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SASAKIAN ϕ-SYMMETRIC SPACES*

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1. Introduction. It is known that a Sasakian manifold which is at the same time a locally symmetric space is a space of constant curvature (Okumura [6]). This fact means that a symmetric space condition is too strong for a Sasakian manifold. In this note, we introduce a notion of Sasakian ϕ-symmetric space which is an analogous notion of Hermitian symmetric space, and discuss about its properties.

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2. Definition of Sasakian locally ϕ-symmetric space. Let M be a $(2n+1)$-dimensional Sasakian manifold with structure tensors ϕ, ξ, η and g:

$$\text{(2.1)} \qquad \begin{cases} \phi^2 X = -X + \eta(X)\xi \\ \eta(\xi) = 1 \end{cases}$$

$$\text{(2.2)} \qquad \begin{cases} g(X, \xi) = \eta(X) \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{cases}$$

$$\text{(2.3)} \qquad \begin{cases} d\eta(X, Y) = g(\phi X, Y) \\ (\nabla_X \phi) Y = \eta(Y)X - g(X, Y)\xi \, , \end{cases}$$

where ∇ is the Riemannian connection for g and X, Y are tangent vectors on M. Let $\tilde{U}$ be a small open neighborhood of $x \in M$ such that the induced Sasakian structure on $\tilde{U}$, denoted by the same letters, is regular. Let $\pi: \tilde{U} \to U = \tilde{U}/\xi$ be a (local) fibering, and let $(J, \bar{g})$ be the induced Kählerian structure on U (cf. Tanno-Baik [10], Ogiue [5]). Let R and $\bar{R}$ be the curvature tensors constructed by g and $\bar{g}$, respectively. For a vector field $\bar{X}$ on U, we denote its horizontal lift (with respect to the connection from η) by $\bar{X}^*$. Then we have, for any vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ on U,

$$\text{(2.4)} \qquad (\bar{\nabla}_{\bar{X}} \bar{Y})^* = \nabla_{\bar{X}^*} \bar{Y}^* - \eta(\nabla_{\bar{X}^*} \bar{Y}^*)\xi \, ,$$

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$$(2.5)\qquad \begin{aligned}&(\bar{R}(\bar{X}, \bar{Y})\bar{Z})^* \\ &= R(\bar{X}^*, \bar{Y}^*)\bar{Z}^* + g(\phi\bar{Y}^*, \bar{Z}^*)\phi\bar{X}^* - g(\phi\bar{X}^*, \bar{Z}^*)\phi\bar{Y}^* \\ &\quad - 2g(\phi\bar{X}^*, \bar{Y}^*)\phi\bar{Z}^* ,\end{aligned}$$

where $\bar{\nabla}$ is the Riemannian connection for $\bar{g}$ (Ogiue [5]). From (2.3), we get

$$(2.6)\qquad \nabla_X\xi = \phi X$$

and hence

$$(2.7)\qquad R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

$$(2.8)\qquad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi .$$

Making use of (2.4)~(2.8), we get

$$(2.9)\qquad ((\bar{\nabla}_{\bar{V}}\bar{R})(\bar{X}, \bar{Y})\bar{Z})^* = -\phi^2[(\nabla_{\bar{V}^*}R)(\bar{X}^*, \bar{Y}^*)\bar{Z}^*]$$

for any vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{V}$ on U. Hence the following definition seems to be quite natural.

DEFINITION. A Sasakian manifold is said to be a locally ϕ-symmetric space if

$$(2.10)\qquad \phi^2[(\nabla_V R)(X, Y)Z] = 0$$

holds for any horizontal vectors X, Y, Z and V, where a horizontal vector means that it is horizontal with respect to the connection form η of the local fibering; namely, a horizontal vector is nothing but a vector which is orthogonal to ξ.

THEOREM 2.1. *A Sasakian manifold is a locally ϕ-symmetric space if and only if each Kählerian manifold, which is a base space of a local fibering, is a Hermitian locally symmetric space.*

EXAMPLE 2.2. Suppose a Sasakian manifold is of constant ϕ-holomorphic sectional curvature. Then the Kählerian manifolds given by local fiberings are of constant holomorphic sectional curvature (Ogiue [5]), and hence Hermitian locally symmetric spaces. Thus a Sasakian manifold which is of constant ϕ-holomorphic sectional curvature is a locally ϕ-symmetric space.

EXAMPLE 2.3. The following example is due to Kato-Motomiya [2].

Let $M = G/G_0$ be a homogeneous space of a semisimple, compact and simply connected Lie group G over a connected, closed subgroup G_0 of G, and assume that the Lie algebra $\mathfrak{G}$ of G has a family $(\mathfrak{G}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{G}$ satisfying the following conditions (i)~(vi):

(i) $\mathfrak{G} = \sum \mathfrak{G}_i$ (direct sum)
(ii) $[\mathfrak{G}_i, \mathfrak{G}_j] \subset \mathfrak{G}_{i+j} + \mathfrak{G}_{|i-j|}$
(iii) $\dim \mathfrak{G}_2 = 1$, and $\mathfrak{G}_i = \{0\}$ if $i > 2$
(iv) There is an element u of $\mathfrak{G}_2$ such that

$$[u, [u, X]] = -X \quad \text{for all} \quad X \in \mathfrak{G}_1$$

(v) $\mathfrak{G}_0$ is the Lie algebra of G_0, and $[\mathfrak{G}_0, \mathfrak{G}_0] = \mathfrak{G}_0$
(vi) $\mathrm{Ad}\,(g)\mathfrak{G}_i = \mathfrak{G}_i$, and $\mathrm{Ad}\,(g)u = u$ for all $g \in G_0$, where $\mathrm{Ad}\,(g)$ denotes the adjoint representation of G_0 in $\mathfrak{G}$.

Then the homogeneous space $M = G/G_0$ admits a G-invariant normal almost contact metric structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ such that

$$\bar{\phi}_0 = \mathrm{ad}_{\mathfrak{M}} u \ , \quad \bar{\xi}_0 = u \ , \quad \bar{\eta}_0 = u^* \quad \text{and} \quad \bar{g} = -\frac{1}{2n} B_{\mathfrak{M}} \ ,$$

where $\mathfrak{M} = \mathfrak{G}_1 + \mathfrak{G}_2$ which is identified with the tangent space $T_0(G/G_0)$ of G/G_0 at $0 = \{G_0\} \in G/G_0$, $\mathrm{ad}_{\mathfrak{M}}\, u$ is the restriction of $\mathrm{ad}\, u$ on $\mathfrak{M}$, u^* is a 1-form on $\mathfrak{M}$ defined by $u^*(u) = 1$ and $u^*(X) = 0$ for all $X \in \mathfrak{G}_1$, $B_{\mathfrak{M}}$ is the restriction of the Killing form B of $\mathfrak{G}$ on $\mathfrak{M}$, and $\dim M = 2n + 1$. This almost contact metric structure satisfies

$$2d\bar{\eta}(X, Y) = \bar{g}(X, \bar{\phi} Y) \ .$$

Hence, if we put

$$\phi = -\bar{\phi} \ , \quad \xi = 2\bar{\xi} \ , \quad \eta = \frac{1}{2}\bar{\eta} \quad \text{and} \quad g = \frac{1}{4}\bar{g} \ ,$$

then (ϕ, ξ, η, g) is a G-invariant Sasakian structure of M. This Sasakian manifold is a principal circle bundle over a Hermitian symmetric space $\tilde{M} = G/H$ with the Kählerian structure (J, Q), where H is the connected Lie subgroup of G with the Lie algebra $\mathfrak{G}_0 + \mathfrak{G}_2$, $J_0 = -\mathrm{ad}_{\mathfrak{G}_1} u$, and $Q_0 = -(1/8n)B_{\mathfrak{G}_1}$. Thus the Sasakian manifold $M = G/G_0$ is a locally ϕ-symmetric space.

REMARK 2.4. In Example 2.3, it turns out that G is simple. Because, since G/H is a compact type Hermitian symmetric space, H has a non-discrete center, say Z; and if G/H is reducible, we have $\dim Z \geqq 2$. On the other hand, the condition (v) implies that $\dim Z = 1$. Hence G/H must be irreducible and hence G must be simple.

EXAMPLE 2.5. Let G/H be an irreducible Hermitian symmetric space so that G is a simple Lie group and H is a compact Lie subgroup of G. In this case, G acts effectively on G/H, H is connected (cf. Helgason [1], p. 214, Thm. 1.1 and p. 305, the proof of Thm. 4.6), G/H is simply connected (cf. Helgason [1], p. 305, Thm. 4.6), and hence we may assume that G is simply connected. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Lie algebras of G and H, respectively, and let

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{M}'$$

be the canonical decomposition. Let B be the Killing form of $\mathfrak{G}$ and let (J, Q) be the Kählerian structure of G/H such that, if G is compact, $Q_0 = (-1/8n)B_{\mathfrak{M}'}$ and if G is noncompact, $Q_0 = (1/8n)B_{\mathfrak{M}'}$, where Q_0 is the restriction of Q to the tangent space $T_0(G/H)$ of G/H at $0 = \{H\} \in G/H$, $\mathfrak{M}'$ is identified with $T_0(G/H)$, $B_{\mathfrak{M}'}$ is the restriction of B to $\mathfrak{M}'$ and $2n$ is the dimension of G/H (cf. Kobayashi-Nomizu [3], p. 250, Prop. 7.4). The center $\mathfrak{Z}$ of $\mathfrak{H}$ is 1-dimensional and there exists $Z_0 \in \mathfrak{Z}$ such that

$$J_0 = -\mathrm{ad}_{\mathfrak{M}'} Z_0 ,$$

where J_0 is the restriction of J to $T_0(G/H)$ and $\mathrm{ad}_{\mathfrak{M}'}$ is the restriction to $\mathfrak{M}'$ of the adjoint representation of $\mathfrak{G}$ in $\mathfrak{G}$ (cf. Kobayashi-Nomizu [3], p. 261, Thm. 9.6). Since H is compact, we have the direct sum decomposition

$$\mathfrak{H} = \mathfrak{G}_0 + \mathfrak{G}_2 ,$$

where $\mathfrak{G}_2$ is the center $\mathfrak{Z}$ of $\mathfrak{H}$, and $\mathfrak{G}_0$ is the ideal $[\mathfrak{H}, \mathfrak{H}]$ of $\mathfrak{H}$ and it is semisimple and compact (cf. Helgason [1], p. 122, Prop. 6.6). If we put $\mathfrak{G}_1 = \mathfrak{M}'$, we get the following:

(i) $\mathfrak{G} = \mathfrak{G}_0 + \mathfrak{G}_1 + \mathfrak{G}_2$ (direct sum)

(ii) $[\mathfrak{G}_i, \mathfrak{G}_j] \subset \mathfrak{G}_{i+j} + \mathfrak{G}_{|i-j|}$, where $\mathfrak{G}_l = \{0\}$ for $l > 2$

(iii) $\dim \mathfrak{G}_2 = 1$

(iv) $[Z_0, [Z_0, X]] = -X$ for all $X \in \mathfrak{G}_1$

(v) $[\mathfrak{G}_0, \mathfrak{G}_0] = \mathfrak{G}_0$

(vi) $\mathrm{Ad}(g)\mathfrak{G}_i = \mathfrak{G}_i$ and $\mathrm{Ad}(g)Z_0 = Z_0$ for all $g \in G_0$, where G_0 is a connected Lie subgroup of H with the Lie algebra $\mathfrak{G}_0$, and Ad is the adjoint representation of G in $\mathfrak{G}$.

PROOF. (i), (iii), (iv) and (v) are trivial by the definitions of $\mathfrak{G}_i$ and Z_0.

$[\mathfrak{G}_0, \mathfrak{G}_0] = \mathfrak{G}_0$, $[\mathfrak{G}_0, \mathfrak{G}_1] \subset [\mathfrak{H}, \mathfrak{M}'] \subset \mathfrak{M}' = \mathfrak{G}_1$, $[\mathfrak{G}_0, \mathfrak{G}_2] = \{0\}$, $[\mathfrak{G}_1, \mathfrak{G}_1] = [\mathfrak{M}', \mathfrak{M}'] \subset \mathfrak{H} = \mathfrak{G}_0 + \mathfrak{G}_2$, $[\mathfrak{G}_1, \mathfrak{G}_2] = [\mathfrak{M}', Z_0] = J_0\mathfrak{M}' = \mathfrak{M}' = \mathfrak{G}_1$ and $[\mathfrak{G}_2, \mathfrak{G}_2] = \{0\}$ show (ii).

Since G_0 is connected, (ii) implies $\mathrm{Ad}(g)\mathfrak{G}_i = \mathfrak{G}_i$ for all $g \in G_0$, and since Z_0 is an element of the center of $\mathfrak{H}$, we get $\mathrm{Ad}(g)Z_0 = Z_0$ for all $g \in G_0$. Thus we get (vi). q.e.d.

Hence, according to Kato-Motomiya [2] (with simple modifications),

G/G_0 admits a G-invariant Sasakian structure (ϕ, ξ, η, g) such that

$$\phi_0 = -\mathrm{ad}_{\mathfrak{M}} Z_0,\ \xi_0 = 2Z_0,\ \eta_0 = \frac{1}{2} Z_0^* ,$$

$$\text{and } g_0 = -\frac{1}{8n} B_{\mathfrak{M}} \text{ if } G \text{ is compact}$$

$$\text{and } g_0 = \frac{1}{8n} B_{\mathfrak{M}} + 2\eta_0 \otimes \eta_0 \text{ if } G \text{ is noncompact} ,$$

where $\mathfrak{M} = \mathfrak{G}_1 + \mathfrak{G}_2$ which is identified with the tangent space $T_0(G/G_0)$, Z_0^* is a 1-form on $\mathfrak{M}$ defined by $Z_0^*(Z_0) = 1$ and $Z_0^*(X) = 0$ for all $X \in \mathfrak{G}_1$. Moreover, this Sasakian manifold G/G_0 is a principal circle bundle over the original Hermitian symmetric space, and hence it is a locally ϕ-symmetric space.

If G is compact (resp. noncompact), we call the homogeneous Sasakian manifold with the above Sasakian structure to be a Sasakian ϕ-symmetric space of the compact (resp. noncompact) type. The precise definitions of the types will be given in the Section 6.

REMARK 2.6. In Example 2.5, if G is noncompact and if we put $g_0' = (1/8n)B_{\mathfrak{M}}$, g_0' induces a G-invariant pseudo-Riemannian metric g' on G/G_0 and (ϕ, ξ, η, g') is a Sasakian structure with a pseudo-Riemannian metric in the sense of Takahashi [7].

REMARK 2.7. Let M be a Sasakian manifold with the property that

$$\phi^2[(\nabla_V R)(X, Y)\xi] = 0 \tag{2.11}$$

holds for any horizontal vectors X, Y and V. Then M is of constant curvature 1. To show it, we need the following lemmas which follow from direct calculations.

LEMMA 2.8 (Tanno [9]). *For any tangent vectors X, Y and Z of M,*

$$R(X, Y)\phi Z = g(\phi X, Z)Y - g(Y, Z)\phi X - g(\phi Y, Z)X + g(X, Z)\phi Y + \phi R(X, Y)Z \tag{2.12}$$

holds good.

Lemmas 2.8 and (2.7) imply the following:

LEMMA 2.9. *For any tangent vectors X, Y and V of M, we get*

$$(\nabla_V R)(X, Y)\xi = g(Y, V)\phi X - g(X, V)\phi Y - \phi R(X, Y)V . \tag{2.13}$$

Now, if X, Y and V are horizontal, then (2.11) implies that the left hand side of (2.13) is a scalar multiple of ξ. On the other hand, the right hand side of (2.13) is horizontal. Hence we get $\phi R(X, Y)V = g(Y, V)\phi X - g(X, V)\phi Y$, which implies that $R(X, Y)V = g(Y, V)X - g(X, V)Y$ holds for any horizontal vectors X, Y and V. Hence M is of constant ϕ-holomorphic sectional curvature 1 and hence of constant curvature 1.

REMARK 2.10. Suppose a Sasakian manifold is a locally ϕ-symmetric space, then (2.10) and (2.13) imply that

(2.14) $$(\nabla_V R)(X, Y)Z = \{g(X, V)g(\phi Y, Z) - g(Y, V)g(\phi X, Z) + g(Z, \phi R(X, Y)V)\}\xi$$

holds good for any horizontal vectors X, Y, Z and V, and vice versa.

3. **ϕ-geodesic symmetry.** A geodesic $\gamma = \gamma(s)$ in a Sasakian manifold M is said to be ϕ-geodesic if its velocity vectors are horizontal; that is, $\eta(\gamma'(s)) = 0$ holds for each s. A local diffeomorphism σ_x of a Sasakian manifold M, $x \in M$, is said to be ϕ-geodesic symmetry at x if, for each ϕ-geodesic $\gamma = \gamma(s)$ such that $\gamma(0)$ lies in the trajectory of ξ passing through x,

(3.1) $$\sigma_x \gamma(s) = \gamma(-s)$$

holds for each s.

REMARK 3.1. Any geodesic $\gamma = \gamma(s)$ with $\eta(\gamma'(0)) = 0$ is a ϕ-geodesic, because the angles of a geodesic and a Killing vector field are constant along the geodesic.

Now, suppose a ϕ-geodesic symmetry σ_x at $x \in M$ be a local automorphism. Let $\tilde{U}$ be an open neighborhood of x with a local fibering $\pi: \tilde{U} \rightarrow U = \tilde{U}/\xi$. Then π induces a geodesic symmetry $\bar{\sigma}_{\pi(x)}$ of U at $\pi(x)$ such that $\bar{\sigma}_{\pi(x)} \circ \pi = \pi \circ \sigma_x$, which is an automorphism of the induced Kählerian structure of U. Hence, making use of Theorem 2.1, we see that if a Sasakian manifold admits a ϕ-geodesic symmetry, which is a local automorphism, at each point, then it is a locally ϕ-symmetric space. The converse also holds good, and its proof will be given in the Section 5. Namely, we get

THEOREM 3.2. *A necessary and sufficient condition for a Sasakian manifold to be a locally ϕ-symmetric space is that it admits a ϕ-geodesic symmetry, which is a local automorphism, at every point.*

REMARK 3.3. We can similarly define a notion of ϕ-geodesic symmetry for a K-contact Riemannian manifold. (2.4) and Remark 3.1 also hold for a K-contact Riemannian manifold. Suppose a ϕ-geodesic symmetry of a K-contact Riemannian manifold is a local automorphism. Then the induced geodesic symmetry of the base space of the local fibering is an automorphism of the induced almost Kählerian structure. Hence a K-contact Riemannian manifold with a ϕ-geodesic symmetry, which is an automorphism, at every point is a Sasakian manifold.

4. **Sectional curvature.** Let $\gamma = \gamma(s)$ be a ϕ-geodesic of a Sasakian manifold M, and let X_0 be a horizontal vector of M at $x_0 = \gamma(0)$. Let $X = X(s)$ and $Z = Z(s)$ be the parallel vector fields along γ such that $X(0) = X_0$ and $Z(0) = \phi\gamma'(0)$. Put $f(s) = g(X(s), \xi)$. Then we get $f'(s) = g(X(s), \phi\gamma'(s))$ and $f''(s) = -g(X(s), \xi) = -f(s)$. Hence, since $f(0) = 0$, we get $f(s) = \alpha \sin s$ for some constant α. Since $\gamma'(s)$ and $X(s)$ are parallel along γ, $g(X(s), \gamma'(s))$ is constant along γ, say c. Let $Y = Y(s)$ be a vector field along γ defined by

$$Y(s) = \phi X(s) + \alpha(\cos s - 1)\gamma'(s) - c\phi\gamma'(s) + cZ(s) . \tag{4.1}$$

Then we get

$$\begin{aligned} Y(0) &= \phi X_0 - c\phi\gamma'(0) + c\phi\gamma'(0) = \phi X_0 , \\ \nabla_{\gamma'} Y &= (\nabla_{\gamma'}\phi)X - \alpha \sin s\gamma' - c(\nabla_{\gamma'}\phi)\gamma' \\ &= \eta(X)\gamma' - g(\gamma', X)\xi - \alpha \sin s\gamma' + cg(\gamma', \gamma')\xi = 0 . \end{aligned}$$

Hence $Y(s)$ is the parallel translate of ϕX_0 along γ.

LEMMA 4.1. *If a horizontal vector Y_0 is orthogonal to $\gamma'(0)$ and $\phi\gamma'(0)$, then its parallel translate $Y(s)$ along γ is also horizontal.*

PROOF. Since Y_0 is horizontal and orthogonal to $\gamma'(0)$ and $\phi\gamma'(0)$, there exists a horizontal vector X_0 such that $\phi X_0 = Y_0$ and it is orthogonal to $\gamma'(0)$ and $\phi\gamma'(0)$. Let $X(s)$ and $Y(s)$ be the parallel translates of X_0 and Y_0, respectively, along γ. Then $Y(s)$ is given by (4.1) with $c = 0$, and hence it is horizontal along γ. q.e.d.

On the other hand, (2.3) and (2.6) imply the following:

LEMMA 4.2. *The parallel translate $Z(s)$ of $\phi\gamma'(0)$ along γ is given by*

$$Z(s) = \cos s\phi\gamma'(s) + \sin s\xi . \tag{4.2}$$

Now, let $\{X_0, Y_0\}$ be orthonormal horizontal vectors such that each of them is equal to $\phi\gamma'(0)$ or orthogonal to $\phi\gamma'(0)$. Let $X(s)$ and $Y(s)$ be parallel translates of X_0 and Y_0 along γ, respectively, and let $\bar{X}(s)$ and $\bar{Y}(s)$ be their orthogonal projections to η-section:

$$\begin{aligned} \bar{X}(s) &= X(s) - \eta(X(s))\xi , \\ \bar{Y}(s) &= Y(s) - \eta(Y(s))\xi . \end{aligned}$$

Then, from Lemmas 4.1 and 4.2, we see that $\{\bar{X}(s), \bar{Y}(s)\}$ span a 2-plane

$\mathscr{P}(s)$ for $-\pi/2 < s < \pi/2$. Let $K(s) = K(\mathscr{P}(s))$, $-\pi/2 < s < \pi/2$, be the sectional curvature for $\mathscr{P}(s)$.

(i) Suppose X_0 and Y_0 are orthogonal to $\phi\gamma'(0)$. Then Lemma 4.1 implies $\bar{X}(s) = X(s)$ and $\bar{Y}(s) = Y(s)$. Hence we get

$$\begin{aligned}\frac{d}{ds}K(s) &= \frac{d}{ds}g(R(X(s), Y(s))Y(s), X(s)) \\ &= g((\nabla_{\gamma'(s)}R)(\bar{X}(s), \bar{Y}(s))\bar{Y}(s), \bar{X}(s)) .\end{aligned} \tag{4.3}$$

(ii) Suppose X_0 is orthogonal to $\phi\gamma'(0)$ and $Y_0 = \phi\gamma'(0)$. Then Lemma 4.2, (2.6), (2.7) and (2.8) imply

$$\begin{aligned}\frac{d}{ds}K(s) &= \frac{d}{ds}g(R(X(s), \phi\gamma'(s))\phi\gamma'(s), X(s)) \\ &= g((\nabla_{\gamma'(s)}R)(X(s), \phi\gamma'(s))\phi\gamma'(s), X(s)) \\ &\quad + g(R(X(s), (\nabla_{\gamma'(s)}\phi)\gamma'(s))\phi\gamma'(s), X(s)) \\ &\quad + g(R(X(s), \phi\gamma'(s))(\nabla_{\gamma'(s)}\phi)\gamma'(s), X(s)) \\ &= g((\nabla_{\gamma'(s)}R)(\bar{X}(s), \bar{Y}(s))\bar{Y}(s), \bar{X}(s))/\cos^2 s .\end{aligned} \tag{4.4}$$

Hence, if a Sasakian manifold is locally ϕ-symmetric, then the sectional curvature $K(s) = K(\mathscr{P}(s))$ is constant along γ for $-\pi/2 < s < \pi/2$. The converse also holds. Namely, we get the following theorem:

THEOREM 4.3. *A Sasakian manifold is a locally ϕ-symmetric space if and only if the sectional curvature $K(s) = K(\mathscr{P}(s))(-\pi/2 < s < \pi/2)$, defined as above, is constant for any orthonormal, horizontal vectors X_0 and Y_0 such that each of them is equal to, or orthogonal to $\phi\gamma'(0)$, and along any ϕ-geodesic $\gamma = \gamma(s)$.*

PROOF. Suppose the sectional curvature $K(s)$ is constant along any ϕ-geodesic. Then

$$g((\nabla_Z R)(X, Y)Y, X) = 0 \tag{4.5}$$

holds for any horizontal vectors X, Y and Z such that each of X and Y is equal to, or orthogonal to ϕZ. In particular, we get

$$g((\nabla_{e_i}R)(e_j, e_k)e_k, e_j) = 0 , \quad i, j, k = 1, 2, \cdots, 2n \tag{4.6}$$

for an arbitrary orthonormal basis $\{e_1, e_2, \cdots, e_{2n}, \xi\}$ of a tangent space of the Sasakian manifold in consideration. It is sufficient to show that (4.5) (or (4.6)) implies that

$$g((\nabla_{e_1}R)(e_j, e_k)e_l, e_m) = 0 \tag{4.7}$$

holds good for $j, k, l, m = 1, 2, \cdots, 2n$.

STEP I. First of all, we assume $e_2 = \phi e_1$ and $e_4 = \phi e_3$, and show (4.7) for $j, k, l, m = 1, 2, 3, 4$. For this purpose, it is sufficient to show that the following hold good for $\alpha, \beta, \gamma, \delta = 3, 4$:

(a) $g((\nabla_{e_1}R)(e_1, e_2)e_1, e_2) = 0$
(b) $g((\nabla_{e_1}R)(e_1, e_2)e_1, e_\alpha) = 0$
(c) $g((\nabla_{e_1}R)(e_1, e_2)e_2, e_\alpha) = 0$
(d) $g((\nabla_{e_1}R)(e_1, e_2)e_\alpha, e_\beta) = 0$
(e) $g((\nabla_{e_1}R)(e_1, e_\alpha)e_1, e_\beta) = 0$
(f) $g((\nabla_{e_1}R)(e_1, e_\alpha)e_2, e_\beta) = 0$
(g) $g((\nabla_{e_1}R)(e_1, e_\alpha)e_\beta, e_\gamma) = 0$
(h) $g((\nabla_{e_1}R)(e_2, e_\alpha)e_2, e_\beta) = 0$
(i) $g((\nabla_{e_1}R)(e_2, e_\alpha)e_\beta, e_\gamma) = 0$
(j) $g((\nabla_{e_1}R)(e_3, e_4)e_3, e_4) = 0$.

According to (4.6), (a) and (j) are trivial. In the following, the indices $\{i, j, k, l\}$ and $\{\alpha, \beta, \gamma, \delta\}$ range $\{1, 2\}$ and $\{3, 4\}$, respectively.

(4.6) implies

$$g((\nabla_{e_i \pm e_j}R)(e_i \pm e_j, e_\alpha)e_\alpha, e_i \pm e_j) = 0 . \tag{4.8}$$

Making use of (4.6), (4.8) reduces to

$$g((\nabla_{e_i}R)(e_i, e_\alpha)e_\alpha, e_j) \pm g((\nabla_{e_j}R)(e_i, e_\alpha)e_\alpha, e_j) = 0 , \tag{4.9}$$

and hence we get

$$g((\nabla_{e_i}R)(e_i, e_\alpha)e_\alpha, e_j) = 0 . \tag{4.10}$$

Applying the 2nd Bianchi identity to $g((\nabla_{e_i}R)(e_\alpha, e_j)e_i, e_\alpha) = 0$, which is equivalent to (4.10), we get $g((\nabla_{e_\alpha}R)(e_j, e_i)e_i, e_\alpha) + g((\nabla_{e_j}R)(e_i, e_\alpha)e_i, e_\alpha) = 0$. Since the second term of the left hand side of the last equation vanishes by (4.6), we get $g((\nabla_{e_\alpha}R)(e_j, e_i)e_i, e_\alpha) = 0$. Thus we get

$$g((\nabla_{e_i}R)(e_\alpha, e_\beta)e_\beta, e_i) = 0 . \tag{4.11}$$

(4.6) implies $g((\nabla_{e_i}R)(e_j, e_\alpha + e_\beta)(e_\alpha + e_\beta), e_j) = 0$, which implies

$$g((\nabla_{e_i}R)(e_j, e_\alpha)e_\beta, e_j) = 0 . \tag{4.12}$$

Applying the Bianchi's second identity to (4.12), we get

$$g((\nabla_{e_j}R)(e_\alpha, e_i)e_\beta, e_j) + g((\nabla_{e_\alpha}R)(e_i, e_j)e_\beta, e_j) = 0 . \tag{4.13}$$

Applying (4.6) to orthonormal vectors $1/\sqrt{2}(e_1 + e_3)$, $1/\sqrt{2}\phi(e_1 + e_3) = 1/\sqrt{2}(e_2 + e_4)$, $1/\sqrt{2}(e_1 - e_3)$, $1/\sqrt{2}\phi(e_1 - e_3) = 1/\sqrt{2}(e_2 - e_4)$, we get

$$g((\nabla_{e_1 \pm e_3}R)(e_1 - e_3, e_2 - e_4)(e_2 - e_4), e_1 - e_3) = 0 , \tag{4.14}$$

$$g((\nabla_{e_1 \pm e_3}R)(e_1 - e_3, e_2 + e_4)(e_2 + e_4), e_1 - e_3) = 0 , \tag{4.15}$$

$$g((\nabla_{e_1 \pm e_3}R)(e_1 + e_3, e_2 - e_4)(e_2 - e_4), e_1 + e_3) = 0 . \tag{4.16}$$

Making use of (4.6), (4.11) and the property of the curvature tensor R to the effect $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$, (4.14) reduces to

$$
\begin{aligned}
(4.17)\qquad & -g((\nabla_{e_1}R)(e_1, e_2)e_2, e_3) - g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) \\
& +g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) + g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) \\
& -g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) \mp g((\nabla_{e_3}R)(e_1, e_2)e_4, e_1) \\
& \pm g((\nabla_{e_3}R)(e_1, e_2)e_4, e_3) \pm g((\nabla_{e_3}R)(e_1, e_4)e_2, e_3) \\
& \mp g((\nabla_{e_3}R)(e_1, e_4)e_4, e_3) \mp g((\nabla_{e_3}R)(e_3, e_2)e_4, e_3) = 0 .
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
(4.18)\qquad & -g((\nabla_{e_1}R)(e_1, e_2)e_2, e_3) - g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) \\
& +g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) + g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) \\
& -g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0 .
\end{aligned}
$$

Since (4.15) is nothing but (4.14) by replacing e_4 with $-e_4$, we get

$$
\begin{aligned}
(4.19)\qquad & -g((\nabla_{e_1}R)(e_1, e_2)e_2, e_3) + g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) \\
& -g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) - g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) \\
& +g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0 .
\end{aligned}
$$

(4.18) and (4.19) imply

$$(4.20)\qquad g((\nabla_{e_1}R)(e_1, e_2)e_2, e_3) = 0 ,$$

$$
\begin{aligned}
(4.21)\qquad & -g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) + g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) \\
& +g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) - g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0 .
\end{aligned}
$$

(4.20) implies

$$(4.22)\qquad g((\nabla_{e_1}R)(e_1, e_2)e_2, e_4) = 0 .$$

Similarly, (4.16) reduces to

$$
\begin{aligned}
(4.23)\qquad & -g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) - g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) \\
& -g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) - g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0 ,
\end{aligned}
$$

where we have used (4.20). Making use of (4.21) and (4.23), we get

$$(4.24)\qquad g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) + g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0 .$$

Applying (4.6) to the orthonormal vectors

$$
\begin{aligned}
& 1/\sqrt{3}(e_1 + \sqrt{2}e_3),\ 1/\sqrt{3}\phi(e_1 + \sqrt{2}e_3) = 1/\sqrt{3}(e_2 + \sqrt{2}e_4) , \\
& 1/\sqrt{3}(\sqrt{2}e_1 - e_3),\ 1/\sqrt{3}\phi(\sqrt{2}e_1 - e_3) = 1/\sqrt{3}(\sqrt{2}e_2 - e_4) ,
\end{aligned}
$$

we get

$$(4.25)\quad g((\nabla_{e_1+\sqrt{2}e_3}R)(\sqrt{2}e_1 - e_3, \sqrt{2}e_2 - e_4)(\sqrt{2}e_2 - e_4), \sqrt{2}e_1 - e_3) = 0 ,$$

$$(4.26)\quad g((\nabla_{\sqrt{2}e_1-e_3}R)(\sqrt{2}e_1 - e_3, \sqrt{2}e_2 - e_4)(\sqrt{2}e_2 - e_4), \sqrt{2}e_1 - e_3) = 0 .$$

They respectively reduce to

$$\begin{aligned}(4.27)\qquad & -2\sqrt{2}\,g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) + 2g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) \\ & + 2g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) - \sqrt{2}\,g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) \\ & + \sqrt{2}\{-2\sqrt{2}\,g((\nabla_{e_3}R)(e_1, e_2)e_4, e_1) + 2g((\nabla_{e_3}R)(e_1, e_2)e_4, e_3) \\ & \quad + 2g((\nabla_{e_3}R)(e_1, e_4)e_2, e_3) - \sqrt{2}\,g((\nabla_{e_3}R)(e_3, e_2)e_4, e_3)\} = 0\ ,\end{aligned}$$

$$\begin{aligned}(4.28)\qquad & \sqrt{2}\{-2\sqrt{2}\,g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) + 2g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) \\ & \quad + 2g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) - \sqrt{2}\,g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3)\} \\ & - \{-2\sqrt{2}\,g((\nabla_{e_3}R)(e_1, e_2)e_4, e_1) + 2g((\nabla_{e_3}R)(e_1, e_2)e_4, e_3) \\ & \quad + 2g((\nabla_{e_3}R)(e_1, e_4)e_2, e_3) - \sqrt{2}\,g((\nabla_{e_3}R)(e_3, e_2)e_4, e_3)\} = 0\ ,\end{aligned}$$

where we have used (2.20). If we make (4.27) + (4.28) $\times \sqrt{2}$, we get

$$\begin{aligned}(4.29)\qquad & -2\sqrt{2}\,g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) + 2g((\nabla_{e_1}R)(e_1, e_2)e_4, e_3) \\ & +2g((\nabla_{e_1}R)(e_1, e_4)e_2, e_3) - \sqrt{2}\,g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0\ .\end{aligned}$$

Making use of (4.21) and (4.29), we get

$$(4.30)\qquad 2g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) - \sqrt{2}\,g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0\ .$$

Hence, making use of (4.24) and (4.30), we get

$$(4.31)\qquad g((\nabla_{e_1}R)(e_1, e_2)e_4, e_1) = 0\ ,$$

$$(4.32)\qquad g((\nabla_{e_1}R)(e_3, e_2)e_4, e_3) = 0\ ,$$

and hence we get

$$(4.33)\qquad g((\nabla_{e_1}R)(e_1, e_2)e_3, e_1) = 0\ ,$$

$$(4.34)\qquad g((\nabla_{e_1}R)(e_4, e_2)e_3, e_4) = 0\ .$$

Now, (4.11) and (4.34) imply that the second term of the left hand side of (4.13) vanishes. Hence we get

$$(4.35)\qquad g((\nabla_{e_j}R)(e_i, e_\alpha)e_\beta, e_j) = 0\ .$$

Now, (b) follows from (4.31) and (4.33). (c) follows from (4.20) and (4.22). (d) follows from (4.35) with the 1st Bianchi identity. (e) follows from (4.12) or (4.35). (f) follows from (4.35). (g) follows from (4.11). (h) follows from (4.12) or (4.35). (i) follows from (4.32) and (4.34).

STEP II. To prove the general case, we use some special technical terms, which are suggested by Prof. S. Tanno. A ϕ-invariant 2-plane is spaned by a horizontal vector X and ϕX, denoted by $\mathscr{P}(X)$. Tangent vectors which belong to the same ϕ-invariant 2-plane are called to be relatives. Tangent vectors which belong to mutually orthogonal ϕ-invariant 2-planes are called to be strangers to each other. For example, a tangent vector which is orthogonol to $\mathscr{P}(X)$ is called to be a stranger to X, and X and ϕX are relatives with X and also with ϕX.

As in Step I, we assume that $e_2 = \phi e_1$. To prove the general case, it is sufficient to show the following:

(k) $g((\nabla_{e_1}R)(X, Y)Z, W) = 0$ for any vectors X, Y, Z and W which are orthogonal to $e_2 = \phi e_1$,

(l) $g((\nabla_{e_1}R)(X, Y)Z, e_i) = 0$ for $i = 1, 2$, X, Y and Z are strangers to e_1 and all of them are relatives,

(m) $g((\nabla_{e_1}R)(X, Y)Z, e_i) = 0$ for $i = 1, 2$, X, Y and Z are strangers to e_1 and all of them are strangers to each other,

(n) $g((\nabla_{e_1}R)(X, Y)Z, e_i) = 0$ for $i = 1, 2$, X, Y and Z are strangers to e_1, X and Y are relatives and Z is a stranger to X (and hence to Y),

(o) $g((\nabla_{e_1}R)(X, Y)Z, e_i) = 0$ for $i = 1, 2$, X, Y and Z are strangers to e_1, Y and Z are relatives and X is a stranger to Y (and hence to Z),

(p) $g((\nabla_{e_1}R)(X, Y)e_2, e_1) = 0$, where X and Y are relatives and they are strangers to e_1,

(q) $g((\nabla_{e_1}R)(X, Y)e_2, e_1) = 0$, where X and Y are strangers to each other and they are strangers to e_1,

(r) $g((\nabla_{e_1}R)(X, e_i)Y, e_j) = 0$ for $i, j = 1, 2$, where X and Y are relatives and they are strangers to e_1,

(s) $g((\nabla_{e_1}R)(X, e_i)Y, e_j) = 0$ for $i, j = 1, 2$, where X and Y are strangers to each other and they are strangers to e_1,

(t) $g((\nabla_{e_1}R)(e_i, e_j)e_k, X) = 0$ for $i, j, k = 1, 2$ and X is a stranger to e_1.

First of all, we shall prove (k). Let X, Y, Z and W be orthogonal to $e_2 = \phi e_1$. Then (4.5) implies that $g((\nabla_{e_1}R)(X, Y + Z)(Y + Z), X) = 0$ and $g((\nabla_{e_1}R)(X + W, Y)Y, X + W) = 0$ hold, which respectively imply

$$g((\nabla_{e_1}R)(X, Y)Z, X) = 0\,, \tag{4.36}$$

$$g((\nabla_{e_1}R)(X, Y)Y, W) = 0\,. \tag{4.37}$$

(4.36) implies $g((\nabla_{e_1}R)(X + W, Y)Z, X + W) = 0$, which becomes $g((\nabla_{e_1}R)(X, Y)Z, W) + g((\nabla_{e_1}R)(W, Y)Z, X) = 0$. Applying the 1st Bianchi identity to the 1st term of the left hand side of the last equation, we get

$$2g((\nabla_{e_1}R)(W, Y)Z, X) + g((\nabla_{e_1}R)(Y, Z)W, X) = 0\,. \tag{4.38}$$

(4.37) implies $g((\nabla_{e_1}R)(X, Y + Z)(Y + Z), W) = 0$, which becomes

$$g((\nabla_{e_1}R)(X, Y)Z, W) + g((\nabla_{e_1}R)(X, Z)Y, W) = 0\,;$$

that is, we get

$$(4.39)\qquad g((\nabla_{e_1}R)(Y, Z)W, X) = -g((\nabla_{e_1}R)(Y, W)Z, X) = g((\nabla_{e_1}R)(W, Y)Z, X)\,.$$

Hence (4.38) and (4.39) imply $3g((\nabla_{e_1}R)(W, Y)Z, X) = 0$, which proves (k).

(l), (p), (r) and (t) are contained in Step I. (m) and (q) follow from (k) by applying the 2nd Bianchi identity. Next, we shall show (s).

Let X and Y be strangers to e_1. Then (4.5) implies

$$g((\nabla_{e_i}R)(X+Y, e_j)e_j, X+Y) = 0\,,$$

which reduces to

$$(4.40)\qquad g((\nabla_{e_i}R)(X, e_j)e_j, Y) = 0 \quad \text{for} \quad i, j = 1, 2\,.$$

On the other hand, we have

$$(4.41)\qquad g((\nabla_{e_1}R)(X, e_2)Y, e_1) = -g((\nabla_X R)(e_2, e_1)Y, e_1) - g((\nabla_{e_2}R)(e_1, X)Y, e_1)\,.$$

The first term of the right hand side of (4.41) vanishes by (k) and the second term vanishes by (4.40). This fact together with (4.40) proves (s).

Finally, we shall prove (o), which with the 1st Bianchi identity implies (n). Let f_1 and $f_2 = \phi f_1$ be strangers to e_1 and let X be a stranger to e_1 and f_1. We want to show $g((\nabla_{e_1}R)(X, f_i)f_j, e_k) = 0$ for $i, j, k = 1, 2$. Since we have

$$g((\nabla_{e_1}R)(X, f_i)f_j, e_k) = -g((\nabla_X R)(f_i, e_1)f_j, e_k) - g((\nabla_{f_i}R)(e_1, X)f_j, e_k)\,,$$

and since the first term of the right hand side of the last equation vanishes by (k), it is sufficient to show that

$$(4.42)\qquad g((\nabla_{f_i}R)(e_1, X)e_k, f_j) = 0 \quad \text{for} \quad i, j, k = 1, 2$$

holds good. If $i = j$, (k) implies (4.42). Thus it is sufficient to show that

$$(4.43)\qquad g((\nabla_{f_1}R)(e_1, X)e_i, f_2) = 0 \quad \text{for} \quad i = 1, 2$$

holds good. Since $e_1 + f_1$ and $\phi(e_1 + f_1) = e_2 + f_2$ are relatives, (t) implies

$$(4.44)\qquad g((\nabla_{e_1+f_1}R)(e_1 + f_1, e_2 + f_2)(e_2 + f_2), X) = 0\,.$$

Making use of (k), (q), (s) and (t), (4.44) reduces to

$$(4.45)\qquad g((\nabla_{e_1}R)(f_1, e_2)f_2, X) + g((\nabla_{e_1}R)(f_1, f_2)e_2, X) + g((\nabla_{f_1}R)(e_1, e_2)f_2, X) + g((\nabla_{f_1}R)(e_1, f_2)e_2, X) = 0\,.$$

Similarly, (s) and (q) respectively imply

$$(4.46)\qquad g((\nabla_{e_1+f_1}R)(e_1 - f_1, e_2 + f_2)(e_2 + f_2), X) = 0\,,$$

$$(4.47)\qquad g((\nabla_{e_1+f_1}R)(e_1+f_1,\ e_2+f_2)(e_2-f_2),\ X)=0\ ,$$

and they respectively reduce to

$$(4.48)\qquad -g((\nabla_{e_1}R)(f_1,\ e_2)f_2,\ X)-g((\nabla_{e_1}R)(f_1,\ f_2)e_2,\ X) + g((\nabla_{f_1}R)(e_1,\ e_2)f_2,\ X)+g((\nabla_{f_1}R)(e_1,\ f_2)e_2,\ X)=0\ ,$$

$$(4.49)\qquad -g((\nabla_{e_1}R)(f_1,\ e_2)f_2,\ X)+g((\nabla_{e_1}R)(f_1,\ f_2)e_2,\ X) -g((\nabla_{f_1}R)(e_1,\ e_2)f_2,\ X)+g((\nabla_{f_1}R)(e_1,\ f_2)e_2,\ X)=0\ .$$

(4.45) and (4.48), and (4.45) and (4.49) respectively imply

$$(4.50)\qquad g((\nabla_{e_1}R)(f_1,\ e_2)f_2,\ X)+g((\nabla_{e_1}R)(f_1,\ f_2)e_2,\ X)=0\ ,$$

$$(4.51)\qquad g((\nabla_{e_1}R)(f_1,\ e_2)f_2,\ X)+g((\nabla_{f_1}R)(e_1,\ e_2)f_2,\ X)=0\ .$$

Now, (4.50) and (4.51) imply

$$(4.52)\qquad g((\nabla_{e_1}R)(f_1,\ f_2)e_2,\ X)=g((\nabla_{f_1}R)(e_1,\ e_2)f_2,\ X)\ ,$$

and (4.51), (4.45) and (4.50) imply

$$(4.53)\qquad g((\nabla_{e_1}R)(f_1,\ e_2)f_2,\ X)=g((\nabla_{f_1}R)(e_1,\ f_2)e_2,\ X)\ .$$

Applying the 1st Bianchi identity to the both sides of (4.52), we get

$$(4.54)\qquad g((\nabla_{e_1}R)(f_2,\ e_2)f_1,\ X)+g((\nabla_{e_1}R)(e_2,\ f_1)f_2,\ X) =g((\nabla_{f_1}R)(e_2,\ f_2)e_1,\ X)+g((\nabla_{f_1}R)(f_2,\ e_1)e_2,\ X)\ .$$

Taking account of (4.53), (4.54) reduces to

$$(4.55)\qquad g((\nabla_{e_1}R)(f_2,\ e_2)f_1,\ X)=g((\nabla_{f_1}R)(e_2,\ f_2)e_1,\ X)\ .$$

The left hand side of (4.55) becomes

$$\begin{aligned} &g((\nabla_{e_1}R)(f_2,\ e_2)f_1,\ X)\\ &\quad =g((\nabla_{e_1}R)(f_1,\ X)f_2,\ e_2)=-g((\nabla_{f_1}R)(X,\ e_1)f_2,\ e_2)-g((\nabla_X R)(e_1,\ f_1)f_2,\ e_2)\\ &\quad =-\ g((\nabla_{f_1}R)(e_1,\ X)e_2,\ f_2)\ ,\end{aligned}$$

where we have used (k) to see that $g((\nabla_X R)(e_1,\ f_1)f_2,\ e_2)$ vanishes. Thus (4.55) reduces to

$$(4.56)\qquad g((\nabla_{f_1}R)(e_1,\ X)e_2,\ f_2)=0\ .$$

On the other hand, taking account of (k), we get

$$(4.57)\qquad \begin{aligned} &g((\nabla_{f_1}R)(e_1,\ X)e_1,\ f_2)\\ &\quad =-g((\nabla_{e_1}R)(X,\ f_1)e_1,\ f_2)-g((\nabla_X R)(f_1,\ e_1)e_1,\ f_2)\\ &\quad =0\ .\end{aligned}$$

(4.56) and (4.57) imply (4.43), which completes the proof of (o). q.e.d.

5. Equivalent conditions for a locally ϕ-symmetric space and miscellaneous properties. Let $M(\phi, \xi, \eta, g)$ be a Sasakian manifold. The following Lemmas 5.1 and 5.2 are equivalent forms of Lemmas 2.8 and 2.9, respectively.

LEMMA 5.1. *For any tangent vectors X, Y and Z of M, we get*

$$\begin{aligned}(5.1)\quad & R(\phi X, Y)Z + R(X, \phi Y)Z \\ &= g(Y, Z)\phi X - g(\phi X, Z)Y - g(X, Z)\phi Y + g(\phi Y, Z)X .\end{aligned}$$

LEMMA 5.2. *For any tangent vectors X, Z and V of M, we get*

$$(5.2)\qquad (\nabla_V R)(X, \xi)Z = g(\phi V, Z)X - g(X, Z)\phi V - R(X, \phi V)Z .$$

LEMMA 5.3. *For any horizontal vectors X, Y and Z of M, we get*

$$(5.3)\qquad (\nabla_\xi R)(X, Y)Z = 0 .$$

PROOF. Let X^*, Y^* and Z^* be ξ-invariant horizontal vector field extensions of X, Y and Z, respectively. Since X^* is ξ-invariant, we get

$$(5.4)\qquad \nabla_\xi X^* = \nabla_{X^*}\xi = \phi X^* .$$

Now, since ξ is a Killing vector field, we have $\mathscr{L}_\xi R = 0$. On the other hand, making use of the invariance of X^*, Y^* and Z^* by ξ, (2.6), (5.4) and Lemmas 2.8 and 5.1, we get

$$\begin{aligned}&(\mathscr{L}_\xi R)(X^*, Y^*)Z^* \\ &\quad = [\xi, R(X^*, Y^*)Z^*] \\ &\quad = \nabla_\xi(R(X^*, Y^*)Z^*) - \nabla_{R(X^*,Y^*)Z^*}\xi \\ &\quad = (\nabla_\xi R)(X^*, Y^*)Z^* + R(\phi X^*, Y^*)Z^* + R(X^*, \phi Y^*)Z^* \\ &\qquad + R(X^*, Y^*)\phi Z^* - \phi R(X^*, Y^*)Z^* \\ &\quad = (\nabla_\xi R)(X^*, Y^*)Z^* .\end{aligned}$$

Hence we get the conclusion.

THEOREM 5.4. *A necessary and sufficient condition for a Sasakian manifold to be locally ϕ-symmetric is that*

$$\begin{aligned}(5.5)\quad & (\nabla_V R)(X, Y)Z \\ &= \{g(X, V)g(\phi Y, Z) - g(Y, V)g(\phi X, Z) + g(\phi R(X, Y)V, Z)\}\xi \\ &\quad + \eta(X)\{-g(\phi V, Z)Y + g(Y, Z)\phi V + R(Y, \phi V)Z\} \\ &\quad + \eta(Y)\{g(\phi V, Z)X - g(X, Z)\phi V - R(X, \phi V)Z\} \\ &\quad + \eta(Z)\{g(Y, V)\phi X - g(X, V)\phi Y - \phi R(X, Y)V\}\end{aligned}$$

holds good for any tangent vectors X, Y, Z and V.

PROOF. If X, Y, Z and V are horizontal, then (5.5) reduces to (2.14). Hence the sufficiency holds good. Conversely, suppose a Sasakian manifold is locally ϕ-symmetric. Let X, Y, Z and V be arbitrary tangent vectors. We compute $(\nabla_{\phi^2 V}R)(\phi^2 X, \phi^2 Y)\phi^2 Z$ in two ways.

First of all, taking account of (2.2), (2.14) implies

$$\begin{aligned}(\nabla_{\phi^2 V}R)&(\phi^2 X, \phi^2 Y)\phi^2 Z \\ &= [\{g(X, V) - \eta(X)\eta(V)\}g(\phi Y, Z) - \{g(Y, V) \\ &\quad - \eta(Y)\eta(V)\}g(\phi X, Z) + g(\phi Z, R(\phi^2 X, \phi^2 Y)\phi^2 V)]\xi \,.\end{aligned}$$

On the other hand, making use of (2.7) and (2.8), we get

$$\begin{aligned}R(\phi^2 X&, \phi^2 Y)\phi^2 V \\ &= R(-X + \eta(X)\xi, -Y + \eta(Y)\xi)(-V + \eta(V)\xi) \\ &= -R(X, Y)V + \eta(Y)\eta(V)X - \eta(X)\eta(V)Y \\ &\quad + \{\eta(X)g(Y, V) - \eta(Y)g(X, V)\}\xi \,.\end{aligned}$$

Hence we get

$$\begin{aligned}(5.6)\quad (\nabla_{\phi^2 V}R)&(\phi^2 X, \phi^2 Y)\phi^2 Z \\ &= \{g(X, V)g(\phi Y, Z) - g(Y, V)g(\phi X, Z) - g(\phi Z, R(X, Y)V)\}\xi \,.\end{aligned}$$

Secondly, since $\phi^2 X$, $\phi^2 Y$ and $\phi^2 Z$ are horizontal, Lemma 5.3 implies

$$\begin{aligned}(\nabla_{\phi^2 V}R)&(\phi^2 X, \phi^2 Y)\phi^2 Z \\ &= -(\nabla_V R)(\phi^2 X, \phi^2 Y)\phi^2 Z \,.\end{aligned}$$

Making use of Lemmas 2.9 and 5.2, we get

$$\begin{aligned}-(\nabla_V R)&(\phi^2 X, \phi^2 Y)\phi^2 Z \\ &= -(\nabla_V R)(-X + \eta(X)\xi, -Y + \eta(Y)\xi)(-Z + \eta(Z)\xi) \\ &= (\nabla_V R)(X, Y)Z \\ &\quad - \eta(Z)\{g(Y, V)\phi X - g(X, V)\phi Y - \phi R(X, Y)V\} \\ &\quad - \eta(Y)\{g(\phi V, Z)X - g(X, Z)\phi V - R(X, \phi V)Z\} \\ &\quad + \eta(Y)\eta(Z)\{\eta(V)\phi X - \phi R(X, \xi)V\} \\ &\quad - \eta(X)\{-g(\phi V, Z)Y + g(Y, Z)\phi V + R(Y, \phi V)Z\} \\ &\quad + \eta(X)\eta(Z)\{-\eta(V)\phi Y + \phi R(Y, \xi)V\} \,.\end{aligned}$$

Since the 4th and 6th terms of the right hand side of the last equation vanish, we get

$$\begin{aligned}(5.7)\qquad (\nabla_{\phi^2 V}R)&(\phi^2 X, \phi^2 Y)\phi^2 Z \\ &= (\nabla_V R)(X, Y)Z \\ &\quad - \eta(X)\{-g(\phi V, Z)Y + g(Y, Z)\phi V + R(Y, \phi V)Z\}\end{aligned}$$

$$- \eta(Y)\{g(\phi V, Z)X - g(X, Z)\phi V - R(X, \phi V)Z\}$$
$$-\eta(Z)\{g(Y, V)\phi X - g(X, V)\phi Y - \phi R(X, Y)V\} .$$

Comparing (5.6) and (5.7), we get (5.5). q.e.d.

K. Motomiya [4] studied a special linear connection, which is a special one introduced by M. Okumura [6], on a non-degenerate normal almost contact manifold, and Kato-Motomiya [2] proved that this linear connection on the homogeneous manifold G/G_0 treated in Example 2.3 is identical with the canonical linear connection of the second kind and that the curvature tensor for this linear connection is parallel. Motomiya [4] also derived a necessary and sufficient condition for the curvature tensor of this linear connection to be parallel in connection with the curvature tensor of the Riemannian connection. In the following, we shall prove that this condition is identical with the condition for a Sasakian manifold to be locally ϕ-symmetric.

First of all, we give a definition of the M. Okumura's linear connection (M-connection, for short) in the case of a Sasakian manifold. Let M be a Sasakian manifold with structure tensors ϕ, ξ, η and g. Let ∇ be the Riemannian connection for g. Let r be an arbitrary fixed real number, and let A be a tensor field of type (1, 2) defined by

$$A(X)Y = d\eta(X, Y)\xi + r\eta(X)\phi Y - \eta(Y)\phi X . \tag{5.8}$$

The M-connection $\tilde{\nabla}$ is now defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X)Y . \tag{5.9}$$

By direct calculations we see that the tensor fields ϕ, ξ, η, g and A are parallel with respect to the M-connection $\tilde{\nabla}$. Let $\tilde{R}$ and R be the curvature tensors for $\tilde{\nabla}$ and ∇, respectively. Then we get

$$\tilde{R}(X, Y)Z = R(X, Y)Z + B(X, Y)Z , \tag{5.10}$$

where

$$\begin{aligned} B(X, Y)Z &= A(A(X)Y)Z - A(A(Y)X)Z - A(X)A(Y)Z + A(Y)A(X)Z \\ &= \eta(Z)\{\eta(X)Y - \eta(Y)X\} + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad + 2rg(\phi X, Y)\phi Z + \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi , \end{aligned} \tag{5.11}$$

and hence we get

$$\begin{aligned} (\tilde{\nabla}_V\tilde{R})(X, Y)Z = (\nabla_V R)(X, Y)Z & \\ + A(V)R(X, Y)Z - R(A(V)X, Y)Z & \\ - R(X, A(V)Y)Z - R(X, Y)A(V)Z . & \end{aligned} \tag{5.12}$$

Thus a necessary and sufficient condition for $\tilde{\nabla}\tilde{R} = 0$ is that

$$(5.13) \qquad (\nabla_V R)(X, Y)Z = -A(V)R(X, Y)Z + R(A(V)X, Y)Z + R(X, A(V)Y)Z + R(X, Y)A(V)Z$$

holds for any tangent vectors X, Y, Z and V. By a straightforward calculation, we see that the right hand side of (5.13) is nothing but the right hand side of (5.5). Hence we get

THEOREM 5.5. *A necessary and sufficient condition for a Sasakian manifold to be locally ϕ-symmetric is that*

$$(5.14) \qquad \tilde{\nabla}\tilde{R} = 0$$

holds good, where $\tilde{\nabla}$ is the M-connection defined by (5.8) *and* (5.9), *and $\tilde{R}$ is the curvature tensor for $\tilde{\nabla}$.*

Let X and Y be linearly independent tangent vectors of a Sasakian manifold. The M-sectional curvature $M(X, Y)$ of the 2-plane spaned by X and Y is by definition

$$(5.15) \qquad M(X, Y) = g(\tilde{R}(X, Y)Y, X)/g(X, X)g(Y, Y) - g(X, Y)^2 .$$

THEOREM 5.6. *If $r = 1$, then the M-sectional curvature of the compact (resp. noncompact) type Sasakian ϕ-symmetric space G/G_0 in Example 2.5 is positive (resp. negative) semidefinite.*

PROOF. Since the M-connection $\tilde{\nabla}$ for $r = 1$ is the canonical linear connection of the second kind (Kato-Motomiya [2]), we have

$$(5.16) \qquad \tilde{R}(X, Y)Z = -[[X, Y]_{\mathfrak{G}_0}, Z] \quad \text{for all} \quad X, Y, Z \in \mathfrak{M} .$$

Let X and Y be orthonormal horizontal vectors. If G/G_0 is of the compact type, we get

$$(5.17)_1 \qquad \begin{aligned} M(X, Y) &= g(\tilde{R}(X, Y)Y, X) \\ &= \frac{-1}{8n} B_{\mathfrak{M}}(-[[X, Y]_{\mathfrak{G}_0}, Y], X) \\ &= \frac{1}{8n} B([X, Y]_{\mathfrak{G}_0}, [Y, X]) \\ &= \frac{-1}{8n} B([X, Y]_{\mathfrak{G}_0}, [X, Y]_{\mathfrak{G}_0}) \geqq 0 , \end{aligned}$$

and if G/G_0 is of the noncompact type, we get

$$(5.17)_{\text{ii}} \qquad \begin{aligned} M(X, Y) &= g(\tilde{R}(X, Y)Y, X) \\ &= \frac{1}{8n}B_{\mathfrak{M}}(-[[X, Y]_{\mathfrak{G}_0}, Y], X) \\ &= \frac{1}{8n}B([X, Y]_{\mathfrak{G}_0}, [X, Y]_{\mathfrak{G}_0}) \leqq 0 \,, \end{aligned}$$

because the Killing form B is negative definite on $\mathfrak{G}_0$ and $B(\mathfrak{G}_0, \mathfrak{G}_2) = \{0\}$.

On the other hand, since we have $\tilde{\nabla}\xi = 0$, we get $\tilde{R}(X, Y)\xi = 0$ for any tangent vectors X and Y, and hence we get

$$(5.18) \qquad M(X, \xi) = 0$$

for any tangent vector X. Moreover, since we have $\tilde{R}(X, \xi) = 0$ (Kato-Motomiya [2]), we get the conclusion. q.e.d.

THEOREM 5.7. *If $r = -1$, then the ϕ-holomorphic M-sectional curvature of a Sasakian manifold of constant ϕ-holomorphic sectional curvature k is $\gtreqqless 0$ according to $k \gtreqqless -3$.*

PROOF. The curvature tensor of the Sasakian manifold of constant ϕ-holomorphic sectional curvature k is given by

$$\begin{aligned} R(X, Y)Z = \frac{k+3}{4}&\{g(Y, Z)X - g(X, Z)Y\} \\ + \frac{k-1}{4}&[\eta(Z)\{\eta(X)Y - \eta(Y)X\} \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi] \,. \end{aligned}$$

Hence, for any horizontal unit vector X, the ϕ-holomorphic M-sectional curvature is given by

$$\begin{aligned} M(X, \phi X) &= g(\tilde{R}(X, \phi X)\phi X, X) \\ &= \frac{k+3}{4} + 3\left(\frac{k-1}{4} + 1\right) \\ &= k + 3 \,. \end{aligned}$$

q.e.d.

Now, we shall prove the necessity of Theorem 3.2.

Let M be a Sasakian manifold. We consider the M-connection $\tilde{\nabla}$ on M. The torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is by definition

$$(5.19) \qquad \begin{aligned} \tilde{T}(X, Y) &= A(X)Y - A(Y)X \\ &= 2d\eta(X, Y)\xi + (r+1)\{\eta(X)\phi Y - \eta(Y)\phi X\} \,, \end{aligned}$$

and it is parallel with respect to $\tilde{\nabla}$. Now, suppose M is a locally ϕ-

symmetric space. Then, from Theorem 5.5, we see that the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is parallel. Let x be an arbitrary point of M. Let $\{e_1, e_2, \cdots, e_{2n}, \xi_x\}$ be an orthonormal basis of $T_x(M)$. We define a linear isomorphism σ_0: $T_x(M) \to T_x(M)$ by

$$(5.20) \qquad \begin{cases} \sigma_0(e_i) = -e_i\ , & 1 \leqq i \leqq 2n\ , \\ \sigma_0(\xi_x) = \xi_x\ . \end{cases}$$

Then, from (5.19), we see that $\tilde{T}(\sigma_0 e_i, \sigma_0 e_j) = \sigma_0 \tilde{T}(e_i, e_j)$ and $\tilde{T}(\sigma_0 e_i, \sigma_0 \xi_x) = \sigma_0 \tilde{T}(e_i, \xi_x)$ hold good for $1 \leqq i, j \leqq 2n$. Hence we get

$$(5.21) \qquad \sigma_0 \tilde{T}_x = \tilde{T}_x\ .$$

On the other hand, (2.7) implies that $R(e_i, e_j)e_k$ is horizontal, where R is the Riemannian curvature tensor, and (5.11) implies that $B(e_i, e_j)e_k$ is horizontal. Hence (5.10) implies that $\tilde{R}(e_i, e_j)e_k$ is horizontal, and hence we get $\tilde{R}(\sigma_0 e_i, \sigma_0 e_j)\sigma_0 e_k = \sigma_0 R(e_i, e_j)e_k$ for $1 \leqq i, j, k \leqq 2n$. Since we have $\tilde{R}(X, Y)\xi = 0$ and $\tilde{R}(X, \xi) = 0$ as mentioned in the proof of Theorem 5.6, we get $\tilde{R}(\sigma_0 e_i, \sigma_0 e_j)\sigma_0 \xi_x = \sigma_0 \tilde{R}(e_i, e_j)\xi_x$, $\tilde{R}(\sigma_0 e_i, \sigma_0 \xi_x)\sigma_0 e_k = \sigma_0 \tilde{R}(e_i, \xi_x)e_k$ and $\tilde{R}(\sigma_0 e_i, \sigma_0 \xi_x)\sigma_0 \xi_x = \sigma_0 \tilde{R}(e_i, \xi_x)\xi_x$. Hence we get

$$(5.22) \qquad \sigma_0 \tilde{R}_x = \tilde{R}_x\ .$$

Hence, according to Lemma 1.2 of Chapter IV in Helgason [1], for example, we see that the local diffeomorphism σ_x, defined by

$$\begin{aligned}(5.23)\quad \sigma_x(x_1, x_2, \cdots, x_{2n}, z) &= \sigma_x(\mathrm{Exp}_x\,(x_1 e_1 + x_2 e_2 + \cdots + x_{2n}e_{2n} + z\xi_x)) \\ &= \mathrm{Exp}_x\,(x_1(-e_1) + x_2(-e_2) + \cdots + x_{2n}(-e_{2n}) + z\xi_x) \\ &= (-x_1,\ -x_2, \cdots, -x_{2n}, z)\end{aligned}$$

on a normal coordinate neighborhood U with a normal coordinate system $(x_1, x_2, \cdots, x_{2n}, z)$ determined by $\{e_1, e_2, \cdots, e_{2n}, \xi_x\}$, is a local affine transformation with respect to $\tilde{\nabla}$. Since $(\sigma_{x*})_x = \sigma_0$, we get $(\sigma_{x*})_x \circ \phi_x = \phi_x \circ (\sigma_{x*})_x$ and $(\sigma_{x*})_x \xi_x = \xi_x$. Hence, since σ_x is a locall affine transformation and since ϕ and ξ are parallel, we see that $\sigma_{x*} \circ \phi = \phi \circ \sigma_{x*}$ and $\sigma_{x*}\xi = \xi$ hold good (cf. Lemma 3.4 in Kato-Motomiya [2]). Thus the following Lemma, which is due to S. Tanno, implies that σ_x is a local automorphism, and hence it is a ϕ-geodesic symmetry at x. Hence the necessity of Theorem 3.2 is proved.

LEMMA 5.8 (Tanno [8]). *Let $M(\phi, \xi, \eta, g)$ be a contact Riemannian manifold. If a diffeomorphism f of M leaves the structure tensor ϕ invariant, then there exists a positive constant α such that*

$$f_*\xi = \alpha\xi,\ f^*\eta = \alpha\eta\ ,$$
$$(f^*g)(X, Y) = \alpha g(X, Y) + \alpha(\alpha - 1)\eta(X)\eta(Y)\ .$$

6. Sasakian globally ϕ-symmetric space. In this section, every manifold is assumed to be connected.

Let M be a Sasakian manifold with structure tensors ϕ, ξ, η and g. If any ϕ-geodesic symmetry of M is extendable as a global automorphism of M, and if the Killing vector field ξ generates the 1-parameter group of global transformations, which are automatically automorphisms of M, we call the Sasakian manifold M to be a globally ϕ-symmetric space.

Let M be a Sasakian globally ϕ-symmetric space. Then the group $A(M)$ of all automorphisms of M is a transitive Lie transformation group of M. Since a homogeneous Sasakian manifold is regular, M is a principal G^1-bundle over a Kählerian manifold B, where G^1 is a 1-dimensional Lie group which is isomorphic to the 1-parameter group of global transformations generated by ξ. Since each ϕ-geodesic symmetry of M is extendable as a global automorphism, each geodesic symmetry of B is extendable as a global automorphism. Hence B is a Hermitian globally symmetric space.

THEOREM 6.1. *A Sasakian globally ϕ-symmetric space is a principal G^1-bundle over a Hermitian globally symmetric space.*

Suppose a Sasakian locally ϕ-symmetric space M is complete and simply connected. Let $\tilde{\nabla}$ be the M-connection for $r = 1$ on M. Then, since the Riemannian connection of M is complete, the M-connection $\tilde{\nabla}$ is complete. The torsion tensor field $\tilde{T}$ and the curvature tensor field $\tilde{R}$ are parallel (Theorem 5.4). Hence, according to Corollary 7.9 of Chapter VI in Kobayashi-Nomizu [3], vol. I, we see that any local affine transformation (with respect to $\tilde{\nabla}$) of M is extendable as a global one. In particular, any ϕ-geodesic symmetry, which is a local automorphism by the assumption, is extendable as a global automorphism. This proves that M is a globally ϕ-symmetric space.

THEOREM 6.2. *A complete and simply connected Sasakian locally ϕ-symmetric space is a globally ϕ-symmetric space.*

Let M be a Sasakian locally ϕ-symmetric space and let x be an arbitrary point of M. According to Theorem 4 in Kato-Motomiya [2], there exist a homogeneous Sasakian manifold G/G_0 and a local isomorphism from a neighborhood of x onto a neighborhood of the origin $0 = \{G_0\}$ of G/G_0. According to Theorem 6.2, since the homogeneous Sasakian manifold G/G_0 is a complete locally ϕ-symmetric space, the universal covering manifold of G/G_0 is a globally ϕ-symmetric space. Hence we get

THEOREM 6.3. *A Sasakian locally ϕ-symmetric space is locally iso-*

morphic to a Sasakian globally ϕ-symmetric space.

Let M be a Sasakian globally ϕ-symmetric space. According to Theorem 6.1, M is a principal G^1-bundle over a Hermitian globally symmetric space B. M is said to be of the compact type, noncompact type, or Euclidean type according to the type of the Hermitian globally symmetric space B.

THEOREM 6.4. *Let M be a Sasakian globally ϕ-symmetric space. We consider the M-connection $\tilde{\nabla}$ for $r = -1$ on M.*

(i) *If M is of the compact type, then M has the M-sectional curvature everywhere $\geqq 0$.*

(ii) *If M is of the noncompact type, then M has the M-sectional curvature everywhere $\leqq 0$.*

(iii) *If M is of the Euclidean type, then M has the M-sectional curvature everywhere $= 0$.*

PROOF. Let $\pi: M \to B = M/\xi$ be the fibering and let $\bar{R}$ be the Riemannian curvature tensor for the induced Riemannian structure of B. Then, if X, Y and Z are horizontal vectors of M, (2.5), (5.10) and (5.11) imply that

$$\tilde{R}(X, Y)Z = (\bar{R}(\pi_* X, \pi_* Y)\pi_* Z)^* \tag{6.1}$$

holds good. On the other hand, as mentioned in the proof of Theorem 5.8, we have $\tilde{R}(X, Y)\xi = 0$ and $\tilde{R}(X, \xi)Y = 0$ for any tangent vectors X and Y. Hence, according to Theorem 3.1 of Chapter V in Helgason [1], for example, we get the conclusion.

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