eta, \langle,\rangle)$ of dimension $m+1$, where we have put $m = 2n$. Let S denote the Ricci tensor of M. Then we have

$$(1.4) \qquad \begin{gathered} S(X, \xi) = m\eta(X), \qquad S(\varphi X, \varphi Y) = S(X, Y) - m\eta(X)\eta(Y), \\ S(\varphi X, Y) = -S(X, \varphi Y). \end{gathered}$$

We denote by Q the Ricci operator of M defined by $\langle QX, Y\rangle = S(X, Y)$. Then equations (1.4) imply

$$(1.5) \qquad Q\xi = m\xi, \qquad Q\varphi X = \varphi QX.$$

The Ricci tensor S of a Sasakian manifold M satisfies (see [7], and (1.2) in [9])

$$(1.6) \qquad \begin{aligned} \nabla_Z(S)(X, Y) = {} & \nabla_X(S)(Y, Z) + \nabla_{\varphi Y}(S)(\varphi X, Z) + \eta(X)S(\varphi Y, Z) + \\ & + 2\eta(Y)S(\varphi X, Z) - m\eta(X)\langle\varphi Y, Z\rangle - 2m\eta(Y)\langle\varphi X, Z\rangle. \end{aligned}$$

If the Ricci tensor S of M is of the form

$$S(X, Y) = a\langle X, Y\rangle + b\eta(X)\eta(Y),$$

where a and b are constants, then M is called an *η-Einstein manifold.*

A plane section in the tangent space $T_x(M)$ at x of a Sasakian manifold M is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature $K(X, \varphi X)$ with respect to a φ-section determined by a vector X is called a *φ-sectional curvature.* It is easily verified that if a Sasakian manifold has a φ-sectional curvature c which does not depend on the φ-section at each point, then c is a constant in the manifold. If a Sasakian manifold has the constant φ-sectional curvature c, then the curvature tensor R of M is given by

$$(1.7)\quad R(X,Y)Z = \frac{1}{4}(c+3)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) +$$
$$+\frac{1}{4}(c-1)\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X,Z\rangle\eta(Y)\xi - \langle Y,Z\rangle\eta(X)\xi +$$
$$+\langle\varphi Y, Z\rangle\varphi X + \langle\varphi Z, X\rangle\varphi Y - 2\langle\varphi X, Y\rangle\varphi Z\big).$$

2. Contact Bochner curvature tensor and Ricci tensor. Let M be an $(m+1)$-dimensional ($m = 2n$) Sasakian manifold. Then the *contact Bochner curvature tensor* B of M is defined by

$$(2.1)\quad B(X,Y) = R(X,Y) + \frac{1}{m+4}\big(QY\wedge X - QX\wedge Y + Q\varphi Y\wedge\varphi X -$$
$$-Q\varphi X\wedge\varphi Y + 2\langle Q\varphi X, Y\rangle\varphi + 2\langle\varphi X, Y\rangle Q\varphi +$$
$$+\eta(Y)\varphi X\wedge\xi + \eta(X)\xi\wedge\varphi Y\big) - \frac{k+m}{m+4}(\varphi Y\wedge\varphi X - 2\langle\varphi X, Y\rangle\varphi) -$$
$$-\frac{k-4}{m+4}Y\wedge X + \frac{k}{m+4}\big(\eta(Y)\xi\wedge X + \eta(X)Y\wedge\xi\big),$$

where $k = (r+m)/(m+2)$, r denotes the scalar curvature of M, and $(X\wedge Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$.

Definition. If the Ricci tensor S of a Sasakian manifold M satisfies $\nabla_X(S)(\varphi Y, \varphi Z) = 0$ for any vector fields X, Y and Z on M, then we say that the Ricci tensor S of M is *η-parallel*.

If the Ricci tensor S of M is η-parallel, then we have [7]

$$(2.2)\quad \nabla_X(S)(Y,Z) = m\big(\langle\varphi X, Y\rangle\eta(Z) + \langle\varphi X, Z\rangle\eta(Y)\big) +$$
$$+\eta(Y)S(X,\varphi Z) + \eta(Z)S(X,\varphi Y).$$

From (2.2) we see that if S is η-parallel, then the scalar curvature r and $\mathrm{Tr}Q^2$, where Tr denotes the trace of the operator, are constant. Taking the covariant differentiation of (2.1) and contraction, we have the following (see [9], (2.4))

LEMMA 2.1. *Let M be a Sasakian manifold with constant scalar curvature. If the contact Bochner curvature tensor vanishes, then the Ricci tensor S of M is η-parallel.*

LEMMA 2.2 (Matsumo and Chūman [9]). *Let M be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M is an η-Einstein manifold, then M is of constant φ-sectional curvature.*

LEMMA 2.3 (Kon [7]). *The Ricci tensor S of a Sasakian manifold M is η-parallel if and only if*

$$(2.3)\quad \langle\nabla Q, \nabla Q\rangle = 2\,\mathrm{Tr}Q^2 + 2m^3 + 2m^2 - 4mr.$$

Proof. By using a φ-basis $E_1, \dots, E_{m+1}$ $(E_{n+t} = \varphi E_t,\ E_{m+1} = \xi)$, we obtain

$$(2.4)\quad \langle \nabla Q, \nabla Q\rangle = \sum_{i,j=1}^{m+1} \langle \nabla_{E_i}(Q)E_j,\ \nabla_{E_i}(Q)E_j\rangle$$
$$= \sum_{i=1}^{m+1}\sum_{j=1}^{m} \langle \nabla_{E_i}(Q)E_j,\ \nabla_{E_i}(Q)E_j\rangle + \sum_{i=1}^{m+1} \langle \nabla_{E_i}(Q)\xi,\ \nabla_{E_i}(Q)\xi\rangle$$
$$= \sum_{i=1}^{m+1}\sum_{j=1}^{m} \langle \nabla_{E_i}(Q)\varphi E_j,\ \nabla_{E_i}(Q)\varphi E_j\rangle + \sum_{i=1}^{m+1} \langle \nabla_{E_i}(Q)\xi,\ \nabla_{E_i}(Q)\xi\rangle$$
$$= 2\,\mathrm{Tr}\,Q^2 + 2m^3 + 2m^2 - 4mr + T,$$

where we have put

$$T = \sum_{i=1}^{m+1}\sum_{j=1}^{m} \langle \varphi\nabla_{E_i}(Q)E_j,\ \varphi\nabla_{E_i}(Q)E_j\rangle.$$

On the other hand, we can easily see that the Ricci tensor S of M is η-parallel if and only if $T = 0$. Thus we have our assertion.

If we take a suitable φ-basis $E_1, \dots, E_{m+1}$ $(\varphi E_t = E_{n+t},\ E_{m-1} = \xi)$, by using (1.4), the Ricci operator Q of M is represented by the matrix form

$$Q = \left[\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_m & \\ \hline & 0 & & m \end{array}\right].$$

In the following, we put

$$H = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$$

which is a symmetric (m, m)-matrix. Then we have

$$(2.5)\qquad r = \mathrm{Tr}\,Q = \mathrm{Tr}\,H + m, \qquad \mathrm{Tr}\,Q^2 = \mathrm{Tr}\,H^2 + m^2.$$

By (2.5) and Lemma 2.3, the Ricci tensor S is η-parallel if and only if

$$(2.6)\qquad \langle \nabla Q, \nabla Q\rangle = 2\,\mathrm{Tr}\,H^2 - 4m\,\mathrm{Tr}\,H + 2m^3.$$

Now we define a (1, 1)-*tensor* A of M by setting

$$AX = QX - aX - b\eta(X)\xi$$

for any vector field X on M, where a and b are constant such that $a+b=m$ and $r=(m+1)a+b$. A Sasakian manifold M is an η-Einstein manifold if and only if $A=0$. Moreover, by (2.5) we have

$$\mathrm{Tr}\,A^2=\mathrm{Tr}\,H^2-\frac{1}{m}(\mathrm{Tr}\,H)^2=\frac{1}{m}\sum_{i>j}(\lambda_i-\lambda_j)^2. \tag{2.7}$$

Consequently, we see that M is η-Einstein if and only if $\lambda_i=\lambda_j$ for all i,j $(i,j=1,\dots,m)$.

In the next place, we prepare the following

LEMMA 2.4. *For a symmetric (m,m)-matrix H, we have*

$$\begin{aligned}(2.8)\quad &\frac{1}{m-1}\sum_{i}\sum_{j\neq k}(\lambda_i+2)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)\\ &=\left[m\,\mathrm{Tr}\,H^3-\frac{2m-1}{m-1}\mathrm{Tr}\,H\cdot\mathrm{Tr}\,H^2+\frac{1}{m-1}(\mathrm{Tr}\,H)^3\right]+\frac{2m(m-2)}{m-1}\mathrm{Tr}\,A^2.\end{aligned}$$

Proof. By a straightforward computation we have (cf. [19], Lemma 4)

$$\begin{aligned}&\frac{1}{m-1}\sum_{i}\sum_{j\neq k}\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)\\ &\qquad=m\,\mathrm{Tr}\,H^3-\frac{2m-1}{m-1}\mathrm{Tr}\,H\cdot\mathrm{Tr}\,H^2+\frac{1}{m-1}(\mathrm{Tr}\,H)^3.\end{aligned}$$

On the other hand, we also have

$$\frac{2}{m-1}\sum_{i}\sum_{j\neq k}(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)=\frac{2m(m-2)}{m-1}\mathrm{Tr}\,A^2.$$

From these equations we obtain (2.8).

In the sequel, we define the *contact Ricci tensor* L by setting

$$L(X,Y)=S(X,Y)+2\langle X,Y\rangle-(m+2)\eta(X)\eta(Y) \tag{2.9}$$

for any vector fields X and Y on M. Clearly, L is symmetric. Putting $L(X,Y)=\langle GX,Y\rangle$, we define the *contact Ricci operator* G. For a suitable basis, G is represented by a matrix form

$$G=\left[\begin{array}{ccc|c}\lambda_1+2 & & & \\ & \ddots & & 0\\ & & \lambda_m+2 & \\ \hline & 0 & & 0\end{array}\right]=\left[\begin{array}{c|c}H & 0\\ \hline 0 & 0\end{array}\right]+2\left[\begin{array}{c|c}I & 0\\ \hline 0 & 0\end{array}\right],$$

where I denotes the identity matrix.

Remark. Let M be a regular Sasakian manifold of dimension $m+1$. If M/ξ denotes the set of orbits of ξ, then M/ξ is a real m-dimensional Kähler manifold (cf. [2], [12], and [15]). Then there exists a fibering $\pi\colon M \to M/\xi$. Let X^* and Y^* be the horizontal lifts of X and Y, respectively, over M/ξ with respect to the connection η. Then the Ricci tensor S' of M/ξ is given by

$$(S'(X, Y))^* = S(X^*, Y^*)+2\langle X^*, Y^*\rangle. \tag{2.10}$$

The horizontal space is spanned by $\{\varphi X\colon X \in T_x(M)\}$ at each point $x \in M$. If we consider

$$\begin{aligned} S(\varphi X, \varphi Y)+2\langle\varphi X, \varphi Y\rangle &= S(X, Y)+2\langle X, Y\rangle-(m+2)\eta(X)\eta(Y) \\ &= L(X, Y), \end{aligned}$$

by (1.2) and (1.4) we can see that the contact Ricci tensor L corresponds to the Ricci tensor S' of M/ξ. On the other hand, by (2.10) we see that the Ricci tensor S' of M/ξ is positive semi-definite (negative semi-definite) if and only if L is positive semi-definite (negative semi-definite), that is, all eigenvalues λ_i of the matrix H satisfy $\lambda_i \geqslant -2$ ($\lambda_i \leqslant -2$). And M/ξ is Einstein if and only if M is η-Einstein. Moreover, the Ricci tensor of M/ξ is parallel if and only if the Ricci tensor of M is η-parallel (see [7]).

In the following, put

$$P = m\operatorname{Tr}H^3-\frac{2m-1}{m-1}\operatorname{Tr}H\cdot\operatorname{Tr}H^2+\frac{1}{m-1}(\operatorname{Tr}H)^3+ \\ +\frac{2m(m-2)}{m-1}\operatorname{Tr}A^2. \tag{2.11}$$

Then we obtain

LEMMA 2.5. *If the contact Ricci operator G of a Sasakian manifold M is positive semi-definite* (*respectively, negative semi-definite*), *then $P \geqslant 0$* (*respectively, $P \leqslant 0$*).

Proof. Let λ_i $(i=1, \ldots, m)$ be eigenvalues of H. Then, by (2.8),

$$P = \frac{1}{m-1}\sum_i\sum_{j\neq k}(\lambda_i+2)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k).$$

Let a_i $(i=1, \ldots, m)$ be eigenvalues of G such that $a_i = \lambda_i+2$ for all i. Then P is represented by

$$P = \frac{1}{m-1}\sum_i\sum_{j\neq k}a_i(a_i-a_j)(a_i-a_k).$$

If G is negative semi-definite, i.e., $a_i \leqslant 0$, we can put $a_m \leqslant \ldots \leqslant a_2 \leqslant a_1 \leqslant 0$. Then taking three arbitrary indices i, j and k such that $k < j < i$, we have

$$\begin{aligned} a_i(a_i - a_j)(a_i - a_k) + a_j(a_j - a_i)(a_j - a_k) + a_k(a_k - a_i)(a_k - a_j) \\ = a_i(a_i - a_j)(a_i - a_k) + (a_j - a_k)^2(a_j + a_k - a_i) \leqslant 0 . \end{aligned}$$

Similarly, if G is positive semi-definite, we have $P \geqslant 0$.

3. Theorems. Let M be an $(m+1)$-dimensional Sasakian manifold with constant scalar curvature. First of all, we compute the (restricted) Laplacian for the Ricci tensor S of M (cf. [6], [7] and [19]).

By (1.6) we have

$$\begin{aligned} (3.1) \quad \nabla^2(S)(X, Y) &= \sum_{i=1}^{m+1} \nabla_{E_i} \nabla_{E_i}(S)(X, Y) \\ &= \sum_{i=1}^{m+1} \big[\big(R(E_i, X)S\big)(E_i, Y) + \big(R(E_i, \varphi Y)S\big)(E_i, \varphi X)\big] - 4S(X, Y) + \\ &\quad + 4m\langle X, Y\rangle + (3r - 3m^2 - 3m)\eta(X)\eta(Y). \end{aligned}$$

Taking a φ-basis $\{E_i\}$ ($\varphi E_i = E_{n+i}$, $E_{m+1} = \xi$), by (3.1) we have

$$\begin{aligned} (3.2) \quad \langle \nabla^2 Q, Q\rangle &= \sum_{j=1}^{m+1} \nabla^2(S)(E_j, QE_j) = \sum_{j=1}^{m} \nabla^2(S)(E_j, QE_j) + \nabla^2(S)(\xi, Q\xi) \\ &= 2\sum_{i,j=1}^{m} \big(R(E_i, E_j)S\big)(E_i, QE_j) + 2\sum_{j=1}^{m} \big(R(\xi, E_j)S\big)(\xi, QE_j) - \\ &\quad - 4\,\mathrm{Tr}H^2 + 4m\,\mathrm{Tr}H + \nabla^2(S)(\xi, Q\xi). \end{aligned}$$

Now we assume that the contact Bochner curvature tensor of M vanishes. Then by (1.3), (1.4), (1.5), (2.1) and (2.5) we have the equations

$$\begin{aligned} (3.3) \quad & 2\sum_{i,j=1}^{m} \big(R(E_i, E_j)S\big)(E_i, QE_j) \\ &= -2\sum_{i,j=1}^{m} \big[S\big(R(E_i, E_j)E_i, QE_j\big) + S\big(E_i, R(E_i, E_j)QE_j\big)\big] \\ &= \frac{2}{m+4}(m\,\mathrm{Tr}H^3 - \mathrm{Tr}H \cdot \mathrm{Tr}H^2) + \frac{k-4}{m+4}[2m\,\mathrm{Tr}H^2 - 2(\mathrm{Tr}H)^2], \end{aligned}$$

where we have put $k = (\mathrm{Tr}H + 2m)/(m+2)$,

$$(3.4) \qquad 2\sum_{j=1}^{m} \big(R(\xi, E_j)S\big)(\xi, QE_j) = 2\,\mathrm{Tr}H^2 - 2m\,\mathrm{Tr}H,$$

$$(3.5) \qquad \nabla^2(S)(\xi, Q\xi) = 2m\,\mathrm{Tr}H - 2m^3.$$

Substituting (3.3), (3.4) and (3.5) into (3.2), we obtain

$$\langle \nabla^2 Q, Q\rangle = \frac{2}{m+4}(m\mathrm{Tr}H^3 - \mathrm{Tr}H\cdot\mathrm{Tr}H^2) - \tag{3.6}$$

$$-\frac{k-4}{m+4}[m\,\mathrm{Tr}H^2 - 2(\mathrm{Tr}H)^2] - 2\,\mathrm{Tr}H^2 + 4m\,\mathrm{Tr}H - 2m^3.$$

On the other hand, by the assumptions, the Ricci tensor S of M is η-parallel. Then $\mathrm{Tr}Q^2$ is a constant. Therefore, by (2.6), we obtain

$$\langle \nabla^2 Q, Q\rangle = \frac{1}{2}\Delta\,\mathrm{Tr}Q^2 - \langle \nabla Q, \nabla Q\rangle = -\langle \nabla Q, \nabla Q\rangle \tag{3.7}$$

$$= -2\,\mathrm{Tr}H^2 + 4m\,\mathrm{Tr}H - 2m^3.$$

By (3.6) and (3.7) we have

$$\frac{2m}{m+4}\mathrm{Tr}H^3 - \frac{4(m+1)}{(m+2)(m+4)}\mathrm{Tr}H\cdot\mathrm{Tr}H^2 + \tag{3.8}$$

$$+\frac{2}{(m+2)(m+4)}(\mathrm{Tr}H)^3 + \frac{4m}{m+2}\mathrm{Tr}H^2 - \frac{4}{m+2}(\mathrm{Tr}H)^2 = 0.$$

Using (2.8) and (2.11), we can rewrite equation (3.8) in the form of (3.9):

LEMMA 3.1. *Let M be an $(m+1)$-dimensional Sasakian manifold with constant scalar curvature. If the contact Bochner curvature tensor of M vanishes, then*

$$P + \frac{3m}{(m-1)(m+2)}\mathrm{Tr}G\cdot\mathrm{Tr}A^2 = 0, \tag{3.9}$$

where G is the contact Ricci operator and $\mathrm{Tr}G = \mathrm{Tr}H + 2m$.

THEOREM 1. *Let M be an $(m+1)$-dimensional Sasakian manifold with constant scalar curvature and vanishing contact Bochner curvature tensor. If the contact Ricci tensor of M is positive semi-definite or negative semi-definite, then M is of constant φ-sectional curvature.*

Proof. Let us assume that the contact Ricci tensor of M is positive semi-definite. Then Lemma 2.5 shows that $P \geqslant 0$. On the other hand, $\mathrm{Tr}G \geqslant 0$. If $\mathrm{Tr}G = 0$, by the assumption we have $G = 0$, and hence M is η-Einstein. If $\mathrm{Tr}G \neq 0$, by (3.9) we must have $\mathrm{Tr}A^2 = 0$, and hence M is η-Einstein. Therefore, Lemma 2.2 shows that M is of constant φ-sectional curvature. Similarly, if G is negative semi-definite, we have $P \leqslant 0$

and $\mathrm{Tr}G \leqslant 0$, and M is an η-Einstein manifold. Thus M is of constant φ-sectional curvature.

Remark. For Theorem A, we can see the following

THEOREM 2. *Let M be a real m-dimensional Kähler manifold with constant scalar curvature and vanishing Bochner curvature tensor. If the Ricci tensor of M is positive semi-definite or negative semi-definite, then M is of constant holomorphic sectional curvature.*

Proof. By the assumptions we see that the Ricci tensor of M is parallel (see Matsumoto [8]). Therefore, using equation (3.4) in Yano and Ishihara [19], we have our assertion by the quite similar method to that in the proof of Theorem 1.

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