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SATO GRASSMANNIANS FOR GENERALIZED TATE SPACES

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Abstract. We generalize the concept of Sato Grassmannians of locally linearly compact topological vector spaces (Tate spaces) to the Beilinson category of the "locally compact objects", or Generalized Tate Spaces, of an exact category. This allows us to extend the Kapranov dimensional torsor Dim and determinantal gerbe Det to generalized Tate spaces and unify their treatment in the determinantal torsor. We then introduce a class of exact categories, that we call partially abelian exact, and prove that if the base category is so, then Dim and Det are multiplicative in admissible short exact sequences of generalized Tate spaces. We then give a cohomological interpretation of these results in terms of the Waldhausen K-theoretical space of the Beilinson category. Our approach can be iterated and we define analogous concepts for the successive categories of n-dimensional (generalized) Tate spaces. In particular we show that the category of Tate spaces is partially abelian exact, so we can extend the results for Dim and Det obtained for 1-Tate spaces to 2-Tate spaces, and provide a new interpretation in the context of algebraic K-theory of results of Kapranov, Arkhipov-Kremnizer and Frenkel-Zhu.

CONTENTS

1. Introduction.

Let k be a field, and consider the Tate space $V = k((t))$. For such a space V, the group $GL(V)$ (sometimes called the "Japanese group" $GL(\infty)$) has properties which are quite different from those of the naively defined group $GL_\infty = \bigcup GL(n)$. In particular, it is typically disconnected, with $\pi_0(GL(V)) = \boldsymbol{Z}$. This has been interpreted by Kapranov in [12] in terms of the *dimensional torsor* $\mathrm{Dim}(V)$, naturally associated with V, which gives rise to a class in $H^1(GL(V), \boldsymbol{Z}) = \mathrm{Hom}(GL(V), \boldsymbol{Z})$. Kapranov also proves that,

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for all Tate spaces V, the dimensional torsor $\mathrm{Dim}(V)$ is multiplicative with respect to admissible short exact sequences of Tate spaces. A similar result is also proved for the determinantal gerbe $\mathrm{Det}(V)$.

In the language of exact categories, Kapranov's results amount to the consideration of the dimensional torsor $\mathrm{Dim}(V)$ and the determinantal gerbe $\mathrm{Det}(V)$ for the objects V of the exact category of Tate spaces $\mathcal{T} = \underleftrightarrow{\lim} \mathbf{Vect}_0(k)$ (see Section 5), where $\mathbf{Vect}_0(k)$ is the category of finite dimensional vector spaces over the field k.

In this paper we propose a topological interpretation of the theory of the dimensional torsor $\mathrm{Dim}(V)$ and the determinantal gerbe $\mathrm{Det}(V)$. This is achieved through a generalization of these concepts to the Beilinson category $\underleftrightarrow{\lim} \mathcal{A}$, where $\mathcal{A}$ is an exact category. Objects of $\underleftrightarrow{\lim} \mathcal{A}$ will serve as categorical generalizations of Tate spaces referred to in the title of this article. We prove the multiplicativity of $\mathrm{Dim}(V)$, and sketch the analogous theory for the determinantal gerbe $\mathrm{Det}(V)$, under the extra assumption that $\mathcal{A}$ has pullbacks of admissible monomorphisms and pushouts of admissible epimorphisms. We call such categories "partially abelian exact", since they can equivalently be described as exact categories such that, for any morphism f which is the composite of an admissible monomorphism followed by an admissibe epimorphism, f can be written in a unique way (up to isomorphisms) as the composite of an admissible epimorphism followed by an admissible monomorphism.

This new setting, which employs exact categories of generalized Tate spaces replacing the category $\mathcal{T}$, finds a natural interpretation in the framework of higher algebraic K-theory. In fact, we interpret Dim and Det and their corresponding "multiplicative" properties as cohomological invariants of the Waldhausen space $S(\underleftrightarrow{\lim} \mathcal{A})$, the fundamental K-theoretical space of the exact category $\underleftrightarrow{\lim} \mathcal{A}$. In turn this interpretation yields that these constructions can be seen as "first and second step" of a delooping relation between $S(\mathcal{A})$, understood as a space of "finite-dimensional (or *discrete*)" objects, and $S(\underleftrightarrow{\lim} \mathcal{A})$, understood as a space of "semi-infinite dimensional (or *locally compact*)" objects. We next apply our theory to the category $\mathcal{T}_2 = \underleftrightarrow{\lim}(\mathcal{T}) = \underleftrightarrow{\lim}\,\underleftrightarrow{\lim} \mathbf{Vect}_0(k)$, whose objects can be called 2-*Tate spaces*. For example, for a field k, the space $k((t))((s))$ is a 2-Tate space over k. Study of 2-Tate spaces was recently taken up by Arkhipov and Kremnizer in [1] and by Frenkel and Zhu in [6], in connection with representations of double loop groups. In the same order of ideas, Gaitsgory and Kazhdan have recently provided a categorical framework for the study of the representations of the group $G(\boldsymbol{F})$, where G is reductive and $\boldsymbol{F}$ is a 2-dimensional local field [7]. In a recent paper [5], Drinfeld defined the notion of dimensional torsor in a more general situation of modules over a commutative ring, and defined the étale local notion of Tate module.

Our results provide a categorical foundation for such study. The peculiarity of our approach can be described as follows. Various authors (e.g., [1], [6]) have proposed the construction of the determinantal gerbe $\mathrm{Det}(V)$, when V is either a 1- or a 2-Tate space. These authors start with the construction of $\mathrm{Det}(V)$ when V is a 1-dimensional Tate space, and then they lift their construction to the 2-dimensional case, by generalizing each 1-dimensional concept (and corresponding proposition) introduced to the new context. In our theory, we propose a K-theoretical interpretation of this 1-dimensional vs. 2-dimensional interplay. We achieve this through (1) the systematic use of the language of iterated ind/pro-categories over a partially abelian exact category $\mathcal{A}$, and (2) the study of the behaviour of the Waldhausen space of an exact category $\mathcal{A}$ under the functor $\underset{\longleftrightarrow}{\lim}$ of Beilinson. This point of view allows us to produce a *single* construction which accounts for both cases. Indeed, we recover the theory of 1-dimensional Tate space when we let $\mathcal{A} = \mathbf{Vect}_0(k)$, and the theory of 2-dimensional Tate spaces when we let $\mathcal{A} = \mathcal{T}$. This is made possible by the fact (proved in Theorem 5.5) that the category $\mathcal{T}$ is partially abelian exact, so our theory can be applied to $\mathcal{T}$. In detail, we obtain a shift map from the n-th cohomology of $S(\mathcal{A})$ to the $(n+1)$-th cohomology of $S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$ for low dimensions n. More exactly, for $n = 1$ and $\mathcal{A} = \mathbf{Vect}_0(k)$, we start with a 1-cocycle on $S(\mathcal{A})$ which gives rise to a 2-cocycle on $S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$ which in turns provide a 3-cocycle on $S(\underset{\longleftrightarrow}{\lim}\underset{\longleftrightarrow}{\lim}\mathcal{A})$; for $n = 2$ the same argument gives, starting from a 2-cocycle on $S(\mathcal{A})$, a 3-cocycle on $S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$, which in turn gives a 4-cocycle on $S(\underset{\longleftrightarrow}{\lim}\underset{\longleftrightarrow}{\lim}\mathcal{A})$, i.e., a 2-gerbe on the objects of the category $\mathcal{T}_2$ of 2-Tate spaces. In this paper we provide a detailed treatment of the constructions on the category $\mathcal{T}$, and sketch the constructions for the category $\mathcal{T}_2$, whose details will be spelled out in a subsequent paper.

In order to generalize the dimensional torsor and the determinantal gerbe to the objects X of the Beilinson category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, for $\mathcal{A}$ exact, we introduce an appropriate concept of Grassmannians for $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, which generalizes the *Sato Grassmannians*, originally defined by Sato in [19] for the category of Tate spaces. Our definition uses the language of ind/pro-objects on $\mathcal{A}$, which has the advantage to allow us to define formally in the same way the Grassmannians for all the iterated categories $\underset{\longleftrightarrow}{\lim}\underset{\longleftrightarrow}{\lim}\mathcal{A}, \dots, \underset{\longleftrightarrow}{\lim}^n\mathcal{A}$. We then study the behavior of the Grassmannian of an object X with respect to admissible short exact sequences of $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, when $\mathcal{A}$ is partially abelian exact. This allows us to define the *determinantal torsor* $\mathcal{D}$ for the objects of the Beilinson category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$. This is a torsor defined over a certain Picard category $\mathcal{P}$. When $\mathcal{P} = V(\mathcal{A})$, the symmetric category of virtual objects on $\mathcal{A}$ defined by Deligne (cf. [4]), the determinantal torsor $\mathcal{D}(X)$ encloses the datum of the $K_0(\mathcal{A})$-torsor $\mathrm{Dim}(X)$ and of the $K_1(\mathcal{A})$-gerbe $\mathrm{Det}(X)$. In particular, for $\mathcal{A} = \mathbf{Vect}_0(k)$, they are $K_0(\mathcal{A}) = \mathbf{Z}$ and $K_1(\mathcal{A}) = k^*$, and this construction provides a unified treatment of the Kapranov $\mathbf{Z}$-dimensional torsor $\mathrm{Dim}(V)$ and k^*-gerbe $\mathrm{Det}(V)$, and extends the theory of [12] to the general K-theoretic setting.

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2. The Waldhausen space. In this section and in the next, we refer to Appendix A and B for the basic material on exact categories and Picard categories, and to [8] and [21] for the terminology relative to simplicial categories.

2.1. The Waldhausen S-construction. Given an exact category, Waldhausen [21] associates to it a simplicial category $S_\bullet(\mathcal{A})$, whose geometric realization (as defined e.g. in [8] or in [9]) $S(\mathcal{A})$ provides a topological model for the K-theory of $\mathcal{A}$, i.e., $K_i(\mathcal{A}) = \pi_{i+1}S(\mathcal{A})$ (see [22]).

DEFINITION 2.1. Let $\mathcal{A}$ be an exact category and $n \geq 0$ an integer. The category $S_n(\mathcal{A})$ is defined as the category whose objects are data $\{\underline{a}\}$ consisting of

- objects $a_{ij} \in \mathcal{A}$, given for each (i, j) with $0 \leq i \leq j \leq n$.
- morphisms $\phi_{ij}^{kl} : a_{ij} \to a_{kl}$, given for $i \leq k$, $j \leq l$ (we shall write $(i, j) \leq (k, l)$)

such that the following conditions are satisfied.

(1) For all (i, j, k), with $i \leq j \leq k$,

$$a_{ij} \xrightarrow{\phi_{ij}^{ik}} a_{ik} \xrightarrow{\phi_{ik}^{jk}} a_{jk}$$

is an admissible short exact sequence of $\mathcal{A}$.

(2) If $(i, j) \leq (k, l) \leq (m, n)$, we have a commutative diagram

$$\phi_{ij}^{mn} = \phi_{kl}^{mn}\phi_{ij}^{kl} .$$

A *morphism* between two objects $\underline{a} \to \underline{b}$ of $S_n(\mathcal{A})$ is by definition a collection of isomorphisms $a_{ij} \xrightarrow{\sim} b_{ij}$, for all $i \leq j$, making the resulting diagrams commutative.

In particular, $a_{ii} = 0$ and we see that $\{\underline{a}\}$ gives rise to a *rigidified admissible filtration* of objects of $\mathcal{A}$ of length n, i.e., a sequence $\underline{a} = 0 \hookrightarrow a_1 \hookrightarrow a_2 \hookrightarrow \cdots \hookrightarrow a_n$ of n admissible monomorphisms toghether with a compatible choice of an object a_{ij}, in the isomorphism class of each quotient a_j/a_i, for $i \leq j$ such that there is a commutative diagram

(2.2)
$$\begin{tikzcd}
a_{01} \arrow[r, hook] & a_{02} \arrow[r, hook] \arrow[d, two heads] & a_{03} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{0n} \arrow[d, two heads] \\
 & a_{12} \arrow[r, hook] & a_{13} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{1n} \arrow[d, two heads] \\
 & & a_{23} \arrow[r, hook] & \cdots \arrow[r, hook] & a_{2n} \arrow[d, two heads] \\
 & & & & \vdots \arrow[d, two heads] \\
 & & & & a_{n-1,n}
\end{tikzcd}$$

whose horizontal arrows are admissible monomorphisms and the vertical arrows are admissible epimorphisms.

For each $n \geq 0$, we define a functor $\partial_0 : S_n(\mathcal{A}) \to S_{n-1}(\mathcal{A})$ by erasing the top row of (2.2) and reindexing. Then, $\partial_0(\underline{a}) = a_{12} \hookrightarrow \cdots \hookrightarrow a_{1n}$, with $\partial_0(\underline{a})_{i,j} = a_{i+1,j+1}$; we define a functor $\partial_i : S_n(\mathcal{A}) \to S_{n-1}(\mathcal{A})$ for all $0 < i \leq n$ by erasing the row $a_{i,*}$ and the column $a_{*,i}$.

The functors $s_i : S_n(\mathcal{A}) \to S_{n+1}(\mathcal{A})$, for $0 \leq i \leq n$ are defined by doubling the object a_i in $\underline{a}$. Then, we have the following proposition.

PROPOSITION 2.3 (cf. [21]). *The system $(S_n(\mathcal{A}), \partial_i, s_j)$ forms a simplicial category $S_\bullet(\mathcal{A})$.*

Next, the geometric realization of $S_\bullet(\mathcal{A})$ is constructed as follows. Since $S_\bullet(\mathcal{A})$ is a simplicial category, we consider the geometric realizations $|S_n(\mathcal{A})|$ of the categories $S_n(\mathcal{A})$. These form a simplicial topological space $BS_\bullet(\mathcal{A})$; we then take the geometric realization of $BS_\bullet(\mathcal{A})$, and write it $S(\mathcal{A})$. Thus, $S(\mathcal{A}) = |S_\bullet(\mathcal{A})|$. This space is called the *Waldhausen space* associated with the exact category $\mathcal{A}$. Notice that the simplicial space $BS_\bullet(\mathcal{A})$ is a bisimplicial set, and the space $S(\mathcal{A})$ can be interpreted as the geometric realization of this bisimplicial set.

REMARK 2.4. The geometric realization $S(\mathcal{A})$ is thus constructed out of the (p, q)-bisimplices $\Delta^p \times \Delta^q$ glued together along the face maps of the bisimplicial set $S_\bullet(\mathcal{A})$. The bisimplices of dimension less than or equal to 3 are parametrized as follows:

- $\Delta^0 \times \Delta^0$: only one point (basepoint) $*$ in $S(A)$.
- $\Delta^1 \times \Delta^0$: one for each object $\{a\}$ of $\mathcal{A}$; geometrically, this gives rise in $S(\mathcal{A})$ to a loop (embedded circle) at $*$ which we denote by $|a|$.
- $\Delta^1 \times \Delta^1$: one for each isomorphism $\{a \xrightarrow{\sim} b\}$ of $\mathcal{A}$, giving rise to a homotopy between the loops $|a| \sim |b|$, hence to an element of $\pi_2(S(A), *)$.
- $\Delta^2 \times \Delta^0$: one for each admissible short exact sequence $\{\sigma : a' \hookrightarrow a \twoheadrightarrow a''\}$. Geometrically, 2-simplexes as in Figure 1.
- $\Delta^2 \times \Delta^1$: one for each isomorphism of admissible short exact sequences $\{\sigma_0 \xrightarrow{\sim} \sigma_1 : \sigma_0, \sigma_1 \in S_{2,0}(\mathcal{A})\}$. Geometrically, the filled prism whose bottom is the 2-simplex $|\sigma_0|$ and whose top is the 2-simplex $|\sigma_1|$.
- $\Delta^1 \times \Delta^2$: one for each composable pair of isomorphisms of $\mathcal{A}$: $\{a \xrightarrow{\sim} b \xrightarrow{\sim} c\}$.
- $\Delta^3 \times \Delta^0$: one for each rigidified admissible filtration of length 2 of $\mathcal{A}$ $\{\tau : a_1 \hookrightarrow a_2 \hookrightarrow a_3\}$. Geometrically, the filled tetrahedron generated by the a_i's as in Figure 2.

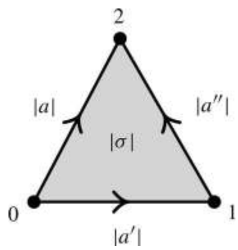

FIGURE 1.

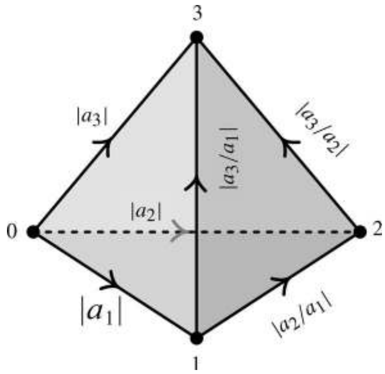

FIGURE 2.

- and so on.

In particular the space $S(\mathcal{A})$ is connected, since $|S_0(\mathcal{A})| = *$.

2.2. Iteration of the S-construction and delooping. In [22] Waldhausen proves also that the space $S(\mathcal{A})$ admits a delooping. Such delooping is constructed as the geometric realization $SS(\mathcal{A})$ of a bisimplicial category $S_\bullet S_\bullet(\mathcal{A})$, obtained by "iterating" the S-construction. Roughly speaking, the (p, q)-bisimplexes of $S_\bullet S_\bullet(\mathcal{A})$ are (p, q)-rigidified admissible bifiltrations of objects of $\mathcal{A}$. By this expression we mean a commutative diagram

$$\begin{tikzcd}
a_{11} \arrow[r, hook] \arrow[d, hook] & a_{12} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & a_{1q} \arrow[d, hook] \\
\vdots & \vdots & \cdots & \vdots \\
\vdots \arrow[d, hook] & \vdots \arrow[d, hook] & \cdots & \vdots \arrow[d, hook] \\
a_{p1} \arrow[r, hook] & a_{p2} \arrow[r, hook] & \cdots \arrow[r, hook] & a_{pq}
\end{tikzcd}$$

such that each horizontal and vertical arrow is an admissible monomorphism, and rigidified similarly to Definition 2.1. We refer again to [22] for details. The clasifying space $SS(\mathcal{A}) = |S_\bullet S_\bullet(\mathcal{A})|$ is thus the geometric realization of a trisimplicial set. We get $S(\mathcal{A}) = \Omega SS(\mathcal{A})$, and it is possible to furtherly iterate the S-construction to obtain an n-simplicial category $S^n_\bullet(\mathcal{A})$, and prove that $S(\mathcal{A}) = \Omega^{n-1}(S^n(\mathcal{A}))$. As a corollary, we have that $S(\mathcal{A})$ is an infinite loop space.

Note that every object $\underline{a}$ of $S_n(\mathcal{A})$ gives an object $\alpha(\underline{a})$ of $S_nS_1(\mathcal{A})$ and an object $\beta(\underline{a})$ of $S_1S_n(\mathcal{A})$ (bifiltrations going purely horizontally or purely vertically). We have therefore two maps of the suspension

$$\Sigma S(\mathcal{A}) \to SS(\mathcal{A})$$

both adjoint to the delooping isomorphism

$$S(\mathcal{A}) \to \Omega SS(\mathcal{A}).$$

On the level of cells, each (p, q)-cell σ of $S(\mathcal{A})$ gives rise to a $(p, 1, q)$-cell $\alpha(\sigma)$ and to a $(1, p, q)$-cell $\beta(\sigma)$ of $SS(\mathcal{A})$. Notice that up to dimension 4, all cells of $SS(\mathcal{A})$ are obtained in this way except for the cells of the following type:

- $\Delta^2 \times \Delta^2 \times \Delta^0$: one for each diagram of objects of $\mathcal{A}$:

(2.5)
$$\begin{tikzcd}
x_1^1 \arrow[r, hook] \arrow[d, hook] & x^1 \arrow[r, two heads] \arrow[d, hook] & x_2^1 \arrow[d, hook] \\
x_1 \arrow[r, hook] \arrow[d, two heads] & x \arrow[r, two heads] \arrow[d, two heads] & x_2 \arrow[d, two heads] \\
x_1^2 \arrow[r, hook] & x^2 \arrow[r, two heads] & x_2^2
\end{tikzcd}$$

whose rows and columns are admissible short exact sequences.

REMARK 2.6. It is important to notice that in the diagram (2.5) one has to impose the admissibility of the sequences of the quotients. Namely, for general exact categories $\mathcal{A}$ this condition does not descend from the admissibility of the monomorphisms which appear in the top left square.

2.3. Determinantal theories on exact categories with values on Picard categories. Let $\mathcal{A}$ be an exact category and $\mathcal{P}$ a symmetric Picard category.

DEFINITION 2.7. A $\mathcal{P}$-*valued determinantal theory* on $\mathcal{A}$ is a pair (h, λ), where h is a functor $S_1(\mathcal{A}) \to \mathcal{P}$ such that $h(0) = \mathbf{1}$ and λ is a system of isomorphisms given for all admissible short exact sequences $\sigma = a' \hookrightarrow a \twoheadrightarrow a''$ of $\mathcal{A}$

$$\lambda_\sigma : h(a') \otimes h(a'') \xrightarrow{\sim} h(a)$$

which are natural with respect to isomorphisms of admissible short exact sequences. These data are required to satisfy the following condition.

- For all admissible filtration of length 2 of $\mathcal{A}$, $a_1 \hookrightarrow a_2 \hookrightarrow a_3$ with a compatible choice of quotients, we have a commutative diagram

$$\begin{tikzcd} h(a_1) \otimes h\left(\frac{a_2}{a_1}\right) \otimes h\left(\frac{a_3}{a_2}\right) \arrow[rr, "1\otimes\lambda"] \arrow[d, "\lambda\otimes 1"'] & & h(a_1) \otimes h\left(\frac{a_3}{a_1}\right) \arrow[d, "\lambda"] \\ h(a_2) \otimes h\left(\frac{a_3}{a_2}\right) \arrow[rr, "\lambda"'] & & h(a_3) \end{tikzcd} \tag{2.8}$$

(where we have omitted for simplicity the associator of $\mathcal{P}$).

A *morphism* of determinantal theories $(h, \lambda) \to (h', \lambda')$ is a collection of morphisms $\{f_i : h(a_i) \to h'(a_i)\}$ of $\mathcal{P}$, such that, for all admissible short exact sequences $a' \hookrightarrow a \twoheadrightarrow a''$, the diagram

$$\begin{tikzcd} h(a') \otimes h(a'') \arrow[r, "\lambda"] \arrow[d, "f_{a'}\otimes f_{a''}"'] & h(a) \arrow[d, "f_a"] \\ h'(a') \otimes h'(a'') \arrow[r, "\lambda'"] & h'(a) \end{tikzcd}$$

commutes.

It is clear that every morphism of determinantal theories is an isomorphism.

REMARKS 2.9. (1) From the functoriality of h it follows that if $f : a \xrightarrow{\sim} b$ is an isomorphism, and $\sigma : a \xrightarrow{\sim} b \twoheadrightarrow 0$ (resp. $\sigma : 0 \hookrightarrow a \xrightarrow{\sim} b$), one has $\lambda_\sigma = h(f) : h(a) = h(a) \otimes h(0) \xrightarrow{\sim} h(b)$.

(2) The conditions defining a determinantal theory on $\mathcal{A}$ can be interpreted as conditions that h must satisfy on the simplices of dimension at most 3 of the simplicial Waldhausen category $S_\bullet(\mathcal{A})$. Indeed, notice in the first place that h is a functor $S_1(\mathcal{A}) \to \mathcal{P}$. Next, referring to the description of low-dimensional bisimplexes given in Section 2.1, h is completely determined as a map which sends:

- bisimplexes of type $\Delta^0 \times \Delta^p (= * \text{ in } S(\mathcal{A})) \to$ the null object $\mathbf{1}$,
- bisimplexes of type $\Delta^1 \times \Delta^0 \to$ objects of $\mathcal{P}$,
- bisimplexes of type $\Delta^1 \times \Delta^1 \to$ isomorphisms of $\mathcal{P}$,
- bisimplexes of type $\Delta^1 \times \Delta^2 \to$ compositions of isomorphisms of $\mathcal{P}$,
- bisimplexes of type $\Delta^2 \times \Delta^0 \to$ isomorphisms of type λ_σ of $\mathcal{P}$,
- bisimplexes of type $\Delta^2 \times \Delta^2 \to$ diagrams expressing the naturality of λ_σ in $\mathcal{P}$,
- bisimplexes of type $\Delta^3 \times \Delta^0 \to$ commutative diagrams of type (2.8).

DEFINITION 2.10. Let $\mathcal{A}$ be an exact category and $\mathcal{P}$ a Picard symmetric category. The category (groupoid) $\mathrm{Det}(\mathcal{A}, \mathcal{P})$ whose objects are the $\mathcal{P}$-valued determinantal theories on $\mathcal{A}$ and morphisms the isomorphisms of determinantal theories is called the category of *$\mathcal{P}$-valued determinantal theories on $\mathcal{A}$*.

2.4. Symmetric vs. non-symmetric determinantal theories. We introduce now the "symmetric versions" of the notions of determinantal theory and of $\mathrm{Det}(\mathcal{A}, \mathcal{P})$, which will be central in the developement of our theory, as follows:

DEFINITION 2.11. Let $\mathcal{P}$ be a symmetric Picard category, with symmetry σ. A $\mathcal{P}$-*valued symmetric determinantal theory* on $\mathcal{A}$ is a $\mathcal{P}$-valued determinantal theory (h, λ) on $\mathcal{A}$, such that, for all diagrams of type (2.5), the diagram

(2.12)

$$\begin{tikzcd}[column sep=small]
h(x_1^1) \otimes h(x_2^1) \otimes h(x_1^2) \otimes h(x_2^2) \arrow[rr, "1\otimes\sigma\otimes 1"] \arrow[d, "\lambda\otimes\lambda"'] & & h(x_1^1) \otimes h(x_1^2) \otimes h(x_2^1) \otimes h(x_2^2) \arrow[d, "\lambda\otimes\lambda"] \\
h(x^1) \otimes h(x^2) \arrow[dr, "\lambda"'] & & h(x_1) \otimes h(x_2) \arrow[dl, "\lambda"] \\
& h(x) &
\end{tikzcd}$$

is commutative.

A morphism of symmetric determinantal theories is defined as in the general case.

DEFINITION 2.13. If $\mathcal{P}$ is a symmetric Picard category, we define $\mathrm{Det}_\sigma(\mathcal{A}, \mathcal{P})$ to be the groupoid whose objects are the symmetric $\mathcal{P}$-valued determinantal theories on $\mathcal{A}$, and whose morphisms are the morphisms of determinantal theories.

REMARK 2.14. Thus, the datum of a symmetric determinantal theory is equivalent to a collection of data on the cells of $SS(\mathcal{A})$ up to dimension 4, which (h, λ) must satisfy. Indeed, all such cells come from those of $S(\mathcal{A})$, except for those of type $\Delta^2 \times \Delta^2 \times \Delta^0$, for which we impose the additional condition (2.12). Notice also that (2.12) implies (2.8), if we let the left column in (2.5) to be the admissible short exact sequence $x_1^1 = x_1 \twoheadrightarrow 0$.

The next proposition, which is a reformulation of a result due to Deligne (cf. [4, 4.8]) will be useful to perform computations.

PROPOSITION 2.15. *Let (h, λ) be a determinantal theory on $\mathcal{A}$, with values in the symmetric Picard category $\mathcal{P}$ with symmetry σ. Then (h, λ) is symmetric if and only if for each pair of objects a, b of $\mathcal{A}$, the diagram*

(2.16)
$$\begin{tikzcd}
h(a) \otimes h(b) \arrow[r, "\lambda"] \arrow[dr, "\sigma"'] & h(a \oplus b) = h(b \oplus a) \arrow[d, "\lambda^{-1}"] \\
& h(b) \otimes h(a)
\end{tikzcd}$$

commutes.

PROOF. We add the details to the argument sketched by Deligne. Since $\mathcal{A}$ is an exact category, it is closed under pushouts of admissible monomorphisms. Hence the diagram

$x_1 \hookrightarrow x_1^1 \hookleftarrow x^1$ admits a pushout, which we denote by $x^1 + x_1$, and in the resulting square

$$\begin{tikzcd} x_1^1 \arrow[r, hook] \arrow[d, hook] & x^1 \arrow[d, hook] \\ x_1 \arrow[r, hook] & x^1 + x_1 \end{tikzcd}$$

all the morphisms are admissible monomorphisms. The arrow $x \twoheadrightarrow x_2^2$ is an admissible epimorphism, and it is easy to see that its cokernel is the arrow $x^1 + x_1 \hookrightarrow x$ induced by the pushout diagram. Therefore the second arrow is an admissible monomorphism and thus

$$x^1 + x_1 \hookrightarrow x \twoheadrightarrow x_2^2$$

is an admissible short exact sequence in $\mathcal{A}$. It follows that $x_2^2 \xrightarrow{\sim} \dfrac{x}{x^1 + x_1}$.

We can consider then the admissible filtrations

$$x_1^1 \hookrightarrow x^1 \hookrightarrow x^1 + x_1 \hookrightarrow x$$
$$x_1^1 \hookrightarrow x_1 \hookrightarrow x^1 + x_1 \hookrightarrow x$$

of x in $\mathcal{A}$. In particular, from $x_1^1 \hookrightarrow x^1 \hookrightarrow x$ and $x_1^1 \hookrightarrow x_1 \hookrightarrow x$ we obtain that the diagram

$$\begin{tikzcd} h(x_1^1) \otimes h(x_2^1) \otimes h(x^2) \arrow[r, "1\otimes\lambda"] \arrow[d, "\lambda\otimes 1"'] & h(x_1^1) \otimes h\left(\dfrac{x}{x_1^1}\right) & h(x_1^1) \otimes h(x_1^2) \otimes h(x_2) \arrow[l, "1\otimes\lambda"'] \arrow[d, "\lambda\otimes 1"] \\ h(x^1) \otimes h(x^2) \arrow[r, "\lambda"'] & h(x) & h(x_1) \otimes h(x_2) \arrow[l, "\lambda"'] \end{tikzcd} \tag{2.17}$$

is commutative.

On the other hand, from $x_1^1 \hookrightarrow x^1 + x_1 \hookrightarrow x$, and observing that $\dfrac{x^1 + x_1}{x_1^1} \xrightarrow{\sim} x_2^1 \oplus x_1^2$, we obtain that the diagram

$$\begin{tikzcd} h(x_1^1) \otimes h(x_2^1 \oplus x_1^2) \otimes h(x_2^2) \arrow[r, "1\otimes\lambda"] \arrow[d, "\lambda\otimes 1"'] & h(x_1^1) \otimes h\left(\dfrac{x}{x_1^1}\right) \arrow[d, "\lambda"] \\ h(x^1 + x_1) \otimes h(x_2^2) \arrow[r, "\lambda"'] & h(x) \end{tikzcd}$$

commutes. Similarly, taking quotients of the first filtration above by x_1^1 we obtain that the diagram

$$\begin{tikzcd} h(x_2^1) \otimes h(x_1^2) \otimes h(x_2^2) \arrow[r, "1\otimes\lambda"] \arrow[d, "\lambda\otimes 1"'] & h(x_2^1 \oplus x_1^2) \otimes h(x_2^2) \arrow[d, "\lambda"] \\ h(x_1^1) \otimes h(x_2^2) \arrow[r, "\lambda"'] & h\left(\dfrac{x}{x_1^1}\right) \end{tikzcd}$$

also commutes. Since h is symmetric, the latter diagram, tensorized with $h(x_1^1)$ and compared with the diagram (2.17), yields the diagram (2.12). This proves the "if" clause of the statement. For the "only if" part, we observe that the commutative diagram (2.16) is just the case $x_1^1 = x_2^2 = 0$ of the commutative diagram (2.12). Thus, the proposition is proved. □

EXAMPLES 2.18. (1) Let k be a field and $\mathcal{A} = \mathbf{Vect}_0(k)$ the abelian category of finite dimensional vector spaces on k. Let $G = k^*$, and $\mathcal{P} = \mathrm{Tors}(G)$. For an object $V \in \mathbf{Vect}_0(k)$, let us denote by $\Lambda^{\max}$ the top exterior power of V. Then we have a G-torsor

$$\det(V) = \Lambda^{\max} - \{0\},$$

called the *determinantal space* of V. For every short exact sequences of vector spaces $V' \hookrightarrow V \twoheadrightarrow V''$, we have natural identifications

$$\lambda_{V',V,V''}\colon\ \det(V') \otimes \det(V'') \to \det(V).$$

The collection $\{\det(V), \lambda\}_{V\in\mathbf{Vect}_0(k)}$ forms a determinantal theory on $\mathcal{A}$ (see [12]). This determinantal theory is non-symmetric.

(1′) (Sketch) The non-symmetric determinantal theory $\det(V)$ has a symmetric analog. Let us consider the category $\mathrm{Pic}_k^{\mathbf{Z}}$ (see Appendix). For any V in $\mathbf{Vect}_0(k)$, let $\det(V)$ be the graded 1-dimensional vector space consisting of the top exterior power $\Lambda^{\max}(V)$ in degree $\dim(V)$. Then, the correspondence $V \mapsto \det(V)$ is a symmetric determinantal theory with values in $\mathrm{Pic}_k^{\mathbf{Z}}$.

(2) *The universal determinantal theory.* The geometric description of the bisimplexes of $S(\mathcal{A})$ of dimension less than or equal to 3 has a natural interpretation in terms of the *universal determinantal theory*. This is a determinantal theory with values in the category of virtual objects (cf. Appendix B). Namely, $(h^u, \lambda^u)\colon \mathcal{A} \to V(\mathcal{A})$ is defined as follows. Referring to the notations used in Remark 2.4, for all object $a \in \mathcal{A}$, $h^u(a)$ is the loop $|a|$ of $S(\mathcal{A})$, interpreted as on object of $V(\mathcal{A})$. Given $\sigma : a' \hookrightarrow a \twoheadrightarrow a''$, $|\sigma|$ is a homotopy class of homotopies between the composition of the loops $|a'| * |a''|$ and $|a|$, and it can be interpreted as an arrow

$$\lambda_\sigma^u = |\sigma|\colon h^u(a') \otimes h^u(a'') \xrightarrow{\sim} h^u(a).$$

of $V(\mathcal{A})$.

We claim that the pair (h^u, λ^u) defines a symmetric determinantal theory. Indeed, let τ be $a_1 \hookrightarrow a_2 \hookrightarrow a_3$. Interpret $|\tau|$ (see Figure 2) as a class of homotopies between the composition of the even faces of $|\tau|$, as in Figure 3, i.e., the arrow of $V(\mathcal{A})$:

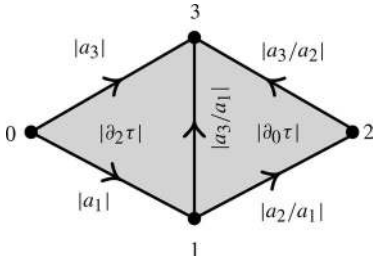

FIGURE 3. Even composition.

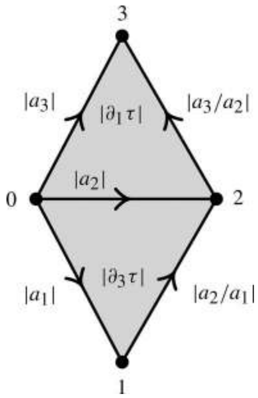

FIGURE 4. Odd composition.

$$h^u(a_3) \xrightarrow{(1\otimes\partial_0(\tau))\partial_2(\tau)} h^u(a_1) \otimes h^u\left(\frac{a_2}{a_1}\right) \otimes h^u\left(\frac{a_3}{a_2}\right)$$

and the composition of the odd faces of $|\tau|$, as in Figure 4, i.e., the arrow of $V(\mathcal{A})$:

$$h^u(a_3) \xrightarrow{(1\otimes\partial_1(\tau))\partial_3(\tau)} h^u(a_1) \otimes h^u\left(\frac{a_2}{a_1}\right) \otimes h^u\left(\frac{a_3}{a_2}\right).$$

Thus, $|\tau|$ yields the commutativity, up to an isomorphism α of associativity of $V(\mathcal{A})$, of a diagram of type (2.8), with $h = h^u$ and $\lambda = \lambda^u$. Similar interpretations give the functoriality of h and the naturality of λ^u_σ with respect to σ. Thus (h^u, λ^u) is a determinantal theory. A direct application of Proposition 2.15 shows that it is symmetric. We call it the *universal determinantal theory* on $\mathcal{A}$. This terminology is justified by the following theorem, which in the symmetric case is due to Deligne (cf. [4, 4.3]), and which explains how to reconstruct any $\mathcal{P}$-valued determinantal theory from the universal determinantal theory.

THEOREM 2.19. (1) *Let $\mathcal{P}$ be a Picard category and* $\mathrm{Fun}^{\otimes}(V(\mathcal{A}), \mathcal{P})$ *the category of Picard functors $V(\mathcal{A}) \to \mathcal{P}$. Then there exists an equivalence of categories*

$$\mathrm{Det}(\mathcal{A}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(V(\mathcal{A}), \mathcal{P}) .$$

(2) *Let $\mathcal{P}$ be a symmetric Picard category, and* $\mathrm{Fun}^{\otimes}_{\sigma}(V(\mathcal{A}), \mathcal{P})$ *the category of symmetric Picard functors $V(\mathcal{A}) \to \mathcal{P}$. Then there exists an equivalence of categories*

$$\mathrm{Det}_{\sigma}(\mathcal{A}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}_{\sigma}(V(\mathcal{A}), \mathcal{P}) .$$

3. Sato Grassmannians in an exact category. Let $\mathcal{A}$ be an exact category and let us consider its Beilinson category $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ of locally compact objects over $\mathcal{A}$ (or *generalized Tate spaces*; see Appendix A). Fix any object $X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}$. We sketch a general theory of the Grassmannian of the generalized Tate space X, which generalizes the concept of the Sato Grassmannian of a Tate space X, when $\mathcal{A} = \mathbf{Vect}_0(k)$.

DEFINITION 3.1. The *Sato Grassmannian* of the object X is the set $\Gamma(X)$ of all the admissible subobjects $[V \hookrightarrow X]$, such that $V \in \mathrm{Pro}^a(\mathcal{A})$ and $X/V \in \mathrm{Ind}^a(\mathcal{A})$.

In other words, for a subobject of X, the statement "$[V \hookrightarrow X] \in \Gamma(X)$" means that there is an admissible short exact sequence of $\underset{\longleftrightarrow}{\lim} \mathcal{A}$:

$$V \hookrightarrow X \twoheadrightarrow \frac{X}{V}$$

such that V is in $\mathrm{Pro}^a(\mathcal{A})$ and X/V is in $\mathrm{Ind}^a(\mathcal{A})$.

In such a situation, and when the class of the monomorphism $m : V \hookrightarrow X$ is known, we shall simply say that V is in $\Gamma(X)$. Let thus $X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}$ be given through a specific ind-pro system $\{X_i\}$, $X =$ "$\underset{\longrightarrow}{\lim}$"$_{i \in I} X_i$. The existence of the monomorphism $m : V \hookrightarrow X$ implies the existence of an $i \in I$ and an admissible monomorphism of $\mathrm{Pro}^a(\mathcal{A})$: $m_i : V \hookrightarrow X_i$. By composing with the structure maps of the ind-system $\{X_i\}$, we obtain that there is an admissible monomorphism $m_j : V \hookrightarrow X_j$ for all $j \geq i$. Then we can write the quotient X/V as

$$\text{“}\underset{\substack{\longrightarrow \\ i \in I}}{\lim}\text{”} \left(\frac{X_i}{V} \right) .$$

The condition expressed in the definition implies that this is a strict admissible ind-system of $\mathcal{A}$. Therefore, each quotient object X_i/V is in $\mathcal{A}$.

THEOREM 3.2. *Let $X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}$ and $V \hookrightarrow W$ be an admissible monomorphism in* $\mathrm{Pro}^a(\mathcal{A})$, *with $W \in \Gamma(X)$. Then $V \in \Gamma(X)$ if and only if $W/V \in \mathcal{A}$.*

PROOF. ("If" part) Let $V \hookrightarrow W \hookrightarrow X$ be the composition of two admissible monomorphisms. We want to show that $X/V \in \mathrm{Ind}^a(\mathcal{A})$.

We get the diagram

(3.3)
$$\begin{tikzcd} V \arrow[r] & W \arrow[r] \arrow[d] & X \arrow[d] \\ & \frac{W}{V} \arrow[r] & \frac{X}{V} \arrow[d] \\ & & \frac{X}{W} \end{tikzcd}$$

where the horizontal arrows are admissible monomorphisms, and the vertical ones admissible epimorphisms. In particular, we get an admissible short exact sequence $W/V \hookrightarrow X/V \twoheadrightarrow X/W$ in $\varprojlim\!\!\!\!\!\longleftrightarrow \mathcal{A}$, with $W/V \in \mathcal{A}$ and $X/W \in \mathrm{Ind}^a(\mathcal{A})$, since $W \in \Gamma(X)$. But $\mathrm{Ind}^a(\mathcal{A})$ is closed under extensions in $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ (cf. [17]), hence it follows $X/V \in \mathrm{Ind}^a(\mathcal{A})$, i.e., $[V \hookrightarrow X] \in \Gamma(X)$.

("Only if" part) It is clear from the same diagram. □

3.1. Partially abelian exact categories.

DEFINITION 3.4. (1) Let $(\mathcal{A}, \mathcal{E})$ be an exact category. We say that $\mathcal{A}$ is *closed under admissible intersections*, or simply that $\mathcal{A}$ satisfies the admissible intersection condition (AIC), if any pair of admissible monomorphisms with the same target, $a' \hookrightarrow a \hookleftarrow a''$ has a pullback p in $\mathcal{A}$, and in the resulting diagram

$$\begin{tikzcd} p \arrow[r, hook] \arrow[d, hook] & a' \arrow[d, hook] \\ a'' \arrow[r, hook] & a \end{tikzcd}$$

all the morphisms are admissible monomorphisms.

(2) Dually, we say that $\mathcal{A}$ satisfies $(\mathrm{AIC})^o$, if any pair of admissible epimorphisms with the same source: $b \twoheadrightarrow b'$, $b \twoheadrightarrow b''$, has a pushout q in $\mathcal{A}$, and in the resulting diagram

$$\begin{tikzcd} b \arrow[r] \arrow[d] & b' \arrow[d] \\ b'' \arrow[r] & q \end{tikzcd}$$

all the morphisms are admissible epimorphisms.

LEMMA 3.5. *Let $(\mathcal{A}, \mathcal{E})$ be closed under admissible intersections. Consider the pullback diagram of the admissible monomorphisms $a \hookrightarrow c$, $b \hookrightarrow c$:*

$$\begin{tikzcd} p \arrow[r, hook, "i"] \arrow[d, hook] & a \arrow[d, hook] \\ b \arrow[r, hook, "i'"'] & c\,. \end{tikzcd} \tag{3.6}$$

Let $j = \operatorname{coker}(i)$ and $j' = \operatorname{coker}(i')$ be admissible epimorphisms. Then, there exists a unique (not necessarily admissible) monomorphism m'' of $\mathcal{A}$, making the diagram

$$\begin{tikzcd} p \arrow[r, hook, "i"] \arrow[d, hook] & a \arrow[r, "j"] \arrow[d, hook] & q \arrow[d, hook, dashed, "m''"] \\ b \arrow[r, hook, "i'"'] & c \arrow[r, "j'"'] & d \end{tikzcd}$$

commutative.

PROOF. The above lemma holds in any abelian category. It is thus valid in the abelian envelope $\mathcal{F}$ of $\mathcal{A}$. In particular, m'' is a monomorphism of $\mathcal{A}$. □

LEMMA 3.7. *In the situation of Lemma* 3.5, *let us extend the diagram* (3.6) *to a* 3×3 *diagram*

$$\begin{tikzcd} p \arrow[r, hook, "i"] \arrow[d, hook, "m'"'] & a \arrow[r, "j"] \arrow[d, hook, "m"] & q \arrow[d, hook, "m''"] \\ b \arrow[r, hook, "i'"'] \arrow[d, "e'"'] & c \arrow[r, "j'"'] \arrow[d, "e"] & d \arrow[d, "e''"] \\ r \arrow[r, "i''"'] & s \arrow[r, "j''"'] & t \end{tikzcd} \tag{3.8}$$

by passing to the cokernels in the abelian envelope $\mathcal{F}$, where (m'', e'') and (i'', j'') are short exact sequences in $\mathcal{F}$, while (i, j), (i', j'), (m, e), (m', e') are admissible short exact sequences in $\mathcal{A}$. In this case, the bottom right square is a pushout diagram.

PROOF. The proof is a direct verification of the universal property of pushouts relative to the bottom right square. □

PROPOSITION 3.9. *If $\mathcal{A}$ satisfies both* (*AIC*) *and* (*AIC*)^o, *then in diagram* (3.8) *m'' is an admissible monomorphism and e'' an admissible epimorphism. As a result*, (3.8) *represents an object of the category $S_2S_2(\mathcal{A})$, of the delooping $S_\bullet S_\bullet(A)$ of $S_\bullet(\mathcal{A})$ (cf. Subsection* 2.2).

PROOF. From Lemma 3.7 we know that the diagram

$$\begin{tikzcd} c \arrow[r, "j'"] \arrow[d, "e"'] & d \arrow[d, "e''"] \\ s \arrow[r, "j''"'] & t \end{tikzcd}$$

is a pushout diagram of the admissible epimorphisms e and j'. Since $\mathcal{A}$ satisfies (AIC)^o, it follows that $t \in \mathcal{A}$ and that j'', e'' are admissible epimorphisms. Therefore, from Lemma A.1, $m'' = \ker(e'')$ is an admissible monomorphism. $\square$

DEFINITION 3.10. An exact category $(\mathcal{A}, \mathcal{E})$ is called *partially abelian exact* (PAE) if every arrow f which is the composite of an admissible monomorphism followed by an admissible epimorphism can be factored in a unique way as the composite of an admissible epimorphism followed by an admissible monomorphism.

For example, an abelian category is partially abelian exact. In Section 5 we shall give an example of an exact category which is not partially abelian exact.

THEOREM 3.11. *The category* $(\mathcal{A}, \mathcal{E})$ *is partially abelian exact if and only if* $\mathcal{A}$ *satisfies both* (*AIC*) *and* (*AIC*)^o.

PROOF. We first show that if $\mathcal{A}$ satisfies (AIC) and (AIC)^o, then $\mathcal{A}$ is partially abelian exact. Let f be the composite $x \overset{m}{\hookrightarrow} y \overset{e}{\twoheadrightarrow} z$ of an admissible mono m followed by an admissible epi e. Let $k \hookrightarrow y$ be the kernel of e, which is an admissible monomorphism, and consider the pullback p of $k \hookrightarrow y$ and m. We obtain the diagram

$$\begin{tikzcd} p \arrow[r, hook] \arrow[d, hook, "m'"'] & k \arrow[d, hook] \\ x \arrow[r, hook, "m"'] \arrow[d, "e'"'] & y \arrow[d, "e"] \\ z' \arrow[r, dashed, "h"'] & z, \end{tikzcd}$$

in which $m' : p \hookrightarrow x$ is an admissible monomorphism and $e' = \mathrm{coker}(m') \in \mathcal{A}$. From Lemma 3.5 there exists a unique admissible monomorphism $h : z' \hookrightarrow z$ for which the bottom square commutes. Thus, $x \overset{e'}{\twoheadrightarrow} z' \overset{h}{\hookrightarrow} z$ is the required factorization of f.

Conversely, suppose that $\mathcal{A}$ is partially abelian exact. We first show that $\mathcal{A}$ satisfies (AIC). Let $k \hookrightarrow y \hookleftarrow x$ be a diagram of admissible monomorphisms of $\mathcal{A}$. Let $z: = y/k$ and apply the factorization condition to the composite $x \hookrightarrow y \twoheadrightarrow z$. Then we obtain the diagram

$$\begin{tikzcd} & k \arrow[d, hook, "m"] \\ x \arrow[r, hook] \arrow[d, "e'"'] & y \arrow[d, "e"] \\ y \arrow[r, hook, "i''"'] & z \end{tikzcd}$$

where e' is an admissible epimorphism and i'' an admissible monomorphism.

From the universal property of $m = \ker(e)$, we obtain a unique morphism $k' \to k$ for which the diagram

$$\begin{tikzcd} k' \arrow[r, "i"] \arrow[d, hook, "m'"'] & k \arrow[d, hook, "m"] \\ x \arrow[r, hook] \arrow[d, "e'"'] & y \arrow[d, "e"] \\ y \arrow[r, hook, "i''"'] & z \end{tikzcd} \tag{3.12}$$

is commutative.

It is clear that i is a monomorphism in $\mathcal{F}$, hence in $\mathcal{A}$, and that the top square is cartesian. We want to prove that i is an admissible monomorphism. Consider the admissible epimorphism $j' = \mathrm{coker}(i')$, and the epimorphism $j = \mathrm{coker}(i)$ in $\mathcal{F}$. Since the top square of (3.12) is cartesian in $\mathcal{F}$, we obtain, from Lemma 3.5, a unique monomorphism $m''\colon k/k' \hookrightarrow y/x$ in $\mathcal{F}$ making the diagram

$$\begin{tikzcd} k' \arrow[r, hook, "i"] \arrow[d, hook] & k \arrow[r, "j"] \arrow[d, hook, "m"] & k/k' \arrow[d, hook, "m''"] \\ x \arrow[r, hook, "i'"'] & y \arrow[r, "j'"'] & y/x \end{tikzcd}$$

commutative.

Let us write f the composite $j' \cdot m$ in the previous diagram. Since m is an admissible monomorphism, and j' an admissible epimorphism, we can factor f as a composition $k \overset{a}{\twoheadrightarrow} z \overset{b}{\hookrightarrow} y/x$ where a is an admissible epimorphism and b an admissible monomorphism. In the abelian envelope $\mathcal{F}$ we thus obtain two decompositions of f as an admissible epimorphism followed by an admissible monomorphism. Since in an abelian category every arrow has an essentially unique such decomposition, it must be $k/k' \overset{\sim}{\to} z$, and j is an admissible epimorphism. Since $i = \ker(j)$, it follows from Lemma A.1 that i is an admissible monomorphism, as required.

By duality, $\mathcal{A}$ satisfies also (AIC)^o. This concludes the proof of the theorem. $\square$

3.2. Grassmannians and intersections. In this section and the next we clarify the behavior of $\Gamma(X)$ under admissible short exact sequences of $\varprojlim\!\!\!\!\!\varinjlim \mathcal{A}$. The main result is Theorem 3.29, which roughly speaking allows us to lift an element $U \in \Gamma(X)$ along admissible monomorphisms $Y \hookrightarrow X$ and to project it along admissible epimorphisms $X \twoheadrightarrow Z$ of $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ to elements of the Grassmannians of Y and Z, respectively, under the assumption that $\mathcal{A}$ is partially abelian exact. We start by showing that $\Gamma(X)$ is closed under the operation of taking the intersection of two elements, under the condition that $\mathcal{A}$ satisfies (AIC).

THEOREM 3.13. *Let $\mathcal{A}$ be an exact category satisfying* (*AIC*). *Let $X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}$ and $[U \hookrightarrow X], [V \hookrightarrow X] \in \Gamma(X)$. For all $m\colon U \hookrightarrow X, n\colon V \hookrightarrow X$ in their respective equivalence classes*, *the diagram*

$$U \xhookrightarrow{m} X \xhookleftarrow{n} V \tag{3.14}$$

can be completed to a pullback diagram in $\underset{\longleftrightarrow}{\lim} \mathcal{A}$

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\ V \arrow[r, hook] & X \end{tikzcd} \tag{3.15}$$

such that $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ are admissible monomorphisms, *and*, *after composing the arrows of* (3.15), *we get $[U \cap V \hookrightarrow X]$ in $\Gamma(X)$.*

LEMMA 3.16. *Let $\mathcal{A}$ be an exact category satisfying* (*AIC*), *and $\mathcal{F}$ its abelian envelope. Suppose that we have two pullback diagrams*

$$\begin{tikzcd} P \arrow[r, hook, "i_1"] \arrow[d, hook, "j_1"'] & A \arrow[d, hook, "j_2"] \\ B \arrow[r, hook, "i_2"'] & Z \end{tikzcd}, \qquad \begin{tikzcd} P'' \arrow[r, hook, "k_1"] \arrow[d, hook, "l_1"'] & A'' \arrow[d, hook, "l_2"] \\ B'' \arrow[r, hook, "k_2"'] & Z'' \end{tikzcd}, \tag{3.17}$$

where all the morphisms are admissible monomorphisms, and there are admissible epimorphisms $e\colon A \twoheadrightarrow A''$, $f\colon Z \twoheadrightarrow Z''$, $g\colon B \twoheadrightarrow B''$ *for which we have a commutative diagram*

(3.18)
$$\begin{tikzcd}
 & P \arrow[rr, hook] \arrow[dd, hook] & & A \arrow[dl, two heads, "e"] \arrow[dd, hook] \\
P'' \arrow[rr, hook] \arrow[dd, hook] & & A'' \arrow[dd, hook] & \\
 & B \arrow[rr, hook] \arrow[dl, two heads, "g"] & & Z \arrow[dl, two heads, "f"] \\
B'' \arrow[rr, hook] & & Z'' &
\end{tikzcd}$$

such that the square

(3.19)
$$\begin{tikzcd}
B \arrow[r, hook] \arrow[d, "g"'] & Z \arrow[d, "f"] \\
B'' \arrow[r, hook] & Z''
\end{tikzcd}$$

is admissible and cartesian. Then there exists a unique morphism $r\colon P \to P''$ *for which the above cubic diagram commutes, and* r *is an admissible epimorphism.*

PROOF. The existence and uniqueness of r is a consequence of the universal property of the pullback P''. We now prove that r is an admissible epimorphism, by showing that r is an epimorphism of $\mathcal{F}$ whose kernel is in $\mathcal{A}$, and then applying Lemma A.1.

Let thus consider the diagram (3.18) in the abelian envelope $\mathcal{F}$. To prove that r is an epimorphism of $\mathcal{A}$, we use a diagram-chase argument.

Suppose, therefore, that an element a is given in P''. We want to construct a preimage of a through r.

Construct, from a, the following elements: $d = k_1(a) \in A''$; $c = l_1(a) \in B''$ and $e = k_2(c) \in Z''$. Then, lift e to a preimage $\tilde{e}$ in Z, which exists since f is surjective. From the cartesianity of the diagram (3.19), we get a unique element $b \in B$ such that $i_2(b) = e$ and $g(b) = c$.

Next, consider the preimages $\tilde{d}$ of d in A. If there exists a $\tilde{d}$ such that $j_1(\tilde{d}) = \tilde{e}$, then, from the cartesianity of the left square in (3.17), we obtain a unique element $\tilde{x} \in P$ for which $i_1(\tilde{x}) = \tilde{d}$ and $j_1(\tilde{x}) = b$. Thus, $r(\tilde{x})$ is the unique preimage of d in P, and r is surjective in this case.

Suppose, on the other hand, that $j_2(\tilde{d}) \neq \tilde{e}$ for all preimage $\tilde{d}$ of d in A. In this case, pick any $j_2(\tilde{d})$ in Z. We get

$$f(j_2(\tilde{d}) - \tilde{e}) = (f \cdot j_2)(\tilde{d}) - f(\tilde{e}) = e - e = 0\,.$$

For a given $\tilde{e}' = j_2(\tilde{d})$ in Z, and from the cartesianity of the diagram (3.19), we obtain a unique element $\tilde{b}$ in B, such that $i_1(\tilde{b}) = \tilde{e}'$ and $g(\tilde{b}) = c$. Now, the cartesianity of the left square in (A.2) yields again a unique element $\tilde{x}$ in P for which $j_1(\tilde{x}) = \tilde{b}$ and $i_1(\tilde{x}) = \tilde{d}$. We thus have again

$$k_1 \cdot r(\tilde{x}) = (e \cdot i_1)(\tilde{x}) = d\,,$$

and we reduce ourselves to the previous case. Then, $r(\tilde{x}) = a$, and r is an epimorphism.

We now prove that r is an admissible epimorphism. For this, we consider the following double cubic diagram, which is the extension of the cubic diagram (3.18) to the kernels of the epimorphisms there involved.

Let $\gamma = \ker(g)$, $\varepsilon = \ker(e)$, $\phi = \ker(f)$ be admissible monomorphisms, and $s = \ker(r)$, the kernel of r in $\mathcal{F}$. Then, we get the cubic diagram

$$\begin{tikzcd}
& A' \arrow[rr, hook] \arrow[dd, hook, "\varepsilon"] && Z' \arrow[dd, hook, "\phi"] \\
P' \arrow[ur, hook] \arrow[rr, hook, "n"] \arrow[dd, hook, "s"'] && B' \arrow[ur, hook] \arrow[dd, hook] \\
& A \arrow[rr, hook, "\gamma"] \arrow[dd, two heads, "e"] && Z \arrow[dd, two heads, "f"] \\
P \arrow[ur, hook] \arrow[rr, hook, "j_1"] \arrow[dd, two heads, "r"'] && B \arrow[ur, hook] \arrow[dd, two heads] \\
& A'' \arrow[rr, hook, "g"] && Z'' \\
P'' \arrow[ur, hook] \arrow[rr, hook, "l_1"'] && B'' \arrow[ur, hook]
\end{tikzcd}$$

in the category $\mathcal{F}$. In this diagram the arrows composing the top square are monomorphisms induced by the universal properties of the kernels involved. The columns (ε, e), (ϕ, f), (γ, g) are admissible short exact sequences of $\mathcal{A}$, while the column (s, r) is a short exact sequence of $\mathcal{F}$. In order to prove that r is an admissible epimorphism, we shall prove that $s = \ker(r)$ is an admissible monomorphism. The claim will then follow from Lemma A.1. It will be enough to show that the square

(3.20)
$$\begin{tikzcd}
P' \arrow[r, hook] \arrow[d, hook, "s"'] & B' \arrow[d, hook, "\gamma"] \\
P \arrow[r, hook, "j_1"'] & B
\end{tikzcd}$$

is cartesian in $\mathcal{F}$. In fact, this will imply that it is the pullback square of two admissible monomorphisms of $\mathcal{A}$, i.e., γ and j_1, thus, from the (AIC) condition and Lemma A.2, the square is cartesian in $\mathcal{A}$ and s is an admissible monomorphism.

We shall use also in this case a diagram-chase argument. Let $v \in P$ and $u \in B'$ be two elements such that $j_1(v) = \gamma(u) = w$ in B. This element w is sent by g to 0 of B'', since $g(w) = g \cdot \gamma(u)$ and (γ, g) is an admissible short exact sequence. Thus, $l_1 \cdot r(v) = 0$. But l_1 is a monomorphism, so $r(v) = 0$.

It follows that v belongs to the kernel of r, hence to the image of s. Let $x \in P'$ be the unique element such that $s(x) = v$. The cartesianity of (3.20) is proved if we can show that $n(x) = u$. But this is clear. Actually, since all the morphisms involved in diagram (3.20) are monomorphisms, and since $\gamma(u) = w = j_1 \cdot s(x)$, n must send x into the unique preimage of w in B', i.e., u. Thus, (3.20) is cartesian in $\mathcal{F}$. Then r is an admissible epimorphism and the proof of the lemma is complete. □

We notice that the proof of the claim that r is an epimorphism also proves the following corollary.

COROLLARY 3.21. *The resulting admissible square*

$$\begin{tikzcd} P'' \arrow[r, hook] \arrow[d, two heads, "r"'] & A'' \arrow[d, two heads] \\ P \arrow[r, hook] & A \end{tikzcd}$$

in diagram (3.18) *is cartesian.*

Using Lemma A.10 and Corollary 3.21, we obtain the following proposition.

PROPOSITION 3.22. *Let U, V, Z be objects of* $\mathrm{Pro}^a(\mathcal{A})$, *with admissible monomorphisms $U \hookrightarrow Z$ and $V \hookrightarrow Z$, which can be expressed as ladders of cartesian admissible squares of $\mathcal{A}$. Then the pullback $U \times_Z V$ exists in* $\mathrm{Pro}^a(\mathcal{A})$ *and in the resulting diagram*

$$\begin{tikzcd} U \times_Z V \arrow[r] \arrow[d] & U \arrow[d, "m"] \\ V \arrow[r, "n"'] & Z \end{tikzcd},$$

the morphisms $U \times_Z V \to U$ and $U \times_Z V \to V$ are admissible monomorphisms which can be expressed as ladders of cartesian admissible squares of $\mathcal{A}$.

Let us now consider the case for $\underleftrightarrow{\lim}\,\mathcal{A}$. Let $X \in \underleftrightarrow{\lim}\,\mathcal{A}$ and suppose $X = \text{“}\underrightarrow{\lim}\text{”}_j X_j$, for $X_j \in \mathrm{Pro}^a(\mathcal{A})$. Let $U \in \mathrm{Pro}^a(\mathcal{A})$, and let $m : U \hookrightarrow X$ be an admissible monomorphism in $\underleftrightarrow{\lim}\,\mathcal{A}$. Since U in $\underleftrightarrow{\lim}\,\mathcal{A}$ is represented as a trivial ind-system of $\mathrm{Pro}^a(\mathcal{A})$, the datum of m is equivalent to the datum of the existence of an index j and an admissible monomorphism $U \hookrightarrow X_j$ in $\mathrm{Pro}^a(\mathcal{A})$.

Then, if $[U \hookrightarrow X], [V \hookrightarrow X] \in \Gamma(X)$, there are indices j_1, j_2 for which any pair of representatives $m\colon U \hookrightarrow X$, $n\colon V \hookrightarrow X$ are given in components by admissible monomorphisms of $\mathrm{Pro}^a(\mathcal{A})$, $U \hookrightarrow X_{j_1}$, $V \hookrightarrow X_{j_2}$, as ladders of cartesian squares in $\mathcal{A}$. By taking $j = \max(j_1, j_2)$, we can assume, without loss of generality, $j_1 = j_2 = j$.

The next lemma follows from Proposition 3.22.

LEMMA 3.23. *With the same notation as above, the object* $U \times_{X_j} V$ *is a pullback in* $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ *of the diagram* (3.14).

We shall denote the object "$\underset{\longrightarrow}{\lim}$"$_j U \times_{X_j} V$ by $U \cap V$. We now can prove Theorem 3.13.

PROOF OF THEOREM 3.13. In Lemma 3.23 we have proved the existence, under the assumptions of the Theorem, of a pullback square (3.15), where $U \cap V \in \mathrm{Pro}^a(\mathcal{A})$, and $U \cap V \hookrightarrow U$, $U \cap V \hookrightarrow V$ are admissible monomorphisms from Proposition 3.22. It remains to prove that $[U \cap V \hookrightarrow X] \in \Gamma(X)$.

This can be achieved by the consideration of the induced admissible short exact sequence in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$

$$\frac{U}{U \cap V} \hookrightarrow \frac{X}{U \cap V} \twoheadrightarrow \frac{X}{U}\,.$$

Since from Proposition 3.22 the monomorphism $U \cap V \hookrightarrow U$ can be expressed as a ladder of cartesian squares, the quotient $U/(U \cap V)$ is in $\mathcal{A}$.

On the other hand, since $[U \hookrightarrow X]$ is in $\Gamma(X)$, X/U is in $\mathrm{Ind}^a(\mathcal{A})$. Thus, in the above short exact sequence, the first and the last term are in $\mathrm{Ind}^a(\mathcal{A})$, which, being closed under extensions in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, forces $X/(U \cap V)$ to be also in $\mathrm{Ind}^a(\mathcal{A})$. Then $[U \cap V \hookrightarrow X]$ is in $\Gamma(X)$, and the theorem is proved. □

3.3. Grassmannians and short exact sequences. We now discuss the behavior of Sato Grassmannians under admissible short exact sequences in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$.

PROPOSITION 3.24. *Let* $\mathcal{A}$ *be an exact category satisfying* (*AIC*). *Let* $m\colon X \hookrightarrow Y$ *be an admissible monomorphism in* $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, *and* $[U \hookrightarrow Y]$ *an element of* $\Gamma(Y)$. *Then the diagram*

(3.25)
$$\begin{tikzcd} X \arrow[r, hook, "m"] & Y \\ & U \arrow[u, hook] \end{tikzcd}$$

can be completed to a pullback diagram

(3.26)
$$\begin{tikzcd} X \arrow[r, hook, "m"] & Y \\ U \cap X \arrow[u, hook] \arrow[r, hook] & U \arrow[u, hook] \end{tikzcd},$$

where the object $U \cap X$ is in $\mathrm{Pro}^a(\mathcal{A})$, *all the maps are admissible monomorphisms*, *and the resulting composition* $[U \cap X \hookrightarrow X]$ *is in* $\Gamma(X)$.

PROOF. Straightify m. Then, for all j, we can represent m by a system of monomorphisms

$$\begin{tikzcd} \cdots \arrow[r, hook] & X_{i-1} \arrow[r, hook] \arrow[d, hook] & X_i \arrow[r, hook] \arrow[d, hook] & X_{i+1} \arrow[r, hook] \arrow[d, hook] & \cdots \\ \cdots \arrow[r, hook] & Y_{i-1} \arrow[r, hook] & Y_i \arrow[r, hook] & Y_{i+1} \arrow[r, hook] & \cdots \end{tikzcd}$$

of $\mathrm{Pro}^a(\mathcal{A})$. Since $U \in \mathrm{Pro}^a(\mathcal{A})$, the existence of the admissible monomorphism $U \hookrightarrow Y$ in $\underleftrightarrow{\lim}\,\mathcal{A}$ is equivalent to the existence of an i and of an admissible monomorphism $U \hookrightarrow Y_i$ of $\mathrm{Pro}^a(\mathcal{A})$. For this monomorphism we have the diagram, in $\mathrm{Pro}^a(\mathcal{A})$:

$$X_i \hookrightarrow Y_i \hookleftarrow U\,. \tag{3.27}$$

We straightify this diagram by writing, in components: $X_i = \text{“}\varprojlim\text{”}_{j \in J} X_j$, $Y_i = \text{“}\varprojlim\text{”}_{j \in J} Y_j$, $U = \text{“}\varprojlim\text{”}_{j \in J} U_j$. We then obtain a diagram of objects of $\mathcal{A}$

$$\begin{tikzcd} \cdots \arrow[r, two heads] & X_{j+1} \arrow[r, two heads] \arrow[d, hook] & X_j \arrow[r, two heads] \arrow[d, hook] & X_{j-1} \arrow[r, two heads] \arrow[d, hook] & \cdots \\ \cdots \arrow[r, two heads] & Y_{j+1} \arrow[r, two heads] & Y_j \arrow[r, two heads] & Y_{j-1} \arrow[r, two heads] & \cdots \\ \cdots \arrow[r, two heads] & U_{j+1} \arrow[r, two heads] \arrow[u, hook] & U_j \arrow[r, two heads] \arrow[u, hook] & U_{j-1} \arrow[r, two heads] \arrow[u, hook] & \cdots \end{tikzcd}.$$

In this diagram, the horizontal arrows are admissible epimorphisms, the vertical arrows admissible monomorphisms, and the square corresponding to the morphism $U \hookrightarrow Y$ are cartesian.

We then construct, for each j, the pullback of

$$X_j \overset{m}{\hookrightarrow} Y_j \overset{n}{\hookleftarrow} U_j$$

which exists since $\mathcal{A}$ satisfies (AIC). We are now in the hypotheses of Lemma 3.16; its application gives us a strictly admissible pro-system $\{X_j \times_{Y_j} U_j\}_j$, and then we obtain an object “$\varprojlim$”$_{j \in J} X_j \times_{Y_j} U_j$, which is a pullback in $\mathrm{Pro}^a(\mathcal{A})$ of the diagram (3.27).

Let us denote this pullback by $X_i \times_{Y_i} U$. For all $i \le j$ we have a canonical map of corresponding pullbacks, induced by the diagram

(3.28)
$$\begin{tikzcd}
 & X_i \times_{Y_i} U \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, hook] & & X_i \arrow[dl, hook] \arrow[dd, hook] \\
X_j \times_{Y_j} U \arrow[rr, hook, crossing over] \arrow[dd, hook] & & X_j & \\
 & U \arrow[rr, hook] \arrow[dl, "id"] & & Y_i \arrow[dl, hook] \\
U \arrow[rr, hook] & & Y_j \arrow[from=uu, hook, crossing over] &
\end{tikzcd},$$

and it is clear that such an arrow is a monomorphism. Then, the object "$\varinjlim$"$_{i \in I}(X_i \times_{Y_i} U)$ is the pullback of the diagram (3.25). We shall denote this object by $U \cap X$.

A priori, $U \cap X$ is an object of $\operatorname{Ind}\operatorname{Pro}^a(\mathcal{A})$. However, for each i, we have from the above cubic diagram an admissible monomorphism in $\operatorname{Pro}^a(\mathcal{A})$:

$$X_i \times_{Y_i} U \overset{m_i}{\hookrightarrow} U .$$

Therefore, the admissible monomorphisms $\{m_i\}$ form an inductive system of admissible monomorphisms, which gives rise to an admissible monomorphism

$$\text{“}\varinjlim_{i \in I}\text{”}(X_i \times_{Y_i} U) \hookrightarrow U .$$

But U is in $\operatorname{Pro}^a(\mathcal{A})$, and then the object $U \cap X =$ "$\varinjlim$"$_{i \in I}(X_i \times_{Y_i} U)$ also belongs to $\operatorname{Pro}^a(\mathcal{A})$. We therefore get a cartesian diagram of type (3.26), where all the morphisms are admissible monomorphisms, and $U \cap X \in \operatorname{Pro}^a(\mathcal{A})$.

It is left to prove that $[U \cap X \hookrightarrow X]$ is in $\Gamma(X)$. We argue as in the proof that $U \cap X$ is in $\operatorname{Pro}^a(\mathcal{A})$. Since the above square is cartesian, we get, on the quotients, a monomorphism

$$\frac{X}{U \cap X} \hookrightarrow \frac{Y}{U} .$$

A priori, the object $X/(U \cap X)$ is in $\operatorname{Ind}^a \operatorname{Pro}^a(\mathcal{A})$. But since $[U \hookrightarrow Y] \in \Gamma(Y)$, Y/U is in $\operatorname{Ind}^a(\mathcal{A})$. Thus, $X/(U \cap X)$ is in $\operatorname{Ind}^a(\mathcal{A})$, i.e., $[U \cap X \hookrightarrow X] \in \Gamma(X)$, and Proposition 3.24 is proved. □

Thus, for an admissible monomorphims $X \hookrightarrow Y$ in $\varprojlim\!\!\!\!\varinjlim\, \mathcal{A}$, and for a given $U \in \Gamma(Y)$, in order to prove that the "intersection" $U \cap X$ is an element of the Grassmannians of X, it is sufficient to assume that $\mathcal{A}$ satisfies (AIC). However, to make sure that the quotient $U/(U \cap X)$ is an element of the Grassmannians of the quotient object Y/X, we need also the dual condition (AIC)^o. This is the content of the next statement.

THEOREM 3.29. *Let $\mathcal{A}$ be a partially abelian exact category. Let $X \hookrightarrow Y \twoheadrightarrow Z$ be an admissible short exact sequence of* $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, *and let* $[U \hookrightarrow Y]$ *in* $\Gamma(Y)$ *be given. Then we have a commutative diagram*

(3.30)
$$\begin{tikzcd} U \cap X \arrow[r, hook] \arrow[d, hook] & U \arrow[r, two heads] \arrow[d, hook] & \dfrac{U}{U \cap X} \arrow[d, hook, "m"] \\ X \arrow[r, hook] & Y \arrow[r, two heads] & Z \end{tikzcd},$$

in which the top sequence is an admissible short exact sequence of $\mathrm{Pro}^a(\mathcal{A})$, *such that the arrow* $U/(U \cap X) \hookrightarrow Z$ *is an admissible monomorphism and* $[U/(U \cap X) \hookrightarrow Z]$ *is in* $\Gamma(Z)$.

TERMINOLOGY. On the operations in Theorem 3.29, we shall say that U has been *lifted to* X along the admissible monomorphism $X \hookrightarrow Y$, and that U has been *projected to* Z along the corresponding epimorphism $Y \twoheadrightarrow Z$.

PROOF. Let us keep the same notations as in the proof of Proposition 3.24. As we have seen, the diagram (3.26) is constructed from the diagrams (3.27) of $\mathrm{Pro}^a(\mathcal{A})$, by forming the limit "$\varinjlim$"$_{i\in I}(X_i \times_{Y_i} U)$, which is still an object of $\mathrm{Pro}^a(\mathcal{A})$. Let us take the quotients of the horizontal monomorphisms and get the following diagram, where the horizontal sequences are admissible short exact sequences:

(3.31)
$$\begin{tikzcd} U \times_{Y_i} X_i \arrow[r, hook] \arrow[d, hook] & U \arrow[r, two heads] \arrow[d, hook] & \dfrac{U}{U \times_{Y_i} X_i} \\ X_i \arrow[r, hook] & Y_i \arrow[r, two heads] & Z_i \end{tikzcd}.$$

We now prove the existence of an admissible monomorphism $m_i : U/(U \times_{Y_i} X_i) \hookrightarrow Z_i$ making (3.31) commutative.

As in Proposition 3.24, write $X_i =$ "$\varprojlim$"$_{j\in J} X_{i,j}$, $Y_i =$ "$\varprojlim$"$_{j\in J} Y_{i,j}$, $U =$ "$\varprojlim$"$_{j\in J} U_j$, with $X_{i,j}$, $Y_{i.j}$, U_j objects of $\mathcal{A}$.

For all j we obtain cartesian diagrams

$$\begin{tikzcd} U_j \times_{Y_{i,j}} X_{i,j} \arrow[r, hook] \arrow[d, hook] & U_j \arrow[d, hook] \\ X_{i,j} \arrow[r, hook] & Y_{i,j} \end{tikzcd}.$$

Since $\mathcal{A}$ is partially abelian exact, we can apply Proposition 3.9, and we obtain commutative diagrams

$$\begin{tikzcd} U_j \times_{Y_{i,j}} X_{i,j} \arrow[r, hook] \arrow[d, hook] & U_j \arrow[r, two heads] \arrow[d, hook] & \dfrac{U_j}{U_j \times_{Y_{i,j}} X_{i,j}} \arrow[d, hook, "m_{i,j}"] \\ X_{i,j} \arrow[r, hook] & Y_{i,j} \arrow[r, two heads] & \dfrac{Y_{i,j}}{X_{i,j}} \end{tikzcd}$$

for all j, where the arrows $m_{i,j}$ are admissible monomorphisms.

Taking projective limits, we get an admissible monomorphism in $\mathrm{Pro}^a(\mathcal{A})$:

$$m_i = \text{“}\varprojlim_{j\in J}\text{”} m_{i,j} : \text{“}\varprojlim_{j\in J}\text{”} \frac{U_j}{U_j \times_{Y_{i,j}} X_{i,j}} \hookrightarrow \text{“}\varprojlim_{j\in J}\text{”} \frac{Y_{i,j}}{X_{i,j}} .$$

Notice that

$$\text{“}\varprojlim_{j\in J}\text{”} \frac{U_j}{U_j \times_{Y_{i,j}} X_{i,j}} = \frac{U}{U \times_{Y_i} X_i} \quad \text{and} \quad \text{“}\varprojlim_{j\in J}\text{”} \frac{Y_{i,j}}{X_{i,j}} = Z_i .$$

Thus, for all i we have an admissible monomorphism m_i making the diagram

$$\begin{tikzcd} U \times_{Y_i} X_i \arrow[r, hook] \arrow[d, hook] & U \arrow[r, two heads] \arrow[d, hook] & \dfrac{U}{U \times_{Y_i} X_i} \arrow[d, hook, "m_i"] \\ X_i \arrow[r, hook] & Y_i \arrow[r, two heads] & Z_i \end{tikzcd} \tag{3.32}$$

commutative.

We then repeat the same argument, with this time taking inductive limits of the diagram (3.32). When applying “$\varinjlim$”$_{i\in I}$ to the left square of (3.32) we get, as in Proposition 3.24, the commutative diagram (3.26). When it is applied to the right square, we get an arrow

$$m = \text{“}\varinjlim_{i\in I}\text{”} m_i : \text{“}\varinjlim_{i\in I}\text{”} \frac{U}{U \times_{Y_i} X_i} \hookrightarrow \text{“}\varinjlim_{i\in I}\text{”} \frac{Y_i}{X_i} ,$$

i.e., an admissible monomorphism $m : U/(U \cap X) \hookrightarrow Z$, for which Diagram 3.30 is commutative. This proves the first assertion of the theorem. It is then left to prove that $[U/(U \cap X) \hookrightarrow Z]$ is in $\Gamma(Z)$. Let us repeat the same argument, this time to the columns of Diagram 3.31, that is, we take this time the quotients in the vertical direction. We obtain a commutative diagram

$$\begin{tikzcd}
U \cap X \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\
X \arrow[r, hook] \arrow[d, two heads] & Y \arrow[d, two heads] \\
\frac{X}{U \cap X} \arrow[r, hook, "i"'] & \frac{Y}{U},
\end{tikzcd}$$

where i is an admissible monomorphism of $\mathrm{Ind}^a(\mathcal{A})$. Then, we get a commutative diagram

$$\begin{tikzcd}
U \cap X \arrow[r, hook] \arrow[d, hook] & U \arrow[r, two heads] \arrow[d, hook] & \frac{U}{U \cap X} \arrow[d, hook, "m"] \\
X \arrow[r, hook] \arrow[d, two heads] & Y \arrow[r, two heads] \arrow[d, two heads] & Z \arrow[d, two heads, "e"] \\
\frac{X}{U \cap X} \arrow[r, hook, "i"'] & \frac{Y}{U} \arrow[r, two heads, "j"'] & Q \quad ,
\end{tikzcd}$$

in which all the rows and columns are admissible short exact sequences, Q is the common quotient and the bottom right square is a pushout square of admissible epimorphisms, as it can be seen by the application of Proposition 3.9, or by dualizing Proposition 3.24.

Since $[U \cap X \hookrightarrow X] \in \Gamma(X)$ and $[U \hookrightarrow Y] \in \Gamma(Y)$, we get that $X/(U \cap X)$ and Y/U are in $\mathrm{Ind}^a(\mathcal{A})$. Hence their quotient Q is in $\mathrm{Ind}^a(\mathcal{A})$. But Q is also the quotient $Z/(U/(U \cap X))$, which is thus in $\mathrm{Ind}^a(\mathcal{A})$. This shows that $U/(U \cap X) \in \Gamma(Z)$, and the proof of the theorem is complete. □

COROLLARY 3.33. *Let $\mathcal{A}$ be a partially abelian exact category and $X_1 \hookrightarrow X_2$ an admissible monomorphism of $\varprojlim_{\leftrightarrow} \mathcal{A}$. Suppose we have a pullback diagram*

(3.34)
$$\begin{tikzcd}
U_1' \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[d, hook] \\
U_2' \arrow[r, hook] & U_2
\end{tikzcd}$$

of admissible monomorphisms in $\mathrm{Pro}^a(\mathcal{A})$, *where* $U_1, U_1' \in \Gamma(X_1)$, $U_2, U_2' \in \Gamma(X_2)$. *Then we have an induced commutative diagram*

$$\begin{tikzcd}
U_1' \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[r, two heads] \arrow[d, hook] & U_1'' \arrow[d, hook] \\
U_2' \arrow[r, hook] \arrow[d, two heads] & U_2 \arrow[r, two heads] \arrow[d, two heads] & U_2'' \arrow[d, two heads] \\
\frac{U_2'}{U_1'} \arrow[r, hook] & \frac{U_2}{U_1} \arrow[r, two heads] & \frac{U_2''}{U_1''},
\end{tikzcd}$$

where all the rows and columns are admissible short exact sequences. In particular, the bottom row and, symmetrically, the right column, is an admissible short exact sequence in $\mathcal{A}$.

4. The determinantal torsor on the Waldhausen space $S(\mathrm{lim}\mathcal{A})$.

See Appendix B for the concepts related to torsors and gerbes used throughout this section.

4.1. The dimensional torsor.

DEFINITION 4.1. Let $\mathcal{A}$ be an exact category and G an abelian group.

(a) A function $\chi : Ob\ \mathcal{A} \to G$ is called a *dimensional theory on* $\mathcal{A}$ if, for any admissible short exact sequence $a' \hookrightarrow a \twoheadrightarrow a''$ of $\mathcal{A}$, the equality: $\chi(a) = \chi(a') + \chi(a'')$ holds.

(b) Let χ be a dimensional theory and $X \in \underleftrightarrow{\lim}\,\mathcal{A}$. A χ*-relative dimensional theory* is a map $d : \Gamma(X) \to G$ such that, for all admissible monomorphisms $U \hookrightarrow V$ between elements $U, V \in \Gamma(X)$, we have

$$d(V) = d(U) + \chi\left(\frac{V}{U}\right). \tag{4.2}$$

(c) Given a dimensional theory χ, we denote by $\mathrm{Dim}_\chi(X)$ the set of all χ-relative dimensional theories on X.

As a consequence of (a), we have $\chi(0) = 0$, and $\chi(a) = \chi(a')$ whenever $a \xrightarrow{\sim} a'$. In particular, if $U \xrightarrow{\sim} V$ in $\Gamma(X)$, then $d(U) = d(V)$ for all $d \in \mathrm{Dim}_\chi(X)$. Moreover, let $K_0(\mathcal{A})$ be the Grothendieck group of the exact category $\mathcal{A}$. From the universal property of $K_0(\mathcal{A})$, the datum of a dimension theory $\chi : Ob\ \mathcal{A} \to G$ is equivalent to the datum of a homomorphism $u_\chi : K_0(A) \to G$.

PROPOSITION 4.3. *Let* $\mathcal{A}$ *be an exact category satisfying* (*AIC*). *Then*, $\mathrm{Dim}_\chi(X)$ *is a* G*-torsor.*

SKETCH OF PROOF. We first define an action $G \times \mathrm{Dim}_\chi(X) \xrightarrow{*} \mathrm{Dim}_\chi(X)$ by letting $(g * d)(U) := g + d(U)$ for all $g \in G$, $d \in Dim_\chi(X)$, and $U \in \Gamma(X)$. It is immediate that this datum defines an action of G on $\mathrm{Dim}_\chi(X)$. We prove that it is free and transitive. To see this, let $d,\ d' \in Dim_\chi(X)$ and fix $U \in \Gamma(X)$. Write $g := d(U) - d'(U)$. Then it is enough to prove that this g does not depend on U. The argument works as follows. Let $U \hookrightarrow U'$ be in $\Gamma(X)$. We have $d(U') = d(U) + \chi\left(U'/U\right)$, $d'(U') = d'(U) + \chi\left(U'/U\right)$.

Thus, since G is abelian, it follows that $d(U') - d'(U') = d(U) - d'(U) = g$. When U' is any element of $\Gamma(X)$, Theorem 3.13 shows that $U \cap U'$ is in $\Gamma(X)$. The consideration of the diagram (3.15) proves that it is $d(U') - d(U) = g$ also in this case. □

4.2. The universal dimensional torsor. It is possible to introduce a dimensional torsor which is "universal" in the sense that it depends only on the category $\mathcal{A}$ via the Grothendieck group $K_0(\mathcal{A})$. This dimensional torsor will be denoted by $\mathrm{Dim}(X)$.

DEFINITION 4.4. Let $\psi : Ob\ \mathcal{A} \to K_0(\mathcal{A})$ be the dimensional theory sending each $a \in Ob\ \mathcal{A}$ to its class $[a] \in K_0(\mathcal{A})$. We shall call this function ψ the *universal dimensional theory* on $\mathcal{A}$. We then define $\mathrm{Dim}(X) := \mathrm{Dim}_\psi(X)$. Thus, $\mathrm{Dim}(X)$ is the $K_0(\mathcal{A})$-torsor associated with the identity on $K_0(\mathcal{A})$.

If $\chi : Ob\ \mathcal{A} \to G$ is any other dimensional theory, and u_χ the corresponding group morphism $K_0(\mathcal{A}) \to G$, we have the following proposition.

PROPOSITION 4.5. *In the above situation, we have* $u_{\chi*}(\mathrm{Dim}(X)) = \mathrm{Dim}_\chi(X)$.

EXAMPLE 4.6. *The Kapranov Dimensional torsor* $\mathrm{Dim}(V)$. Let $\mathcal{A} = \mathbf{Vect}_0(k)$. We have $\varprojlim\limits_{\longleftrightarrow} \mathcal{A} = \mathcal{T}$, the category of Tate spaces. Let $V \in \mathcal{T}$ be a Tate space. Since $K_0(\mathcal{A}) = \mathbf{Z}$, $\mathrm{Dim}(V)$ is a $\mathbf{Z}$-torsor. This torsor is the *Kapranov dimensional torsor* associated to the Tate space V, defined by Kapranov in [12].

4.3. Dimensional torsors form a symmetric determinantal theory. We now study the behavior of the dimensional torsor with respect to admissible short exact sequences of $\underset{\longleftrightarrow}{\lim}\ \mathcal{A}$, where $\mathcal{A}$ is a partially abelian exact category.

Let $\mathcal{A}$ be a partially abelian exact category and χ be a dimensional theory on $\mathcal{A}$ with values in an abelian group G. Consider, in $\underset{\longleftrightarrow}{\lim}\ \mathcal{A}$, any admissible short exact sequence: $X' \hookrightarrow X \twoheadrightarrow X''$. Let $[U \hookrightarrow X] \in \Gamma(X)$. Since $\mathcal{A}$ is partially abelian exact, from Theorem 3.29 we get the admissible short exact sequence of $\mathrm{Pro}^a(\mathcal{A})$

$$U \cap X' \hookrightarrow U \twoheadrightarrow \frac{U}{U \cap X'},$$

where $[U \cap X' \hookrightarrow X'] \in \Gamma(X')$ and $\left[U/(U \cap X') \hookrightarrow X''\right] \in \Gamma(X'')$.

Next, let $d' \in \mathrm{Dim}_\chi(X)$ and $d'' \in \mathrm{Dim}_\chi(X'')$. Define

$$d(U) := d'(U \cap X') + d''\left(\frac{U}{U \cap X'}\right) \tag{4.7}$$

for all $[U \hookrightarrow X] \in \Gamma(X)$.

THEOREM 4.8. (1) *The map $d : \Gamma(X) \to G$ defined in* (4.7) *is a χ-relative dimensional theory on X.*

(2) *The induced map* $\mu : \mathrm{Dim}_\chi(X') \times \mathrm{Dim}_\chi(X'') \to \mathrm{Dim}(X)$ *given by* $\mu(d', d'') = d$ *descends to a* (*iso-*)*morphism of* G*-torsors*

$$\mu_{X',X,X''} : \mathrm{Dim}_\chi(X') \otimes \mathrm{Dim}_\chi(X'') \to \mathrm{Dim}_\chi(X)\,,$$

for which the pair $(\mathrm{Dim}_\chi(X), \mu)_{X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}}$ *is a symmetric determinantal theory on* $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ *with values in the Picard category* $\mathrm{Tors}(G)$.

PROOF. (1) Let $U_1 \hookrightarrow U_2 \hookrightarrow X$ with $[U_1 \hookrightarrow X],\ [U_2 \hookrightarrow X] \in \Gamma(X)$.

We have the relations

$$d(U_2) := d'(U_2 \cap X') + d''\left(\frac{U_2}{U_2 \cap X'}\right),$$

$$d(U_1) := d'(U_1 \cap X') + d''\left(\frac{U_1}{U_1 \cap X'}\right),$$

$$d'(U_2 \cap X') := d'(U_1 \cap X') + \chi\left(\frac{U_2 \cap X'}{U_1 \cap X'}\right)$$

and

$$d''\left(\frac{U_2}{U_2 \cap X'}\right) := d''\left(\frac{U_1}{U_1 \cap X'}\right) + \chi\left(\frac{\dfrac{U_2}{U_2 \cap X'}}{\dfrac{U_1}{U_1 \cap X'}}\right).$$

It is enough to prove that $d(U_2) = d(U_1) + \chi(U_2/U_1)$. This results from the above relations thanks to the commutativity of G and because the sequence

$$\frac{U_2 \cap X'}{U_1 \cap X'} \hookrightarrow \frac{U_2}{U_1} \twoheadrightarrow \frac{\dfrac{U_2}{U_2 \cap X'}}{\dfrac{U_1}{U_1 \cap X'}}$$

is an admissible short exact sequence of $\mathcal{A}$ from Theorem 3.29. Thus, since χ is defined on $K_0(\mathcal{A})$, we have

$$\chi\left(\frac{U_2}{U_1}\right) = \chi\left(\frac{U_2 \cap X'}{U_1 \cap X'}\right) + \chi\left(\frac{\dfrac{U_2}{U_2 \cap X'}}{\dfrac{U_1}{U_1 \cap X'}}\right).$$

Substituting this equality in the expression obtained for $d(U_2) - d(U_1)$, we get $\chi(U_2/U_1)$, as claimed.

(2) To check that μ descends to a (iso)morphism of torsors, it is enough to check that $\mu(gd', d'') = \mu(d', gd'')$ for all $g \in G$. But this is immediate from the definition of μ and the commutativity of G.

In order to prove that $(\mathrm{Dim}_\chi(X), \mu)_{X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}}$ is a determinantal theory we need to show that the isomorphisms μ are natural with respect to isomorphisms of admissible short exact sequences of $\underset{\longleftrightarrow}{\lim} \mathcal{A}$, and that the diagram (2.8) commutes for $h(X) = \mathrm{Dim}(X)$, $a_i = X_i$ for $i = 1, 2, 3$ and $\lambda = \mu$. We shall need a topological lemma about the Grassmannians.

Consider the following diagram in $\varinjlim\!\!\!\!\!\!\longleftrightarrow \mathcal{A}$, where the horizontal arrows are admissible monomorphisms and the vertical ones the corresponding cokernels:

(4.9)
$$\begin{tikzcd} X_1 \arrow[r, hook] & X_2 \arrow[r, hook] \arrow[d, two heads] & X_3 \arrow[d, two heads] \\ & \frac{X_2}{X_1} \arrow[r, hook] & \frac{X_3}{X_1} \arrow[d, two heads] \\ & & \frac{X_3}{X_2} . \end{tikzcd}$$

We are given three dimensional theories, d_1, d_{21}, d_{32} on X_1, X_2/X_1 and X_3/X_2, respectively. The commutativity of the diagram (2.8) is equivalent to the equality

$$\mu(\mu(d_1, d_{21}), d_{32}) = \mu(d_1, \mu(d_{21}, d_{32}))\,,$$

as dimensional theories on X_3.

Suppose that $[U \hookrightarrow X_3] \in \Gamma(X_3)$ is given. We construct first $\mu(\mu(d_1, d_{21}), d_{32})$, by applying (4.7).

We first lift U along $X_2 \hookrightarrow X_3$ to $U_2 = U \cap X_2 \in \Gamma(X_2)$. We then project U_2 along $X_3 \twoheadrightarrow X_3/X_2$ to an element $U_{32} = U/U_2 \in \Gamma\,(X_3/X_2)$. The element U_2 is then lifted along $X_1 \hookrightarrow X_2$ to $U_1 = U \cap X_1 \in \Gamma(X_1)$, and then projected along $X_1 \twoheadrightarrow X_2/X_1$ to the element

$$U_{21} = \frac{U_2}{U_1} = \frac{U \cap X_2}{U \cap X_1} \in \Gamma\left(\frac{X_2}{X_1}\right).$$

We thus can write

$$\mu(\mu(d_1, d_{21}), d_{32}) = d_1(U_1) + d_{21}(U_{21}) + d_{32}(U_{32})\,.$$

We similarly construct $\mu(d_1, \mu(d_{21}, d_{32}))$ as follows. We first lift U to the same $U_1 \in \Gamma(X_1)$, since pullbacks are unique up to a unique isomorphism. We then project U along $X_3 \twoheadrightarrow X_3/X_1$, to obtain $U/U_1 \in \Gamma\,(X_3/X_1)$. This element is then lifted along $X_2/X_1 \hookrightarrow X_3/X_1$ to the element

$$U'_{21} = \frac{U}{U_1} \cap \frac{X_2}{X_1} \in \Gamma\left(\frac{X_2}{X_1}\right),$$

and then projected along $X_3/X_1 \hookrightarrow X_3/X_2$ to the element $U'_{32} = (U/U_1)/U'_{21}$ in $\Gamma\,(X_3/X_2)$. We thus have

$$\mu(d_1, \mu(d_{21}, d_{32}))(U) = d_1(U_1) + d_{21}(U'_{21}) + d_{32}(U'_{32})\,.$$

The proof of (4) is then an immediate consequence of the following lemma.

LEMMA 4.10. *In the above situation, we have equalities $U_{21} = U'_{21}$ in $\Gamma\,(X_2/X_1)$ and $U_{32} = U'_{32}$ in $\Gamma\,(X_3/X_2)$.*

PROOF. Both equalities are general properties which hold in any abelian category. The first equality, in set-theoretical terms, reads

$$\frac{U \cap X_2}{U \cap X_1} = \ker \left\{ \frac{U}{U \cap X_1} \to \frac{X_3}{X_2} \right\}$$

as subobjects in

$$\frac{X_2}{X_1} = \ker \left\{ \frac{X_3}{X_1} \twoheadrightarrow \frac{X_3}{X_2} \right\} .$$

The second equality is a consequence of the first, since $(U/U_1)/(U_2/U_1) = U/U_2$, and the lemma is proved. $\square$

It remains to check the symmetry of $\mathrm{Dim}_\chi(X)$. This is easily done directly using Proposition 2.15. The proof of Theorem 4.8 is now complete. $\square$

4.4. Cohomological interpretation of $\mathrm{Dim}(X)$ **in terms of the Waldhausen space of** $\lim \mathcal{A}$**.** We refer to multiplicative torsors of degree n over the bisimplicial set determined by $S_\bullet(\mathcal{A})$ simply as multiplicative torsors of degree n over $S(\mathcal{A})$.

For $i = 0$ the relation $\pi_{i+1}(S(\mathcal{A})) = K_i(\mathcal{A})$ gives $\pi_1(S(\mathcal{A})) = H_1(S(\mathcal{A}), \mathbf{Z}) = K_0(\mathcal{A})$ (the Grothendieck group of the category $\mathcal{A}$). So the universal dimensional theory on $\mathcal{A}$ gives rise to a class $\zeta \in H^1(S(\mathcal{A}), K_0(\mathcal{A}))$.

EXAMPLE 4.11. It is immediate from the definitions that a 0-multiplicative G-torsor on $S(\mathcal{A})$ is a dimensional theory, and that a 1-multiplicative G-torsor on $S(\mathcal{A})$ is a determinantal theory on $\mathcal{A}$.

It is then possible to re-interpret Theorem 4.8 in terms of the Waldhausen space of $\underset{\longleftrightarrow}{\lim} \mathcal{A}$, as follows.

THEOREM 4.12. *Let $\mathcal{A}$ be a partially abelian exact category. Let G be an abelian group and $\chi : K_0(\mathcal{A}) \to G$ a homomorphism. Therefore, the collection* $\{\mathrm{Dim}(X); X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}\}$ *is a multiplicative G-torsor on* $S(\underset{\longleftrightarrow}{\lim} \mathcal{A})$.

COROLLARY 4.13. *The class ζ in* $H^1(S(\mathcal{A}), K_0(\mathcal{A}))$ *gives rise to a cohomology class in* $H^2(S(\underset{\longleftrightarrow}{\lim} \mathcal{A}), K_0(\mathcal{A}))$.

This result can be interpreted as a "first step delooping" between the first cohomology of $S(\mathcal{A})$ and the second cohomology of $S(\underset{\longleftrightarrow}{\lim} \mathcal{A})$.

PROOF. The theorem is a restatement of Theorem 4.8. The claims proved there are equivalent to the statement that $\{\mathrm{Dim}(X); X \in \underset{\longleftrightarrow}{\lim} \mathcal{A}\}$ is a multiplicative torsor. The corollary follows when we let $G = K_0(\mathcal{A})$. Then, from Theorem B.7, the induced multiplicative $K_0(\mathcal{A})$-torsor $\mathrm{Dim}(X)$ represents an element of $H^2(S(\underset{\longleftrightarrow}{\lim} \mathcal{A}), K_0(\mathcal{A}))$. $\square$

4.5. The determinantal torsor $\mathcal{D}_h(X)$**.** Given a Tate space V, we generalize the construction of the determinantal gerbe $\mathrm{Det}(V)$ (cf. [12]) in two directions: in the first place we consider any generalized Tate space X, provided that the exact base category $\mathcal{A}$ is partially abelian exact; secondly, instead of a gerbe over an abelian group, we shall produce a torsor

$\mathcal{D}$ over a symmetric Picard category $\mathcal{P}$, which gives rise to the gerbe Det when we restrict to $\pi_1(\mathcal{P})$, and to the torsor Dim when restricted to $\pi_0(\mathcal{P})$.

Let $\mathcal{A}$ be an exact category and X an object of $\varprojlim\!\!\!\!\!\varinjlim \mathcal{A}$. Let (h, λ) be a determinantal theory on $\mathcal{A}$ with values in a symmetric Picard category $\mathcal{P}$.

DEFINITION 4.14. An *h-relative determinantal theory Δ on X* is the datum consisting of a pair (Δ, δ), where Δ is a function $\Delta : \Gamma(X) \to \mathrm{Ob}\,\mathcal{P}$ with the following properties.

(1) For all admissible monomorphism $U \hookrightarrow V$ in $\Gamma(X)$, there is an isomorphism

$$\delta_{U,V} : \Delta(U) \otimes h\left(\frac{V}{U}\right) \xrightarrow{\sim} \Delta(V),$$

natural with respect to isomorphisms of admissible short exact sequences $U \hookrightarrow V \twoheadrightarrow V/U$.

(2) For all filtrations of length 2 of admissible monomorphisms in $\Gamma(X)$, $U_1 \hookrightarrow U_2 \hookrightarrow U_3$, there is a commutative diagram

$$\begin{tikzcd} \Delta(U_1) \otimes h\left(\frac{U_2}{U_1}\right) \otimes h\left(\frac{U_3}{U_2}\right) \arrow[r, "1\otimes\lambda"] \arrow[d, "\delta_{U_1,U_2}\otimes 1"'] & \Delta(U_1) \otimes h\left(\frac{U_3}{U_1}\right) \arrow[d, "\delta_{U_1,U_3}"] \\ \Delta(U_2) \otimes h\left(\frac{U_3}{U_2}\right) \arrow[r, "\delta_{U_2,U_3}"'] & \Delta(U_3) \end{tikzcd} \tag{4.15}$$

where, as before, we have omitted the associator for simplicity.

A *morphism of h-relative determinantal theories* $f : (\Delta, \delta) \to (\Delta', \delta')$ is a collection of isomorphisms of $\mathcal{P}$, $\{f_U : \Delta(U) \to \Delta'(U)\}_{U\in\Gamma(X)}$, such that, for $U \hookrightarrow V$ in $\Gamma(X)$, the diagram

$$\begin{tikzcd} \Delta(U) \otimes h\left(\frac{V}{U}\right) \arrow[r, "\delta_{U,V}"] \arrow[d, "f_U\otimes 1"'] & \Delta(V) \arrow[d, "f_V"] \\ \Delta'(U) \otimes h\left(\frac{V}{U}\right) \arrow[r, "\delta'_{U,V}"] & \Delta'(V) \end{tikzcd}$$

commutes. It is clear that any such morphism is invertible, hence an isomorphism.

DEFINITION 4.16. Let $\mathcal{P}$ be a Picard category, and let X be an object in $\varprojlim\!\!\!\!\!\varinjlim \mathcal{A}$ as before. We denote by $\mathcal{D}_h(X, \mathcal{P})$, or simply $\mathcal{D}_h(X)$ if no confusion arises, the category (groupoid) whose objects are h-relative determinantal theories on X with values in $\mathcal{P}$ and morphisms are the morphisms of determinantal theories.

THEOREM 4.17. *If the exact category $\mathcal{A}$ satisfies (AIC), then $\mathcal{D}_h(X)$ is a $\mathcal{P}$-torsor.*

PROOF. For all objects $a \in \mathcal{P}$, $(\Delta, \delta) \in \mathcal{D}_h(X)$, and U in $\Gamma(X)$, we define an action $\otimes$ of $\mathcal{P}$ on $\mathcal{D}_h(X)$ by $\mathcal{P} \times \mathcal{D}_h(X) \to \mathcal{D}_h(X)$, $\quad (a, \Delta)(U) \mapsto a \otimes \Delta(U)$.

Let us fix (Δ_0, δ_0) and consider the induced functor

$$\mathcal{P} \xrightarrow{-\otimes\Delta_0} \mathcal{D}_h(X)\,, \quad b \mapsto b \otimes \Delta_0\,.$$

We prove that this functor is an equivalence of categories. We first show that it is essentially surjective.

Let $(\Delta, \delta) \in \operatorname{Ob}\mathcal{D}_h(X)$ be given. We shall prove the existence of an object $a \in \mathcal{P}$ and an isomorphism $\Delta \xrightarrow{\sim} a \otimes \Delta_0$ of determinantal theories. Choose $[U \hookrightarrow X] \in \Gamma(X)$. In $\mathcal{P}$, consider the isomorphism naturally defined in $\mathcal{P}$,

$$\Delta(U) \xrightarrow{\sim} \Delta(U) \otimes 1 \xrightarrow{\sim} \Delta(U) \otimes (\Delta_0(U)^* \otimes \Delta_0(U)) \xrightarrow{\alpha} (\Delta(U) \otimes \Delta_0(U)^*) \otimes \Delta_0(U)\,,$$

where the first is the isomorphism given by 1 as a null object of $\mathcal{P}$ and the second is the isomorphism of duality for objects of $\mathcal{P}$. We let $a := \Delta(U)\otimes\Delta_0(U)^*$, and we write the above composition as $f_U : \Delta(U) \xrightarrow{\sim} a \otimes \Delta_0(U)$. We have to show the following: for all $W_1 \hookrightarrow W_2$ in $\Gamma(X)$, there are isomorphisms $f_{W_1} : \Delta(W_1) \xrightarrow{\sim} a \otimes \Delta_0(W_1)$ and $f_{W_2} : \Delta(W_2) \xrightarrow{\sim} a \otimes \Delta_0(W_2)$ for which the diagram

(4.18)
$$\begin{tikzcd}
\Delta(W_1) \otimes h\left(\dfrac{W_2}{W_1}\right) \arrow[r, "\delta_{W_1,W_2}"] \arrow[d, "f_{W_1}\otimes 1"'] & \Delta(W_2) \arrow[d, "f_{W_2}"] \\
a \otimes \Delta_0(W_1) \otimes h\left(\dfrac{W_2}{W_1}\right) \arrow[r, "1\otimes\delta_{0W_1,W_2}"] & a \otimes \Delta_0(W_2)
\end{tikzcd}$$

is commutative.

We start by defining f_V for all V with $U \hookrightarrow V$. In this case, f_V is defined as the dotted arrow of the diagram below, i.e., as the composite of the isomorphisms represented by full arrows as the already defined morphisms:

(4.19)
$$\begin{tikzcd}
\Delta(U) \otimes h\left(\dfrac{V}{U}\right) \arrow[r, "\delta_{U,V}"] \arrow[d, "f_U\otimes 1"'] & \Delta(V) \arrow[d, dashed, "f_V"] \\
a \otimes \Delta_0(U) \otimes h\left(\dfrac{V}{U}\right) \arrow[r, "1\otimes\delta_{0U,V}"] & a \otimes \Delta_0(V)\,.
\end{tikzcd}$$

Similarly one defines f_V if $V \hookrightarrow U$.

Next, let $W \in \Gamma(X)$. To define f_W, we consider the diagram

(4.20)
$$\begin{tikzcd}
U \cap W \arrow[r, hook] \arrow[d, hook] & W \arrow[d, hook] \\
U \arrow[r, hook] & X
\end{tikzcd}$$

in $\varinjlim\limits_{\longleftrightarrow} \mathcal{A}$ with $[W \cap U] \in \Gamma(X)$, whose existence follows from Theorem 3.13. We apply the diagram (4.19) to $V = W \cap U$ and $U = U$. This defines $f_{W \cap U}$.

From $f_{W \cap U}$ we can define $f_W \colon \Delta(W) \to a \otimes \Delta_0(W)$ using again the diagram (4.19). In this situation $U = U \cap W$ and $V = W$. This defines f_W for all $W \in \Gamma(X)$.

It therefore remains to prove that for all $W_1 \hookrightarrow W_2$ in $\Gamma(X)$, the diagram (4.18) commutes, where f_{W_1} and f_{W_2} have been constructed according to the above procedure.

Let us first consider two elements $V_1, V_2 \in \Gamma(X)$ such that $U \hookrightarrow V_1 \hookrightarrow V_2$. From the isomorphism

$$\lambda \colon h\left(\frac{V_1}{U}\right) \otimes h\left(\frac{V_2}{V_1}\right) \xrightarrow{\sim} h\left(\frac{V_2}{U}\right),$$

we obtain the commutative diagram

$$\begin{tikzcd}
\Delta(V_1) \otimes h\left(\frac{V_2}{V_1}\right) \arrow[r, "\delta^{-1}\otimes 1"] \arrow[d, "f_{V_1}\otimes 1"'] & \Delta(U) \otimes h\left(\frac{V_1}{U}\right) \otimes h\left(\frac{V_2}{V_1}\right) \arrow[r, "1\otimes\lambda"] \arrow[d, "f_U\otimes 1"] & \Delta(U) \otimes h\left(\frac{V_2}{U}\right) \arrow[r, "\delta"] \arrow[d, "f_U\otimes 1"] & \Delta(V_2) \arrow[d, "f_{V_2}"] \\
a \otimes \Delta_0(V_1) \otimes h\left(\frac{V_2}{V_1}\right) \arrow[r, "1\otimes\delta_0^{-1}\otimes 1"] & a \otimes \Delta_0(U) \otimes h\left(\frac{V_1}{U}\right) \otimes h\left(\frac{V_2}{V_1}\right) \arrow[r, "1\otimes\lambda"] & a \otimes \Delta_0(U) \otimes h\left(\frac{V_2}{U}\right) \arrow[r, "\delta"] & a \otimes \Delta_0(V_2).
\end{tikzcd} \tag{4.21}$$

This diagram allows us to express f_{V_2} in terms of f_{V_1}. We have a similar diagram when $V_1 \hookrightarrow V_2 \hookrightarrow U$.

Now let $W_1 \hookrightarrow W_2$. Then, $U \cap W_1 \hookrightarrow U \cap W_2 \hookrightarrow U$ and $U \cap W_1 \hookrightarrow W_1 \hookrightarrow W_2$ are admissible filtrations in $\Gamma(X)$.

From the diagram (4.21), applied to the first filtration, we obtain a diagram of type (4.19), with $U \cap W_1$ as U and $U \cap W_2$ as V. We compose this diagram with the diagram defining f_{W_2}. We get

$$\begin{tikzcd}
\Delta(U \cap W_1) \otimes h\left(\frac{U \cap W_2}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{U \cap W_2}\right) \arrow[r] \arrow[d, "f_{U\cap W_1}\otimes 1"'] & \Delta(W_2) \arrow[d, "f_{W_2}"] \\
a \otimes \Delta_0(U \cap W_1) \otimes h\left(\frac{U \cap W_2}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{U \cap W_2}\right) \arrow[r] & a \otimes \Delta_0(W_2).
\end{tikzcd}$$

On the other hand, we have isomorphisms

$$h\left(\frac{U \cap W_2}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{U \cap W_2}\right) \xrightarrow{\sim} h\left(\frac{W_2}{U \cap W_1}\right)$$

from the first filtration and

$$h\left(\frac{W_2}{U \cap W_1}\right) \xrightarrow{\sim} h\left(\frac{W_1}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{W_1}\right)$$

from the second. Thus,

$$h\left(\frac{U \cap W_2}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{U \cap W_2}\right) \xrightarrow{\sim} h\left(\frac{W_1}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{W_1}\right).$$

The above diagram can thus be rewritten as

$$\begin{tikzcd}
\Delta(U \cap W_1) \otimes h\left(\frac{W_1}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{W_1}\right) \arrow[r] \arrow[d, "f_{U\cap W_1}\otimes 1"'] & \Delta(W_2) \arrow[d, "f_{W_2}"] \\
a \otimes \Delta_0(U \cap W_1) \otimes h\left(\frac{W_1}{U \cap W_1}\right) \otimes h\left(\frac{W_2}{W_1}\right) \arrow[r] & a \otimes \Delta_0(W_2)\,.
\end{tikzcd}$$

Composing this diagram with the diagram (4.19) defining f_{W_1}, we finally get the diagram (4.18), thus proving that the functor $\mathcal{P} \to \mathcal{D}_h(X)$ is essentially surjective.

We sketch the proof that the functor is full. This amounts to show that for all object of $\mathcal{P}$, the map

$$\mathrm{Hom}_{\mathcal{P}}(a, b) \to \mathrm{Hom}_{\mathcal{D}_h(X)}(a \otimes \Delta_0, b \otimes \Delta_0)\,, \quad h \mapsto h \otimes 1_{\Delta_0}$$

is surjective. We shall consider only the case $a = b = 1$, since the general case is treated with the obvious modifications.

Let be $f \in \mathrm{Aut}(1 \otimes \Delta_0) \xrightarrow{\sim} \mathrm{Aut}(\Delta_0)$. Let us choose $U \in \Gamma(X)$ and $f_U : \Delta_0(U) \xrightarrow{\sim} \Delta_0(U)$ be given. There is a unique $g : 1 \to 1$ making the following diagram commute:

$$\begin{tikzcd}
\Delta_0(U) \otimes \Delta_0(U)^{-1} \arrow[r, "f_0\otimes 1_{\Delta_0(U)^{-1}}"] \arrow[d] & \Delta_0(U) \otimes \Delta_0(U)^{-1} \arrow[d] \\
1 \arrow[r, dashed, "g"'] & 1
\end{tikzcd} .$$

For this g we have $g \otimes 1_{\Delta_0(U)} = f_U : 1 \otimes \Delta_0(U) \to 1 \otimes \Delta_0(U)$. It is sufficient to prove that for all $V \in \Gamma(X)$ the arrow $g \otimes 1_{\Delta_0(V)}$ coincides with f_V. This will imply that $f = g \otimes 1_{\Delta_0}$, and hence the functor is full.

Let V be an element of $\Gamma(X)$. Consider the case $U \hookrightarrow V$. In this case f_V is the unique arrow making the diagram

$$\begin{tikzcd}
1 \otimes \Delta_0(U) \otimes h\left(\frac{V}{U}\right) \arrow[r, "1\otimes\delta_0"] \arrow[d, "f_U\otimes 1"'] & 1 \otimes \Delta_0(V) \arrow[d, "f_V"] \\
1 \otimes \Delta_0(U) \otimes h\left(\frac{V}{U}\right) \arrow[r, "1\otimes\delta_0"] & 1 \otimes \Delta_0(V)
\end{tikzcd}$$

commutative. From the diagram for the identity on $(\varDelta_0, \delta_0)$, applying $g : 1 \to 1$ and using the bifunctoriality of $\otimes$, we get the diagram

$$\begin{tikzcd} 1 \otimes \varDelta_0(U) \otimes h\left(\dfrac{V}{U}\right) \arrow[r, "1\otimes\delta_0"] \arrow[d, "g\otimes 1_{\varDelta_0(U)}\otimes 1"'] & 1 \otimes \varDelta_0(V) \arrow[d, "g\otimes 1_{\varDelta_0(V)}"] \\ 1 \otimes \varDelta_0(U) \otimes h\left(\dfrac{V}{U}\right) \arrow[r, "1\otimes\delta_0"] & 1 \otimes \varDelta_0(V) \end{tikzcd} ,$$

and the equality $g \otimes 1_{\varDelta_0(U)} = f_U$. It follows that $g \otimes 1_{\varDelta_0(V)}$ coincides with f_V, as claimed. The proof for the general case of an element $W \in \varGamma(X)$ follows the same pattern as the proof that $-\otimes \varDelta_0$ is essentially surjective. Thus the functor is full. Injectivity of the map is obvious, so the functor is also faithful and so an equivalence. □

4.6. Examples: the gerbe of determinantal theories Det(V)**.** As a corollary of Proposition B.13 and of Theorem 4.17, we have the following

PROPOSITION 4.22. *Let $\mathcal{A}$ be an exact category satisfying* (*AIC*), *h a determinantal theory on $\mathcal{A}$ with values in a Picard category $\mathcal{P}$, and $(\varDelta, \delta)$ an object of $\mathcal{D}_h(X)$. Then $\mathcal{D}_h(X)_{(\varDelta,\delta)}$ is a $\pi_1(\mathcal{P})$-gerbe.*

With the same step-by-step method used to prove in Theorem 4.17 the existence of an isomorphism of h-relative determinantal theories, we can prove the following lemma.

LEMMA 4.23. *If $\mathcal{P}$ is a connected category, then $\mathcal{D}_h(X)$ is a connected groupoid.*

Thus, if $\mathcal{P}$ is connected, for all $(\varDelta, \delta)$ we have $\mathcal{D}_h(X)_{(\varDelta,\delta)} = \mathcal{D}_h(X)$, which is then a $\pi_1(\mathcal{P})$-gerbe.

EXAMPLES 4.24. (1) The Kapranov gerbe of determinantal theories Det(V). Let k be a field, $\mathcal{A} = \mathbf{Vect}_0(k)$, and $\mathcal{P}$ the category $\mathbf{Vect}_1(k)$ of 1-dimensional vector spaces over k. The category $\mathcal{P}$ is obviously a connected Picard category with $\pi_1(\mathcal{P}) = k^*$. Let h be the determinantal theory on $\mathcal{A}$ which associates to each finite dimensional space its determinantal space (as described in the example in Subsection 2.3). Finally, let $V \in \underset{\longleftrightarrow}{\lim}\,\mathbf{Vect}_0(k)$ be a Tate space. From Lemma 4.23, $\mathrm{Det}(V) := \mathcal{D}_h(V)$ is connected and thus a k^*-gerbe. It is called the *gerbe of determinantal theories* of the Tate space V, which was introduced by Kapranov in [12].

(2) Let $\mathcal{A}$ be an exact category satisfying (AIC), G an abelian group and $\mathcal{P} = \mathrm{Tors}(G)$. Let h be a determinantal theory on $\mathcal{A}$ with values on $\mathcal{P}$. The category $\mathcal{P}$ is a connected Picard category, so we have $\mathcal{D}_h(X)_{(\varDelta,\delta)} = \mathcal{D}_h(X)$, for each such h and any determinantal theory $(\varDelta, \delta)$. Since $\pi_1(\mathcal{P}) = G$, $\mathcal{D}_h(X)$ is a G-gerbe. In this case we shall employ also the notation $\mathrm{Det}_h(X)$, to emphasize that this is really the Kapranov G-gerbe of determinantal theories of a generalized Tate space X.

4.7. The universal $\mathcal{D}(X)$. In analogy with the case of the dimensional torsor $\mathrm{Dim}(X)$, it is possible to define a "universal" determinantal torsor $\mathcal{D}(X)$ over Picard categories.

DEFINITION 4.25. Let $\mathcal{A}$ be an exact category satisfying (AIC) and X an object of $\underleftrightarrow{\lim}\,\mathcal{A}$. The *universal determinantal torsor* is the $V(\mathcal{A})$-torsor $\mathcal{D}(X) := \mathcal{D}_{h^u}(X, V(\mathcal{A}))$ associated with the symmetric universal determinantal theory on $\mathcal{A}$, where h^u is defined in Subsection 2.3.

REMARK 4.26. It is possible to characterize $\mathcal{D}(X)$ by an appropriate 2-categorical universal property. We postpone the precise statement and the discussion of this topic to a later paper.

EXAMPLE 4.27. Let $\mathcal{A} = \mathbf{Vect}_0(k)$, and $V \in \underleftrightarrow{\lim}\,\mathcal{A} = \mathcal{T}$ a Tate space. In this case, the category of virtual objects $\mathcal{P} = V(\mathcal{A})$ has $\pi_0(\mathcal{P}) = \mathbf{Z}$, and thus it is not connected. Since $\pi_1(\mathcal{P}) = k^*$, the universal determinantal $V(\mathcal{A})$-torsor $\mathcal{D}_{h^u}(V)$ is a non-connected groupoid. Each of its connected components $\mathcal{D}(V)_{(\Delta,\delta)}$ is a k^*-gerbe, and all of these components compose a set (indexed by $\mathbf{Z}$) of copies of the Kapranov k^*-gerbe $\mathrm{Det}(V)$.

4.8. Cohomological interpretation of $\mathrm{Det}(X)$ in terms of the Waldhausen space of $\lim \mathcal{A}$.

THEOREM 4.28. *Let G be an abelian group and $\mathcal{A}$ a partially abelian exact category. Let $(h, \lambda) \in \mathrm{Det}_\sigma(\mathcal{A}, \mathrm{Tors}(G))$ be a symmetric determinantal theory on $\mathcal{A}$ with values in the Picard category of G-torsors. Then, the collection* $\{\mathrm{Det}_h(X);\ X \in \underleftrightarrow{\lim}\,\mathcal{A}\}$, *defined in* 4.24 (2), *is a multiplicative G-gerbe of degree* 1 on $S(\underleftrightarrow{\lim}\,\mathcal{A})$.

SKETCH OF PROOF. The proof of this theorem is similar to, although considerably longer than, the proof of Theorem 4.12. We emphasize only the most salient points. As already noticed, (h, λ) can be interpreted as a symmetric multiplicative torsor on $\mathcal{A}$. The core of the proof consists in showing the existence of an equivalence of G-gerbes $\mu : \mathrm{Det}_h(X') \otimes \mathrm{Det}_h(X'') \to \mathrm{Det}_h(X)$ for an admissible short exact sequence $X' \hookrightarrow X \twoheadrightarrow X''$ in $\underleftrightarrow{\lim}\,\mathcal{A}$.

For an admissible filtration $U_1 \hookrightarrow U_2 \hookrightarrow U_3$ in $\Gamma(X)$, consider the induced commutative diagram

$$\begin{tikzcd}
U_1' \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[r, two heads] \arrow[d, hook] & U_1'' \arrow[d, hook] \\
U_2' \arrow[r, hook] \arrow[d, hook] & U_2 \arrow[r, two heads] \arrow[d, hook] & U_2'' \arrow[d, hook] \\
U_3' \arrow[r, hook] \arrow[d, hook] & U_3 \arrow[r, two heads] \arrow[d, hook] & U_3'' \arrow[d, hook] \\
X' \arrow[r, hook] & X \arrow[r, two heads] & X''
\end{tikzcd}$$

whose left squares are pullbacks and the horizontal rows are admissible short exact sequences. Then from Theorem 3.29 $U''_1 \hookrightarrow U''_2 \hookrightarrow U''_3$ is an admissible filtration in $\Gamma(X'')$. Since $\mathcal{A}$ is partially abelian exact, we can use Corollary 3.33, and thus we obtain an induced commutative diagram

(4.29)
$$\begin{tikzcd}
\frac{U'_3}{U'_2} \arrow[r, hook] & \frac{U_3}{U_2} \arrow[r, two heads] & \frac{U''_3}{U''_2} \\
\frac{U'_3}{U'_1} \arrow[r, hook] \arrow[u, two heads] & \frac{U_3}{U_1} \arrow[r, two heads] \arrow[u, two heads] & \frac{U''_3}{U''_1} \arrow[u, two heads] \\
\frac{U'_2}{U'_1} \arrow[r, hook] \arrow[u, hook] & \frac{U_2}{U_1} \arrow[r, two heads] \arrow[u, hook] & \frac{U''_2}{U''_1} \arrow[u, hook]
\end{tikzcd}$$

whose rows and columns are admissible short exact sequences.

Let $(\Delta', \delta') \in \mathrm{Det}_h(X')$ and $(\Delta'', \delta'') \in \mathrm{Det}_h(X'')$. Let Δ be a function $\Gamma(X) \to \mathrm{Tors}(G)$ defined as $\Delta(U_1) \colon = \Delta'(U'_1) \otimes \Delta''(U''_1)$ and δ an arrow $\Delta(U_1) \otimes h\,(U_2/U_1) \to \Delta(U_2)$, defined as the composition

$$\begin{tikzcd}
\Delta(U_1) \otimes h\left(\frac{U_2}{U_1}\right) = \Delta'(U'_1) \otimes \Delta''(U''_1) \otimes h\left(\frac{U_2}{U_1}\right) \arrow[r, "1\otimes\lambda^{-1}"] \arrow[dr, "\delta"'] & \Delta'(U'_1) \otimes \Delta''(U''_1) \otimes h\left(\frac{U'_2}{U'_1}\right) \otimes h\left(\frac{U''_2}{U''_1}\right) \arrow[d, "1\otimes\sigma\otimes 1"] \\
 & \Delta'(U'_1) \otimes h\left(\frac{U'_2}{U'_1}\right) \otimes \Delta''(U''_1) \otimes h\left(\frac{U''_2}{U''_1}\right) \arrow[d, "\delta'\otimes\delta''"] \\
 & \Delta'(U'_2) \otimes \Delta''(U''_2) = \Delta(U_2)
\end{tikzcd}.$$

We claim that (Δ, δ) is an object of $\mathrm{Det}_h(X)$. This amounts to show that for this pair the diagram (4.15) is commutative.

The proof consists in the construction of the diagram (4.15) by tensorizing the analogous diagrams for the determinantal theories (Δ', δ') and (Δ'', δ''), by the use of the definitions of Δ and δ. The resulting tensor product of the diagrams is equal to (4.15), provided that the diagram

$$\begin{tikzcd}
h\left(\frac{U'_2}{U'_1}\right) \otimes h\left(\frac{U'_3}{U'_2}\right) \otimes h\left(\frac{U''_2}{U''_1}\right) \otimes h\left(\frac{U''_3}{U''_2}\right) \arrow[rr, "1\otimes\sigma\otimes 1"] \arrow[d, "\lambda\otimes\lambda"] & & h\left(\frac{U'_2}{U'_1}\right) \otimes h\left(\frac{U''_2}{U''_1}\right) \otimes h\left(\frac{U'_3}{U'_2}\right) \otimes h\left(\frac{U''_3}{U''_2}\right) \arrow[d, "\lambda\otimes\lambda"] \\
h\left(\frac{U'_3}{U'_1}\right) \otimes h\left(\frac{U''_3}{U''_1}\right) \arrow[dr, "\lambda"'] & & h\left(\frac{U_2}{U_1}\right) \otimes h\left(\frac{U_3}{U_2}\right) \arrow[dl, "\lambda"] \\
 & h\left(\frac{U_3}{U_1}\right) &
\end{tikzcd}$$

commutes. But (h, λ) is symmetric, and hence this diagram commutes from Definition 2.11, applied to the diagram (4.29). Therefore μ is well defined on the objects. The proof that μ is a multiplicative equivalence in the sense of Definition B.8 is straightforward. Thus the theorem follows. □

Since such a determinantal theory h can be interpreted as a multiplicative G-torsor of degree 1 on $S_\bullet(\mathcal{A})$, it determines a class in $H^2(S(\mathcal{A}), G)$. Then, from Theorem B.9, we obtain the following, which is the analog of Corollary 4.13:

COROLLARY 4.30. *The class* $[h] \in H^2(S(\mathcal{A}), G)$ *gives rise to a cohomology class in* $H^3(S(\varprojlim\!\!\!\!\!\varinjlim\ \mathcal{A}), G)$.

The corollary has an interpretation analogous to that of Corollary 4.13, as the "second step delooping" of the cohomology of $S(\mathcal{A})$ in terms of the cohomology of $S(\underset{\longleftrightarrow}{\lim}\, \mathcal{A})$.

5. Applications. Tate spaces and the iteration of the dimensional torsor. In this section we focus on the abelian category $\mathcal{A} = \mathbf{Vect}_0(k)$ of finite dimensional vector spaces over a field k.

5.1. Tate spaces.

DEFINITION 5.1. Let k be a field. The category $\mathcal{T} := \underset{\longleftrightarrow}{\lim} \mathbf{Vect}_0(k)$ is called the category of *Tate vector spaces* over k.

Let us denote by $\mathcal{L}_0$ the category of linearly compact topological k-vector spaces and by $\mathcal{L}$ the category of locally linearly compact topological k-vector spaces and their morphisms, as introduced in [15, II.27.1 and II.27.9]. We recall the following lemma and propositions, whose proofs can be found in [17].

LEMMA 5.2. *There are equivalences of categories*

$$\Phi_0 : \mathrm{Pro}(\mathbf{Vect}_0(k)) \xrightarrow{\sim} \mathrm{Pro}^s(\mathbf{Vect}_0(k)) \xrightarrow{\sim} \mathcal{L}_0\,.$$

In particular, the category $\mathcal{L}_0$ *is an abelian category.*

PROPOSITION 5.3. *There is an equivalence of categories* $\Phi : \mathcal{T} \xrightarrow{\sim} \mathcal{L}$, *whose restriction to the category* $\mathrm{Pro}(\mathbf{Vect}_0(k))$ *is* Φ_0.

As a consequence of Proposition 5.3, $\mathcal{L}$ is endowed with a structure of an exact category, and it is self-dual (see Proposition A.8).

PROPOSITION 5.4. (a) *Under the identification of Proposition* 5.3, *the class of admissible monomorphisms of* $\mathcal{L}$ *coincides with the class of its closed embeddings.*

(b) *Similarly, the class of admissible epimorphisms in* $\mathcal{L}$ *coincides with the class of continuous surjective morphisms* $p : B \to C$, *such that the canonical bijection* $B/\ker(p) \to C$ *is a homeomorphism.*

The above proposition, which is proved in [17], allows us to identify $\mathcal{T}$ and $\mathcal{L}$. We also recall that the category $\mathcal{T}$ is not abelian. For example, the inclusion $k[t] \hookrightarrow k[[t]]$ is a non-admissible monomorphisms in $\mathcal{T}$.

THEOREM 5.5. *The category $\mathcal{T}$ is partially abelian exact.*

PROOF. From the equivalence $\mathcal{T} \xrightarrow{\sim} \mathcal{L}$ of Propositions 5.3 and 5.4, the closure of $\mathcal{T}$ under admissible intersections is clear, since the intersection of two closed subspaces of a space $X \in \mathcal{T}$ is closed. Thus $\mathcal{T}$ satisfies (AIC). The dual condition (AIC)^o comes from this fact because of the self-duality of $\mathcal{T}$. □

5.2. Sato Grassmannians. The concept of Sato Grassmannian, introduced for any generalized Tate space X in Definition 3.1, coincides with the concept of semi-infinite Grassmannian in the case X is a Tate vector space (i.e., when $\mathcal{A} = \mathbf{Vect}_0(k)$).

PROPOSITION 5.6. *Let X be an object of $\mathcal{T}$. Then the Sato Grassmannian of X coincides with the set $G(X)$ of open, linearly compact subspaces of X.*

PROOF. (i) We first prove that $\Gamma(X) \subset G(X)$. Let $U \in \Gamma(X)$. By definition, $U \in \mathrm{Pro}^s(\mathbf{Vect}_0(k)) \xrightarrow{\sim} \mathcal{L}_0$, so U is a linearly compact subspace of X. Next, since U is closed in X, the projection $X \twoheadrightarrow X/U$ is a continuous map. Since $X/U \in \mathrm{Ind}^s(\mathbf{Vect}_0(k)) \sim \mathbf{Vect}(k)$, it follows that X/U is a discrete space. Thus, $U = \pi^{-1}(0)$ is open.

(ii) We now show that $G(X) \subset \Gamma(X)$. Let $V \in G(X)$ be an open, linearly compact subspace of X. Since V is linearly compact, $V \in \mathrm{Pro}^s(\mathbf{Vect}_0(k))$. Also, V is closed in X. So from [15, 27.8], the inclusion $V \hookrightarrow X$ is a closed embedding, and hence an admissible monomorphism. Since V is open, and is a linear subspace of X, we obtain that V is a nuclear subspace. Thus, from [15, 25.8(c)], the quotient X/V is discrete, i.e., is an object of $\mathrm{Ind}^s(\mathbf{Vect}_0(k))$. It follows that $V \in \Gamma(X)$, and we are done. □

5.3. 2-Tate spaces.

DEFINITION 5.7. Let k be a field. The category $\mathcal{T}_2 = \varprojlim\!\!\!\!\leftrightarrow \mathcal{T}$ is called the category of 2-*Tate spaces* over k.

The category $\mathcal{T}_2$ is thus in a natural way an exact category, and it is of course possible to further iterate the functor $\underleftrightarrow{\lim}$ and define, for all n, the exact category $\mathcal{T}_n = \underleftrightarrow{\lim}\mathcal{T}_{n-1} = \underleftrightarrow{\lim}^n \mathbf{Vect}_0(k)$ of *n-Tate spaces*, but in this paper we shall be only concerned with 2-Tate spaces. We remark however that our definition of n-Tate spaces coincides with that of Arkhipov and Kremnizer, in [1].

5.4. Iteration. Since the category $\mathcal{T}$ is partially abelian exact from Theorem 5.5, it is possible to extend the results on Dim and Det of the previous sections to the objects of the category $\mathcal{T}_2$ of 2-Tate spaces.

Let $\Xi \in \mathcal{T}_2$ be a 2-Tate space. We shall denote, from now on, by $\mathrm{Dim}^{(1)}$ the universal dimensional $\mathbf{Z}$-torsor Dim over the category $\mathcal{T}$ constructed in Section 4.2. As we have seen, the collection of $\mathrm{Dim}^{(1)}(V)$, for $V \in \mathrm{Ob}\,\mathcal{T}$ forms a symmetric determinantal theory.

THEOREM 5.8. *It is possible to define a* $(\mathbf{Z}$*-tors*$)$*-torsor* (*i.e., a* $\mathbf{Z}$*-gerbe*), *associated to the object* $\Xi \in \mathcal{T}_2$, *as* $\mathrm{Dim}^{(2)}(\Xi) := \mathrm{Det}_{\mathrm{Dim}^{(1)}}(\Xi)$.

The gerbe $\mathrm{Dim}^{(2)}$ is multiplicative with respect to admissible short exact sequences of $\mathcal{T}_2$.

It is also possible to define $\mathrm{Det}^{(2)}(\Xi)$, in analogy with $\mathrm{Dim}^{(2)}(\Xi)$, as the universal determinantal 2-gerbe of Ξ over the universal determinantal 1-gerbe $\mathrm{Det}^{(1)}(X) = \mathrm{Det}(X)$ on $\mathcal{T}$. It results a multiplicative 2-gerbe $\mathrm{Det}^{(2)}(\Xi) := \mathrm{Det}_{\mathrm{Det}^{(1)}}(\Xi)$ over k^*.

This theory coincides with the theory of gerbel theories and 2-gerbes contained in [1] and [6]. We postpone to a forthcoming paper a more detailed proof of this equivalence.

Appendix A. Exact categories and locally compact objects.

A.1. Generalities on exact categories. Let $\mathcal{A}$ be an exact category, in the sense of Quillen [18], and $\mathcal{F}$ its abelian envelope. We recall some facts from [17].

LEMMA A.1. *Let $f : a \twoheadrightarrow b$ be an epimorphism of $\mathcal{F}$ with $a, b \in \mathcal{A}$. Then f is an admissible epimorphism of $\mathcal{A}$ if and only if* $\ker(f)$ *is in $\mathcal{A}$. Dually, a monomorphism $g : c \hookrightarrow d$ of $\mathcal{F}$ with c, d in $\mathcal{A}$ is an admissible monomorphism of $\mathcal{A}$ if and only if* $\operatorname{coker}(g)$ *is in $\mathcal{A}$.*

LEMMA A.2. *A pullback diagram in the category $\mathcal{A}$ remains a pullback diagram in the category $\mathcal{F}$.*

DEFINITION A.3. An *admissible subobject* of an object $a \in \mathcal{A}$ is a class of admissible monomorphisms $a' \hookrightarrow a$ modulo the equivalence relation given by $(a' \hookrightarrow a) \sim (a'' \hookrightarrow a)$ if and only if there exists an isomorphism $a' \xrightarrow{\sim} a''$ such that

$$\begin{tikzcd} a' \arrow[rr] \arrow[dr, hook] & & a'' \arrow[dl, hook'] \\ & a & \end{tikzcd}$$

is commutative.

We recall that a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & V \end{tikzcd}$$

is said to be *admissible* if the horizontal arrows are admissible monomorphisms and the vertical ones are admissible epimorphisms. If such a square is cartesian, it is also cocartesian, and vice versa.

A.2. Ind/Pro-exact categories and the Beilinson category. We refer to the papers [2], [3], [10], [17] for the background on the language of ind-pro objects and the Beilinson category $\underleftrightarrow{\lim}\,\mathcal{A}$ which is the natural setting of the concepts we are going to introduce.

DEFINITION A.4 ([3], [17]). The category $\mathrm{Pro}^a(\mathcal{A})$ (resp. $\mathrm{Ind}^a(\mathcal{A})$) of *strictly admissible pro-objects* (resp. *ind-objects*) of $\mathcal{A}$ is the subcategory of $\mathrm{Pro}(\mathcal{A})$ (resp. $\mathrm{Ind}(\mathcal{A})$) whose objects have structure morphisms which are *admissible* epimorphisms (resp. monomorphisms). With an abuse of language, we shall refer to $\mathrm{Pro}^a(\mathcal{A})$, $\mathrm{Ind}^a(\mathcal{A})$ simply as the categories of *strict* ind- and pro-objects of $\mathcal{A}$.

DEFINITION A.5 ([3], [17]). The *Beilinson category* of the exact category $\mathcal{A}$ is the category denoted by $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ defined as the full subcategory of $\mathrm{Ind}^a\,\mathrm{Pro}^a(\mathcal{A})$ whose objects are formal limits "$\underset{\longrightarrow}{\lim}$"$_j$"$\underset{\longleftarrow}{\lim}$"$_i X_{i,j}$, for $(i, j) \in \mathbf{Z} \times \mathbf{Z}$, with $i \le j$, and for which the squares

(A.6)
$$\begin{tikzcd} X_{i'j} \arrow[r] \arrow[d] & X_{i'j'} \arrow[d] \\ X_{ij} \arrow[r] & X_{ij'} \end{tikzcd}$$

defined for $i \le i'$, $j \le j'$, are cartesian (and thus they are automatically cocartesian). The objects of such category will also be called *generalized Tate spaces*.

Objects of the Beilinson category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ provide a model for local compactness in the linear context (cf. [17]), generalizing the case of locally linearly compact vector spaces to exact categories. For this reason $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ is also referred to sometimes as the *category of locally compact objects* over the exact category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$.

LEMMA A.7 (cf. [17]). *When* $(\mathcal{A}, \mathcal{E})$ *is exact, the categories* $\mathrm{Ind}(\mathcal{A})$, $\mathrm{Pro}(\mathcal{A})$, $\mathrm{Ind}^a_{\aleph_0}(\mathcal{A})$, $\mathrm{Pro}^a_{\aleph_0}(\mathcal{A})$ *and* $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ *inherit in a natural way the structure of exact categories.*

We also recall the following from [3]:

PROPOSITION A.8. *For any exact category* $\mathcal{A}$, $(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})^o = \underset{\longleftrightarrow}{\lim}(\mathcal{A}^o)$.

In particular, $\mathbf{Vect}_0(k) = \mathbf{Vect}_0(k)^o$, and so the category $\mathcal{T}$ is self-dual. In this paper, we shall bound ourselves to countable ind- and pro-categories. We recall the exact structures of the categories $\mathrm{Ind}(\mathcal{A})$, $\mathrm{Pro}(\mathcal{A})$ and $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, by specifying the classes of their admissible mono/epimorphisms, as worked out in [17]:

LEMMA A.9. *Let* $m : X \hookrightarrow Y$ *be an admissible monomorphism of* $\mathrm{Ind}(\mathcal{A})$. *Then for every ind-representation of the objects* X *and* Y, *say* $X =$ "$\underset{\longrightarrow}{\lim}$"$_{i\in I} X_i$ *and* $Y =$ "$\underset{\longrightarrow}{\lim}$"$_{j\in J} Y_j$, m *can be written in components as* $\{m^i_j\}$ *in such a way that for every* i *there is a* j *and an admissible monomorphism* $m^i_j : X_i \hookrightarrow Y_j$ *of* $\mathcal{A}$. *Similarly, let* $n : A \hookrightarrow B$ *be an admissible monomorphism of* $\mathrm{Pro}(\mathcal{A})$. *Then for every pro-representation of the objects* A *and* B, *say* $A =$ "$\underset{\longleftarrow}{\lim}$"$_{i\in I} A_i$ *and* $B =$ "$\underset{\longleftarrow}{\lim}$"$_{j\in J} B_j$, n *can be written in components as* $\{n^i_j\}$ *in such a way that for every* j *there is an* i *and an admissible monomorphism* $n^i_j : A_i \hookrightarrow B_j$ *of* $\mathcal{A}$.

As a consequence of the previous lemma, in [17, 4.19], we obtain the following lemma.

LEMMA A.10 (Straightification of admissible monomorphisms). *Let $m : X \hookrightarrow Y$ be an admissible monomorphism of* Ind($\mathcal{A}$). *Then it is possible to express X and Y as ind-systems $X = $ "$\varinjlim$"$_{i\in I} X_i$ and $Y = $ "$\varinjlim$"$_{i\in I} Y_i$, and $m = $ "$\varinjlim$"$_{i\in I} m_i$, where for each $i \in I$, $m_i : X_i \hookrightarrow Y_i$ is an admissible monomorphism of $\mathcal{A}$. Similarly, let $n : A \hookrightarrow B$ an admissible monomorphism of* $\mathrm{Pro}^a(\mathcal{A})$. *It is possible to express A and B as pro-systems $A = $ "$\varprojlim$"$_{i\in I} A_i$ and $B = $ "$\varprojlim$"$_{i\in I} B_i$, and $n = $ "$\varprojlim$"$_{i\in I} n_i$, where for each $i \in I$, $n_i : A_i \hookrightarrow B_i$ is an admissible monomorphism of $\mathcal{A}$.*

The analogous propositions for admissible epimorphisms of $\mathrm{Ind}^a(\mathcal{A})$ and $\mathrm{Pro}^a(\mathcal{A})$ follow from the ones above.

We call an object X of Ind($\mathcal{A}$) (resp. $Y \in$ Pro($\mathcal{A}$)) a *stabilizing object* if it can be expressed as $X = $ "$\varinjlim$"$_{i\in I} X_i$ for a set of objects X_i (resp. as $Y = $ "$\varprojlim$"$_{j\in J} Y_j$ for a set of objects Y_j), for which there exists an i_0 such that the morphisms $\cdots \to X_{i-1} \to X_{i+1} \to X_{i+1} \to \cdots$ are all isomorphisms for $i \geq i_0$. (resp. for which there exists a j_0 such that the morphisms $\cdots \to Y_{j+1} \to Y_j \to Y_{j-1} \to \cdots$ are isomorphisms). It is clear that a stabilizing object in Ind($\mathcal{A}$) (resp., Pro($\mathcal{A}$)) is isomorphic to an object of $\mathcal{A}$.

PROPOSITION A.11. *Let $m : X \hookrightarrow Y$ be an admissible monomorphism in* $\mathrm{Pro}^a(\mathcal{A})$. *Then the quotient Y/X is isomorphic to an object of $\mathcal{A}$ if and only if m is representable by a ladder of cartesian squares.*

The proof follows from a chase-diagram argument, and it is left to the reader.

COROLLARY A.12. *Let X be an object of $\underset{\longleftrightarrow}{\lim}\, \mathcal{A}$, and $X = $ "$\varinjlim$"$_{j\in J} X_j$, for an ind-system of objects $\{X_j\}$ in* $\mathrm{Pro}^a(\mathcal{A})$. *Then, for $j < j'$, the quotient $X_{j'}/X_j$ is in $\mathcal{A}$.*

DEFINITION A.13. An object of $\underset{\longleftrightarrow}{\lim}\, \mathcal{A}$ is said to be *compact* if it is isomorphic to an object of $\mathrm{Pro}^a(\mathcal{A})$ and *discrete* if it is isomorphic to an object of $\mathrm{Ind}^a(\mathcal{A})$.

PROPOSITION A.14. *If an object Z is both compact and discrete, then Z is isomorphic to an object of $\mathcal{A}$.*

Appendix B. Multiplicative torsors and gerbes. Let G be an abelian group. By the term *G-torsor* we mean a set T with an action of G which is free and transitive (sometimes referred to in the literature as "torsor over a point". We refer to [12] for the basic definitions and properties of the category of G-torsors. Here we recall that the category Tors(G) of the torsors over G is a *symmetric Picard category* (see [4] for the definition and the main properties of Picard categories), with the monoidal structure given by the tensor product of torsors. The Picard category Tors(G) is strictly symmetric. The dual of a G-torsor T is the G-torsor Hom(T, G). It is clear that Tors(G) is a connected Picard category, i.e., $\pi_0(\mathrm{Tors}(G)) = 0$. Futhermore, we have $\pi_1(\mathrm{Tors}(G)) = G$.

Similarly, we mean, by a *G-gerbe* $\mathfrak{g}$, a connected groupoid, such that for all pair of objects x, y of $\mathfrak{g}$, the set of morphisms $\mathrm{Hom}_{\mathfrak{g}}(x, y)$ is given a structure of G-torsor, and the composition of morphisms is a G-bilinear map. We refer again to [12] for the main properties of the 2-category of gerbes.

B.1. The category of virtual objects of an exact category. Following Deligne [4], we associate to each exact category $\mathcal{A}$ a symmetric Picard category $V(\mathcal{A})$, called the *category of virtual objects* of $\mathcal{A}$. Here is a slightly modified version of Deligne's construction:

• An object of $V(\mathcal{A})$ is a loop of $S(\mathcal{A})$, $\gamma\colon [0,1] \mapsto S(\mathcal{A})$ (cf. Subsection 2.1), with $\gamma(0) = \gamma(1) = *$.

• A morphism $\gamma_1 \to \gamma_2$ is a homotopy class rel $*$ of homotopies from γ_1 to γ_2.

The composite of two morphisms $\gamma_1 \xrightarrow{[F]} \gamma_2 \xrightarrow{[G]} \gamma_3$ is defined as the class of the homotopy $F * G : \gamma_1 \to \gamma_3$. Since $F * (G * H) \sim (F * G) * H$, the composition of arrows is associative and $V(\mathcal{A})$ is a category. The category $V(\mathcal{A})$ is a Picard category, with the tensor product on objects $\gamma_1 \otimes \gamma_2$ defined as the composite of loops $\gamma_1 * \gamma_2$. The associativity constraint is given by the class of the standard homotopy of loops $\gamma_1 * (\gamma_2 * \gamma_3) \sim (\gamma_1 * \gamma_2) * \gamma_3$. The unit object is the constant loop at 0. Further, $V(\mathcal{A})$ admits a symmetry which makes it into a symmetric Picard category. To see this, consider the direct sum $\oplus$ in the exact category $\mathcal{A}$. The operation $\oplus$ makes $S(\mathcal{A})$ into an H-space, whose sum will be still denoted by $\oplus$, commutative up to all higher homotopies. This defines a commutativity constraint on $V(\mathcal{A})$, via

$$\gamma_1 * \gamma_2 \sim \gamma_1 \oplus \gamma_2 \sim \gamma_2 \oplus \gamma_1 \sim \gamma_2 * \gamma_1 \,,$$

cf. [4, 4.2.2]. It is not difficult to see that the above isomorphism of $V(\mathcal{A})$ makes $V(\mathcal{A})$ into a symmetric Picard category, with $\pi_0(V(\mathcal{A})) = K_0(\mathcal{A})$ and $\pi_1(V(\mathcal{A})) = K_1(\mathcal{A})$.

In general, $V(\mathcal{A})$ is not strictly symmetric. When $\mathcal{A} = \mathbf{Vect}_0(k)$, the symmetric Picard category $V(\mathcal{A})$ is equivalent to the category $\mathrm{Pic}_k^{\mathbf{Z}}$ of $\mathbf{Z}$-graded 1-dimensional vector spaces over k. This is a symmetric Picard category, with symmetry given as follows. Suppose that L has degree a and M has degree b. Then, for all $x \in L$ and $y \in M$, we define $\sigma_{x,y}\colon x \otimes y \to (-1)^{ab} y \otimes x$. The equivalence $V(\mathcal{A}) \sim \mathrm{Pic}_k^{\mathbf{Z}}$ is mentioned in [5, 5.5.1].

B.2. Multiplicative torsors with degree. The concept of multiplicative (bi-)torsor has been introduced by Grothendieck in connection with the problem of the description of the second cohomology group of an abelian group G, and the classification of the central extensions of a group by an abelian group in [11]. Although introduced in the context of group cohomology, Grothendieck's definition can be easily reworked for the more general case of simplicial sets, which is the context in which we, at first, shall use this notation.

A pasting rule. Let $\mathcal{C}$ be the category of torsors over an abelian group G. This is a monoidal, strictly symmetric category. We denote by c its symmetry. We introduce a notation which shall help us to write in a compact form some particular compositions of morphisms of $\mathcal{C}$.

Let $f : A \to B_1 \otimes B_2$ and $g : B_1 \otimes B_3 \to C$ be two morphisms of $\mathcal{C}$. Since $\mathrm{dom}(g) \neq \mathrm{cod}(f)$, these morphisms cannot be composed. However, we can define a new morphism, by "pasting together" f and g, as follows:

$$A \otimes B_3 \xrightarrow{f\otimes B_3} B_1 \otimes B_2 \otimes B_3 \xrightarrow{c\otimes B_3} B_2 \otimes B_1 \otimes B_3 \xrightarrow{B_2\otimes g} B_2 \otimes C\,. \tag{B.1}$$

We shall denote such a composition by "$g \cdot f$". Similarly, one defines "$h \cdot g \cdot f$", and so on.

Simplicial sets and pasting of torsor morphisms. Let $(\Sigma_\bullet, \partial_i, s_i)$ be a simplicial set (as defined e.g. in [8] and [9]) and G an abelian group. Let T_ρ be a G-torsor for all $\rho \in \Sigma_{n-1}$ and $\alpha_\sigma : \bigotimes T_{\partial_{2i}(\sigma)} \to \bigotimes T_{\partial_{2i+1}(\sigma)}$ an isomorphism of G-torsors for all $\sigma \in \Sigma_n$. Let $\tau \in \Sigma_{n+1}$. It is possible to construct the following composition, that we shall call the *even composition* of the α's:

$$E_\tau := \text{``} \cdots \cdot \alpha_{\partial_2(\tau)} \cdot \alpha_{\partial_0(\tau)} \text{''},$$

and the similarly defined *odd composition*, as

$$O_\tau := \text{``}\alpha_{\partial_1(\tau)} \cdot \alpha_{\partial_3(\tau)} \cdot \cdots \text{''},$$

which are extended, respectively, to all the even/odd factors $\alpha_{\partial_i(\tau)}$ in the indicated order.

The Street decomposition of a simplex. Following Street [20] we introduce a useful way to decompose the boundary of a simplex σ of a simplicial set. Let $\sigma \in \Sigma_n$ and $\partial_+(\sigma) = \{\partial_{2i}(\sigma)\}$, $\partial_-(\sigma) = \{\partial_{2i+1}(\sigma)\}$. Then, we put (cf. [13])

$$\partial_{++}(\sigma) = \bigcup \partial_+ \partial_{2i}(\sigma), \qquad \partial_{+-}(\sigma) = \bigcup \partial_+ \partial_{2i+1}(\sigma), \tag{B.2}$$

$$\partial_{-+}(\sigma) = \bigcup \partial_- \partial_{2i}(\sigma), \qquad \partial_{--}(\sigma) = \bigcup \partial_- \partial_{2i+1}(\sigma). \tag{B.3}$$

LEMMA B.4. *For all $\sigma \in \Sigma_n$, we have*: $\partial_{++}(\sigma) = \partial_{--}(\sigma)$ *and* $\partial_{+-}(\sigma) = \partial_{-+}(\sigma)$.

PROOF. In fact, the lemma is just a restatement of the simplicial identities $\partial_i \partial_j = \partial_{j-1} \partial_i$, $(i \leq j)$. We leave the details to the reader. □

As a consequence, we have the following proposition.

PROPOSITION B.5. *For all $n \geq 1$ the domain of the even composition coincides with the domain of the odd composition*, *and similarly for the target.*

PROOF. The same argument used to prove Lemma B.4 shows that the domain of the even composition is $\partial_{++}(\sigma)$ and the domain of the odd composition is $\partial_{--}(\sigma)$. The Lemma B.4 thus gives the identity of the domains. For the targets one similarly shows that the target of the even composition is $\partial_{+-}(\sigma)$ and the target of the odd composition is $\partial_{-+}(\sigma)$ and uses again Lemma B.4 to prove their identity. □

Multiplicative torsors of degree n. Because of the above proposition, the following makes sense:

DEFINITION B.6. Let $n \geq 1$. A *multiplicative G-torsor of degree* $(n-1)$ *on a simplicial set* $\Sigma_\bullet$ is a datum $T = \{T_\rho, \alpha_\sigma\}$ consisting of

- a G-torsor T_ρ for all $\rho \in \Sigma_{n-1}$,
- an isomorphism of G-torsors $\alpha_\sigma : \bigotimes T_{\partial_{2i}(\sigma)} \to \bigotimes T_{\partial_{2i+1}(\sigma)}$ for all $\sigma \in \Sigma_n$,
- an identity $E_\tau = O_\tau$ for all $\tau \in \Sigma_{n+1}$.

One defines similarly the concept of multiplicative torsor of degree n for a bisimplicial (trisimplicial, etc.) set. Let $\{T_\rho, \alpha_\sigma\}$ and $\{T'_\rho, \alpha'_\sigma\}$ be two multiplicative G-torsors of degree

$n-1$. A *morphism* between them is a collection of morphisms of the underlying G-torsors $f_\rho : T_\rho \to T'_\rho$, defined for all $\rho \in \Sigma_{n-1}$, such that, for all $\sigma \in \Sigma_n$, the diagram

$$\begin{tikzcd} \bigotimes T_{\partial_{2i}(\sigma)} \arrow[r, "\alpha_\sigma"] \arrow[d, "\otimes f_{\partial_{2i}(\sigma)}"'] & \bigotimes T_{\partial_{2i+1}(\sigma)} \arrow[d, "\otimes f_{\partial_{2i+1}(\sigma)}"] \\ \bigotimes T'_{\partial_{2i}(\sigma)} \arrow[r, "\alpha'_\sigma"'] & \bigotimes T'_{\partial_{2i+1}(\sigma)} \end{tikzcd}$$

is commutative.

The collection of the multiplicative G-torsors of degree n over $\Sigma_\bullet$ and their morphisms forms a category (groupoid), that we shall denote by $\mathrm{Mult}_n(\Sigma_\bullet, G)$. It is clear that the tensor product of the underlying torsors induces a strictly symmetric tensor product also for objects of $\mathrm{Mult}_n(\Sigma_\bullet, G)$, which in turns induces a strictly symmetric Picard category structure on $\mathrm{Mult}_n(\Sigma_\bullet, G)$, whose symmetry is defined as in $\mathrm{Tors}(G)$. Moreover, we have the following, whose proof we postpone to a later paper.

THEOREM B.7. (a) *The Picard group* $\pi_0(\mathrm{Mult}_n(\Sigma_\bullet, G))$ *is isomorphic to the* $(n+1)$*-st cohomology group* $H^{n+1}(\Sigma_\bullet, G)$.

(b) *The group* $\pi_1(\mathrm{Mult}_n(\Sigma_\bullet, G))$ *is isomorphic to* $Z^n(\Sigma_\bullet, G)$, *the group of simplicial* n*-cocycles.*

For example, a determinantal theory with values in the category $\mathrm{Tors}(G)$ is the same thing as a multiplicative G-torsor of degree 1 on the Waldhausen space $S(\mathcal{A})$.

B.3. Multiplicative G-gerbes. In analogy with the concept of "multiplicative G-torsor of degree n" over a simplicial (bisimplicial, trisimplicial, . . .) set $\Sigma_\bullet$, it is also possible to introduce the concept of *multiplicative G-gerbe of degree n* over $\Sigma_\bullet$. Since in this work we will only use multiplicative gerbes of degree 1, we shall bound ourselves to this case.

DEFINITION B.8. Let $\Sigma_\bullet$ be a simplicial set and G an abelian group. A *multiplicative G-gerbe*, is the datum $(\mathfrak{g}, \alpha, \beta)$ consisting of

(1) a G-gerbe $\mathfrak{g}_\rho$ for all $\rho \in \Sigma_1$,

(2) an equivalence of G-gerbes $\alpha_\sigma : \mathfrak{g}_{\partial_2(\sigma)} \otimes \mathfrak{g}_{\partial_0(\sigma)} \xrightarrow{\sim} \mathfrak{g}_{\partial_1(\sigma)}$ for all $\sigma \in \Sigma_2$,

(3) for all $\tau \in \Sigma_3$, a diagram, involving the $\alpha_{\partial\tau}$'s, commuting up to a natural isomorphism β_τ which can be written according to our pasting rule (B.1) as β_τ : "$\alpha_{\partial_2(\tau)} \cdot \alpha_{\partial_0(\tau)}$"$\simeq$ "$\alpha_{\partial_1(\tau)} \cdot \alpha_{\partial_3(\tau)}$",

(4) for all $\upsilon \in \Sigma_4$, a cubic commutative diagram involving the $\beta_{\partial\upsilon}$'s, which can be written as "$\beta_{\partial_4\upsilon} \cdot \beta_{\partial_2\upsilon} \cdot \beta_{\partial_0\upsilon}$"$=$ "$\beta_{\partial_1\upsilon} \cdot \beta_{\partial_3\upsilon}$".

Similarly, one defines multiplicative gerbes on bisimplicial (trisimplicial, etc.) sets.

It is possible to associate to a multiplicative gerbe of degree 1 a multiplicative torsor of degree 2.

THEOREM B.9. *A multiplicative G-gerbe* $(\mathfrak{g}, \alpha, \beta)$ *induces a multiplicative G-torsor of degree* 2 *on* $\Sigma_\bullet$.

SKETCH OF PROOF. Let us consider a 2-simplex $\sigma \in \Sigma_2$. Choose elements $x_0 \in \mathfrak{g}_{\partial_0\sigma}$, $x_1 \in \mathfrak{g}_{\partial_1\sigma}$ and $x_2 \in \mathfrak{g}_{\partial_2\sigma}$. We define a G-torsor T_σ, associated to σ, as $T_\sigma := \mathrm{Hom}_{\mathfrak{g}_{\partial_1\sigma}}(\alpha_\sigma(x_0 \otimes x_2), x_1)$. Condition (3) of the definition of a multiplicative gerbe implies, for all $\tau \in \Sigma_3$, the existence of an isomorphism of G-torsors $\mu_\tau : T_{\partial_0\tau} \otimes T_{\partial_2\tau} \to T_{\partial_1\tau} \otimes T_{\partial_3\tau}$, and condition (4) shows that for all $\upsilon \in \Sigma_4$, the isomorphisms $\mu_{\partial\upsilon}$ satisfy condition (3) of the definition of a multiplicative torsor of degree 2. Thus, (T, μ) is a multiplicative torsor of degree 2. $\square$

From this theorem it follows that a multiplicative gerbe of degree 1 gives rise to a class in $H^3(\Sigma_\bullet, G)$. More in general, a multiplicative gerbe of degree n induces a multiplicative torsor of degree $n+1$ and thus it determines a class in $H^{n+2}(\Sigma_\bullet, G)$.

B.4. Torsors over a Picard category. We recall here a generalization of the concept of a torsor to Picard categories, which is discussed in a more general setting by Drinfeld in [5].

DEFINITION B.10. Let $(\mathcal{P}, \otimes, \alpha, \sigma, \mathbf{1})$ be a symmetric Picard category. A *torsor* over $\mathcal{P}$ is a groupoid $\mathcal{T}$, together with a bifunctor $\otimes : \mathcal{P} \times \mathcal{T} \to \mathcal{T}$, $(x, a) \mapsto x \otimes a$, wth the following properties.

- There are natural isomorphisms $\beta_{x_1,x_2,a} : x_1 \otimes (x_2 \otimes a) \to (x_1 \otimes x_2) \otimes a$ for which the following diagram is commutative:

$$\begin{tikzcd} x_1 \otimes (x_2 \otimes (x_3 \otimes a)) \arrow[r, "1_{x_1} \otimes \beta_{x_2,x_3,a}"] \arrow[d, "\beta_{x_1,x_2,x_3\otimes a}"'] & x_1 \otimes ((x_2 \otimes x_3) \otimes a) \arrow[r, "\beta_{x_1,x_2\otimes x_3,a}"] & (x_1 \otimes (x_2 \otimes x_3)) \otimes a \arrow[d, "\alpha_{x_1,x_2,x_3} \otimes 1_a"] \\ (x_1 \otimes x_2) \otimes (x_3 \otimes a) \arrow[rr, "\beta_{x_1\otimes x_2,x_3,a}"'] & & ((x_1 \otimes x_2) \otimes x_3) \otimes a . \end{tikzcd}$$

- For all objects $a \in \mathcal{T}$ there is a natural isomorphism

$$\lambda_a : \mathbf{1} \otimes a \xrightarrow{\sim} a \tag{B.11}$$

compatible with the associativity constraint and with the isomorphism β.

- For all objects $a \in \mathcal{T}$, the induced functor $\mathcal{P} \xrightarrow{-\otimes a} \mathcal{T}$ is an equivalence of categories.

Let $\mathcal{T}$ be a $\mathcal{P}$-torsor. In $\mathrm{Ob}(\mathcal{T})$ we introduce an equivalence relation by letting $x \sim y$ if there is an isomorphism $x \to y$. We denote by $[x]$ the equivalence class of the object x. Let $\pi_0(\mathcal{T}) := \mathrm{Ob}(\mathcal{T})/\sim$. The proof of the following is trivial.

PROPOSITION B.12. *Let $\mathcal{T}$ be a $\mathcal{P}$-torsor. Then there is an induced action of the group $\pi_0(\mathcal{P})$ on the set $\pi_0(\mathcal{T})$, which makes $\pi_0(\mathcal{T})$ into a torsor over the abelian group $\pi_0(\mathcal{P})$.*

Now let $\mathcal{P}$ be a symmetric Picard category and $\mathcal{T}$ a torsor over $\mathcal{P}$. Let $x \in \mathrm{Ob}\,\mathcal{T}$ and denote by $\mathcal{T}_x$ the connected component of $\mathcal{T}$ containing x. Thus, $\mathcal{T}_x$ is in particular a connected groupoid. We have the following, whose proof we leave to the reader.

PROPOSITION B.13. *For all $x \in \mathrm{Ob}\,\mathcal{T}$, we have that $\mathcal{T}_x$ is a $\pi_1(\mathcal{T})$-gerbe.*

In particular, for each object $x \in \mathcal{T}$ the gerbes $\mathcal{T}_x$ are pairwise equivalent. Thus, Proposition B.13 together with Proposition B.12, imply that the datum of a torsor over a Picard category encloses the datum of a torsor (over $\pi_0(\mathcal{P})$) and of a gerbe (over $\pi_1(\mathcal{P})$).

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