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SATURATION OF THE FROISSART BOUND BY CROSSING SYMMETRIC AND UNITARY AMPLITUDES

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A B S T R A C T

A class of amplitudes is constructed that saturates the Froissart bound and satisfies crossing symmetry and all inelastic unitarity inequalities.

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1. INTRODUCTION

The Froissart bound of the elastic scattering amplitude[1] has been derived on the basis of axiomatic quantum field theory[2] and there is no question that any specific field theory of hadrons has to obey this bound. The proof can be given on the basis of rather few properties of the amplitude[2] and it might be possible that this bound can be improved if more constraints (which are satisfied by quantum field theory) are taken into account. This problem is not only of academic interest, since recent experimental results on pp, $p\bar{p}$ and πp scattering are compatible with the saturation of this bound[3]. In fact one can expect more experimental information about this question pretty soon from $p\bar{p}$ collisions at the CERN-SPS.

The aim of this paper is to show that the Froissart bound cannot be improved on the basis of the general constraints analyticity, crossing symmetry and inelastic unitarity alone. The proof is done by the construction of $\pi\pi$ scattering amplitudes which saturate the Froissart bound. The $\pi\pi$ scattering process is chosen for two reasons:

1) it has maximal constraints due to crossing symmetry, and
2) the particles have no spin.

Concerning the experiments it is desirable to construct amplitudes for spin 1/2 particles. This is possible with essentially the same methods as presented here and will be done in another publication[4].

The scattering amplitude $A(s,t)$ is written with the usual Mandelstam variables s, t and u, the mass of the particles is renormalized to unity, and therefore $s + t + u = 4$. The amplitude is holomorphic and polynomially bounded

$$|A(s,t)| \leq \text{const}\,(1 + |s| + |t|)^{n} \tag{1.1}$$

in the domain

$$\mathbb{C}^2_{cut} = \{ (s,t) \in \mathbb{C}^2 \mid s \notin [4,\infty),\ t \notin [4,\infty),\ s+t \notin (-\infty, 0] \}$$

Crossing symmetry is the total symmetry in the variables s, t and $u = 4 - s - t$,

$$A(s,t) = A(t,s) = A(s,u) = A(u,s) = A(t,u) = A(u,t) \tag{1.2}$$

(The discussion of isospin crossing symmetry is postponed to Section 4.)

The partial wave decomposition on the physical cut $s \geq 4$ is written as

$$A(s+i0,t) = \sqrt{\frac{s}{s-4}} \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell\left(1+\frac{2t}{s-4}\right) \tag{1.3}$$

with

$$a_\ell(s) = [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^{0} A(s+i0,t)\, P_\ell\left(1+\frac{2t}{s-4}\right) dt \tag{1.4}$$

Unitarity of the S matrix imposes the inequalities

$$\operatorname{Im} a_\ell(s) \geq |a_\ell(s)|^2 \quad \text{for } \ell = 0,2,4,\dots \text{ and } s \geq 4 . \tag{1.5}$$

In this paper the existence of amplitudes is shown which have the following properties

a) Mandelstam analyticity and polynomial boundedness (1.1), the boundary values on the cuts are Hölder continuous,

b) symmetry with respect to all channels (1.2) (or $\pi\pi$ crossing symmetry),

c) unitarity (1.5),

d) saturation of the Froissart bound, i.e.,

$$\operatorname{Im} A(s+i0, t=0) \geq \text{const } s\,(\log s)^2 \quad \text{if } s \geq s_1 > 4 . \tag{1.6}$$

The only general condition which can be formulated with the elastic amplitude alone and which has not been incorporated, is exact elastic unitarity

$$\operatorname{Im} a_\ell(s) = |a_\ell(s)|^2 \quad \text{for } \ell = 0,2,4,\dots \text{ if } 4 \leq s \leq 16 . \tag{1.7}$$

So far all examples which saturate the Froissart bound only partly fulfil the constraints (a), (b) and (c). A review of these attempts is given in Ref. 5) and a more recent result can be found in Ref. 6). It should be stressed that the main problem comes from the saturation of the Froissart bound. Examples of amplitudes which satisfy the axiomatic constraints (a), (b) and (c) alone are known[7),8)] and even elastic unitarity (1.7) can be included[9)]. But in these publications the maximal increase of the amplitude is $\operatorname{Im} A(s + io, t = 0) \sim s \log s$ if $s \to \infty$.

This paper is organized as follows. In Section 2 an $s - u$ symmetric amplitude is constructed which satisfies (a), (b) and (d) and has uniformly bounded partial waves. The unitarity inequalities (1.5) are derived in Section 3 for energies $s \geq s_1 > 4$. The extension to crossing symmetric amplitudes, either total symmetry in s, t and u or crossing with charged pions follows the arguments of Ref. 7) and is presented in Section 4.

2. AN $s - u$ SYMMETRIC AMPLITUDE

In this section a construction of an $s - u$ symmetric amplitude $F(s,u)$ [*)] is given which satisfies Mandelstam analyticity, polynomial boundedness and the estimates

$$\operatorname{Im} F(s+io, 4-s) \geq c \; s(\log s)^2 \qquad \text{if} \quad s \geq 6 \tag{2.1}$$

and

$$|F(s+io, u)| \leq c' \cdot \mathcal{X}_{2,3}(s,t) \qquad \text{if} \quad s \geq 4 \quad \text{and} \quad 4-s \leq t \leq 0 \tag{2.2}$$

with some positive constants c and c' and the function

$$\mathcal{X}_{m,n}(s,t) = \begin{cases} s(\log s)^m (1+\sqrt{-t}\,\log s)^{-n} & \text{if } -1 \leq t \leq 0 \,, \\ s^p |t|^{-\frac{1+p}{2}} \quad \text{with } 0<p<1 & \text{if } t \leq -1 \,. \end{cases} \tag{2.3}$$

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*) The arguments of a function do not in general refer to a symmetrization. The only exception is $F(s,u)$.

The estimate (2.2) implies a uniform bound on all partial waves. As a first step of the construction a smoothed logarithm is defined

$$L(s) = \int_4^6 \sigma(x) \log(x-s)\, dx \tag{2.4}$$

for $s \in \mathbb{C}_{cut}$ with a bounded positive weight function $\sigma(x) \geq 0$, $x \in \mathbb{R}$, $\sigma(x) = 0$ if $x < 4$ or $x > 6$, which is normalized to $\int \sigma(x) dx = 1$. The boundary values on the cut, $L(s \pm io)$, $s \geq 4$, are Hölder continuous functions with any index $0 < \mu < 1$. The symmetrized function

$$H(s,t) = L(s) + L(u) - L(t+1) \tag{2.5}$$

is holomorphic in $(s,t) \in \mathbb{C}^2_{cut}$. In the physical region of the s channel it can be written as

$$H(s+io,t) = h(s,t) - i\pi\, \vartheta(s) \tag{2.6}$$

with the real functions

$$\begin{aligned} h(s,t) = \int \sigma(x) \log|s-x|\, dx\ + \\ + L(4-s-t) - L(t+1) \end{aligned} \tag{2.7}$$

and

$$\vartheta(s) = \int_4^s \sigma(x)\, dx \qquad \text{if} \quad s \geq 4 \,. \tag{2.8}$$

For the construction of this section it would be sufficient to take the simpler function $L(s) + L(u)$, but the sum (2.5) has the essential advantage that the partial waves of the real part $h(s,t)$ are positive,

$$h(s,t) = \sqrt{\frac{s}{s-4}} \sum_{\ell=0}^{\infty} (2\ell+1)\, h_\ell(s)\, P_\ell\left(1+\frac{2t}{s-4}\right) \tag{2.9}$$

with

$$\begin{cases} h_0(s) > \log(s-6) & \text{if } s > 7 \,, \\ h_\ell(s) > 0 & \text{if } \ell = 1,2,\ldots \text{ and } s > 4 \,. \end{cases} \tag{2.10}$$

The proof of this estimate follows from a simple calculation which can be found in Ref. 7). The function $H(s,t)$ satisfies the following inequalities

$$\begin{cases} |\operatorname{Im} H(s,t)| \le 3\pi \\ |\operatorname{Re} H(s,t)| \le \log \dfrac{(1+|s|)(1+|s|+|t|)}{1+|t|} + \text{const} \end{cases} \tag{2.11}$$

$$\text{for } (s,t) \in \mathbb{C}^2_{cut}$$

and

$$|H(s,t) - 2\log(1+|s|)| \le \text{const} \tag{2.12}$$
$$\text{for } s \in \mathbb{C}_{cut} \quad \text{and} \quad |t| \le 1 .$$

The starting point for the amplitude $F(s,u)$ is the function

$$\mathcal{G}(s,t,z) = -\frac{1}{\cos\frac{\pi}{2}z} \exp\left[\frac{1+z}{2} H(s,t)\right] . \tag{2.13}$$

This function is holomorphic in z if $|z|<1$ and it is holomorphic in s and t in the Mandelstam domain. In the physical region it can be written as [see (2.6)-(2.8)],

$$\mathcal{G}(s+io,t,z) = \left(\operatorname{tg}\frac{\pi}{2}z + i\right) \exp\left[\frac{1+z}{2} h(s,t)\right] \tag{2.14}$$
$$\text{if } s > 6 .$$

Let $\gamma(\xi)$ be a real analytic function defined by the integral

$$\gamma(\xi) = \frac{\xi}{\pi} \int_{x_0}^{\infty} \frac{\gamma_0(x)}{x(x-\xi)}\, dx \tag{2.15}$$

with a positive Hölder continuous imaginary part $\operatorname{Im}\gamma(x+io) = \gamma_0(x) \ge 0$ on the cut $x > x_0 \ge 2$ and the uniform bound

$$|\gamma(\xi)| \le \delta < 1 \qquad \text{for } \xi \in \mathbb{C}_{cut} . \tag{2.16}$$

Then $\mathcal{G}(s,t,\gamma(t))$ is an $s-u$ symmetric amplitude with the asymptotics

$$\mathcal{G}(s+i0,t,\gamma(t)) \simeq \left(\operatorname{tg}\frac{\pi}{2}\gamma(t)+i\right) s^{1+\gamma(t)} \tag{2.17}$$

$$\text{if} \quad s\to\infty \quad \text{and} \quad t \ \text{fixed}.$$

In the next step this function is modified to an amplitude which has a $\sqrt{-t}\ \log s$ shrinking of the forward peak. [Such a behaviour is necessary to saturate the Froissart bound[10].] The amplitude

$$\tilde{\mathcal{G}}(s,t;\gamma) = \frac{1}{\sqrt{t}}\left[\mathcal{G}(s,t,\gamma(\sqrt{t})) - \mathcal{G}(s,t,\gamma(-\sqrt{t}))\right] \tag{2.18}$$

develops two complex conjugate Regge trajectories (if $t<0$) of simple poles at angular momenta $\ell = 1+\gamma(\pm i\sqrt{-t})$. The $\sqrt{t}$ cut is cancelled in (2.18) and $\tilde{\mathcal{G}}$ is a real analytic function in the Mandelstam domain. The cut of $\gamma(\xi)$ may start at $x_0 = 2$ to produce a correct t cut in (2.18).

From (2.15) it follows that $\operatorname{Re}\gamma(i\tau) = \operatorname{Re}\ (-i\tau)$ is a monotonically decreasing function in the interval $0 \leq \tau < \infty$ with values from $\gamma(0)=0$ to $\gamma(\infty)<0$. For large values of $s>0$ and fixed $t<0$ the imaginary part of (2.18) behaves like

$$\operatorname{Im}\tilde{\mathcal{G}}(s+i0,t;\gamma) \simeq 2\, s^{1+\bar{\gamma}(t)}\,\frac{\sin[a(t)\log s]}{\sqrt{-t}} \tag{2.19}$$

with
$$\bar{\gamma}(t) = \operatorname{Re}\gamma(i\sqrt{-t}) < 0 \quad \text{and}$$
$$a(t) = \operatorname{Im}\gamma(i\sqrt{-t}).$$

An exact upper bound is (for details see the Appendix)

$$|\tilde{\mathcal{G}}(s+i0,t;\gamma)| \leq \text{const}\ \mathcal{K}_{1,1}(s,t) \tag{2.20}$$

$$\text{if} \quad s\geq 4 \quad \text{and} \quad 4-s \leq t \leq 0.$$

The amplitude $\tilde{\mathfrak{g}}(s,t;\gamma)\cdot H(s,t)$ would saturate the Froissart bound, but it violates unitarity, see Ref. 6). To improve the shrinking of the forward peak the function

$$\hat{g}(s,t) = \int_{\lambda_1}^{\lambda_2} \rho(\lambda)\, \tilde{g}(s,t;\lambda\gamma)\, d\lambda \tag{2.21}$$

is defined with a smooth positive weight function $\rho(\lambda)$ which satisfies

$$\begin{cases} \rho(\lambda) \in C^3(\mathbb{R}) , \\ \rho(\lambda) \geq 0 , \quad \int \rho(\lambda)\, d\lambda = 1 , \\ \rho(\lambda) = 0 \quad \text{if } \lambda < \lambda_1 \text{ or } \lambda > \lambda_2 \\ \qquad\qquad \text{with } 0 < \lambda_1 < \lambda_2 < \delta^{-1} . \end{cases} \tag{2.22}$$

The range of the λ integration is restricted to an interval $[\lambda_1,\lambda_2]$ such that the essential inequalities for the trajectory function (2.15) $\gamma(\xi)$ apply to $\lambda\gamma(\xi)$: $\mathrm{Im}\, \lambda\gamma(x + i0) \geq 0$ and $|\lambda\gamma(\xi)| \leq 1 - \varepsilon$ with some $\varepsilon > 0$ if $\lambda_1 \leq \lambda \leq \lambda_2$. The λ integration in (2.21) leads to a Fourier-Laplace integral which is investigated in the Appendix. Essentially it transforms the term $\sin[a(t) \log s]$ in Eq. (2.19) into a rapidly decreasing function of $\sqrt{-t} \log s$ yielding a stronger shrinking of the forward peak. The forward amplitude $\hat{\mathfrak{g}}(s,t = 0) = \tilde{\mathfrak{g}}(s,t = 0;\gamma)$ is not changed by the integration. An upper bound for $\hat{\mathfrak{g}}$ is

$$\begin{aligned} |\hat{g}(s+i0,t)| &\leq \text{const}\ \mathcal{K}_{1,3}(s,t) \\ &\text{if } s \geq 4 \text{ and } 4-s \leq t \leq 0 . \end{aligned} \tag{2.23}$$

Let $\beta(t)$ be a real analytic function with a cut $t \geq 4$ and the constraints

$$\begin{cases} \mathrm{Im}\,\beta(t+i0) \text{ Hölder continuous and} \\ \mathrm{Im}\,\beta(t+i0) \geq 0 \qquad\qquad \text{if } t \geq 4 , \\ \beta(\infty) \geq 0 , \\ |\beta(t)| \leq \text{const} \qquad\qquad \text{if } t \in \mathbb{C}_{\text{cut}} \end{cases} \tag{2.24}$$

then the amplitude

$$F(s,u) = \beta(t) \cdot H(s,t) \cdot \hat{g}(s,t) \tag{2.25}$$

satisfies $s - u$ crossing symmetry and both the conditions (2.1) and (2.2). The partial waves of $F(s,u)$

$$F(s+i0,u) = \sqrt{\frac{s}{s-4}} \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell\left(1+\frac{2t}{s-4}\right) \tag{2.26}$$

are uniformly bounded

$$|f_\ell(s)| \le c_1 \quad , \quad \ell = 0,1,2,\ldots \quad , \quad s \ge 4 \tag{2.27}$$

as a consequence of the estimate (2.2).

3. UNITARITY

3.1 General considerations

The proof that the amplitude (2.25) $F(s,u)$ satisfies (up to normalization) the unitarity inequalities (1.5) for energies $s \ge s_1 \ge 4$ follows the arguments of Ref. 7). As an essential step linear unitarity relations

$$|\mathrm{Re}\, f_\ell(s)| \le c_2 \,\mathrm{Im}\, f_\ell(s)$$
$$\ell = 0,1,2,\ldots \quad , \quad s \ge s_1 \tag{3.1}$$

are derived for all partial waves. The quadratic inequalities then follow from the bound (2.27) and from (3.1)

$$(\mathrm{Im}\, f_\ell)^2 \le c_1 \,\mathrm{Im}\, f_\ell \quad ,$$
$$(\mathrm{Re}\, f_\ell)^2 \le c_1 |\mathrm{Re}\, f_\ell| \le c_1 c_2 \,\mathrm{Im}\, f_\ell$$

hence

$$|f_\ell(s)|^2 \le c_1(1+c_2)\,\mathrm{Im}\, f_\ell(s)$$
$$\ell = 0,1,2,\ldots \quad , \quad s \ge s_1 \ . \tag{3.2}$$

The constant $c_1(1 + c_2)$ can be absorbed by a normalization $F \to c \cdot F$. The class of functions $\phi(s,t)$ which satisfies α) $\phi(s,t)$ is Hölder continuous in $s > 4$ and holomorphic in $t \in \mathbb{C} \setminus (-\infty,-s] \cup [4,\infty)$,

β) the partial wave amplitudes of

$$\phi(s,t) = \sqrt{\frac{s}{s-4}} \sum_{\ell=0}^{\infty} (2\ell+1)\, \varphi_\ell(s)\, P_\ell\left(1+\frac{2t}{s-4}\right)$$

are positive, $\varphi_\ell(s) \geq 0$, for $s \geq s_1$,

is denoted by $\mathcal{A}$, or more precisely by $\mathcal{A}(s_1)$. If $\phi_1(s,t) \in \mathcal{A}$ and $\phi_2(s,t) \in \mathcal{A}$ then the functions $\phi_1\phi_2$ and $\alpha\phi_1 + \beta\phi_2$, $\alpha \geq 0$, $\beta \geq 0$, are also elements of $\mathcal{A}$. As a consequence of this simple statement more complicated constructions are possible, e.g., if $\phi(s,t) \in \mathcal{A}$ then also $\exp \phi(s,t) \in \mathcal{A}$.

For functions which satisfy the conditions α) a partial order relation $\phi_1 \prec \phi_2$ can be defined by

$$\phi_1(s,t) \prec \phi_2(s,t) \quad \text{if} \quad \phi_2(s,t) - \phi_1(s,t) \in \mathcal{A}$$

This relation is preserved by the addition of any function $\psi(s,t)$ which is only submitted to the constraint α):

$$\phi_1 \prec \phi_2 \quad \rightsquigarrow \quad \phi_1 + \psi \prec \phi_2 + \psi$$

and by multiplication with a function $\chi(s,t) \in \mathcal{A}$

$$\phi_1 \prec \phi_2 \quad \rightsquigarrow \quad \chi \cdot \phi_1 \prec \chi \cdot \phi_2 \quad .$$

The unitarity relations (3.1) are equivalent to

$$-c_2 \,\mathrm{Im}\, F(s+i0,u) \prec \mathrm{Re}\, F(s+i0,u) \prec c_2 \,\mathrm{Im}\, F(s+i0,u) \quad \text{if} \quad s \geq s_1 \,. \tag{3.3}$$

3.2 Unitarity of $\widetilde{\mathfrak{G}}$

As a first step unitarity is derived for the amplitude (2.18) $\widetilde{\mathfrak{G}}(s,t;\gamma)$. As a consequence of the estimate (2.20) the partial waves $g_\ell(s)$ of $\widetilde{\mathfrak{G}}(s + i0, t;\gamma)$ are uniformly bounded

$$|g_\ell(s)| \leq c_3 \qquad , \ell = 0,1,2,\ldots \quad , \quad s \geq 4 , \tag{3.4}$$

and the remaining problem is to derive the inequalities

$$\left| \operatorname{Re} g_\ell(s) \right| \leq C_4 \operatorname{Im} g_\ell(s) \tag{3.5}$$
$$\text{for} \quad \ell = 0,1,2,\ldots \quad \text{and} \quad s \geq s_1 > 4.$$

The function (2.14) $\mathcal{G}(s,t,\gamma(\sqrt{t}))$ can be expanded for energies $s > 6$ as

$$\mathcal{G} = \left(\operatorname{tg} \tfrac{\pi}{2}\gamma + i \right) \exp\left[\tfrac{1+\omega}{2} \, h(s,t) \right] \times$$
$$\times \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \left(h(s,t) \right)^n \left(\gamma - \omega \right)^n \tag{3.6}$$

with $\gamma = \gamma(\sqrt{t})$ and $\omega = \gamma(\infty) < 0$. Hence the real and the imaginary parts of (2.18) $\widetilde{\mathcal{G}}$ are given by the series

$$\operatorname{Im} \widetilde{\mathcal{G}}(s+i0, t; \gamma) = \exp\left[\tfrac{1+\omega}{2} \, h(s,t) \right] \times$$
$$\times \sum_{n=1}^{\infty} \frac{2^{-n}}{n!} \left(h(s,t) \right)^n \, \Phi_n(t) \tag{3.7}$$

$$\operatorname{Re} \widetilde{\mathcal{G}}(s+i0, t; \gamma) = \exp\left[\tfrac{1+\omega}{2} \, h(s,t) \right] \times$$
$$\times \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \left(h(s,t) \right)^n \, \Psi_n(t) \tag{3.8}$$

with

$$\Phi_n(t) = \frac{1}{\sqrt{t}} \left[a_n(\sqrt{t}) - a_n(-\sqrt{t}) \right], \tag{3.9}$$

$$a_n(\xi) = \left[\gamma(\xi) - \omega \right]^n, \tag{3.10}$$
$$n = 1,2,\ldots \quad ,$$

and
$$\Psi_n(t) = \frac{1}{\sqrt{t}}\left[b_n(\sqrt{t}) - b_n(-\sqrt{t})\right] \tag{3.11}$$

$$\begin{cases} b_0(\xi) = \operatorname{tg}\frac{\pi}{2}\gamma(\xi) - \operatorname{tg}\frac{\pi}{2}\omega \\ b_n(\xi) = [\gamma(\xi)-\omega]^n \operatorname{tg}\frac{\pi}{2}\gamma(\xi) \\ n = 1,2,\ldots \end{cases} \tag{3.12}$$

The function $h(s,t)$ has positive partial waves if $s > 7$, [see Eq. (2.10)] and since $\omega > -1$ [see Eq. (2.16)] also $\exp[(1+\omega/2)\, h(s,t)] \in \mathcal{A}$. The proof of the inequalities (3.5) now follows by the estimation of terms with the same index n in the series (3.7) and (3.8). [Only the $n = 0$ term in (3.7) has to be treated separately.] It is sufficient to derive the relations

$$\begin{cases} -\frac{c_4}{2}\Phi_1(t) \prec \Psi_{0,1}(t) \prec \frac{c_4}{2}\Phi_1(t), \\ -c_4\Phi_n(t) \prec \Psi_n(t) \prec c_4\Phi_n(t), \\ n = 2,3,\ldots . \end{cases} \tag{3.13}$$

Since $1 \prec h(s,t)$ if $s \geq s_1$ with $\log(s_1 - 6) \geq 1$ the first line of (3.13) yields

$$-c_4\, h\cdot\Phi_1 \prec \Psi_0 + h\cdot\Psi_1 \prec c_4\, h\cdot\Phi_1$$

The estimate of the terms with $n \geq 2$ follows from the second line of (3.13) by multiplication with $h^n \in \mathcal{A}$. Hence the relations (3.13) imply

$$-c_4 \operatorname{Im}\tilde{g} \prec \operatorname{Re}\tilde{g} \prec c_4 \operatorname{Im}\tilde{g} \qquad \text{if} \quad s \geq s_1 \geq 6+e, \tag{3.14}$$

which is equivalent to Eq. (3.5).

To derive the relations (3.13) one has to consider the functions (3.9) and (3.11) with the rather unpleasant structure

$$\Phi(t) = \frac{1}{\sqrt{t}}\left[a(\sqrt{t}) - a(-\sqrt{t})\right] \tag{3.15}$$

in more detail [see Ref. 6)]. The function $a(\xi)$ is normalized to $a(\infty) = 0$ and satisfies the dispersion relation

$$a(\xi) = \frac{1}{\pi} \int_{2}^{\infty} \frac{\alpha(x)}{x - \xi} \, dx \tag{3.16}$$

with a real Hölder continuous $\alpha(x)$. The function $A(t)$ is then real analytic in $\mathbb{C} \setminus [4,\infty)$ with the absorptive part

$$\Phi_t(t) = \frac{1}{\sqrt{t}} \, \alpha(\sqrt{t}) \quad , \quad t \geq 4 \, . \tag{3.17}$$

If $\alpha(x) \in \mathcal{L}^2(\mathbb{R})$ then also $a(i\tau) \in \mathcal{L}^2(\mathbb{R}_\tau)$ and the Fourier-Laplace transform (along the imaginary axis)

$$\varphi(b) = \mathcal{F}[a(i\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(i\tau) \, e^{-ib\tau} \, d\tau \tag{3.18}$$

exists as a function $\varphi(b) \in \mathcal{L}^2(\mathbb{R})$. It vanishes if $b < 0$ since $a(\xi)$ is holomorphic in the half plane $\mathrm{Re}\,\xi < 0$ and for $b > 0$ the transform (3.18) leads to the Laplace integral

$$\varphi(b) = \frac{1}{\pi} \int_{2}^{\infty} \alpha(x) \, e^{-bx} \, dx \tag{3.19}$$

The inverse Fourier transform yields

$$\Phi(t) = \frac{2}{\sqrt{-t}} \int_{0}^{\infty} \varphi(b) \sin\left(b\sqrt{-t}\right) db \tag{3.20}$$

The partial wave projections f_ℓ of (3.20) are calculated as [see Ref. 11) Eq. (7.244.1)]

$$f_\ell(s) = \frac{\pi}{\sqrt{s}} \int_{0}^{\infty} \varphi(b) \left[J_{\ell+\frac{1}{2}}\left(\frac{b\sqrt{s-4}}{2}\right)\right]^2 db \tag{3.21}$$

The kernel $[J_{\ell+1/2}(\ldots)]^2$ is positive, therefore the partial waves (3.21) of the function (3.20) $\Phi(t)$ are positive if the Fourier transform (3.18) is a positive function.

Under the condition (2.16) and the additional technical assumption

$$\operatorname{Im} \gamma(x+io) \in \mathcal{L}^2(\mathbb{R}) \tag{3.22}$$

the above arguments apply to the functions (3.9) and (3.11). Since $\operatorname{Im} \gamma(x+io) \geq$ ≥ 0 the imaginary parts of (3.10) and (3.12) satisfy the inequalities

$$\begin{cases} |\operatorname{Im} b_0(x+io)| \leq c \cdot \operatorname{Im} a_1(x+io) \\ |\operatorname{Im} b_1(x+io)| \leq c \cdot \operatorname{Im} a_1(x+io) \end{cases} \tag{3.23}$$

with some positive constant c. These inequalities are transferred to the Laplace integrals (3.19)

$$|\psi_n(b)| \leq c \; \varphi_1(b) \quad , \; n = 0,1 \tag{3.24}$$

with

$$\begin{cases} \varphi_n(b) = \mathcal{F}[a_n(i\tau)] \; , \\ \psi_n(b) = \mathcal{F}[b_n(i\tau)] \; . \end{cases} \tag{3.25}$$

The partial waves of the amplitudes Ψ_0, Ψ_1 and Φ_1 are calculated from the representation (3.21) and the inequalities (3.24) yield the first line of the relations (3.13) with $c_4 = 2c$.

The Fourier transform $\varphi_1(b)$ is an element of $\mathcal{L}^2(\mathbb{R})$ due to the assumption (3.22). But (2.16) allows an additional estimate of the Laplace integral (3.19) [with $\alpha(x) = \operatorname{Im} \gamma(x + io)$]

$$|\varphi_1(b)| \leq \text{const } b^{-1} \exp(-2b)$$

hence

$$\varphi_1(b) \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \; .$$

The Fourier-Laplace transforms of a_n and b_n for $n \geq 2$ can now be calculated as convolutions. Since

$$a_n(\xi) = [a_1(\xi)]^n \quad \text{and} \quad b_n(\xi) = b_0(\xi) \cdot a_n(\xi)$$
$$n = 1,2,\ldots$$

the Fourier transforms are

$$\varphi_{n+1}(b) = \varphi_1(b) * \varphi_n(b) \tag{3.36}$$

and

$$\psi_{n+1}(b) = \psi_1(b) * \varphi_n(b), \quad n = 1, 2, \dots . \tag{3.27}$$

The convolution integrals $\varphi * \psi = \int \varphi(b - b')\psi(b')db'$ exist as functions in $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$. Since $\varphi_1(b) \geq 0$ Eq. (3.26) implies that all $\varphi_n(b)$, $n = 1,2,\dots,$ are positive functions and from (3.24)-(3.27) follow the inequalities

$$|\psi_n(b)| \leq c\, \varphi_n(b) \quad , \quad n = 2, 3, \dots . \tag{3.28}$$

These inequalities can be transferred to the partial waves of (3.9) $\Phi_n(t)$ and (3.11) $\Psi_n(t)$ and the relations (3.13) follow with the constant $c_4 = 2c$. This completes the proof of (3.14).

3.3 Unitarity of F(s,u)

The unitarity relation (3.14) can be derived (with a larger value of the constant c_4) for any amplitude $\tilde{\mathcal{G}}(s,t;\lambda\gamma)$ if the parameter λ is restricted to an interval $0 < \lambda_1 \leq \lambda \leq \lambda_2 < \delta^{-1}$. Therefore this relation remains valid for the integral (2.21) $\hat{\mathcal{G}}(s,t)$.

The amplitude $F(s,u)$ is defined as the product (2.25) $F = \beta(t)H(s,t)\hat{\mathcal{G}}(s,t)$. The factor $\beta(t)$ is easily absorbed in the relation (3.14) since the constraints (2.24) imply $\beta(t) \in \mathcal{A}$.

If the real and the imaginary part of $\beta(t)\hat{\mathcal{G}}(s,t)$ are denoted by M and N

$$\beta(t)\, \hat{\mathcal{G}}(s+io,t) = M(s,t) + i\, N(s,t) \tag{3.29}$$

the result obtained so far is

$$-c_4 N(s,t) < M(s,t) < c_4 N(s,t) \tag{3.30}$$

The additional factor (2.6) $H(s + io,t)$ also has a real and an imaginary part

$$H(s+io,t) = h(s,t) - i\pi \qquad \text{if } s \geq 7 . \tag{3.31}$$

If s_1 is chosen such that

$$\log(s_1 - 6) \geq \pi(1 + c_4 + c_4^2) \tag{3.32}$$

then the estimate (2.10) of the partial waves of $h(s,t)$ implies

$$\pi(1 + c_4 + c_4^2) < h(s,t) \qquad \text{if } s \geq s_1$$

and together with (3.30) follows

$$(1 + c_4)\pi M + \pi N < hN \qquad \text{if } s \geq s_1 . \tag{3.33}$$

The amplitude

$$\begin{aligned} F &= (h - i\pi)(M + iN) = \\ &= (hM + \pi N) + i(hN - \pi M) \end{aligned}$$

is then estimated as

$$\begin{aligned} -(1 + c_4)(hN - \pi M) &\underset{(3.33)}{<} -c_4 hN - \pi N \underset{(3.30)}{<} \\ &< hM + \pi N \underset{(3.30)}{<} c_4 hN + \pi N \underset{(3.33)}{<} \\ &< (1 + c_4)(hN - \pi M) \end{aligned}$$

which yields the unitarity relation (3.3) with $c_2 = 1 + c_4$ for energies $s \geq s_1$. The value of s_1 is determined by Eq. (3.32).

4. CROSSING SYMMETRIC AMPLITUDES

Crossing symmetry and correct threshold behaviour can be incorporated with a method which has been developed in Ref. 7). The main results of Ref. 7) which are important for this section can be summarized in the following propositions. The amplitudes $G(s,t)$, $H(s,t)$ and $F(s,t)$ have to satisfy the analyticity properties (a) of Section 1. The respective partial waves are denoted by g_ℓ, h_ℓ and f_ℓ.

Proposition 1

If the amplitude $G(s,t)$ is bounded by

$$|G(s+i0,t)| \leq \text{const} \sum_{j=1,2} s^{-\alpha_j-\varepsilon}(1+|t|)^{\alpha_j-\varepsilon}$$

$$\text{with} \quad -1 \leq \alpha_j \leq 1 \quad \text{and} \quad \varepsilon > 0 \tag{4.1}$$

$$\text{for} \quad s \geq s_1 \quad \text{and} \quad t \in \mathbb{C} \setminus (-\infty, -s] \cup [4, \infty)$$

then an amplitude $H(s,t)$ can be found such that $F(s,t) = c \cdot G(s,t) + H(s,t)$ with some constant $c > 0$ satisfies the unitarity inequalities

$$\operatorname{Im} f_\ell(s) \geq |f_\ell(s)|^2 \quad , \ell = 0,1,2,\ldots \tag{4.2}$$

for all energies $s \geq 4$. The amplitude $H(s,t)$ can be majorized by a bound of the type (4.1) and is defined by a once subtracted Mandelstam representation with positive spectral functions.

Proposition 2

If the amplitude $G(s,t)$ satisfies the unitarity inequalities

$$\operatorname{Im} g_\ell(s) \geq |g_\ell(s)|^2 \quad , \ell = 0,1,2,\ldots \tag{4.3}$$

for energies $s \geq s_1 > 4$ then an amplitude $H(s,t)$ can be constructed such that $F(s,t) = c\, G(s,t) + H(s,t)$ (with some constant $c > 0$) satisfies the unitarity inequalities (4.2) for all energies $s \geq 4$. The amplitude $H(s,t)$ can be defined by an unsubtracted Mandelstam representation with positive double spectral function.

The proof of these propositions will not be given here, it can be found in Ref. 7). But a few comments are added.

1) If $G(s,t)$ is symmetric in t and u (only even partial waves contribute) then $H(s,t)$ can be chosen as a totally symmetric function in s, t and u.

2) The bound (4.1) is essentially a bound on unitary amplitudes with positive double spectral function[12]. (It can be improved somewhat but linearly or stronger increasing amplitudes are definitely excluded.)

3) The essential point for the proof of both propositions is that for large angular momenta the Froissart-Gribov representation for the partial waves can be estimated by the contribution of a neighbourhood of the threshold in t. Therefore, by adding an amplitude with a large positive spectral

function, this will dominate the partial waves and one can achieve the inequalities (4.2).

4) It is possible to exploit the quadratic structure of the unitarity relations (4.2) and to obtain a double spectral function for the amplitude $F(s,t)$ with a support only in the domain prescribed by elastic unitarity. In the case of Proposition 1 this is possible without further restrictions, in the case of Proposition 2 this is possible at least for all amplitudes $G(s,t)$ which have a threshold t_0 in the t channel (and u channel) slightly above the elastic threshold, $t_0 > 4$.

To obtain the crossing symmetric amplitude $A(s,t)$ announced in Section 1, the function $F(s,u)$ which was constructed in Section 2 is symmetrized and a symmetric amplitude $H(s,t)$ with positive spectral functions is added.

In Section 3 it has been shown that $F(s,u)$ and therefore also $1/2(F(s,u) + F(s,t))$ satisfies the unitarity inequalities for energies $s \geq s_1$. Following Proposition 2 one can find an s, t and u symmetric amplitude $H_1(s,t)$ such that

$$A_1(s,t) = c_1 \Big(F(s,u) + F(s,t) \Big) + H_1(s,t) \tag{4.4}$$

with some strictly positive constant c_1 obeys the inequalities (1.5) for all energies $s \geq 4$.

In the appendix the crossed term $F(t,u)$ is estimated by

$$|F(t,u)| \leq \text{const}\; s^{-\frac{1}{2}-\frac{p}{2}} \Big[(1+|t|)(s+|t|)\Big]^{\frac{p}{2}} \tag{4.5}$$

with an exponent $p < 1$ for energies $s \geq s_2$ and $t \in \mathbb{C}_{cut}$. But this is exactly a bound of the type (4.1) and following Proposition 1 a symmetric amplitude $H_2(s,t)$ can be found such that with some constant $c_2 > 0$ the amplitude

$$A_2(s,t) = c_2\, F(t,u) + H_2(s,t) \tag{4.6}$$

satisfies the inequalities (1.5).

If A_1 and A_2 are any two amplitudes which fulfil the unitarity relations (1.5) then the sum $\alpha_1 A_2 + \alpha_2 A_2$ will also do it if the constants $\alpha_{1,2}$ are restricted to $\alpha_{1,2} \geq 0$ and $\alpha_1 + \alpha_2 \leq 1$. Now, taking (4.4) and (4.6), these constants are chosen such that $\alpha_1 c_1 = \alpha_2 c_2 = c > 0$ and

$$\begin{aligned} A(s,t) &= \alpha_1 A_1(s,t) + \alpha_2 A_2(s,t) \\ &= c\,\big(F(s,u) + F(s,t) + F(t,u)\big) + H(s,t) \end{aligned} \tag{4.7}$$

with $H = \alpha_1 H_1 + \alpha_2 H_2$ is an amplitude which satisfies all properties (a)-(d) of Section 1. The lower estimate (1.6) is not spoiled by the amplitude $H(s,t)$ because H_1 and H_2 can be chosen to decrease like (4.5) if $s \to \infty$.

The construction of the amplitude (4.7) does not include exact elastic unitarity. But if the threshold of $F(s,u) + F(s,t)$ in the t channel lies slightly above four, a property which can easily be achieved by shifting the thresholds of the basic functions (2.4) $L(s)$, (2.15) $\gamma(\xi)$ and (2.24) $\beta(s)$ to $s_0 > 4$ or $x_0 = \sqrt{s_0} > 2$ [in the case of $\gamma(\xi)$], then $H(s,t)$ can be chosen such that $A(s,t)$ has a double spectral region with a boundary as demanded by elastic unitarity.

The amplitude (4.7) satisfies the simple crossing relation (1.2). There is no difficulty in extending the construction to amplitudes with isospin-1 crossing. The methods of Ref. 7), Section 5, can be applied to the ansatz (2.25) $F(s,u)$ [instead of the Regge ansatz $R(s,u)$ there]. Then the amplitude with isospin-0 exchange in the t channel saturates the Froissart bound and all s channel isospin amplitudes satisfy the unitarity inequalities (1.5).

The Pomeron of the constructed amplitudes is determined by the function (2.25) $F(s,u)$. Characterized as a singularity in the angular momentum plane, it consists of two branch cuts

$$\mathcal{L}^{\pm}(t) = \{\, \ell = 1 + \lambda\,\gamma(\pm\sqrt{t}\,),\ \lambda_1 \le \lambda \le \lambda_2 \,\}$$

where $\gamma(\xi)$ is given in Eq. (2.15). For $t = 0$ these branch cuts degenerate to a pole of third order at the coincidence point $\ell = 1 + \gamma(0) = 1$.

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APPENDIX

In this appendix upper bounds on the constructed amplitudes are derived. The exponential function satisfies the simple inequality

$$|e^{z_1\xi} - e^{z_2\xi}| \le |z_1 - z_2|\,|\xi|\,\left(e^{\mathrm{Re}\, z_1\xi} + e^{\mathrm{Re}\, z_2\xi}\right)$$

for arbitrary complex numbers z_1, z_2 and ξ. If $|z_{1,2}| < \delta < 1$ the function (2.13) $g(s,t;z)$ is therefore estimated by

$$|g(s,t,z_1) - g(s,t,z_2)| \le \mathrm{const}\,(1 + |H(s,t)|)\; e^{\tau}$$

with

$$\tau = \max_{k=1,2} \mathrm{Re}\left(\frac{1+z_k}{2} H(s,t)\right)$$

The bound (2.11) on $H(s,t)$ and the inequality

$$|\gamma(\sqrt{t})| \le \mathrm{const}\left(\frac{|t|}{1+|t|}\right)^{\frac{1}{2}} , \qquad \text{(A.1)}$$

a consequence of $\gamma(0) = 0$ and (2.16), then yield a majorization for (2.18) $\tilde{g}(s,t,\gamma)$

$$|\tilde{g}(s,t,\gamma)| \le \mathrm{const}\,(1+|t|)^{-\frac{1}{2}} \times$$

$$\times \left(\frac{1+|s|^2+|st|}{1+|t|}\right)^{\frac{1+\bar{\gamma}(t)}{2}} \log\,(2+|s|) \qquad \text{(A.2)}$$

with $\quad \bar{\gamma}(t) = \max \mathrm{Re}\,\gamma(\pm\sqrt{t})$.

The definitions (2.21) and (2.25) then lead to

$$|F(s,u)| \le \mathrm{const}\,(1+|t|)^{-\frac{1}{2}} \times$$

$$\times \left(\frac{1+|s|^2+|ts|}{1+|t|}\right)^{\frac{\alpha(t)}{2}} \log^2(2+|s|) \qquad \text{(A.3)}$$

with

$$\alpha(t) = \begin{cases} 1 + \lambda_1 \bar{\gamma}(t) & \text{if} \quad \bar{\gamma}(t) < 0 \\ 1 + \lambda_2 \bar{\gamma}(t) & \text{if} \quad \bar{\gamma}(t) \geq 0 \end{cases}$$

Since $\gamma(\infty) = \omega < 0$ and $0 < \lambda_1 < \lambda_2$ the exponent converges to $\alpha(t) \to 1 + \lambda_1\omega <$ < 1 if $|t| \to \infty$ and (A.3) implies the estimate (4.7) for the crossed term $F(t,u)$.

For better estimates in the physical region $s \geq 4$, $4 - s \leq t \leq 0$ the real part of $H(s + io,t)$ can be majorized by

$$|h(s,t)| \leq \log\left(\frac{s^2}{1+|t|}\right) + \text{const} . \tag{A.4}$$

For any number $a \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, the exponential function is bounded by

$$|e^{az} - e^{az^*}| \leq e^{ax} \frac{4\,|ay|}{1+|ay|}$$

Since

$$\gamma(-i\sqrt{-t}) = \gamma^*(i\sqrt{-t}) \,, \quad t<0 \,,$$

and

$$|\gamma(i\sqrt{-t}) - \gamma(-i\sqrt{-t})| \leq \text{const} \sqrt{-t}$$

the bound (A.2) on $\tilde{g}(s,t;\gamma)$ can be improved to

$$|\tilde{g}(s+io,t;\gamma)| \leq \text{const} \left(\frac{s}{1+\sqrt{-t}}\right)^{1+\bar{\gamma}(t)} \frac{\log s}{1+\sqrt{-t}\,\log s} \tag{A.5}$$
$$\text{if} \quad s \geq 4 \quad \text{and} \quad 4-s \leq t \leq 0 \, .$$

The exponent $\bar{\gamma}(t) = \mathrm{Re}\,\gamma(i\sqrt{-t})$ is strictly negative if $t < 0$ and monotonically decreasing if $-t \to \infty$. Therefore the estimate (2.20) follows from (A.5).

The forward peak of the amplitude (2.21) $\hat{g}(s,t)$ is evaluated with the following theorem.

Theorem:

Let $g(z)$ be holomorphic in $|z| < 1$ and let $\rho(\lambda)$ be a $C^N(\mathbb{R})$ function with support $0 < \lambda_1 \leq \lambda \leq \lambda_2 < \infty$, then

$$G(z,\xi) = \int \rho(\lambda)\, g(\lambda z) \exp((1+\lambda z)\xi)\, d\lambda \tag{A.6}$$

is holomorphic in $|z| < \lambda_2^{-1}$ and $\xi \in \mathbb{C}$ and can be majorized by

$$\left| \left(\frac{\partial}{\partial z}\right)^n G(z,\xi) \right| \leq C_{n,N} (1 + |z\xi|)^{-N} \times$$
$$\times (1+|\xi|)^n \exp\left[(1+\lambda_0 \operatorname{Re} z)\xi\right] \tag{A.7}$$

if $|z| \leq \delta < \lambda_2^{-1}$ and $\operatorname{Re}\xi \geq 0$, $|\operatorname{Im}\xi| \leq \text{const}$,

with $\lambda_0 = \begin{cases} \lambda_1 & \text{if } \operatorname{Re} z \leq 0, \\ \lambda_2 & \text{if } \operatorname{Re} z > 0. \end{cases}$

Proof:

Since

$$\left(\frac{\partial}{\partial z}\right)^n G = \sum_{\nu=0}^{n} \binom{n}{\nu} G_{n,\nu}(z,\xi)\, \xi^{n-\nu}$$

with functions

$$G_{n,\nu}(z,\xi) = \int \lambda^n \rho(\lambda)\, g^{(\nu)}(\lambda z)\, e^{(1+\lambda z)\xi}\, d\lambda$$

which have the same structure as $G(z,\xi)$, it is sufficient to prove (A.7) for $n = 0$. By partial integration there follows for $0 \leq M \leq N$

$$(z\xi)^M G(z,\xi) = \int \rho(\lambda)\, g(\lambda z) \left(\frac{\partial}{\partial\lambda}\right)^M e^{(1+\lambda z)\xi}\, d\lambda$$

$$= (-)^M \int e^{(1+\lambda z)\xi} \left(\frac{\partial}{\partial\lambda}\right)^M \left(\rho(\lambda)\, g(\lambda z)\right) d\lambda\, .$$

The last integral is dominated by the exponential function, hence

$$|(z\xi)^M G(z,\xi)| \leq \mathrm{const}_M \; e^{(1+\lambda_0 \,\mathrm{Re}\, z)\,\xi}$$

$$\text{if } |z| \leq \delta \text{ and } \mathrm{Re}\,\xi \geq 0 \,, \; |\mathrm{Im}\,\xi| \leq \mathrm{const} \,,$$

and (A.7) follows for $n = 0$. This completes the proof.

The estimate (A.7) yields a bound on

$$G(z,\xi) - G(z^*,\xi) = \int_{\tau} \frac{\partial}{\partial u} G(u,\xi)\, du$$

The path τ is chosen as

$$\tau = \{ u \mid |u| = |z| \,, \; \mathrm{Re}\, u \leq \mathrm{Re}\, z \}$$

The length of τ is bounded by $\pi\, |\mathrm{Im}\, z|$. With $n = 1$ it follows from (A.7).

$$|G(z,\xi) - G(z^*,\xi)| \leq \mathrm{const}\, (1 + |z\xi|)^{-N} \times$$

$$\times (1+|\xi|)\, e^{(1+\lambda_0 \,\mathrm{Re}\, z)\cdot \mathrm{Re}\,\xi} \cdot |\mathrm{Im}\, z| \qquad \text{(A.8)}$$

$$\text{if } |z| \leq \delta \text{ and } \mathrm{Re}\,\xi \geq 0 \,, \; |\mathrm{Im}\,\xi| \leq \mathrm{const} .$$

The amplitude $\hat{g}(s,t)$ is defined in (2.21) exactly as

$$\hat{g}(s,t) = \frac{1}{\sqrt{t}} \left[G(\gamma(\sqrt{t}), \tfrac{1}{2} H) - G(\gamma(-\sqrt{t}), \tfrac{1}{2} H) \right]$$

with the function (A.6) $G(z,\xi)$ and

$$g(z) = -\left(\cos \frac{\pi}{2} z \right)^{-1} \,, \quad H = H(s,t) .$$

The inequalities

$$c_1 |t|^{\frac{1}{2}} \leq |\gamma(\sqrt{t})| \leq c_2 |t|^{\frac{1}{2}} \qquad \text{if } |t| \leq 1$$

$$\text{with } 0 < c_1 < c_2 < \infty$$

and (2.11), (2.12) together with (A.8) imply the estimate of the forward peak of $\hat{g}$

$$|\hat{g}(s,t)| \le \text{const } |s|^{1+\lambda_0 \bar{\gamma}(t)} \frac{\log|s|}{(1+\sqrt{|t|}\log s)^N} \tag{A.9}$$

if $|s| \ge 4$ and $|t| \le 1$

If $N = 3$ then the bound (2.23) follows from (A.5) and (A.9).

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