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SATURATION OF MULTIPLIER OPERATORS IN BANACH SPACES

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1. Introduction. Let X be a (real or complex) Banach space with norm $\|\cdot\|_X$ and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. A family $\{L_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of operators in $B[X]$ is called a linear approximation process on X if for every $f \in X$,

$$\lim_{n\to\infty} \|L_{n,\lambda}(f) - f\|_X = 0 \quad \text{uniformly in } \lambda \in \Lambda, \tag{1}$$

where N denotes the set of all natural numbers and Λ is an arbitrary index set ([17]).

In [17] we studied the direct estimates of the rate of convergence of $L_{n,\lambda}(f)$ to f (in the sense of (1)) for linear approximation processes $\{L_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of convolution operators or multiplier operators in $B[X]$. Here we determine the optimal rate of this convergence.

For this purpose, we introduce the following definition.

DEFINITION 1. Let $\mathscr{L} = \{L_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ be a linear approximation process on X. Suppose that there exists a family $\{\theta_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of positive real numbers with $\lim_{n\to\infty} \theta_{n,\lambda} = 0$ uniformly in $\lambda \in \Lambda$, such that every $f \in X$ for which $\|L_{n,\lambda}(f) - f\|_X = o(\theta_{n,\lambda})$ $(n \to \infty)$ uniformly in $\lambda \in \Lambda$ is an invariant element of $\mathscr{L}$, i.e., $L_{n,\lambda}(f) = f$ for all $n \in N$, $\lambda \in \Lambda$, and the set

$$S[X; \mathscr{L}] = \{f \in X; \|L_{n,\lambda}(f) - f\|_X = O(\theta_{n,\lambda}) \quad (n \to \infty) \quad \text{uniformly in } \lambda \in \Lambda\}$$

contains at least one noninvariant element of $\mathscr{L}$. Then $\mathscr{L}$ is said to be saturated with order $(\theta_{n,\lambda})$, and $S[X; \mathscr{L}]$ is called its Favard class or saturation class.

REMARK 1. If, for a sequence $\{L_n\}_{n \in N}$ of operators in $B[X]$ converging strongly to the identity operator, $L_{n,\lambda} = L_n$ for all $n \in N$, $\lambda \in \Lambda$, then this concept coincides with the usual one ([4; p. 434], cf. [2; p. 25], [8], [15]), which was first introduced by Favard for summation methods of Fourier series in a lecture in 1947 (cf. [7]). Nowadays there is a vast literature concerning saturation for various summation processes. Saturation theory for summation processes of abstract Fourier series in a Banach space is treated by Butzer, Nessel and Trebels [5] and by Gopalan [8], and saturation behavior of approximation processes of Voronovskaja-type operators in arbitrary Banach spaces is treated by the author [16] (for detailed bibliographical comments one may refer to [2], [3], [4], [6]).

The problem of saturation is to establish the existence of the saturation order $(\theta_{n,\lambda})$, and to characterize the saturation class $S[X; \mathscr{L}]$ of a given linear approximation process $\mathscr{L}$.

In this paper we study the problems of saturation for linear approximation processes $\mathscr{L} = \{L_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of multiplier operators in $B[X]$. These are discussed in the setting of asymptotic relations of Voronovskaja's type which characterize the saturation class $S[X; \mathscr{L}]$ in terms of relative completions of Banach subspaces of X (cf. [2; Sec. 2.2], [4; Sec. 10.4]).

Consequently, we have the saturation theorem for linear approximation processes on X of convolution operators considered in [17]. We also give applications to the approximation problem of various summation processes of multiplier operators, which are induced by a general method of summability in connection with families of infinite matrices of scalars. This method includes the usual matrix summability, the F-summability (the method of almost convergence) and the F_A-summability of Lorentz [11] (cf. [10], [14]), the A_B-summability of Mazhar and Siddiqi [13] and the $\mathscr{A}$-summability of Bell [1] (cf. [12]).

2. **Regularization processes.** Here we introduce the notion of a regularization process of operators, which may be an essential tool for characterizing the saturation class of linear approximation processes in question satisfying Voronovskaja-type conditions.

Let Z denote the set of all integers, and let $\mathscr{S}$ denote the set of all sequences $\alpha = \{\alpha_j\}_{j\in Z}$ of scalars. With the terminology as in [17] (cf. [5]), let $\{P_j\}_{j\in Z}$ be a total, fundamental sequence of mutually orthogonal projections in $B[X]$. Then with each $f \in X$ one may associate its (formal) Fourier series expansion (with respect to $\{P_j\}$) $f \sim \sum_{j=-\infty}^{\infty} P_j(f)$. An operator $A \in B[X]$ is called a multiplier operator if there exists a sequence $\alpha \in \mathscr{S}$ such that for every $f \in X$, $A(f) \sim \sum_{j=-\infty}^{\infty} \alpha_j P_j(f)$, and the following notation is used:

$$A \sim \sum_{j=-\infty}^{\infty} \alpha_j P_j \, .$$

Let $\{T_t; t \in \boldsymbol{R}\}$, $\boldsymbol{R}$ being the real line, be a family of operators in

$B[X]$ such that $\sup\{\|T_t\|_{B[X]};\ t\in \boldsymbol{R}\}$ is finite and

$$(2)\qquad T_t \sim \sum_{j=-\infty}^{\infty} \exp(\tau_j t)P_j \qquad (t\in \boldsymbol{R}),$$

where $\tau=\{\tau_j\}$ is a sequence in $\mathscr{S}$. We observe that in [17; Proposition 2] it is shown that the family $\{T_t\}$ is a strongly continuous group of operators in $B[X]$ with the infinitesimal generator G with domain $D(G)$ satisfying $G(f)\sim\sum_{j=-\infty}^{\infty}\tau_j P_j(f)$ for every $f\in D(G)$ and that if, with the Cesàro mean operator $\sigma_n=\sum_{j=-n}^{n}\{1-|j|/(n+1)\}P_j$ (of order 1), the sequence $\{\sigma_n\}$ is uniformly bounded, i.e.,

$$(3)\qquad \sup_n \|\sigma_n\|_{B[X]} < \infty ,$$

then $D(G)=\{f\in X;\ g\sim\sum_{j=-\infty}^{\infty}\tau_j P_j(f)$ for some $g\in X\}$. Moreover, with each function $k\in L^1_{2\pi}$ (the Banach space of all 2π-periodic, Lebesgue integrable functions k with the norm $\|k\|_1=(1/2\pi)\int_{-\pi}^{\pi}|k(t)|dt$) and the identity operator $I\in B[X]$, the convolution operator $k*I\in B[X]$ defined by

$$(4)\qquad k*I(f)=k*f=(1/2\pi)\int_{-\pi}^{\pi}k(t)T_t(f)dt \qquad (f\in X),$$

the integral being a Bochner integral, is a multiplier operator such that

$$(5)\qquad k*I\sim\sum_{j=-\infty}^{\infty}\kappa_j P_j,\quad \kappa_j=(1/2\pi)\int_{-\pi}^{\pi}k(t)\exp(\tau_j t)dt .$$

DEFINITION 2. Let M be a linear subspace of X and let $\mathscr{A}$ be a family of operators in $B[X]$. A sequence $\{U_n\}_{n\in N}$ of operators in $B[X]$ which commute with all operators in $\mathscr{A}$ is called a regularization process on M for $\mathscr{A}$ if $U_n(X)\subset M$ for all $n\in N$ and $\lim_{n\to\infty}\|U_n(f)-f\|_X=0$ for every $f\in X$.

REMARK 2. Let M be a linear subspace of X which contains $P_j(X)$ for each $j\in Z$, and let $\mathscr{A}$ be a family of multiplier operators or convolution operators of the form (4) under the assumptions that $\{T_t\}$ is strongly continuous and $P_jT_t=T_tP_j$ for all $j\in Z$, $t\in\boldsymbol{R}$ instead of (2). Let $\{U_n\}_{n\in N}$ be a uniformly bounded sequence of multiplier operators having the expansions $U_n\sim\sum_{j=-\infty}^{\infty}\xi_n(j)P_j$ with $\xi_n(j)=0$ whenever $|j|>n$, and $\lim_{n\to\infty}\xi_n(j)=1$ for each $j\in Z$. Then the sequence $\{U_n\}$ is a regularization process on M for $\mathscr{A}$. Thus if (3) is satisfied, then $\{\sigma_n\}$ is a regularization process on M for $\mathscr{A}$.

3. A saturation theorem. From now on let $\mathscr{L}=\{L_{n,\lambda};\ n\in N,\ \lambda\in\Lambda\}$ be a linear approximation process on X of multiplier operators having the expansions

$$L_{n,\lambda} \sim \sum_{j=-\infty}^{\infty} \omega_{n,\lambda}(j) P_j \qquad (n \in N, \lambda \in \Lambda) .$$

We set

$$Z' = \{j \in Z; \omega_{n,\lambda}(j) = 1 \quad \text{for all } n \in N, \ \lambda \in \Lambda\}$$

and always suppose $Z' \neq Z$. Then the following criterion will be useful in deciding whether the saturation behavior occurs for $\mathscr{L}$.

(S-1) There exists a family $\{\theta_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of positive real numbers with $\lim_{n\to\infty} \theta_{n,\lambda} = 0$ uniformly in $\lambda \in \Lambda$ and a sequence $\phi = \{\phi_j\}_{j \in Z} \in \mathscr{S}$ with $\phi_j \neq 0$ whenever $j \notin Z'$ such that for each $j \in Z$,

$$\lim_{n\to\infty} \theta_{n,\lambda}^{-1}(\omega_{n,\lambda}(j) - 1) = \phi_j \quad \text{uniformly in } \lambda \in \Lambda . \tag{6}$$

PROPOSITION 1. *Suppose* $\mathscr{L}$ *satisfies* (S-1).

(i) *If f and g are elements in X such that* $\lim_{n\to\infty} \| \theta_{n,\lambda}^{-1}(L_{n,\lambda}(f) - f) - g \|_X = 0$ *uniformly in* $\lambda \in \Lambda$, *then the Fourier series expansion of g is given by* $g \sim \sum_{j=-\infty}^{\infty} \phi_j P_j(f)$. *In case* $g = 0$ *we have* $L_{n,\lambda}(f) = f$ *for all* $n \in N, \lambda \in \Lambda$, *i.e., f is an invariant element of* $\mathscr{L}$.

(ii) *There exists a noninvariant element* $f_0 \in X$ *of* $\mathscr{L}$ *such that* $\| L_{n,\lambda}(f_0) - f_0 \|_X = O(\theta_{n,\lambda})$ $(n \to \infty)$ *uniformly in* $\lambda \in \Lambda$.

PROOF. The proof is essentially similar to that of Theorem 6.1 of [5], and so we omit the details.

In view of Part (i) of Proposition 1, we introduce the following subspaces of X associated with sequences in $\mathscr{S}$:

Given a sequence $\psi = \{\psi_j\}_{j \in Z} \in \mathscr{S}$, let $W[X; \psi]$ denote the linear subspace of X consisting of all $f \in X$ for which there exists an element $f_\psi \in X$ such that $f_\psi \sim \sum_{j=-\infty}^{\infty} \psi_j P_j(f)$. Note that f_ψ is uniquely determined by f, since $\{P_j\}$ is total, and so the map $V_\psi : f \to f_\psi$ defines a closed linear operator of $W[X; \psi]$ into X. Furthermore, since $P_j(X) \subset W[X; \psi]$ for each $j \in Z$ and $\{P_j\}$ is fundamental, $W[X; \psi]$ is dense in X. Obviously, (6) implies that for each $f \in P_j(X)$, $j \in Z$,

$$\lim_{n\to\infty} \| \theta_{n,\lambda}^{-1}(L_{n,\lambda}(f) - f) - V_\psi(f) \|_X = 0 \quad \text{uniformly in } \lambda \in \Lambda .$$

This relation suggests the introduction of the following definition.

DEFINITION 3. A family $\{A_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of operators in $B[X]$ is said to satisfy the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$ if there exists a family $\{\alpha_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of positive real numbers with $\lim_{n\to\infty} \alpha_{n,\lambda} = 0$ uniformly in $\lambda \in \Lambda$ and a linear operator L with domain $D(L)$ and range in X such that for every $f \in D(L)$

$$\lim_{n\to\infty} \| \alpha_{n,\lambda}^{-1}(A_{n,\lambda}(f) - f) - L(f) \|_X = 0 \quad \text{uniformly in } \lambda \in \Lambda .$$

REMARK 3. If, for a sequence $\{A_n\}_{n\in N}$ of operators in $B[X]$, $A_{n,\lambda} = A_n$ for all $n \in N$, $\lambda \in \Lambda$, then this concept reduces to that due to the author [16].

If M is a Banach subspace of X with norm $\|\cdot\|_M$, then its relative completion, denoted by $\tilde{M}$, is the set of all $f \in X$ for which there exists a sequence $\{f_n\}_{n\in N}$ of elements in M such that $\sup_n \|f_n\|_M < \infty$ and $\lim_{n\to\infty} \|f_n - f\|_X = 0$. For the basic properties of such spaces, see [2; p. 14 ff.] and [4; Propositions 10.4.2 and 10.4.3]. Note that if V is a closed linear operator with domain $D(V)$ and range in X, then $D(V)$ becomes a Banach subspace of X under the norm $\|\cdot\|_{D(V)}$ defined by $\|f\|_{D(V)} = \|f\|_X + \|V(f)\|_X$ for all $f \in D(V)$.

PROPOSITION 2. *Let $\mathscr{A} = \{A_{n,\lambda};\ n \in N, \lambda \in \Lambda\}$ be a family of operators in $B[X]$ satisfying the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$, and let $f \in X$. Then we have:*

(i) *If L is closed and $f \in \widetilde{D(L)}$, then $\|A_{n,\lambda}(f) - f\|_X = O(\alpha_{n,\lambda})$ $(n \to \infty)$ uniformly in $\lambda \in \Lambda$.*

(ii) *If there exists a regularization process $\{U_n\}_{n\in N}$ on $D(L)$ for $\mathscr{A}$, then the fact that $\|A_{n,\lambda}(f) - f\|_X = O(\alpha_{n,\lambda})$ $(n \to \infty)$ uniformly in $\lambda \in \Lambda$ implies $\sup_n \|U_n(f)\|_{D(L)} < \infty$, thus $f \in \widetilde{D(L)}$ if L is closed.*

PROOF. (i) Since $\mathscr{A}$ satisfies the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$, for each $g \in D(L)$ there exists a natural number n_0 such that $\sup\{\|\alpha_{n,\lambda}^{-1}(A_{n,\lambda}(g) - g)\|_X;\ n \geq n_0, \lambda \in \Lambda\}$ is finite. Thus by the uniform boundedness principle, there exists a constant $C > 0$ such that

$$\alpha_{n,\lambda}^{-1} \| A_{n,\lambda}(g) - g \|_X \leqq C \| g \|_{D(L)} \tag{7}$$

for all $n \geqq n_0$, $\lambda \in \Lambda$ and $g \in D(L)$. We now assume that f belongs to $\widetilde{D(L)}$. Then there exists a sequence $\{f_m\}_{m\in N}$ of elements in $D(L)$ and a constant $C' > 0$ such that $\|f_m\|_{D(L)} \leqq C'$ for all $m \in N$ and $\lim_{m\to\infty} \|f_m - f\|_X = 0$. Replacing g by f_m in (7), and letting m tend to infinity, we have $\|A_{n,\lambda}(f) - f\|_X \leqq CC'\alpha_{n,\lambda}$ for all $n \geqq n_0$, $\lambda \in \Lambda$ and so the assertion (i) is proved.

(ii) Suppose that there exist a constant $K > 0$ and a natural number m_0 such that $\|A_{m,\lambda}(f) - f\|_X \leqq K\alpha_{m,\lambda}$ for all $m \geqq m_0$ and all $\lambda \in \Lambda$. Thus, since $U_n A_{m,\lambda} = A_{m,\lambda} U_n$, we have

$$\begin{aligned} \| \alpha_{m,\lambda}^{-1}\{A_{m,\lambda}(U_n(f)) - U_n(f)\} \|_X &\leqq \| U_n \|_{B[X]} \| \alpha_{m,\lambda}^{-1}(A_{m,\lambda}(f) - f) \|_X \\ &\leqq K \| U_n \|_{B[X]} , \end{aligned}$$

which yields $\|L(U_n(f))\|_X \leqq K\|U_n\|_{B[X]}$, since $U_n(f)$ belongs to $D(L)$ and $\mathscr{A}$ satisfies the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$. Consequently, for all $n \in N$ we have

$$\|U_n(f)\|_{D(L)} = \|U_n(f)\|_X + \|L(U_n(f))\|_X \leqq (\|f\|_X + K)\|U_n\|_{B[X]},$$

and so $\sup_n \|U_n(f)\|_{D(L)}$ is finite since the sequence $\{U_n\}$ is uniformly bounded. Also, $\lim_{n\to\infty} \|U_n(f) - f\|_X = 0$. Hence f belongs to $\widetilde{D(L)}$ if L is closed. The proof is complete.

We are now in a position to establish the saturation theorem for $\mathscr{L}$.

THEOREM 1. *Suppose that $\mathscr{L}$ satisfies the Voronovskaja condition of type $(\theta_{n,\lambda}; V_\phi)$ for some $\phi = \{\phi_j\}_{j\in Z} \in \mathscr{S}$ with $\phi_j \neq 0$ whenever $j \notin Z'$. Then $\mathscr{L}$ is saturated with order $(\theta_{n,\lambda})$, and $W[X;\phi]^\sim \subset S[X;\mathscr{L}]$. If, furthermore, there exists a regularization process $\{U_n\}_{n\in N}$ on $W[X;\phi]$ for $\mathscr{L}$, then $S[X;\mathscr{L}] = W[X;\phi]^\sim = \{f \in X; \|U_n(f)\|_{W[X;\phi]} = O(1)\}$.*

PROOF. This follows from Propositions 1 and 2.

The following condition ensures that $\mathscr{L}$ will satisfy the Voronovskaja condition:

(S-2) There exists a family $\{\theta_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of positive real numbers with $\lim_{n\to\infty} \theta_{n,\lambda} = 0$ uniformly in $\lambda \in \Lambda$, a sequence $\phi = \{\phi_j\}_{j\in Z} \in \mathscr{S}$ and a linear approximation process $\{Q_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ on X of multiplier operators having the expansions

$$Q_{n,\lambda} \sim \sum_{j=-\infty}^{\infty} \gamma_{n,\lambda}(j)P_j \qquad (n \in N, \lambda \in \Lambda) \tag{8}$$

such that

$$\theta_{n,\lambda}^{-1}(\omega_{n,\lambda}(j) - 1) = \phi_j \gamma_{n,\lambda}(j) \tag{9}$$

for all $n \in N$, $j \in Z$, $\lambda \in \Lambda$.

PROPOSITION 3. *Condition* (S-2) *implies that $\mathscr{L}$ satisfies the Voronovskaja condition of type $(\theta_{n,\lambda}; V_\phi)$.*

PROOF. Let $f \in W[X;\phi]$. Then by (8) and (9) we have

$$\begin{aligned} P_j(\theta_{n,\lambda}^{-1}(L_{n,\lambda}(f) - f)) &= \theta_{n,\lambda}^{-1}(\omega_{n,\lambda}(j) - 1)P_j(f) = \phi_j\gamma_{n,\lambda}(j)P_j(f) \\ &= \phi_j P_j(Q_{n,\lambda}(f)) = P_j(V_\phi(Q_{n,\lambda}(f))), \end{aligned}$$

and consequently,

$$\theta_{n,\lambda}^{-1}(L_{n,\lambda}(f) - f) = Q_{n,\lambda}(V_\phi(f)) \tag{10}$$

for all $n \in N$, $\lambda \in \Lambda$, since $\{P_j\}$ is total and V_ϕ commutes with all multiplier operators on $W[X;\phi]$. Thus, since $\{Q_{n,\lambda}\}$ is a linear approximation process on X, (10) implies $\lim_{n\to\infty} \| \theta_{n,\lambda}^{-1}(L_{n,\lambda}(f)-f)- V_{\phi}(f)\|_X = 0$ uniformly in $\lambda \in \Lambda$, and the proposition is proved.

As an immediate consequence of Theorem 1 and Proposition 3, we have the following.

COROLLARY 1. *Suppose that $\mathscr{L}$ satisfies* (S-2) *with $\phi_j \neq 0$ whenever $j \notin Z'$. Then $\mathscr{L}$ is saturated with order $(\theta_{n,\lambda})$, and $W[X;\phi]^{\sim} \subset S[X;\mathscr{L}]$. If, in addition, there exists a regularization process $\{U_n\}_{n\in N}$ on $W[X;\phi]$ for $\mathscr{L}$, then $S[X;\mathscr{L}] = W[X;\phi]^{\sim} = \{f \in X; \|U_n(f)\|_{W[X;\phi]} = O(1)\}$.*

We need the following proposition in order to derive another characterization of the saturation class.

PROPOSITION 4. *Let $\mathscr{A} = \{A_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ be a family of operators in $B[X]$ which commute with P_j for each $j \in Z$, and let $\{U_n\}_{n\in N}$ be a uniformly bounded sequence of multiplier operators having the expansions $U_n \sim \sum_{j=-\infty}^{\infty} \xi_n(j)P_j$ with $\xi_n(j) = 0$ whenever $|j| > n$. Suppose that $\mathscr{A}$ satisfies the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$ and that $P_j(X) \subset D(L)$ for each $j \in Z$. Then the implications* (a)$\Rightarrow$(b)$\Rightarrow$(c) *hold for an element $f \in X$:*

(a) $$\|A_{n,\lambda}(f) - f\|_X = O(\alpha_{n,\lambda}) \qquad (n\to\infty)$$

uniformly in $\lambda \in \Lambda$;

(b) $$\left\| \sum_{j=-n}^{n} \xi_n(j)L(P_j(f))\right\|_X = O(1) ;$$

(c) $$\|U_n(f)\|_{D(L)} = O(1) .$$

If, in addition, $\lim_{n\to\infty} \xi_n(j) = 1$ for each $j \in Z$, and L is closed, then (c) *implies* (a).

PROOF. Since P_j and $A_{m,\lambda}$ commute, we have

$$U_n(A_{m,\lambda}(f) - f) = \sum_{j=-n}^{n} \xi_n(j)P_j(A_{m,\lambda}(f) - f)$$
$$= \sum_{j=-n}^{n} \xi_n(j)\{A_{m,\lambda}(P_j(f)) - P_j(f)\} ,$$

and hence

$$\left\| \sum_{j=-n}^{n} \xi_n(j)\{A_{m,\lambda}(P_j(f)) - P_j(f)\} \right\|_X \leqq \|U_n\|_{B[X]} \|A_{m,\lambda}(f) - f\|_X .$$

From this inequality we conclude that (a) implies (b), since $\{U_n\}$ is uniformly bounded and $\mathscr{A}$ satisfies the Voronovskaja condition of type $(\alpha_{n,\lambda}; L)$ with $P_j(f) \in D(L)$, $j \in Z$.

Next we have $L(U_n(f)) = \sum_{j=-n}^{n} \xi_n(j) L(P_j(f))$, and so

$$\| U_n(f) \|_{D(L)} \leqq \| U_n \|_{B[X]} \| f \|_X + \| \sum_{j=-n}^{n} \xi_n(j) L(P_j(f)) \|_X ,$$

which proves that (b) implies (c), since $\{U_n\}$ is uniformly bounded.

Suppose now that $\lim_{n\to\infty} \xi_n(j) = 1$ for each $j \in Z$. Then $\{U_n\}$ becomes a regularization process on $D(L)$ for $\mathscr{A}$. Thus, if L is closed, then by Proposition 2 (c) implies (a), and the proof is completed.

Proposition 4 yields the following additional characterization of the saturation class of $\mathscr{L}$.

THEOREM 2. *Suppose that* $\mathscr{L}$ *satisfies the Voronovskaja condition of type* $(\theta_{n,\lambda}; V_\phi)$ *for some* $\phi = \{\phi_j\}_{j\in Z} \in \mathscr{S}$ *with* $\phi_j \neq 0$ *whenever* $j \notin Z'$, *and let* $\{U_n\}_{n\in N}$ *be as in Proposition* 4 *with the additional assumption that* $\lim_{n\to\infty} \xi_n(j) = 1$ *for each* $j \in Z$. *Then* $\mathscr{L}$ *is saturated with order* $(\theta_{n,\lambda})$, *and* $S[X; \mathscr{L}] = W[X; \phi]^\sim = V[X; \{U_n\}, \phi]$, *where*

$$V[X; \{U_n\}, \phi] = \left\{ f \in X; \left\| \sum_{j=-n}^{n} \xi_n(j) \phi_j P_j(f) \right\|_X = O(1) \right\} .$$

PROOF. This follows from Theorem 1 and Proposition 4.

As an immediate consequence of Theorem 2 and Proposition 3, we have the following.

COROLLARY 2. *Suppose that* $\mathscr{L}$ *satisfies* (S-2) *with* $\phi_j \neq 0$ *whenever* $j \notin Z'$, *and let* $\{U_n\}$ *be as in Theorem* 2. *Then the conclusion of Theorem* 2 *holds.*

In particular, the uniform boundedness of the Cesàro mean operators σ_n gives the following.

THEOREM 3. *Suppose that* $\mathscr{L}$ *satisfies the Voronovskaja condition of type* $(\theta_{n,\lambda}; V_\phi)$ *for some* $\phi = \{\phi_j\}_{j\in Z} \in \mathscr{S}$ *with* $\phi_j \neq 0$ *whenever* $j \notin Z'$, *and* (3) *is satisfied. Then* $\mathscr{L}$ *is saturated with order* $(\theta_{n,\lambda})$, *and*

$$S[X; \mathscr{L}] = W[X; \phi]^\sim = V[X; \{\sigma_n\}, \phi] .$$

COROLLARY 3. *Suppose that* $\mathscr{L}$ *satisfies* (S-2) *with* $\phi_j \neq 0$ *whenever* $j \notin Z'$, *and* (3) *is satisfied. Then the conclusion of Theorem* 3 *holds.*

4. **Applications.** Let $\{T_t; t \in \boldsymbol{R}\}$ and G be as in Section 2. For $r = 0, 1, 2, \cdots$, the operator G^r is defined inductively by the relations $G^0 = I$, $G^1 = G$,

$$D(G^r) = \{f; f \in D(G^{r-1}) \text{ and } G^{r-1}(f) \in D(G)\}$$

and

$$G^r(f) = G(G^{r-1}(f)),\ f \in D(G^r),\ r = 1, 2, \cdots .$$

In view of (4) and (5) all the results obtained in Section 3 are applicable to linear approximation processes $\mathscr{K} = \{k_{n,\lambda} * I;\ n \in N,\ \lambda \in \Lambda\}$ on X, with $k_{n,\lambda} \in L^1_{2\pi}$, having the expansions

$$k_{n,\lambda} * I \sim \sum_{j=-\infty}^{\infty} \kappa_{n,\lambda}(j) P_j,\quad \kappa_{n,\lambda}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t) \exp(\tau_j t) dt .$$

In particular, we have the following.

THEOREM 4. *Let $\{k_{n,\lambda};\ n \in N, \lambda \in \Lambda\}$ be a family of functions in $L^1_{2\pi}$ such that*

$$\sup\{\|k_{n,\lambda}\|_1;\ n \in N, \lambda \in \Lambda\} < \infty . \tag{11}$$

Suppose that for the family $\mathscr{K}$ the condition (S-2) *holds with $\phi = \{\tau_j^r\}_{j \in Z}$ for some $r \in N$ and $\tau_j \neq 0$ whenever $j \notin Z'$, and that* (3) *is satisfied. Then $\mathscr{K}$ is saturated with order $(\theta_{n,\lambda})$, and $S[X; \mathscr{K}] = \widetilde{D(G^r)} = V[X; \{\sigma_n\}, \{\tau_j^r\}]$.*

PROOF. Since $\{P_j\}$ is fundamental, the conditions (11) and (S-2) imply that $\mathscr{K}$ is a linear approximation process on X. By Proposition 2 of [17] and by induction on r we have $G^r(f) \sim \sum_{j=-\infty}^{\infty} \tau_j^r P_j(f)$ for every $f \in D(G^r)$, and

$$D(G^r) = \left\{f \in X\,;\ g \sim \sum_{j=-\infty}^{\infty} \tau_j^r P_j(f) \text{ for some } g \in X\right\} ,$$

and so $W[X; \phi] = D(G^r)$ and $V_\phi = G^r$, where $\phi = \{\tau_j^r\}$. Thus the desired result follows from Corollary 3.

COROLLARY 4. *Let $\{k_{n,\lambda}\}$ be as in Theorem 4 with the additional assumptions that each $k_{n,\lambda}$ is non-negative and $\lim_{n\to\infty}\{\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))\} = 0$ uniformly in $\lambda \in \Lambda$, where*

$$\hat{k}_{n,\lambda}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t) e^{-ijt} dt \quad (n \in N,\ j \in Z,\ \lambda \in \Lambda)$$

and $\mathrm{Re}(\hat{k}_{n,\lambda}(1))$ denotes the real part of $\hat{k}_{n,\lambda}(1)$. Suppose that for the family $\mathscr{K}$ the condition (S-2) *holds with $\theta_{n,\lambda} = \hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))$ and $\phi = \{\tau_j^r\}_{j \in Z}$ for some $r \in N$ and $\tau_j \neq 0$ whenever $j \notin Z'$, and that* (3) *is satisfied. Then $\mathscr{K}$ is saturated with order $(\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1)))$, and $S[X; \mathscr{K}] = \widetilde{D(G^r)} = V[X; \{\sigma_n\}, \{\tau_j^r\}]$.*

In view of the particular cases $\tau_j = -ij$ and $r = 2$, we make the following remark:

REMARK 4. Let $\{k_{n,\lambda};\ n \in N, \lambda \in \Lambda\}$ be a family of non-negative, even functions in $L^1_{2\pi}$ satisfying $\hat{k}_{n,\lambda}(0) = 1$ for all $n \in N$, $\lambda \in \Lambda$, and $\lim_{n\to\infty}(1-\hat{k}_{n,\lambda}(1)) = 0$ uniformly in $\lambda \in \Lambda$, and let $\tau_j = -ij$, $j \in Z$. Then for the family $\mathscr{K}$, one has several conditions equivalent to (S-1) with $\theta_{n,\lambda}=1-\hat{k}_{n,\lambda}(1)$ and $\phi_j = -j^2$. That is, the following are equivalent:

(i) For each $j \in Z$,

$$\lim_{n\to\infty} (\hat{k}_{n,\lambda}(j) - 1)/(1 - \hat{k}_{n,\lambda}(1)) = -j^2 \quad \text{uniformly in } \lambda \in \Lambda ;$$

(ii) (i) holds for $j = 2$;

(iii) $\int_0^\pi k_{n,\lambda}(t) \sin^4 (t/2)dt = o(1 - \hat{k}_{n,\lambda}(1))$ $(n \to \infty)$ uniformly in $\lambda \in \Lambda$;

(iv) For any fixed δ satisfying $0 < \delta < \pi$,

$$\lim_{n\to\infty} \int_{\delta \leq |t| \leq \pi} k_{n,\lambda}(t)dt = o(1 - k_{n,\lambda}(1)) \quad (n \to \infty)$$

uniformly in $\lambda \in \Lambda$.

The proof of these equivalences is essentially similar to that of Theorem 3.8 in [6], and so we omit the details.

DEFINITION 4. Let $B = \{A^{(\lambda)}; \lambda \in \Lambda\}$ be a family of infinite matrices $A^{(\lambda)} = (a^{(\lambda)}_{nm})_{n,m\geq 0}$ of scalars. A sequence $\{f_n\}$ of elements in X is said to be B-summable to f if

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a^{(\lambda)}_{nm} f_m = f \quad \text{uniformly in } \lambda \in \Lambda , \tag{12}$$

where it is assumed that the series in (12) converge for each n and λ.

We shall now mention some examples.

(1°) If, for some matrix A, $A^{(\lambda)} = A$ for all $\lambda \in \Lambda$, then B-summability is just matrix summability by A. In particular, if for every $\lambda \in \Lambda$, $A^{(\lambda)}$ is the unit matrix, then $\{f_n\}$ is B-summable to f if and only if it converges to f.

(2°) Let $\{\{q^{(\lambda)}_n\}_{n\geq 0}; \lambda \in \Lambda\}$ be a family of sequences of scalars such that $Q^{(\lambda)}_n = \sum_{j=0}^{n} q^{(\lambda)}_j \neq 0$ for all n, λ. Let

$$\begin{aligned} a^{(\lambda)}_{nm} &= q^{(\lambda)}_{n-m}/Q^{(\lambda)}_n \quad \text{for } 0 \leqq m \leqq n \\ &= 0 \quad \text{for } m > n . \end{aligned} \tag{13}$$

Then we call the B-summability $(N, q^{(\lambda)}_m)$-summability.

(3°) Let Λ be a subset of $\boldsymbol{R}$. If each entry $a^{(\lambda)}_{nm}$ is a non-negative continuous function on Λ such that $\sum_{j=0}^{\infty} a^{(\lambda)}_{nj} = 1$ for each n and λ, then we call the B-summability $(W, a^{(\lambda)}_{nm})$-summability. The concrete examples of this type are the following:

(14)
$$\Lambda \subset [0, 1], \quad a_{nm}^{(\lambda)} = \binom{n}{m} \lambda^m (1-\lambda)^{n-m} \quad \text{for } 0 \leqq m \leqq n$$
$$= 0 \quad \text{for } m > n .$$

(15)
$$\Lambda \subset [0, \infty), \quad a_{nm}^{(\lambda)} = \exp(-n\lambda)(n\lambda)^m/m! .$$

(4°) If Λ is the set of all non-negative integers and X is the Banach space of all real or complex numbers, then B-summability reduces to the method of summability considered by Bell [1] (cf. [12]), which not only includes the F-summability (method of almost convergence) and the F_Λ-summability of Lorentz [11] but also includes the A_B-summability of Mazhar and Siddiqi [13].

DEFINITION 5. Let B be as in Definition 4. B is said to be regular if it satisfies the following conditions:

(A-1) For each $m = 0, 1, \cdots$, $\lim_{n\to\infty} a_{nm}^{(\lambda)} = 0$ uniformly in $\lambda \in \Lambda$.

(A-2) $\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} = 1$ uniformly in $\lambda \in \Lambda$.

(A-3) For each $n \in N$, $\lambda \in \Lambda$, $a_n^{(\lambda)} = \sum_{m=0}^{\infty} |a_{nm}^{(\lambda)}| < \infty$, and there exists a natural number n_0 such that $\sup\{a_n^{(\lambda)}; n \geqq n_0, \lambda \in \Lambda\} < \infty$.

Note that if B is positive, i.e., $a_{nm}^{(\lambda)} \geqq 0$ for all n, m, λ and $\sum_{j=0}^{\infty} a_{nj}^{(\lambda)} = 1$ for every n, λ, then conditions (A-2) and (A-3) already hold. For instance, the matrices B defined by (13), (14) and (15), respectively, have these properties.

The basic relationship between the regularity of B and B-summability is the following result which is a generalization of Theorem 2 of [1] to an arbitrary Banach space setting.

PROPOSITION 5. *A family of infinite matrices of scalars, $B = \{(a_{nm}^{(\lambda)}); \lambda \in \Lambda\}$, is regular if and only if it satisfies the following condition:*

(A-4) *Each convergent sequence in X is B-summable to its limit.*

PROOF. It is straightforward that if B is regular, then it satisfies (A-4). Suppose now that (A-4) holds. Let $c(X)$ denote the Banach space of all convergent sequences $\{f_m\}$ of elements in X with norm $\|\{f_m\}\|_{c(X)} = \sup_m \|f_m\|_X$. Let f be a fixed non-zero element in X. For each $j = 0, 1, 2, \cdots$, define the sequence $\{f_m^{(j)}\}$ by $f_m^{(j)} = f$ for $m = j$, and $f_m^{(j)} = 0$ for $m \neq j$. Then $\lim_{m\to\infty} f_m^{(j)} = 0$, and so (A-4) implies $0 = \lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} f_m^{(j)} = \lim_{n\to\infty} a_{nj}^{(\lambda)} f$ uniformly in $\lambda \in \Lambda$. Consequently, for each $j = 0, 1, 2, \cdots$, we have $\lim_{n\to\infty} a_{nj}^{(\lambda)} = 0$ uniformly in $\lambda \in \Lambda$. Next we define the sequence $\{f_m\}$ by $f_m = f$ for all m, and so $\lim_{m\to\infty} f_m = f$. Thus (A-4) implies $f = \lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}^{(\lambda)} f_j = \lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}^{(\lambda)} f$, uniformly in $\lambda \in \Lambda$, and so $\lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}^{(\lambda)} = 1$ uniformly in $\lambda \in \Lambda$. Thus conditions (A-1) and (A-2) are proved.

Finally, we show (A-3). We first prove that for each $n \in N, \lambda \in \Lambda$, $a_n^{(\lambda)} < \infty$. Indeed, if $a_n^{(\lambda)} = \infty$ for some n and λ, then there exists a natural number p and a sequence $\{\varepsilon_j\}$ of positive real numbers such that $\lim_{j\to\infty} \varepsilon_j = 0$ and $\sum_{j=p}^{\infty} \varepsilon_j |a_{nj}^{(\lambda)}| = \infty$. Now, define the sequence $\{g_j\}$ by $g_j = 0$ for $j = 0, 1, \cdots, p-1$, and $g_j = \varepsilon_j \operatorname{sgn} a_{nj}^{(\lambda)} f$ for $j = p, p+1, \cdots$, where $\operatorname{sgn} z = |z|/z$ for every scalar $z \neq 0$, and $\operatorname{sgn} 0 = 0$. Then we have $\lim_{j\to\infty} g_j = 0$ and $\|\sum_{j=0}^{\infty} a_{nj}^{(\lambda)} g_j\|_X = \|f\|_X \sum_{j=p}^{\infty} |\varepsilon_j a_{nj}^{(\lambda)}| = \infty$. This contradicts the convergence of $\sum_{j=0}^{\infty} a_{nj}^{(\lambda)} g_j$. Now, for each $n \in N, \lambda \in \Lambda$ we define the transformation $\Psi_{n,\lambda}$: $c(X) \to X$ by $\Psi_{n,\lambda}(\{f_j\}) = \sum_{j=0}^{\infty} a_{nj}^{(\lambda)} f_j$. $\Psi_{n,\lambda}$ is clearly linear. Since

$$\|\Psi_{n,\lambda}(\{f_j\})\|_X \leqq \sum_{j=0}^{\infty} |a_{nj}^{(\lambda)}| \|f_j\|_X \leqq a_n^{(\lambda)} \|\{f_j\}\|_{c(X)}$$

for all $\{f_j\} \in c(X)$, $\Psi_{n,\lambda}$ is bounded and $\|\Psi_{n,\lambda}\| \leqq a_n^{(\lambda)}$. Actually this inequality is an equality. Indeed, let h be an element in X with $\|h\|_X = 1$. For each $m = 0, 1, 2, \cdots$, we define the sequence $\{h_j^{(m)}\}$ by $h_j^{(m)} = \operatorname{sgn} a_{nj}^{(\lambda)} h$ for $j = 0, 1, \cdots, m$, and $h_j^{(m)} = 0$ for $j = m+1, m+2, \cdots$. Then we have $\lim_{j\to\infty} h_j^{(m)} = 0$ and $\|\{h_j^{(m)}\}\|_{c(X)} = 1$. Thus

$$\|\Psi_{n,\lambda}\| \geqq \left\| \sum_{j=0}^{\infty} a_{nj}^{(\lambda)} h_j^{(m)} \right\|_X = \sum_{j=0}^{\infty} |a_{nj}^{(\lambda)}| ,$$

which yields the desired result. Since B satisfies (A-4), for every $\{f_j\} \in c(X)$ we have $\lim_{n\to\infty} \Psi_{n,\lambda}(\{f_j\}) = \lim_{j\to\infty} f_j$, and by the uniform boundedness principle there exists a natural number n_0 such that

$$\sup \{a_n^{(\lambda)}; n \geqq n_0, \lambda \in \Lambda\} = \sup \{\|\Psi_{n,\lambda}\|; n \geqq n_0, \lambda \in \Lambda\} < \infty ,$$

and (A-3) is proved. Therefore (A-4) implies (A-1), (A-2) and (A-3), and the proof is complete.

If, for an infinite real or complex matrix $A = (a_{nm})$, $(a_{nm}^{(\lambda)}) = (a_{nm})$ for all $\lambda \in \Lambda$, then from Proposition 5 we obtain a generalization of the classical theorem of Silverman-Toeplitz on the regularity of the method of summability by A to an arbitrary Banach space setting. Let $0 < a < b \leqq 1$. Then $B = \{(a_{nm}^{(\lambda)}); a \leqq \lambda \leqq b\}$, $a_{nm}^{(\lambda)}$ being defined by (14), is regular, and so by Proposition 5 it satisfies (A-4). Let $0 \leqq c < d < \infty$. Then $B = \{(a_{nm}^{(\lambda)}); c \leqq \lambda \leqq d\}$, $a_{nm}^{(\lambda)}$ being defined by (15), is regular and thus it satisfies (A-4).

Let $\{L_n\}$ be a uniformly bounded sequence of multiplier operators in $B[X]$ having the expansions

$$L_n \sim \sum_{j=-\infty}^{\infty} \zeta_n(j) P_j , \tag{16}$$

and let $B = \{(a_{nm}^{(\lambda)}); \lambda \in \Lambda\}$ be a family of infinite matrices of scalars such that for each n, λ, $\sum_{m=0}^{\infty} |a_{nm}^{(\lambda)}| < \infty$. For each n, λ we define the operator $A_{n,\lambda}$ of X into itself by

$$(17) \qquad A_{n,\lambda} = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} L_m ,$$

which is a multiplier operator such that

$$(18) \qquad A_{n,\lambda} \sim \sum_{j=-\infty}^{\infty} \zeta_{n,\lambda}(j) P_j , \quad \zeta_{n,\lambda}(j) = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} \zeta_m(j) .$$

Thus all the results obtained in Section 3 are applicable to linear approximation processes $\mathscr{A} = \{A_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of multiplier operators defined by (17), having the expansions (18) with (16). In particular, we have the following.

THEOREM 5. *Let $\{U_n\}$ be as in Theorem 2. Let $\{L_n\}$ be a uniformly bounded sequence of multiplier operators in $B[X]$ having the expansions* (16), *and let $B = \{(a_{nm}^{(\lambda)}); \lambda \in \Lambda\}$ be a family of infinite matrices of non-negative real numbers such that for each n, λ, $\sum_{m=0}^{\infty} a_{nm}^{(\lambda)} = 1$. Assume that $P \neq Z$, where $P = \{j \in Z; \zeta_n(j) = 1$ for all $n \in N\}$. Suppose that there exists a sequence $\{\theta_n\}$ of positive real numbers which is B-summable to zero and a sequence $\phi = \{\phi_j\}_{j \in Z} \in \mathscr{S}$ with $\phi_j \neq 0$ whenever $j \notin P$ such that $\zeta_n(j) - 1 = \theta_n \phi_j$ for all $n \in N$, $j \in Z$. Then the family $\mathscr{A}$ is saturated with order $(\theta_{n,\lambda})$, where $\theta_{n,\lambda} = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} \theta_m$, and*

$$S[X; \mathscr{A}] = W[X; \phi]^{\sim} = V[X; \{U_n\}, \phi] .$$

PROOF. For all $n \in N$, $\lambda \in \Lambda$ and all $j \in Z$, we have

$$(19) \qquad \zeta_{n,\lambda}(j) - 1 = \theta_{n,\lambda} \phi_j ,$$

from which it follows that $\mathscr{A}$ is a linear approximation process on X, since $\{P_j\}$ is fundamental and

$$\sup \{\|A_{n,\lambda}\|_{B[X]}; n \in N, \lambda \in \Lambda\} \leqq \sup_n \|L_n\|_{B[X]} < \infty .$$

Also, (19) implies that $\mathscr{A}$ satisfies (S-2) with $Q_{n,\lambda} = I$. Thus the desired result follows from Corollary 2.

COROLLARY 5. *Let $\{L_n\}$ be a uniformly bounded sequence of multiplier operators in $B[X]$ having the expansions* (16) *with the additional assumption that $\zeta_n(j) = 0$ whenever $|j| > n$, and let B be as in Theorem 5 with the additional assumption that it satisfies* (A-1). *Suppose that there exists a sequence $\{\theta_n\}$ of positive real numbers converging to zero and a sequence $\phi = \{\phi_j\}_{j \in Z} \in \mathscr{S}$ with $\phi_j \neq 0$ whenever $j \notin P$, P being as in Theorem 5, such that $\zeta_n(j) - 1 = \theta_n \phi_j$ for all $n \in N$ and all $j \in Z$.*

Then $\mathscr{A}$ *is saturated with order* $(\theta_{n,\lambda})$, *where* $\theta_{n,\lambda} = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)}\theta_m$, *and* $S[X; \mathscr{A}] = W[X; \phi]^{\sim} = V[X; \{L_n\}, \phi]$.

PROOF. Since B is regular, by Proposition 5 for $X = \boldsymbol{R}$, $\{\theta_n\}$ is B-summable to zero. Therefore the claim of the corollary follows from Theorem 5.

Let $\{b_n\}$ be a sequence of functions in $L_{2\pi}^1$ such that $\sup_n \|b_n\|_1 < \infty$. Then, for each n, λ we have

$$B_{n,\lambda} = \sum_{j=0}^{\infty} a_{nj}^{(\lambda)}(b_j * I) = \Big(\sum_{j=0}^{\infty} a_{nj}^{(\lambda)} b_j\Big) * I\,, \tag{20}$$

which is a multiplier operator in $B[X]$, and so all the results obtained are applicable to linear approximation processes $\mathscr{B} = \{B_{n,\lambda}; n \in \boldsymbol{N}, \lambda \in \Lambda\}$, where each operator $B_{n,\lambda}$ is defined by (20). In particular, in view of Theorems 4 and 5, we have the following.

THEOREM 6. *Suppose that* (3) *is satisfied and* $\tau_j \neq 0$ *whenever* $j \notin \boldsymbol{Q}$, *where*

$$\boldsymbol{Q} = \{j \in \boldsymbol{Z}; \beta_n(j) = 1 \text{ for all } n \in \boldsymbol{N}\}, \quad \boldsymbol{Q} \neq \boldsymbol{Z}$$

and

$$\beta_n(j) = (1/2\pi)\int_{-\pi}^{\pi} b_n(t)\exp(\tau_j t)dt \quad (n \in \boldsymbol{N},\ j \in \boldsymbol{Z})\,.$$

Let B be as in Theorem 5. *Suppose that there exists a sequence* $\{\rho_n\}$ *of positive real numbers which is B-summable to zero such that for some* $r \in \boldsymbol{N}$, $\beta_n(j) - 1 = \rho_n \tau_j^r$ *for all* $n \in \boldsymbol{N}$ *and all* $j \in \boldsymbol{Z}$. *Then* $\mathscr{B}$ *is saturated with order* $(\rho_{n,\lambda})$, *where* $\rho_{n,\lambda} = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)}\rho_m$, *and*

$$S[X; \mathscr{B}] = \widetilde{D(G^r)} = V[X; \{\sigma_n\}, \{\tau_j^r\}]\,.$$

COROLLARY 6. *Let* $\{b_n\}$ *be as above with the additional assumption that each* b_n *is non-negative. Suppose that* (3) *is satisfied and* $\tau_j \neq 0$ *whenever* $j \notin \boldsymbol{Q}$, $\boldsymbol{Q}$ *being as in Theorem* 6.

(i) *Let B as in Theorem* 5. *If the hypothesis of Theorem* 6 *is satisfied with* $\rho_n = \hat{b}_n(0) - \mathrm{Re}\,(\hat{b}_n(1))$, *then the conclusion of Theorem* 6 *holds.*

(ii) *Let B as in Corollary* 5. *If* $\lim_{n\to\infty}\rho_n = 0$, *where* $\rho_n = \hat{b}_n(0) - \mathrm{Re}\,(\hat{b}_n(1))$ *and for some* $r \in \boldsymbol{N}$, $\beta_n(j) - 1 = \rho_n\tau_j^r$ *for all* $n \in \boldsymbol{N}$ *and all* $j \in \boldsymbol{Z}$, *then the conclusion of Theorem* 6 *holds.*

REMARK 5. For each $n \in \boldsymbol{N}$, $\lambda \in \Lambda$ let $b_{n,\lambda} = \sum_{m=0}^{\infty} a_{nm}^{(\lambda)} b_m$. Then, applying Proposition 1 and Corollary 1 of [17], we have the following statements (i) and (ii), which include the corresponding results of Remark 2 of [17] for the almost convergence.

(i) If $\{b_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ is an approximate identity ([17; Definition 2]), then $\mathscr{B} = \{b_{n,\lambda} * I; n \in N, \lambda \in \Lambda\}$ is a linear approximation process on X.

(ii) Suppose that B is positive and each b_n is non-negative. If $\{\hat{b}_n(0)\}$ and $\{\hat{b}_n(0) - \mathrm{Re}\,(\hat{b}_n(1))\}$ are B-summable to one and zero, respectively, then $\mathscr{B}$ is a linear approximation process on X. Furthermore, applying Theorem 4 of [17] we have a quantitative version of (ii) which estimates the rate of convergence for the methods of B-summability.

These results are applicable to the methods of B-summability of the above-mentioned examples (1°), (2°) and (3°), respectively.

Now as examples of multiplier operators considered in Corollary 5, let us mention the following:

(5°) The typical mean operator R_n^κ of order $\kappa > 0$ is defined by

$$R_n^\kappa = \sum_{j=-n}^{n} \{1 - (|j|/(n+1))^\kappa\} P_j$$

(cf. [5]). Suppose that $\{R_n^\kappa\}$ is uniformly bounded and let $A_{n,\lambda}$ be defined by (17) with $L_m = R_m^\kappa$. Then we have:

(i) Let B as in Theorem 5. If $\{1/(n+1)^\kappa\}$ is B-summable to zero, then the family $\mathscr{A} = \{A_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ is saturated with order $(\sum_{m=0}^{\infty} a_{nm}^{(\lambda)}/(m+1)^\kappa)$, and $S[X; \mathscr{A}] = W[X; \{-|j|^\kappa\}]^\sim = V[X; \{R_n^\kappa\}, \{-|j|^\kappa\}]$.

(ii) Let B as in Corollary 5. Then the conclusion of (i) holds.

(6°) Let $\delta = \{\delta_n\}$ be a sequence of positive real numbers and let $\kappa > 0$. We define the operator $S_n^{(\delta;\kappa)}$ by

$$S_n^{(\delta;\kappa)} = (1/(\delta_n + 1))(\delta_n S_n + R_n^\kappa)\ ,$$

where S_n denotes the n-th partial sum operator, i.e., $S_n = \sum_{j=-n}^{n} P_j$. It is easily seen that

$$S_n^{(\delta;\kappa)} = \sum_{j=-n}^{n} \{1 - |j|^\kappa/((\delta_n + 1)(n+1)^\kappa)\} P_j\ ,$$

which reduces to the arithmetic mean operator $(S_n + \sigma_n)/2$ of S_n and σ_n for $\delta = \{1\}$ and $\kappa = 1$. Statements analogous to parts (i) and (ii) of (5°) may be derived for the sequences $\{S_n^{(\delta;\kappa)}\}$.

REMARK 6. The Cesàro mean operator σ_n^κ of order $\kappa > -1$ is defined by

$$\sigma_n^\kappa = (1/A_n^{(\kappa)}) \sum_{j=-n}^{n} A_{n-|j|}^{(\kappa)} P_j\ , \quad A_n^{(\kappa)} = \binom{n+\kappa}{n}$$

(cf. [5]). Obviously, $\sigma_n^0 = S_n$ and $\sigma_n^1 = \sigma_n$. Note that $\{\sigma_n^\kappa\}$ converges strongly to I if and only if it is uniformly bounded.

In view of Proposition 5, we make the following remark on Example (2°).

REMARK 7. Let $\{\{q_n^{(\lambda)}\}_{n\geq 0};\ \lambda\in\Lambda\}$ be a family of sequences of non-negative real numbers such that $q_0^{(\lambda)}>0$ for all $\lambda\in\Lambda$, and let $B=\{(a_{nm}^{(\lambda)});\ \lambda\in\Lambda\}$, where each entry $a_{nm}^{(\lambda)}$ is defined by (13). Then the following are equivalent:

(i) B is regular;

(ii) $\lim_{n\to\infty} q_n^{(\lambda)}/Q_n^{(\lambda)}=0$ uniformly in $\lambda\in\Lambda$;

(iii) B satisfies (A-4).

By this result we see that $\lim_{n\to\infty}\|f_n-f\|_X=0$ implies

$$\lim_{n\to\infty}\left\|(1/A_n^{(\kappa)})\sum_{j=0}^{n}A_{n-j}^{(\kappa-1)}f_j-f\right\|_X=0$$

uniformly in $\kappa\in(0,a]$, $0<a<\infty$.

As another example of the application of Proposition 5, we consider a modification of the Cesàro mean operators for sequences in X. Let $\{f_n\}$ be a sequence of elements in X, and let

$$C_n^{\kappa}=(1/A_n^{(\kappa)})\sum_{j=0}^{n}A_{n-j}^{(\kappa)}f_j,\quad \kappa>-1,\quad n=0,1,2\cdots.$$

Then, by Proposition 5, we conclude that $\lim_{n\to\infty}\|C_n^{\kappa}-f\|_X=0$ implies $\lim_{n\to\infty}\|C_n^{\kappa+\rho}-f\|_X=0$ uniformly in $\rho\in[a,b]$, $0<a<b<\infty$. In particular, if $\sum_{n=0}^{\infty}f_n=f$, then $\lim_{n\to\infty}C_n^{\rho}=f$ uniformly in $\rho\in[a,b]$.

Next we shall consider the case where X is a homogeneous Banach subspace of $L_{2\pi}^1$. For the definition and examples of such spaces, see [17] (cf. [9; p. 14], [18; p. 206]). Defining the sequence $\{P_j\}_{j\in Z}$ by $P_j(f)(t)=\hat{f}(j)e^{ijt}$, it is obvious that $\{P_j\}$ is a total, fundamental sequence of mutually orthogonal projections in $B[X]$, since $\lim_{n\to\infty}\|\sigma_n(g)-g\|_X=0$ whenever g belongs to X by [9; Theorems 2.11 and 2.12]. Consequently, under this setting all the results obtained in this paper are applicable to homogeneous Banach spaces X.

Besides, in connection with the methods of B-summability in homogeneous Banach subspaces X of $L_{2\pi}^1$ we recast Part (ii) of Remark 5 by the test functions as follows:

Let B and $\{b_n\}$ be as in Part (ii) of Remark 5 and let $u_0(t)=1$, $u_1(t)=\sin t$ and $u_2(t)=\cos t$ for all $t\in \boldsymbol{R}$. Then the following are equivalent:

(i) $\{b_n * f\}$ is B-summable to f for every $f\in X$;

(ii) $\{b_n * u_j\}$ is B-summable to u_j for $j=0,1,2$;

(iii) $\{\hat{b}_n(0)\}$ and $\{\hat{b}_n(0)-\mathrm{Re}\,(\hat{b}_n(1))\}$ are B-summable to one and zero, respectively.

This immediately follows from [17; Theorem 5] and the equivalence of (i) and (ii) extends King and Swetits [10; Theorem 5] on the almost convergence for sequences of positive convolution integral operators on $C_{2\pi}$, the Banach space of all 2π-periodic, real-valued continuous functions on $\boldsymbol{R}$, to the more general methods of B-summability in homogeneous Banach subspaces X of $L^1_{2\pi}$.

Finally, we shall consider the case where X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\{e_n\}_{n\geqq 0}$ be a closed orthonormal system in X, that is, a sequence of elements in X such that the linear subspace of X spanned by $\{e_n\}$ is dense in X and $\langle e_n, e_m\rangle = \delta_{n,m}$ for all $n, m \geqq 0$, where $\delta_{n,m}$ is Kronecker's symbol. Defining the sequence $\{P_j\}_{j\in Z}$ by $P_j(f) = \langle f, e_j\rangle e_j$ for $j \geqq 0$ and $P_j(f) = 0$ for $j < 0$, it is seen that $\{P_j\}$ is a total, fundamental sequence of mutually orthogonal projections in $B[X]$ (cf. [5; Remark in Sec. 2], [17; Remark 8], [19; Sec. 4 of Chapter I]). Consequently, under this setting all the results obtained in this paper are applicable to the saturation problems in Hilbert spaces X.

We now consider the Hilbert space $L^2(E)$ of all measurable, square integrable functions on E, where E is a subset of $\boldsymbol{R}$. Recall that the inner product in this space is defined by

$$\langle f, g\rangle = \int_E f(t)\overline{g(t)}dt \qquad (f, g \in L^2(E)) .$$

We close with the following concrete examples of closed orthonormal systems $\{e_n\}_{n\geqq 0}$ in $L^2(E)$.

(I) *Jacobi system.* Let $E = [-1, 1]$ and $\alpha > -1$, $\beta > -1$. Let

$$e_n(t) = e_n^{(\alpha,\beta)}(t) = h_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(t), \qquad n = 0, 1, 2, \cdots ,$$

where

$$h_n^{(\alpha,\beta)} = \left\{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right\}^{1/2}$$

and $P_n^{(\alpha,\beta)}(t)$ is the Jacobi polynomial (cf. [20; Chapter IV]):

$$\begin{aligned} P_n^{(\alpha,\beta)}(t) &= \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}\{(1-t)^{n+\alpha}(1+t)^{n+\beta}\} \\ &= \sum_{j=0}^{n}\binom{n+\alpha}{j}\binom{n+\beta}{n-j}\{(t-1)/2\}^{n-j}\{(t+1)/2\}^j . \end{aligned}$$

The following particular selections α and β carry special names.
$\alpha = 0$, $\beta = 0$: Legendre system.
$\alpha = -1/2$, $\beta = -1/2$: Chebyshev system of the first kind.

$\alpha = 1/2$, $\beta = 1/2$: Chebyshev system of the second kind.
$\alpha = \beta$: Ultraspherical (Gegenbauer) system.

(II) *Laguerre system.* Let $E = [0, \infty)$ and $\alpha > -1$. Let

$$e_n(t) = e_n^{(\alpha)}(t) = \{n!/\Gamma(\alpha + n + 1)\}^{1/2} \exp(-t/2) t^{\alpha/2} L_n^{(\alpha)}(t) ,$$

where $L_n^{(\alpha)}(t)$ is the Laguerre polynomial (cf. [20; Chapter V]):

$$L_n^{(\alpha)}(t) = (1/n!) \exp(t) t^{-\alpha} \frac{d^n}{dt^n}\{\exp(-t) t^{n+\alpha}\}$$
$$= \sum_{j=0}^{n} \binom{n+\alpha}{n-j} (-t)^j/j! .$$

(III) *Hermite system.* Let $E = \boldsymbol{R}$, and let

$$e_n(t) = (2^n n!)^{-1/2} \pi^{-1/4} \exp(-t^2/2) H_n(t) ,$$

where $H_n(t)$ is the Hermite polynomial (cf. [20; Chapter V]):

$$H_n(t) = (-1)^n \exp(t^2) \frac{d^n}{dt^n} \exp(-t^2)$$
$$= n! \sum_{j=0}^{[n/2]} \{(-1)^j/(j!(n-2j)!)\}(2t)^{n-2j} .$$

REMARK 8. The ultraspherical, Laguerre and Hermite systems in $L^p(E)$ are similarly considered for various values of p, $1 \leqq p < \infty$ and we omit the details (cf. [5], [8], [21]).

(IV) *Bessel system.* Let $E = (0, 1)$ and $\nu > -1$. Let

$$e_n(t) = e_n^{(\nu)}(t) = (2t)^{1/2} J_\nu(\mu_n t)/J_{\nu+1}(\mu_n) ,$$

where $J_\nu(t)$ is the Bessel function (of the first kind), i.e.,

$$J_\nu(t) = (t/2)^\nu \sum_{j=0}^{\infty} \{(-1)^j/(j!\, \Gamma(\nu + j + 1))\}(t/2)^{2j}$$

and $\{\mu_n\}$ is the sequence of positive zeros of $J_\nu(t)$, arranged in ascending order of magnitude (cf. [20; Sec. 1.7.1], [21]). It should be noted that the Bessel series converge in $L^p(0, 1)$ whenever $\nu \geqq -1/2$ and $1 < p < \infty$ ([21; Theorem 4.1]), which establishes the convergence of Dini series in the same spaces ([21; Theorem 7.1]).

(V) *Haar system.* Let $E = [0, 1]$ and let $\{e_n\}$ be the sequence of Haar functions on E defined as follows:

$$e_1(t) = \chi_E(t) ,$$
$$e_n(t) = 2^{m/2}\{\chi_{[0,1)}(2^{m+1}t - 2n + 2) - \chi_{[0,1)}(2^{m+1}t - 2n + 1)\} ,$$
$$(n = 2^m + j,\ m = 0, 1, 2, \cdots;\quad j = 1, 2, \cdots, 2^m) ,$$

where $\chi_F(t)$ denotes the characteristic function of the interval F. It should be noted that the Haar series converge in $L^p(0, 1)$ whenever $1 \leqq p < \infty$ (cf. [19; pp. 13–16]).

(VI) *Walsh system.* Let $E = [0, 1]$. The Rademacher functions are defined by

$$r_0(t) = \chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t),\ r_0(t+1) = r_0(t),\ r_n(t) = r_0(2^n t)\ (n = 1, 2, \cdots) .$$

Let $\{e_n\}$ be the sequence of Walsh functions on E defined as follows:

$$e_0(t) = 1\ ,\quad e_n(t) = r_{n_1}(t) r_{n_2}(t) \cdots r_{n_m}(t)\ ,$$
$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_m}\ ,\quad n_1 > n_2 > \cdots > n_m \geqq 0\ .$$

It should be noted that the Walsh system is orthogonal, fundamental and total in $L^p(0, 1)$ whenever $1 \leqq p < \infty$ (cf. [19; pp. 396–406]).

We make the following final remark.

REMARK 9. Let $\boldsymbol{R}^d$ denote the d-dimensional Euclidean space with elements $x = (x_1, x_2, \cdots, x_d)$ and inner product

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d\ .$$

Let T^d be the d-dimensional torus and Z^d the set of all lattice points in $\boldsymbol{R}^d$, i.e., the d-fold Cartesian product of Z. Let $L^p(T^d)$, $1 \leqq p < \infty$ and $C(T^d)$ be the Banach spaces of all p-th power Lebesgue integrable functions and continuous functions on T^d which are 2π-periodic in each coordinate variable with standard norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ defined by

$$\left\{(2\pi)^{-d} \int_{T^d} |f(x)|^p dx\right\}^{1/p} \quad \text{and} \quad \max\{|f(x)|;\, x \in T^d\}\ ,$$

respectively. Now it is easy to state a strict d-dimensional analogue of homogeneous Banach subspaces of $L^1_{2\pi}$ $(= L^1(T^1))$ and $L^p(T^d)$, $1 \leqq p < \infty$ and $C(T^d)$ are such spaces, respectively (cf. [18; p. 206]). Let X be a homogeneous Banach space of $L^1(T^d)$. Then the total, fundamental sequence $\{P_j\}_{j \in Z}$ of mutually orthogonal projections in $B[X]$ is naturally induced from the Fourier coefficients of $f \in X$ defined by

$$\hat{f}(m) = (2\pi)^{-d} \int_{T^d} f(x) \exp(-im \cdot x) dx\ , \quad m \in Z^d\ ,$$

and we omit the details. Concerning the Fourier series expansions in association with spherical harmonics in the spaces $L^p(S^d)$, $1 \leqq p < \infty$ and $C(S^d)$, where S^d denotes the surface of the unit sphere in $\boldsymbol{R}^d$, one may consult [5; Sec. 8.4].

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