

SBV-like regularity for general hyperbolic systems of conservation laws in one space dimension

STEFANO BIANCHINI AND LEI YU

ABSTRACT. *We prove the SBV regularity of the characteristic speed of the scalar hyperbolic conservation law and SBV-like regularity of the eigenvalue functions of the Jacobian matrix of flux function for general hyperbolic systems of conservation laws. More precisely, for the equation*

$$u_t + f(u)_x = 0, \quad u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega \subset \mathbb{R}^N,$$

we only assume that the flux f is a C^2 function in the scalar case ($N = 1$) and Jacobian matrix Df has distinct real eigenvalues in the system case ($N \geq 2$). Using a modification of the main decay estimate in [8] and the localization method applied in [17], we show that for the scalar equation $f'(u)$ belongs to the SBV space, and for system of conservation laws the i -th component of $D_x \lambda_i(u)$ has no Cantor part, where λ_i is the i -th eigenvalue of the matrix Df .

Keywords: hyperbolic conservation laws, SBV-like regular, wave-front tracking
MS Classification 2010: 35L65, 35D30

1. Introduction

The study of the regularity of solutions to a general hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \quad u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega \subset \mathbb{R}^N \quad (1)$$

with initial data

$$u(t = 0) = u_0 \in \text{BV}(\mathbb{R}, \Omega) \quad (2)$$

is an important topic in the study of hyperbolic equations. In particular, recently there have been interesting advances in the analysis of the structure of the measure derivative $D_x u(t)$ of BV solution to genuinely nonlinear scalar equations and hyperbolic systems. The results obtained are that, in addition

to the BV bounds, the solution enjoys the strong regularity property that no Cantor part in the space derivative of $u(t)$ appears out of a countable set of times [1, 8, 17]: the fact that the measure $D_x u(t)$ has only absolutely continuous and jump part yields by definition that $u(t) \in \text{SBV}(\mathbb{R})$.

The main idea of the proof is to find a positive bounded functional, which is monotonically decreasing in time: then one shows that at each time a Cantor part appears the functional has a jump downward, and hence one concludes that the SBV regularity of u holds outside a countable set of times.

This paper concerns the extension of the results of [8] to the case where the system is only strictly hyperbolic, i.e. no assumption on the nonlinear structure of the eigenvalues λ_i of Df is done. Clearly, by just considering a linearly degenerate eigenvalue, it is fairly easy to see that the solution u itself cannot be in the SBV function space, so the regularity concerns some nonlinear function of u .

We state the main theorems of this paper: in the following a BV function on \mathbb{R} will be considered defined everywhere by taking the right continuous representative.

In the scalar case, one has

THEOREM 1.1. *Suppose that $u \in \text{BV}(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of the scalar conservation law (3). Then there exists a countable set $S \subset \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+ \setminus S$ the following holds:*

$$f'(u(t, \cdot)) \in \text{SBV}_{\text{loc}}(\mathbb{R}).$$

After introducing the definition of i -th component of $D_x \lambda_i(u)$ (see (16)), we have the SBV-like regularity for the system case.

THEOREM 1.2. *Let u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system (6) with small BV norm. Then there exists an at most countable set $S \subset \mathbb{R}^+$ such that i -th component of $D_x \lambda_i(u(t, \cdot))$ has no Cantor part for every $t \in \mathbb{R}^+ \setminus S$ and $i \in \{1, 2, \dots, N\}$.*

Since in the genuinely nonlinear case $u \mapsto \lambda_i(u)$ is invertible along the i -th admissible curves $T_s^i[u]$ (see Theorem 3.2 for the definition), it follows that Theorem 4.1 is an extension of the results contained in [8] (and Theorem 1.1 is an extension of the results contained in [17] when the source is 0). The example contained in Remark 7.2 shows that the results are sharp.

The main point of the paper is the fact that the wave-front tracking approximation for the waves of a genuinely nonlinear family does not essentially differ from the wave-front approximations of genuinely nonlinear systems: in other words, the wave pattern of a genuinely nonlinear characteristic family for a (approximate) solution in a general hyperbolic system has the same structure as if all characteristic families are genuinely nonlinear. Thus the analysis carried out in [8] holds also in this case.

The proof of the above two theorems is done as follows. To introduce the argument in the easiest setting, in Section 2, we give a proof for the SBV regularity of the characteristic speed for the general scalar conservation laws. The proof is just a slight modification of the proof of [17, Theorem 1.1].

As one sees in the proof of Theorem 1.1, the main tool is to obtain the SBV regularity when only one characteristic field is genuinely nonlinear (Corollary 4.2). By inspection, the analysis of [8] relies on the wave-front tracking approximation of [9], which assumes that all characteristic fields are genuinely nonlinear or linearly degenerate. Thus we devote Sections 3.2, Section 5.1 to introduce the wave-front tracking approximation for general systems [3].

The focus of Section 5.2 is the observation that the convergence and regularity estimates of [Theorem 10.4][9] still holds for the i -th component of u_x , under the only assumption that the i -th characteristic field is genuinely nonlinear: these estimates are needed in order to define the i -th (ϵ_1, ϵ_0) -shocks and to pass to the limit the estimates concerning the interaction, cancellation and jump measures. The latter is responsible for the functional controlling the SBV regularity, Theorem 4.1.

After these estimates, for completeness we repeat the proof of the decay of negative waves in Section 6.2. Finally we show how to adapt the strategy of the scalar case in Section 7.

2. The scalar case

In this section, we restrict our attention to the scalar conservation laws and motivate our general strategy with this comparatively simpler situation. Let us consider the entropy solution to the hyperbolic conservation law in one space dimension

$$\begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega \subset \mathbb{R}, f \in C^2(\Omega, \mathbb{R}), \\ u|_{t=0} = u_0 & u_0 \in \text{BV}(\mathbb{R}, \Omega). \end{cases} \quad (3)$$

In [17], it is proved the SBV regularity result for the convex or concave flux case.

LEMMA 2.1. [17] *Suppose $f \in C^2(\mathbb{R})$ and $|f''(u)| > 0$. Let $u \in L^\infty(\mathbb{R})$ be an entropy solution of the scalar conservation law (3). Then there exists a countable set $S \subset \mathbb{R}$ such that for every $\tau \in \mathbb{R}^+ \setminus S$ the following holds:*

$$u(\tau, \cdot) \in \text{SBV}_{\text{loc}}(\mathbb{R}).$$

Further, by Volpert's Chain Rule (see [2, Theorem 3.99]), it follows that $f'(u(\tau, \cdot)) \in \text{SBV}_{\text{loc}}(\mathbb{R})$ for $\tau \in \mathbb{R}^+ \setminus S$: actually, since $f'' \neq 0$, the two conditions $f'(u(\tau, \cdot)) \in \text{SBV}_{\text{loc}}(\mathbb{R})$ and $u(\tau, \cdot) \in \text{SBV}_{\text{loc}}(\mathbb{R})$ are equivalent.

Following the same argument together with the analysis in [17], we can get the SBV regularity of the slope of characteristics for the scalar conservation law with general flux as stated in Theorem 1.1.

Proof of Theorem 1.1. Recall that if $u \in \text{BV}(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution, then by the theory of entropy solutions, it follows that $u(\tau, \cdot) \in \text{BV}(\mathbb{R})$ is well defined for every $\tau \in \mathbb{R}^+$.

Define the sets

$$\begin{aligned} J_\tau &:= \{x \in \mathbb{R} : u(\tau, x-) \neq u(\tau, x+)\}, \\ F_\tau &:= \{x \in \mathbb{R} : f''(u(\tau, x)) = 0\}, \\ C &:= \{(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R} : \xi \in J_\tau \cup F_\tau\}. \end{aligned}$$

Set also $C_\tau := J_\tau \cup F_\tau$ as the τ -section of C .

Since the Cantor part $D^c u(\tau, \cdot)$ of $Du(\tau, \cdot)$ and the jump part $D^j u(\tau, \cdot)$ of $Du(\tau, \cdot)$ are mutually singular, then $|D^c u(\tau, \cdot)|(J_\tau) = 0$. Using the fact that $f''(u(\tau, \cdot)) = 0$ on F_τ , by Volpert's Chain Rule one obtains

$$\begin{aligned} |D^c f'(u(\tau, \cdot))|(C_\tau) &\leq |D^c f'(u(\tau, \cdot))|(J_\tau) + |D^c f'(u(\tau, \cdot))|(F_\tau) \\ &= |f''(u(\tau, \cdot))D^c u(\tau, \cdot)|(J_\tau) + |f''(u(\tau, \cdot))D^c u(\tau, \cdot)|(F_\tau) = 0. \end{aligned}$$

Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \setminus C$. Using the finite speed of propagation and the maximum principle for entropy solutions and the fact that $u(t_0, x)$ is continuous at x_0 by the definition of C , it is possible to find a triangle of the form

$$T(t_0, x_0) := \left\{ (t, x) : |x - x_0| < b_0 - \bar{\lambda}(t - t_0), 0 < t - t_0 < b_0/\bar{\lambda} \right\} \quad (4)$$

such that $|f''(u(t, x))| \geq c_0 > 0$, for any $(t, x) \in T(t_0, x_0)$. Here c_0 depends on (t_0, x_0) and $\bar{\lambda}$ is the maximal speed of propagation, which depends only on the L^∞ -bound of u_{t_0} (and hence only depends on the L^∞ -bound of u by maximal principle).

In particular, in $T(t_0, x_0)$ the solution u of (3) coincides with the solution of the following problem

$$\begin{cases} w_t + f(w)_x = 0, \\ w(t_0, x) = \begin{cases} u(t_0, x) & |x - x_0| < b_0, \\ \frac{1}{2b_0} \int_{x_0 - b_0}^{x_0 + b_0} u(t_0, y) dy & |x - x_0| \geq b_0. \end{cases} \end{cases}$$

By Lemma 2.1, $w(t, \cdot)$ is SBV regular for any $t > t_0$ out of a countable set of times $S(t_0, x_0)$. Write $T_\tau(t_0, x_0) := T(t_0, x_0) \cap \{t = \tau\}$, thus $u(\tau, \cdot)|_{T_\tau(t_0, x_0)}$ and $f'(u(\tau, \cdot))|_{T_\tau(t_0, x_0)}$ are SBV for $\tau \in]t_0, t_0 + b/\bar{\lambda}[\setminus S(t_0, x_0)$.

Let B be the set of all points of $\mathbb{R}^+ \times \mathbb{R} \setminus C$ which are contained in at least one of these triangles. (Notice that $T(t_0, x_0)$ is an open set and does not contain

the point (t_0, x_0) .) Let $\{T(t_i, x_i)\}_{i \in \mathbb{N}}$ be a countable subfamily of the triangles covering B . From the previous observation on the function $u \llcorner_{T(t_i, x_i)}$, the set

$$S_i := \{ \tau : u(\tau, \cdot) \llcorner_{T_\tau(t_i, x_i)} \notin \text{SBV}(T_\tau(t_i, x_i)) \}$$

is at most countable.

Let $C' := \mathbb{R}^+ \times \mathbb{R} \setminus (B \cup C)$ and $S_{C'} := \{ \tau \in \mathbb{R}^+ : \{t = \tau\} \cap C' \neq \emptyset \}$. It is obvious that for every $t' \in \mathbb{R}^+ \setminus S_{C'}$, $x' \in \mathbb{R}$, either there is a triangle $T \in \{T(t_i, x_i)\}_{i \in \mathbb{N}}$ such that $(t', x') \in T$ and $u(t, \cdot) \llcorner_T$ is SBV function out of countable many times or $(t', x') \in C$.

We claim that the set $S_{C'}$ is at most countable. Indeed, it is enough to prove that the set $S_K := \{ \tau \in \mathbb{R}^+ : \{t = \tau\} \cap C' \cap K \neq \emptyset \}$ is at most countable for every compact set $K \subset \mathbb{R}^+ \times \mathbb{R}$ when the triangles $T(t', x')$ have a base of fixed length for every $(t', x') \in C'$: it is fairly simple to see that in this case the set S_K is finite since (t', x') can not be contained in any other $T(t'', x'')$ for $t' \neq t''$ and $(t'', x'') \in C'$.

For any τ not in the countable set

$$S_{C'} \cup \bigcup_{i \in \mathbb{N}} S_i,$$

one obtains the following inequality:

$$\begin{aligned} |D^c f'(u(\tau, \cdot))(\mathbb{R})| &\leq |D^c f'(u(\tau, \cdot))| \left(\bigcup_{i \in \mathbb{N}} T_\tau(t_i, x_i) \right) \\ &\quad + |D^c f'(u(\tau, \cdot))|(C_\tau) = 0. \end{aligned} \tag{5}$$

This concludes the proof. □

By a standard argument in the theory of BV functions, we have the following result.

COROLLARY 2.2. *Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be an entropy solution of the scalar conservation law (3). Then $f'(u) \in \text{SBV}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$.*

The difference is that now the function $f'(u)$ is considered as a function of two variable.

Proof. The starting point is that up to a countable set of times, $Df'(u(t, \cdot))$ has no Cantor part (Theorem 1.1). From the slicing theory of BV function ([2, Theorem 3.107-108]), we know that the Cantor part of the 2-dimensional measure $D_x f'(u)$ is the integral with respect of t of the Cantor part of $Df'(u(t, \cdot))$. This concludes that $D_x f'(u)$ has no Cantor part, i.e. $D_x^c f'(u) = 0$.

By combining Volpert’s Chain Rule and the conservation law (3), one has

$$D_t^c u = -f'(u)D_x^c u.$$

Using Volpert’s rule once again, one obtains

$$D_t^c f'(u) = -f''(u)D_t^c u = -f''(u)f'(u)D_x^c u = -f'(u)D_x^c f'(u) = 0,$$

which concludes that also $D_t f(u)$ has no Cantor part. □

REMARK 2.3. *In [17], it is proved that if f in (3) has only countable many inflection points. i.e. the set*

$$\{u \in \Omega : f''(u) \neq 0\}$$

is at most countable, then the entropy solution of (3) is SBV regular. It is easy to see that for general hyperbolic scalar conservation laws $f \in C^2$ is not enough to obtain the SBV regularity. In fact, we can consider $f' \equiv \text{constant}$, which means (3) degenerates into a linear equation. Then the entropy solution u is not SBV regular unless the initial data u_0 is a SBV function.

3. Notations and settings for general systems

Throughout the rest of the paper, the symbol $\mathcal{O}(1)$ always denotes a quantity uniformly bounded by a constant depending only on the system (1).

3.1. Preliminary notation

Consider the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega \subset \mathbb{R}^N, f \in C^2(\Omega, \mathbb{R}), \\ u|_{t=0} = u_0 & u_0 \in \text{BV}(\mathbb{R}, \Omega). \end{cases} \tag{6}$$

The only assumption is strict hyperbolicity in Ω : the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobi matrix $A(u) = Df(u)$ satisfy

$$\lambda_1(u) < \dots < \lambda_N(u), \quad u \in \Omega.$$

Furthermore, as we only consider the solutions with small total variation, it is not restrictive to assume that Ω is bounded and there exist constants $\{\check{\lambda}_j\}_{j=0}^N$, such that

$$\check{\lambda}_{k-1} < \lambda_k(u) < \check{\lambda}_k, \quad \forall u \in \Omega, k = 1, \dots, N. \tag{7}$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u , such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i = 1, \dots, N. \tag{8}$$

DEFINITION 3.1. For $i = 1, \dots, N$, we say that the i -th characteristic field (or i -th family) is genuinely nonlinear if

$$\nabla \lambda_i(u) \cdot r_i(u) \neq 0 \quad \text{for all } u \in \Omega,$$

and we say that the i -th characteristic field (or i -th family) is linearly degenerate if instead

$$\nabla \lambda_i(u) \cdot r_i(u) = 0 \quad \text{for all } u \in \Omega.$$

In the following, if the i -th characteristic field is genuinely nonlinear, instead of (8) we normalize $r_i(u)$ such that

$$\nabla \lambda_i(u) \cdot r_i(u) \equiv 1. \tag{9}$$

In [7], it is proved that if the total variation of u_0 is sufficiently small, the solutions of the viscous parabolic approximation equations

$$\begin{cases} u_t + f(u)_x = \epsilon u_{xx}, \\ u(0, x) = u_0(x), \end{cases}$$

are uniformly bounded, and the limit of u^ϵ as $\epsilon \rightarrow 0$ is called *vanishing viscosity solution* of (6) and it is a BV function.

3.2. Construction of solutions to the Riemann problem

The Riemann problem is the Cauchy problem (6) with piecewise constant initial data of the form

$$u_0 = \begin{cases} u^L & x < 0, \\ u^R & x > 0. \end{cases} \tag{10}$$

The solution to this problem is the key ingredient for building the front-tracking approximate solution: the basic step is the construction of the admissible *elementary curve* of the k -th family for any given left state u^L .

A working definition of admissible elementary curves can be given by means of the following theorem.

THEOREM 3.2 ([4, 7]). For every $u \in \Omega$ there exist

1. N Lipschitz continuous curves $s \mapsto T_s^k[u] \in \Omega$, $k = 1, \dots, N$, satisfying $\lim_{s \rightarrow 0} \frac{d}{ds} T_s^k[u] = r_k(u)$,
2. N Lipschitz functions $(s, \tau) \mapsto \sigma_s^k[u](\tau)$, with $0 \leq \tau \leq s$, $k = 1, \dots, N$, satisfying $\tau \mapsto \sigma_s^k[u](\tau)$ increasing and $\sigma_0^k[u](0) = \lambda_k(u)$

with the following properties.

When $u^L \in \Omega$, $u^R = T_s^k[u^L]$, for some s sufficiently small, the unique vanishing viscosity solution of the Riemann problem (6)-(10) is defined a.e. by

$$u(t, x) := \begin{cases} u^L & x/t < \sigma_s^k[u^L](0), \\ T_\tau^k[u^L] & x/t = \sigma_s^k[u^L](\tau), \tau \in \mathcal{I}, \\ u^R & x/t > \sigma_s^k[u^L](s). \end{cases}$$

where $\mathcal{I} := \{\tau \in [0, s] : \sigma_s^k[u^L](\tau) \neq \sigma_s^k[u^L](\tau') \text{ for all } \tau' \neq \tau\}$.

REMARK 3.3. If i -th family is genuinely nonlinear, then the Lipschitz curve $T_s^i[\bar{u}]$ can be written as

$$T_s^i[\bar{u}] = \begin{cases} R_i[\bar{u}](s) & s \geq 0, \\ S_i[\bar{u}](s) & s < 0, \end{cases}$$

where $R_i[\bar{u}]$, $S_i[\bar{u}]$ are respectively the rarefaction curve and the Rankine-Hugoniot curve of the i -th family with any given point \bar{u} in Ω . Some certain elementary weak solution, called rarefaction waves and shock waves can be defined along the rarefaction curve and Rankine-Hugoniot curve, for example see [9]. The elementary curve $T_s^i[\bar{u}]$ is parametrized by

$$s = l_i(\bar{u}) \cdot (T_s^i[\bar{u}] - \bar{u}). \tag{11}$$

The vanishing viscosity solution [7] of a Riemann problem for (6) is obtained by constructing a Lipschitz continuous map

$$(s_1, \dots, s_N) \mapsto T_{s_N}^N \left[T_{s_{N-1}}^{N-1} \left[\dots \left[T_{s_1}^1 [u^L] \right] \right] \right] = u^R,$$

which is one to one from a neighborhood of the origin onto a neighborhood of u^L . Then we can uniquely determine intermediate states $u^L = \omega_0, \omega_1, \dots, \omega_N = u^R$, and the wave sizes s_1, s_2, \dots, s_N such that

$$\omega_k = T_{s_k}^k[\omega_{k-1}], \quad k = 1, \dots, N,$$

provided that $|u^L - u^R|$ is sufficiently small.

By Theorem 3.2, each Riemann problem with initial datum

$$u_0 = \begin{cases} \omega_{k-1} & x < 0, \\ \omega_k & x > 0, \end{cases} \tag{12}$$

admits a vanishing viscosity solution u_k , containing a sequence of rarefactions, shocks and discontinuities of the k -th family: we call u_k the k -th elementary

composite wave. Therefore, under the strict hyperbolicity assumption, the general solution of the Riemann problem with the initial data (10) is obtained by piecing together the vanishing viscosity solutions of the elementary Riemann problems given by (6)-(12).

Indeed, from the strict hyperbolicity assumption (7), the speed of each elementary k -th wave in the solution u_k is inside the interval $[\check{\lambda}_{k-1}, \check{\lambda}_k]$ if $s \ll 1$, so that the solution of the general Riemann problem (6)-(10) is then given by

$$u(t, x) = \begin{cases} u^L & x/t < \check{\lambda}_0 \\ u_k(t, x) & \check{\lambda}_{k-1} < x/t < \check{\lambda}_k, k = 1, \dots, N, \\ u^R & x/t > \check{\lambda}_N. \end{cases} \tag{13}$$

REMARK 3.4. *If the characteristic fields are either genuinely nonlinear or linearly degenerate, the admissible solution of Riemann problem (6)-(10) consists of N family of waves. Each family contains either only one shock, one rarefaction wave or one contact discontinuity. However, the general solution of a Riemann problem provided above may contain a countable number of rarefaction waves, shock waves and contact discontinuities.*

3.3. Cantor part of the derivative of characteristic for i -th waves

Recalling the solution (13) to the Riemann problem (6)-(10), let $\tilde{\lambda}_i(u^L, u^R)$ denote the i -th eigenvalue of the average matrix

$$A(u^L, u^R) = \int_0^1 A(\theta u^L + (1 - \theta)u^R)d\theta, \tag{14}$$

and $\tilde{l}_i(u^L, u^R), \tilde{r}_i(u^L, u^R)$ are the corresponding left and right eigenvector satisfying $\tilde{l}_i \cdot \tilde{r}_i = \delta_{ij}$ and $|\tilde{r}_j| \equiv 1$, for every $i, j \in \{1, \dots, N\}$. Define thus

$$\tilde{\lambda}_i(t, x) = \tilde{\lambda}_i(u(t, x-), u(t, x+)), \tag{15a}$$

$$\tilde{r}_i(t, x) = \tilde{r}_i(u(t, x-), u(t, x+)), \tag{15b}$$

$$\tilde{l}_i(t, x) = \tilde{l}_i(u(t, x-), u(t, x+)). \tag{15c}$$

Since the \tilde{r}_i, \tilde{l}_i have directions close to r_i, l_i , one can decompose $D_x u$ into the sum of N measures:

$$D_x u = \sum_{k=1}^N v_k \tilde{r}_k.$$

where $v_i = \tilde{l}_i \cdot D_x u$ is a scalar valued measure which we call as *i -th wave measure* [9].

In the same way we can decompose the a.c. part $D_x^{\text{ac}}u$, the Cantor part $D_x^{\text{c}}u$ and the jump part $D_x^{\text{jump}}u$ of $D_x u$ as

$$D_x^{\text{ac}}u = \sum_{k=1}^N v_k^{\text{ac}} \tilde{r}_k, \quad D_x^{\text{c}}u = \sum_{k=1}^N v_k^{\text{c}} \tilde{r}_k, \quad D_x^{\text{jump}}u = \sum_{k=1}^N v_k^{\text{jump}} \tilde{r}_k.$$

We call v_i^{c} the Cantor part of v_i and denote by

$$v_i^{\text{cont}} := v_i^{\text{c}} + v_i^{\text{ac}} = \tilde{l}_i \cdot (D_x^{\text{c}}u + D_x^{\text{ac}}u)$$

the continuous part of v_i . According to Volpert’s Chain Rule

$$D_x \lambda_i(u) = \nabla \lambda_i(u)(D_x^{\text{ac}}u + D_x^{\text{c}}u) + [\lambda_i(u^+) - \lambda_i(u^-)]\delta_x,$$

and then

$$D_x^{\text{c}}\lambda_i(u) = \nabla \lambda_i \cdot D_x^{\text{c}}u = \sum_k (\nabla \lambda_i \cdot \tilde{r}_k) v_k^{\text{c}}.$$

We define the i -th component of $D_x \lambda_i(u)$ as

$$[D_x \lambda_i(u)]_i := (\nabla \lambda_i \cdot \tilde{r}_i) v_i^{\text{cont}} + [\lambda_i(u^+) - \lambda_i(u^-)] \frac{|v_i^{\text{jump}}(x)|}{\sum_k |v_k^{\text{jump}}(x)|}, \tag{16}$$

and the Cantor part of i -th component of $D_x \lambda_i(u)$ to be

$$[D_x^{\text{c}}\lambda_i(u)]_i := (\nabla \lambda_i \cdot \tilde{r}_i) v_i^{\text{c}}. \tag{17}$$

4. Main SBV regularity argument

Following [8], the key idea to obtain SBV-like regularity for v_i is to prove a decay estimate for the continuous part of v_i . We state here the main estimate of our paper.

THEOREM 4.1. *Consider the general strictly hyperbolic system (6), and suppose that the i -th characteristic field is genuinely nonlinear. Then there exists a finite, non-negative Radon measure μ_i^{ICJ} on $\mathbb{R}^+ \times \mathbb{R}$ such that for $t > \tau > 0$*

$$|v_i^{\text{cont}}(t)|(B) \leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}(B)}{\tau} + \mu_i^{\text{ICJ}}([t - \tau, t + \tau] \times \mathbb{R}) \right\} \tag{18}$$

for all Borel subset B of \mathbb{R} .

Different from [8], we assume only one characteristic field to be genuinely nonlinear and no other requirement on the other characteristic fields. Once Theorem 4.1 is proved, then the SBV argument develops as follows [8].

Suppose at time $t = s$, $v_i(s)$ has a Cantor part. Then there exists a \mathcal{L}^1 -negligible Borel set K with $v_i^{\text{cont}}(s)(K) > 0$ and $D^{\text{jump}}v_i(s)(K) = 0$. Then for all $s > \tau > 0$,

$$0 < |v_i(s)|(K) = |v_i^{\text{cont}}(s)|(K) \leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}^1(K)}{\tau} + \mu_i^{\text{ICJ}}([s - \tau, s + \tau] \times \mathbb{R}) \right\}.$$

Since $\mathcal{L}^1(K) = 0$, we can let $\tau \rightarrow 0$, and deduce that $\mu_i^{\text{ICJ}}(\{s\} \times \mathbb{R}) > 0$. This shows that the Cantor part appears at most countably many times because μ_i^{ICJ} is finite.

Then, we can have the following result which generalizes [8, Corollary 3.2] to the case when only one characteristic field is genuinely nonlinear and no assumption is made on the others.

COROLLARY 4.2. *Let u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system (6), and assume that the i -th characteristic field is genuinely nonlinear. Then $v_i(t)$ has no Cantor part out of a countable set of times.*

As we see in the scalar case, by proving the SBV regularity of the solution under the genuinely nonlinearity assumption of one characteristic field, we can deduce a kind of SBV regularity of the characteristic speed for general systems.

Unlike the scalar case, we do not have the maximum principle to guarantee the small variation of u in the triangle $T(t_0, x_0)$ defined in (4). However, in the system case, we have the following estimates for the vanishing viscosity solutions.

For $a < b$ and $\tau \geq 0$, we denote by $\text{Tot.Var.}\{u(\tau);]a, b[\}$ the total variation of $u(\tau)$ over the open interval $]a, b[$. Moreover, consider the triangle

$$\Delta_{a,b}^{\tau,\eta} := \left\{ (t, x) : \tau < t < (b - a)/2\eta, a + \eta t < x < b - \eta t \right\}.$$

The oscillation of u over $\Delta_{a,b}^{\tau,\eta}$ will be denoted by

$$\text{Osc.}\{u; \Delta_{a,b}^{\tau,\eta}\} := \sup \left\{ |u(t, x) - u(t', x')| : (t, x), (t', x') \in \Delta_{a,b}^{\tau,\eta} \right\}.$$

We have the following results.

THEOREM 4.3 (Tame Oscillation, [7]). *There exists $C' > 0$ and $\bar{\eta} > 0$ such that for every $a < b$ and $\tau \geq 0$, one has*

$$\text{Osc.}\{u; \Delta_{a,b}^{\tau,\bar{\eta}}\} \leq C' \cdot \text{Tot.Var.}\{u(\tau);]a, b[\}.$$

Adapting the proof of the scalar case, we can prove the main Theorem 1.2 of this paper: the proof of this theorem will be done in Section 7.

5. Review of wave-front tracking approximation for general system

To prove Theorem 4.1, we use the front tracking approximation in [3] which extends the one in [9] to the general systems. Since the construction is now standard, we only give a short overview about existence, compactness and convergence of the approximation, pointing to the properties needed in our argument: more precisely, we will only consider how one constructs the approximate wave pattern of the k -th genuinely nonlinear family (Section 5.1.2).

The main point is that, for general systems, the accurate/simplified/crude Riemann solvers for the k -th wave coincides with the approximate/simplified/crude Riemann solvers when all families are genuinely nonlinear (see below for the definition of accurate/simplified/crude Riemann solvers). This means that the wave pattern of the k -th genuinely nonlinear family will have the same structure as if all other families are genuinely nonlinear: by this, we mean that shock-shock interaction generates shocks, the jump in characteristic speed across k -th waves is proportional to their size, and one can thus use the k -component of the derivative of λ_k (16) to measure the total variation of v_k .

5.1. Description of the wave-front tracking approximation

The wave-front tracking approximation is an algorithm which produces piecewise constant approximate solutions to the Cauchy problem (6). Roughly speaking, we first choose a piecewise constant function u_0^ϵ which is a good approximation to the initial data u_0 such that

$$\text{Tot.Var.}\{u_0^\epsilon\} \leq \text{Tot.Var.}\{u_0\}, \quad \|u_0^\epsilon - u_0\|_{L^1} < \epsilon, \quad (19)$$

and u_0^ϵ only has finite jumps. Let $x_1 < \dots < x_m$ be the jump points of u_0^ϵ . For each $\alpha = 1, \dots, m$, we approximately solve the Riemann problem (see Section 3.2, just shifting the center from $(0, 0)$ to $(0, x_\alpha)$) with the initial data given by the jump $[u_0^\epsilon(x_\alpha-), [u_0^\epsilon(x_\alpha+)]$ by a function $w(t, x) = \phi(\frac{x-x_\alpha}{t-t_0})$ where ϕ is a piecewise constant function. The straight lines where the discontinuities are located are called *wave-fronts* (or just *fronts* for shortness). The wave-fronts can be prolonged until they interact with other fronts, then at the interaction point, the corresponding Riemann problem is approximately solved and several new fronts are generated forward. Then one tracks the wave-fronts until they interact with other wave-fronts, etc... In order to avoid the algorithm to produce infinite many wave-fronts in finite time, different kinds of approximate Riemann solvers should be introduced.

5.1.1. Approximate Riemann solver

Suppose at the point (t_1, x_1) a wave-front of size s' belonging to k' -th family interacts from the left with a wave-front of size s'' belonging to k'' -th family for some $k', k'' \in \{1, \dots, N\}$ such that $k' < k''$ and (see Section 3.2 for the definition of T_s^k)

$$u^M = T_{s'}^{k'}[u^L], \quad u^R = T_{s''}^{k''}[u^M].$$

Assuming that $|u^L - u^R|$ sufficiently small, at the interaction point, the Riemann problem with the initial data given by the jump $[u^L, u^R]$ will be solved by approximate Riemann solver. There are two kinds of approximate Riemann solvers defined for interactions between two physical wave-fronts.

- *Accurate Riemann Solver*: It replaces each elementary composite wave of the exact Riemann solution (refers to u_k in (13)) by an approximate elementary wave which is a finite collection of jumps traveling with a speed given by the average speed $\tilde{\lambda}_k$ (see (15a)), and the wave opening (i.e. the difference in speeds between any two consecutive fronts) is less than some small parameter ϵ controlling the accuracy of the approximation.
- *Simplified Riemann Solver*: It only generates approximate elementary waves belonging to k' -th and k'' -th families with corresponding size s' and s'' as the incoming ones if $k' \neq k''$, and approximate elementary waves of size $s' + s''$ belonging to k' -th family if $k' = k''$. The simplified Riemann solver collects the remaining new waves into a single *nonphysical front*, traveling with a constant speed $\hat{\lambda}$, strictly larger than all characteristic speed $\hat{\lambda}$. Therefore, usually the simplified Riemann solver generate less outgoing fronts after interaction than the accurate Riemann solver.

Since the simplified Riemann solver produces nonphysical wave-fronts and they can not interact with each other, one only needs an approximate Riemann solver defined for the interaction between, for example, a physical front of the k -th family with size s , connecting u^M, u^R and a nonphysical front (coming from the left) connecting the left value u^L and u^M traveling with speed $\hat{\lambda}$.

- *Crude Riemann Solver* generates a k -th front connecting u^L and $\tilde{u}^M = T_s^k[u^L]$ traveling with speed $\tilde{\lambda}_i$ and a nonphysical wave-front joining \tilde{u}^M and u^R , traveling with speed $\hat{\lambda}$. In the following, for simplicity, we just say that the non-physical fronts belong to the $(N + 1)$ -th characteristic field.

REMARK 5.1. *We can assume that at each time $t > 0$, at most one interaction takes place, involving exactly two incoming fronts, because we can slightly change the speed of one of the incoming fronts if more than two fronts meet at*

the same point. It is sufficient to require that the error vanishes when $\epsilon \rightarrow 0$. To simplify the analysis, we assume that the fronts satisfy the Rankine-Hugoniot conditions exactly.

5.1.2. The approximate Riemann solvers for genuinely nonlinear waves

If the k -th characteristic family is genuinely nonlinear, the elementary wave u_k is either a shock wave or a rarefaction wave. The key example of the accurate Riemann solver is thus to consider how these two solutions are approximated.

If k -th elementary wave u_k in (13) is just a single shock, for example

$$u_k = \begin{cases} u^L & x/t < \sigma, \\ u^R & x/t > \sigma, \end{cases}$$

where σ is the speed of shock wave, then the approximated k -th wave coincides the exact one (apart from the speed in case, see the above remark).

If u_k is a rarefaction wave of the k -th family connecting the left value u^L and the right value u^R , for example, if $u^R := T_s^k[u^L]$ and

$$u_k = \begin{cases} u^L & x/t < \lambda_k(u^L), \\ T_{s^*}^k[u^L] & x/t \in [\lambda_k(u^L), \lambda_k(u^R)], \quad x/t = \lambda_k(T_{s^*}^k[u^L]), \\ u^R & x/t > \lambda_k(u^R), \end{cases}$$

where $s^* \in [0, s]$. Then the approximation \tilde{u}_k is a rarefaction fan containing several rarefaction fronts. More precisely, we can choose real numbers $0 = s_0 < s_1 < \dots < s_n = s$, and define the points $w_i := T_{s_i}^k[u^L]$, $i = 0, \dots, n$, with the following properties,

$$w_{i+1} = T_{(s_{i+1}-s_i)}^k[w_i],$$

and the wave opening of consecutive wave-fronts are sufficiently small, i.e.

$$\sigma_s^k[u^L](s_{i+1}) - \sigma_s^k[u^L](s_i) \leq \epsilon, \quad \forall i = 0, \dots, n - 1.$$

where the function σ_s^k is defined in Theorem 3.2. We let the jump $[\omega_i, \omega_{i+1}]$ travel with the speed $\tilde{\sigma}_i := \tilde{\lambda}_k(\omega_i, \omega_{i+1})$ (15a), so that the rarefaction fan \tilde{u}_k becomes

$$\tilde{u}_k = \begin{cases} u^L & x/t < \tilde{\sigma}_1, \\ \omega_i & \tilde{\sigma}_i \leq x/t < \tilde{\sigma}_{i+1}, \quad i = 1, \dots, n - 1, \\ u^R & x/t \geq \tilde{\sigma}_n. \end{cases}$$

5.1.3. Interaction potential and BV estimates

Suppose two wave-fronts with size s' and s'' interact. In order to get the estimate on the difference between the size of the incoming waves and the size of the outgoing waves produced by the interaction, we need to define the *amount of interaction* $\mathcal{I}(s', s'')$ between s' and s'' .

When s' and s'' belong to different characteristic families (including $N+1$ -th family), set

$$\mathcal{I}(s', s'') = |s's''|. \tag{20}$$

If s', s'' belong to the same characteristic family, the definition of $\mathcal{I}(s', s'')$ is more complicated (see [3, Definition 3]). We just mention that if s', s'' are the sizes of two shocks which have the same sign, traveling with the speed σ' and σ'' respectively, then the amount of interaction takes the form

$$\mathcal{I}(s', s'') = |s's''||\sigma' - \sigma''|, \tag{21}$$

i.e. the product of the size of the waves times the difference of their speeds (of the order of the angle between the two shocks).

To control the amount of interaction, the following potential is introduced. At each time $t > 0$ when no interaction occurs, and $u(t, \cdot)$ has jumps at x_1, \dots, x_m , we denote by

$$\omega_1, \dots, \omega_m, \quad s_1, \dots, s_m, \quad i_1, \dots, i_m,$$

their left states, signed sizes and characteristic families, respectively: the sign of s_α is given by the respective orientation of $dT_s^k[u]/ds$ and r_k , if the jump at x_α belongs to the k -th family. The Total Variation of u will be computed as

$$V(t) = V(u(t)) := \sum_{\alpha} |s_{\alpha}|.$$

Following [4], we define the *Glimm wave interaction potential* as follows:

$$\begin{aligned} \mathcal{Q}(t) = \mathcal{Q}(u(t)) := & \sum_{\substack{i_{\alpha} > i_{\beta} \\ x_{\alpha} < x_{\beta}}} |s_{\alpha} s_{\beta}| \\ & + \frac{1}{4} \sum_{i_{\alpha} = i_{\beta} < N+1} \int_0^{|s_{\alpha}|} \int_0^{|s_{\beta}|} |\sigma_{s_{\beta}}^{i_{\beta}}[\omega_{\beta}](\tau'') - \sigma_{s_{\alpha}}^{i_{\alpha}}[\omega_{\alpha}](\tau')| d\tau' d\tau''. \end{aligned} \tag{22}$$

Denoting the time jumps of the total variation and the Glimm potential as

$$\Delta V(\tau) = V(\tau+) - V(\tau-), \quad \Delta \mathcal{Q}(\tau) = \mathcal{Q}(\tau+) - \mathcal{Q}(\tau-),$$

the fundamental estimates are the following ([3, Lemma 5]): in fact, when two wave-fronts with size s' , s'' interact,

$$\Delta \mathcal{Q}(\tau) = -\mathcal{O}(1)\mathcal{I}(s', s''), \quad (23a)$$

$$\Delta V(\tau) = \mathcal{O}(1)\mathcal{I}(s', s''). \quad (23b)$$

Thus one defines the *Glimm functional*

$$\Upsilon(t) := V(t) + C_0 \mathcal{Q}(t) \quad (24)$$

with C_0 suitable constant, so that Υ decreases at any interaction. Using this functional, one can prove that ϵ -approximate solutions exist and their total variations are uniformly bounded (see [3, Section 6.1]).

5.1.4. Construction of the approximate solutions and their convergence to exact solution

The construction starts at initial time $t = 0$ with a given $\epsilon > 0$, by taking u_0^ϵ as a suitable piecewise constant approximation of initial data u_0 , satisfying (19). At the jump points of u_0^ϵ , we locally solve the Riemann problem by accurate Riemann solver. The approximate solution u^ϵ then can be prolonged until a first time t_1 when two wave-fronts interact. Again we solve the Riemann problem at the interaction point by an approximate Riemann solver. Whenever the amount of interaction (see Section 5.1.3 for the definition) of the incoming waves is larger than some threshold parameter $\rho = \rho(\epsilon) > 0$, we shall adopt the accurate Riemann solver. Instead, in the case where the amount of interaction of the incoming waves is less than ρ , we shall adopt simplified Riemann solvers. And we will apply the crude Riemann solver if one of the incoming wave-front is non-physical front. One can show that the number of wave-fronts in approximate solution constructed by such algorithm remains finite for all times (see [3, Section 6.2]).

We call such approximate solutions *ϵ -approximate front tracking solutions*. At each time t when there is no interaction, the restriction $u^\epsilon(t)$ is a step function whose jumps are located along straight lines in the (t, x) -plane.

Let $\{\epsilon_\nu\}_{\nu=1}^\infty$ be a sequence of positive real numbers converging to zero. Consider a corresponding sequence of ϵ_ν -approximate front tracking solutions $u^\nu := u^{\epsilon_\nu}$ of (6): it is standard to show that the functions $t \mapsto u^\nu(t, \cdot)$ are uniformly Lipschitz continuous in L^1 norm, and the decay of the Glimm Functional yields that the solutions $u^\nu(t, \cdot)$ have uniformly bounded total variation. Then by Helly's theorem, u^ν converges up to a subsequence in $\mathbb{L}_{\text{loc}}^1(\mathbb{R}^+ \times \mathbb{R})$ to some function u , which is a weak solution of (6).

It can be shown that by the choice of the Riemann Solver in Theorem 3.2, the solution obtained by the front tracking approximation coincides with the unique vanishing viscosity solution [7]. Furthermore, there exists a closed domain $\mathcal{D} \subset L^1(\mathbb{R}, \Omega)$ and a unique distributional solution u , which is a Lipschitz semigroup $\mathcal{D} \times [0, +\infty[\rightarrow \mathcal{D}$ and which for piecewise constant initial data coincides, for a small time, with the solution of the Cauchy problem obtained piecing together the standard entropy solutions of the Riemann problems. Moreover, it lives in the space of BV functions.

For simplicity, the pointwise value of u is its L^1 representative such that the restriction map $t \mapsto u(t)$ is continuous from the right in L^1 and $x \mapsto u(t, x)$ is right continuous from the right.

5.1.5. Further estimates

To each u^ν , we define the *measure μ_ν^I of interaction* and the *measure μ_ν^{IC} of interaction and cancellation* concentrated on the set of interaction points as follows. If two wave-fronts belonging to the families $i, i' \in \{1, \dots, N + 1\}$ with size s', s'' interact at a point P , we define by

$$\begin{aligned} \mu_\nu^I(\{P\}) &:= \mathcal{I}(s', s''), \\ \mu_\nu^{IC}(\{P\}) &:= \mathcal{I}(s', s'') + \begin{cases} |s'| + |s''| - |s' + s''| & i = i', \\ 0 & i \neq i'. \end{cases} \end{aligned} \tag{25}$$

the measure of interaction and the measure of interaction-cancellation.

The wave size estimates ([3, Lemma 1]) yields balance principles for the wave size of approximate solution. More precisely, given a polygonal region Γ with edges transversal to the waves it encounters, denote by $W_{\nu, \text{in}}^{i\pm}, W_{\nu, \text{out}}^{i\pm}$ the positive (+) or negative (-) i -th waves in u^ν entering or exiting Γ , and let $W_{\nu, \text{in}}^i = W_{\nu, \text{in}}^{i+} - W_{\nu, \text{in}}^{i-}, W_{\nu, \text{out}}^i = W_{\nu, \text{out}}^{i+} - W_{\nu, \text{out}}^{i-}$. Then the measure of interaction and the measure of interaction-cancellation control the difference between the amount of exiting i -th waves and the amount of entering i -th waves w.r.t. the region as follows:

$$|W_{\nu, \text{out}}^i - W_{\nu, \text{in}}^i| \leq \mathcal{O}(1)\mu_\nu^I(\Gamma), \tag{26a}$$

$$|W_{\nu, \text{out}}^{i\pm} - W_{\nu, \text{in}}^{i\pm}| \leq \mathcal{O}(1)\mu_\nu^{IC}(\Gamma). \tag{26b}$$

The above estimates are fairly easy consequence of the interaction estimates (23) and the definition of μ_ν^I, μ_ν^{IC} .

By taking a subsequence and using the weak compactness of bounded measures, there exist measures μ^I and μ^{IC} on $\mathbb{R}^+ \times \mathbb{R}$ such that the following weak convergence holds:

$$\mu_\nu^I \rightharpoonup \mu^I, \quad \mu_\nu^{IC} \rightharpoonup \mu^{IC}. \tag{27}$$

5.2. Jump part of i -th waves

The derivative of u^ν is clearly concentrated on polygonal lines, being a piecewise constant function with discontinuities along lines. Suppose the i -th family is genuinely nonlinear. To select the wave fronts belonging to i -th family of u^ν converging to the jump part of u , we use the following definition.

DEFINITION 5.2 (Maximal (ϵ^0, ϵ^1) -shock front). [9] *A maximal (ϵ^0, ϵ^1) -shock front for the i -th family of an ϵ_ν -approximate front-tracking solution u^ν is any maximal (w.r.t. inclusion) polygonal line $(t, \gamma^\nu(t))$ in the (t, x) -plane, $t_0 \leq t \leq t_1$, satisfying:*

- (i) *the segments of γ^ν are i -shocks of u^ν with size $|s^\nu| \geq \epsilon^0$, and at least once $|s^\nu| \geq \epsilon^1$;*
- (ii) *the nodes are interaction points of u^ν ;*
- (iii) *it is on the left of any other polygonal line which it intersects and which have the above two properties.*

Let $M_{(\epsilon^0, \epsilon^1)}^{\nu, i}$ be the number of maximal (ϵ^0, ϵ^1) -shock front for the i -th family. Denote

$$\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i} : \left[t_{(\epsilon^0, \epsilon^1), m}^{\nu, i, -}, t_{(\epsilon^0, \epsilon^1), m}^{\nu, i, +} \right] \rightarrow \mathbb{R}, \quad m = 1, \dots, M_{(\epsilon^0, \epsilon^1)}^{\nu, i},$$

as the maximal (ϵ^0, ϵ^1) -shock fronts for the i -th family in u^ν . Up to a subsequence, we can assume that $M_{(\epsilon^0, \epsilon^1)}^{\nu, i} = \bar{M}_{(\epsilon^0, \epsilon^1)}^i$ is a constant independent of ν because the total variations of u^ν are bounded.

Consider the collection of all maximal (ϵ^0, ϵ^1) -shocks for the i -th family and define

$$\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i} = \bigcup_{m=1}^{\bar{M}_{(\epsilon^0, \epsilon^1)}^i} \text{Graph} \left(\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i} \right),$$

and let $\{\epsilon_k^0\}_{k \in \mathbb{N}}, \{\epsilon_k^1\}_{k \in \mathbb{N}}$ be two sequences satisfying $0 < 2^k \epsilon_k^0 \leq \epsilon_k^1 \searrow 0$.

Up to a diagonal argument and by a suitable labeling of the curves, one can assume that for each fixed k, m the Lipschitz curves $\gamma_{(\epsilon_k^0, \epsilon_k^1), m}^{\nu, i}$ converge uniformly to a Lipschitz curve $\gamma_{(\epsilon_k^0, \epsilon_k^1), m}^i$. Let

$$\mathcal{F}^i := \bigcup_{m, k} \text{Graph} \left(\gamma_{(\epsilon_k^0, \epsilon_k^1), m}^i \right).$$

denote the collection of all these limiting curves in u .

For fixed (ϵ^0, ϵ^1) , we write for shortness

$$\tilde{l}_i^\nu(t, x) := \tilde{l}_i(u^\nu(t, x-), u^\nu(t, x+)) \tag{28}$$

and define

$$v_{i,(\epsilon^0, \epsilon^1)}^{\nu, \text{jump}} := \tilde{l}_i^\nu \cdot u_{x^\perp \mathcal{I}_{(\epsilon^0, \epsilon^1)}^{\nu, i}}. \tag{29}$$

Following the same idea of the proof of [9, Theorem 10.4], the next lemma holds if only the i -th characteristic field is genuinely nonlinear.

LEMMA 5.3. *The jump part of v_i is concentrated on \mathcal{I}^i .*

Moreover there exists a countable set $\Theta \subset \mathbb{R}^+ \times \mathbb{R}$, such that for each point

$$P = (\tau, \xi) = (\tau, \gamma_m^i(\tau)) \notin \Theta$$

where i -th shock curve γ_m^i is approximated by the sequence of (ϵ^0, ϵ^1) -shock fronts $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ of the approximate solutions u^ν , the following holds

$$\lim_{r \rightarrow 0^+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x < \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}(t) \\ (t, x) \in B(P, r)}} |u^\nu(t, x) - u^-| \right) = 0, \tag{30a}$$

$$\lim_{r \rightarrow 0^+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x > \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}(t) \\ (t, x) \in B(P, r)}} |u^\nu(t, x) - u^+| \right) = 0. \tag{30b}$$

Moreover, we can choose a sequence $\{\nu_k\}_{k=1}^\infty$ such that

$$v_i^{\text{jump}} = \text{weak}^* - \lim_k \sum_{i=1}^N v_{i,(\epsilon_k^0, \epsilon_k^1)}^{\nu_k, \text{jump}}. \tag{31}$$

The key argument of the proof is that we can use the tools of the proof of [9, Theorem 10.4] because the wave structure of the i -th genuinely nonlinear family has the following properties:

1. the interaction among two shocks of the i -th family generates only one shock of the i -th family,
2. the strength of i -th waves can be measured by the jump of the i -th characteristic speed λ_i ,
3. the speed of i -th waves is very close to the average of the jump of λ_i across the discontinuity.

These properties are a direct consequence of the behavior of the approximate Riemann solvers on the i -th waves if the i -th family is genuinely nonlinear (Section 5.1.2).

Before proving the lemma, we recall some definitions which will be used in the proof.

DEFINITION 5.4 ([9], Definition 7.2). Let $\hat{\lambda}$ be a constant larger than the absolute value of all characteristic speed. We say a curve $x = y(t)$, $t \in [a, b]$ is space-like if

$$|y(t_2) - y(t_1)| > \hat{\lambda}(t_2 - t_1) \quad \text{for all } a < t_1 < t_2 < b.$$

We recall that a *minimal generalized i -characteristic* is an absolutely continuous curve starting from (t_0, x_0) satisfying the differential inclusion

$$x^\nu(t; t_0, x_0) := \min \left\{ \begin{array}{l} x^\nu(t) : x^\nu(t_0) = x_0, \\ \dot{x}^\nu(t) \in [\lambda_i(u^\nu(t, x(t)+), \lambda_i(u^\nu(t, x(t)-))] \end{array} \right\}$$

for a.e. $t \geq t_0$.

For any given $(T, \bar{x}) \in \mathbb{R}$, we consider the minimal (maximal) generalized i -characteristic through (T, \bar{x}) , defined as

$$\chi^{-(+)}(t) = \min(\max)\{\chi(t) : \chi \text{ is a generalized } i\text{-characteristic, } \chi(T) = \bar{x}\}.$$

From the properties of approximate solutions, we conclude that there is no wave-front of i -th family crossing χ^+ from the left or crossing χ^- from the right.

Sketch of the proof. Let Θ be the set defined by all jump points of the initial datum, the atoms of μ^{IC} (see (27)). For any point $P \in \mathcal{S}^i \setminus \Theta$, if (30a) or (30b) does not hold, then this means that the approximate solutions u^ν have some uniform oscillation: Indeed, if (30a) not true, there exist $P_\nu, Q_\nu \rightarrow P$ and P_ν, Q_ν on the left of $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$, $\overline{P_\nu Q_\nu}$ is space-like such that

$$u(P_\nu) \rightarrow u^-$$

and

$$|u_\nu(P_\nu) - u_\nu(Q_\nu)| \geq \epsilon,$$

for some constant $\epsilon > 0$. It is not restrictive to assume that the direction $\overline{P_\nu Q_\nu} \rightarrow \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ towards $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$. Let $\Lambda_k(\overline{P_\nu Q_\nu})$ be the total wave strength of fronts of k -th family which across the segment $\overline{P_\nu Q_\nu}$. Then, one has $\Lambda_j(\overline{P_\nu Q_\nu}) \geq c\epsilon$ for some $j \in \{1, \dots, d\}$ and some constant $c > 0$. We consider three cases.

- 1 $j > i$, we take the minimal forward generalized j -characteristic χ^+ through P_ν and maximal generalized j -characteristic χ^- through Q_ν .

If χ^+ and χ^- interact with each other at O_ν before hitting $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$, we consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+ and χ^- . Since no fronts can leave Γ_ν through χ^+ and χ^- . By (20) and (25), one obtains that there exists a constant $c_1 > 0$ such that $\mu_\nu^{IC}(\Gamma_\nu) \geq c_1 \epsilon^2$.

If χ^+ interacts with $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ at A_ν and χ^- interact $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ at B_ν , we consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+ , χ^- and $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$. Then

either there exists a constant $0 < c'_0 < 1$ such that $\mu_\nu^{IC}(\Gamma_\nu) > c'_0\epsilon$ or there exists a constant $0 < c''_0 < 1$ such that fronts with total strength larger than $c''_0\epsilon_0$ hit $\overline{A_\nu B_\nu}$. By (20) and the fact that each front on $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ has strength less than $-\epsilon_0$, we determine that $\mu_\nu^{IC}(\overline{\Gamma}_\nu) \geq c_0\epsilon\epsilon_0$ on the closure of Γ_ν .

Thus, let $B(P, r_\nu)$ be a ball with center at P containing Γ_ν with radius $r_\nu \rightarrow 0$ as $\nu \rightarrow 0$. This implies that $\mu^{IC}(\{P\}) > 0$ against the assumption $P \notin \Theta$.

- 2 $j < i$, we consider the minimal backward generalized j -characteristic through the point P_ν and the maximal backward generalized j -characteristic through the point Q_ν . Then by the similar argument for the case $j > i$, we get $\mu^{IC}(\{P\}) > 0$ against the assumptions.
- 3 $j = i$ and for any $j' \neq i$, $1 \leq j' \leq N$, $\Lambda_{j'}(\overline{P_\nu Q_\nu}) \rightarrow 0$ as $\nu \rightarrow \infty$. In this case, suppose that $\overline{P_\nu Q_\nu}$ intersects the curve $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ at B_ν . Because of genuine nonlinearity, the minimal generalized i -characteristic χ through P_ν will hit $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ if no previous large interactions or cancellations occur on $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$. We consider the triangle region Γ_ν bounded by the segment $P_\nu B_\nu$, the curve $\gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}$ and χ . Since no fronts of i th-family can exit from Γ_ν through χ , one obtains $\mu_\nu^{IC}(\Gamma_\nu)$ uniformly positive which contradicts the assumption $\mu^{IC}(P) = 0$.

Therefore, we conclude that (30a) is true. And (30b) is similar to prove.

For $P \notin \mathcal{T}^i \cup \Theta$, if $v_i^{\text{jump}}(P) > 0$, i.e. P is a jump point of u , by the similar argument of Step 8 in the proof of [9, Theorem 10.4], the waves present in the approximate solutions are canceled, and thus $\mu^{IC}(P) > 0$. It is impossible since $P \notin \Theta$. This concludes that v_i^{jump} is concentrated on \mathcal{T}^i , because by (30) the jumps in the approximate solutions are vanishing in a neighborhood of every $P \notin \mathcal{T}^i \cup \Theta$.

We are left with the proof of (31). At jump point $(t, \gamma_{(\epsilon^0, \epsilon^1), m}^i(t)) \in \mathcal{T}^i \setminus \Theta$, according to (30a), (30b), there exist a sequence $(t^\nu, \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}(t^\nu))$ such that

$$(t, \gamma_{(\epsilon^0, \epsilon^1), m}^i(t)) = \lim_{\nu \rightarrow \infty} (t^\nu, \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}(t^\nu)) \tag{32}$$

and its left and right values converges to the left and right values of the jump in $(t, \gamma_{(\epsilon^0, \epsilon^1), m}^i(t))$.

Since $f \in C^2$, by the definition (14) the matrix $A(u^L, u^R)$ depends continuously on the value (u^L, u^R) , and since its eigenvalues are uniformly separated the same continuity holds for its eigenvalues $\tilde{\lambda}_k(u^L, u^R)$, left eigenvectors $\tilde{l}_k(u^L, u^R)$ and right eigenvectors $\tilde{r}_k(u^L, u^R)$. Using the notation (15a) and (28),

one obtains

$$\tilde{l}_i(t, \gamma_{(\epsilon^0, \epsilon^1), m}^i(t)) = \lim_{\nu} \tilde{l}_i^{\nu}(t^{\nu}, \gamma_{(\epsilon^0, \epsilon^1), m}^{\nu, i}(t^{\nu})), \tag{33}$$

and similar limits hold for $\tilde{r}_i, \tilde{\lambda}_i$.

Up to a subsequence $\{\nu_k\}$, from the convergence of the graphs of $\mathcal{T}_{(\epsilon_k^0, \epsilon_k^1)}^{\nu_k, i}$ to \mathcal{T}^i and (30a), (30b), it is fairly easy to prove that

$$Du_{\perp \mathcal{T}^i} = \text{weak}^* - \lim_{k \rightarrow \infty} Du^{\nu_k}_{\perp \mathcal{T}_{(\epsilon_k^0, \epsilon_k^1)}^{\nu_k, i}}. \tag{34}$$

According to (29), (33) and (34), one concludes the weak convergence of $v_{i, (\epsilon_k^0, \epsilon_k^1)}^{\nu_k, \text{jump}}$ to v_i^{jump} . □

6. Proof of Theorem 4.1

6.1. Decay estimate for positive waves

The Glimm Functional for BV functions to general systems has been obtained in [4], and when u is piecewise constant, it reduced to (22): and we will write it as \mathcal{Q} also the formulation of the functional given in [4]. Moreover, for the same constant $C_0 > 0$ of the Glimm Functional $\Upsilon(t)$ (24), the sum $\text{Tot.Var.}(u) + C_0 \mathcal{Q}(u)$ is lower semi-continuous w.r.t the L^1 norm (see [9, Theorem 10.1]).

For any Radon measure μ , we denote $[\mu]^+$ and $[\mu]^-$ as the positive and negative part of μ according to Hahn-Jordan decomposition. The same proof of the decay of the Glimm Functional $\Upsilon(t)$ yields that for every finite union of the open intervals $J = I_1 \cup \dots \cup I_m$

$$[v_i]^{\pm}(J) + C_0 \mathcal{Q}(u) \leq \liminf_{\nu \rightarrow \infty} \{[v_i^{\nu}]^{\pm}(J) + C_0 \mathcal{Q}(u^{\nu})\}, \quad i = 1, \dots, n, \tag{35}$$

as $u^{\nu} \rightarrow u$ in L^1 .

In [9, 10] the authors prove a decay estimate for positive part of the i -th wave measure under the assumption that i -th characteristic field is genuinely nonlinear and the other characteristic fields are either genuinely nonlinear or linearly degenerate. In [12], a sharp decay estimate for positive waves is also given under the same assumptions as those in [9, 10]. By inspection, one can verify that the proof also works (with a little modification) under no assumptions on the nonlinearity on the other characteristic fields, since the essential requirements of strict hyperbolicity and of the controllability of interaction amounts by Glimm Potential still hold: the main variation is that one should replace the original Glimm Potential in [9] with the generalized one given in [4].

We thus state the following theorem, which is the analog of [9, Theorem 10.3].

THEOREM 6.1. *Let the system (1) be strictly hyperbolic and the i -th characteristic field be genuinely non-linear. Then there exists a constant C'' such that, for every $0 \leq s < t$ and every solution u with small total variation obtained as the limit of wave-front tracking approximation, the measure $[v_i(t)]^+$ satisfies*

$$[v_i(t)]^+(B) \leq C'' \left\{ \frac{\mathcal{L}^1}{t-s}(B) + [\mathcal{Q}(s) - \mathcal{Q}(t)] \right\} \tag{36}$$

for every B Borel set in \mathbb{R} .

The estimate (36) gives half of the bound (18). In this section, we always assume that the i -th family is genuinely nonlinear.

6.2. Decay estimate for negative waves

To simplify the notation, we omit the index (ϵ^0, ϵ^1) in $v_{i,(\epsilon^0, \epsilon^1)}^{\nu, \text{jump}}$ in the rest of the proof. In order to get the uniform estimate for the *continuous part* $v_i^{\nu, \text{cont}} := v_i^\nu - v_i^{\nu, \text{jump}}$, we need to consider the distributions

$$\mu_i^\nu := \partial_t v_i^\nu + \partial_x(\tilde{\lambda}_i^\nu v_i^\nu), \quad \mu_i^{\nu, \text{jump}} := \partial_t v_i^{\nu, \text{jump}} + \partial_x(\tilde{\lambda}_i^\nu v_i^{\nu, \text{jump}}).$$

6.2.1. Estimate for the source

Let $y_m : [\tau_m^-, \tau_m^+] \rightarrow \mathbb{R}$, $m = 1, \dots, L^\nu$, be time-parameterized segments whose graphs are the i -th wave-fronts of u^ν and define

$$u_m^L := u(t, y_m(t)-), \quad u_m^R = u(t, y_m(t)+), \quad t \in]\tau_m^-, \tau_m^+[.$$

For any test function $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ one obtains

$$-\int_{\mathbb{R}^+ \times \mathbb{R}} \phi d\mu_i^\nu = \sum_{m=1}^{L^\nu} [\phi(\tau_m^+, y_m(\tau_m^+)) - \phi(\tau_m^-, y_m(\tau_m^-))] \tilde{l}_i \cdot (u_m^R - u_m^L). \tag{37}$$

For any m , since the i -th characteristic field is genuinely nonlinear, one has

$$|\tilde{l}_i(u^L, u^R) - l_i(u^L)| = \mathcal{O}(1)|u_m^R - u_m^L|,$$

where $u_m^R = T_{s_i}^i[u_m^L]$ for some size s_i . Then it follows from (11) that

$$s_i \cong \tilde{l}_i \cdot (u_m^R - u_m^L). \tag{38}$$

Let $\{(t_k, x_k)\}_k$ be the collection of points where the i -th fronts interact. The computation (37) yields that μ_i^ν concentrates on the interaction points, i.e.

$$\mu_i^\nu = \sum_k p_k \delta_{(t_k, x_k)},$$

where p_k is the difference between the strength of the i -th waves leaving at (t_k, x_k) and the i -th waves arriving at (t_k, x_k) . We estimate the quantity p_k depending on the type of interaction:

Since in [8], it is proved that the total size of nonphysical wave-fronts are of the same order of ϵ_ν , when decomposing u_x^ν , we only consider the physical fronts. If at (t_k, x_k) , two physical fronts with i -th component size s'_i, s''_i interact and generate an i -th wave or a rarefaction fan with total size $s_i = \sum_m s_i^m$, from (37) and (38), one has

$$p_k \cong s_i - s'_i - s''_i. \tag{39}$$

Notice that s' or s'' or both may vanish in (39) if one of incoming physical fronts does not belong to the i -th family.

According to the estimate in [3, Lemma 1], the difference of sizes between the incoming and outgoing waves of the same family is controlled by the amount of interaction (see Section 5.1.3), so that one concludes

$$|\mu_i^\nu|(\{(t_k, x_k)\}) \leq \mathcal{O}(1)\mathcal{I}(s_i, s'_i)$$

and thus

$$|\mu_i^\nu|(\{t_k\} \times \mathbb{R}) \leq \mathcal{O}(1)\{\Upsilon^\nu(t_k^-) - \Upsilon^\nu(t_k^+)\}.$$

This yields

$$|\mu_i^\nu|(\mathbb{R}^+ \times \mathbb{R}) \leq \mathcal{O}(1)\Upsilon^\nu(0),$$

i.e. $|\mu_i^\nu|$ is a finite Radon measure.

6.2.2. Estimate for the jump part

Let $\gamma_m^i : [\tau_m^-, \tau_m^+] \rightarrow \mathbb{R}$, $m = 1, \dots, \bar{M}_{(\epsilon^0, \epsilon^1)}^i$, be the curves whose graphs are the segments supporting the fronts of u^ν belonging to $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$, and write

$$u_m^L := u(t, \gamma_m^i(t) -), \quad u_m^R := u(t, \gamma_m^i(t) +), \quad t \in]\tau_m^-, \tau_m^+].$$

For any test function $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ by direct computation one has as in (37)

$$-\int_{\mathbb{R}^+ \times \mathbb{R}} \phi d\mu_i^{\nu, \text{jump}} = \sum_{m=1}^{\bar{M}_{(\epsilon^0, \epsilon^1)}^i} [\phi(\tau_m^+, y_m(\tau_m^+)) - \phi(\tau_m^-, y_m(\tau_m^-))] \tilde{l}_i \cdot (u^R - u^L),$$

which yields

$$\mu_i^{\nu, \text{jump}} = \sum_k q_k \delta_{(t_k, x_k)},$$

where (τ_k, x_k) are the nodes of the jumps in $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$ and the quantities q_k can be computed as follows: if the i -th incoming waves have sizes s' and s'' , and the outgoing i -th shock has size s , then (see [8])

$$q_k \cong \begin{cases} -s' & (t_k, x_k) \text{ terminal point of a front not merging} \\ & \text{into another front,} \\ s & (t_k, x_k) \text{ initial point of a maximal front,} \\ s - s' - s'' & (t_k, x_k) \text{ is a triple point of } \mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}, \\ s - s' & (t_k, x_k) \text{ interaction point of a front with waves} \\ & \text{not belonging to } \mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}. \end{cases} \quad (40)$$

In fact, since $s \leq 0$ on shocks the second case of (40) implies $q_k \leq 0$. For the triple point, one has that

$$q_k \leq \mu_\nu^{\text{IC}}(\tau_k, x_k).$$

When a shock front in $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$ interacts with a front not belonging to $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$, there are three situations:

- It interacts with a rarefaction front of i -th family, then one has by the interaction estimates

$$q_k \leq \mu_\nu^{\text{IC}}(\tau_k, x_k).$$

- It interacts with a front of different family, then also one gets

$$q_k \leq \mu_\nu^{\text{I}}(\tau_k, x_k).$$

- It interacts with a shock of i -th family which does not belong to $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$, then

$$q_k \leq 0.$$

Suppose now that (τ_k, x_k) is a terminal point of an (ϵ^0, ϵ^1) -shock front γ_m . By the definition of (ϵ^0, ϵ^1) -shock, for some $t \leq \tau_k$ the shock front γ_m has size $s_0 \leq -\epsilon^1$, and at (τ_k, x_k) the size s_1 of the outgoing i -th front must be not less than $-\epsilon^0$ as a result of interaction between two wave-fronts belonging to different family or cancellation between two wave-fronts belonging to the same family along γ_k . Hence we obtain

$$\epsilon^1 - \epsilon^0 \leq |s_0| - |s_1| \leq \mathcal{O}(1)\mu_\nu^{\text{IC}}(\gamma_k).$$

This yields

$$\begin{aligned} q_k &\cong -s_1 + (s_1 + q_k) \\ &\leq \frac{\epsilon^0}{\epsilon^1 - \epsilon^0}(\epsilon^1 - \epsilon^0) + \mathcal{O}(1)\mu_\nu^{\text{I}}(t_k, x_k) \leq \frac{\mathcal{O}(1)\epsilon^0}{\epsilon^1 - \epsilon^0}\mu_\nu^{\text{IC}}(\gamma_k) + \mathcal{O}(1)\mu_\nu^{\text{I}}(t_k, x_k). \end{aligned}$$

Since the end points correspond to disjoint maximal i -th fronts, due to genuinely nonlinearity, it follows that

$$\sum_{(t_k, x_k) \text{ end point}} q_k \leq \mathcal{O}(1)\mu_\nu^{\text{IC}}(\mathbb{R}^+ \times \mathbb{R}),$$

so that it is a uniformly bounded measure. We thus conclude that the distribution

$$\bar{\mu}^\nu := -\mu_i^{\nu, \text{jump}} + \mathcal{O}(1)\mu_\nu^{\text{IC}} + \sum_{(t_k, x_k) \text{ end point}} q_k \delta_{(t_k, x_k)}$$

is non-negative, so it is a Radon measure and thus also $\mu_i^{\nu, \text{jump}}$ is a Radon measure.

In order to obtain a lower bound, one considers the Lipschitz continuous test function

$$\phi_\alpha(t) := \chi_{[0, T+\alpha]}(t) - \frac{t-T}{\alpha} \chi_{[T, T+\alpha]}(t), \quad \alpha > 0,$$

which is allowed because v_i^ν is a bounded measure. Since $\bar{\mu}$ is non-negative, one obtains

$$\begin{aligned} \bar{\mu}^\nu([0, T] \times \mathbb{R}) &\leq \int_{\mathbb{R}^+ \times \mathbb{R}} \phi_\alpha d\bar{\mu} \\ &= - \int_{\mathbb{R}^+ \times \mathbb{R}} \phi_\alpha d\mu_i^{\nu, \text{jump}} + \mathcal{O}(1) \int_{\mathbb{R}^+ \times \mathbb{R}} \phi_\alpha d\mu_\nu^{\text{IC}} + \sum_{(t_k, x_k) \text{ end point}} q_k \phi_\alpha(t_k) \\ &\leq \int_{\mathbb{R}^+ \times \mathbb{R}} [(\phi_\alpha)_t + \tilde{\lambda}_i^\nu(\phi_\alpha)_x] d[v_i^{\nu, \text{jump}}(t)] dt + [v_i^{\nu, \text{jump}}(0)](\mathbb{R}) \\ &\quad + \mathcal{O}(1)\mu_\nu^{\text{IC}}([0, T + \alpha] \times \mathbb{R}) \\ &\leq -\frac{1}{\alpha} \int_T^{T+\alpha} [v_i^{\nu, \text{jump}}(t)](\mathbb{R}) dt + [v_i^{\nu, \text{jump}}(0)](\mathbb{R}) + \mathcal{O}(1)\mu_\nu^{\text{IC}}([0, T + \alpha] \times \mathbb{R}). \end{aligned}$$

Letting $\alpha \searrow 0$ and since $[v_i^{\nu, \text{jump}}(\mathbb{R})](0)$ is negative, one concludes

$$\bar{\mu}^\nu([0, T] \times \mathbb{R}) \leq -[v_i^{\nu, \text{jump}}(T)](\mathbb{R}) + \mathcal{O}(1)\mu_\nu^{\text{IC}}([0, T + \alpha] \times \mathbb{R}) \leq \mathcal{O}(1)\Upsilon^\nu(0).$$

We conclude this section by writing the uniform estimate

$$-\mathcal{O}(1)\Upsilon^\nu(0) \leq \mu_i^{\nu, \text{jump}} \leq \mathcal{O}(1)\mu_\nu^{\text{IC}}.$$

In particular, the definitions of the measures $\mu_i^\nu, \mu_i^{\nu, \text{jump}}$ give the following balances for the i -th waves across the horizontal lines:

$$[v_i^\nu(t+)](\mathbb{R}) - [v_i^\nu(t-)](\mathbb{R}) = \mu_i^\nu(\{t\} \times \mathbb{R}), \tag{41a}$$

$$[v_i^{\nu, \text{jump}}(t+)](\mathbb{R}) - [v_i^{\nu, \text{jump}}(t-)](\mathbb{R}) = \mu_i^{\nu, \text{jump}}(\{t\} \times \mathbb{R}). \tag{41b}$$

The limits are taken in the weak topology. Notice that we can always take that $t \mapsto v_i^\nu(t), v_i^{\nu, \text{jump}}(t)$ is right continuous in the weak topology.

6.2.3. Balances of waves in the region bounded by generalized characteristics

Given an interval $I = [a, b]$, we define the region $A_{[a,b]}^{\nu,(t_0,\tau)}$ bounded by the minimal i -th characteristics $a(t), b(t)$ of u^ν starting at (t_0, a) and (t_0, b) by

$$A_{[a,b]}^{\nu,(t_0,\tau)} := \left\{ (t, x) : t_0 < t \leq t_0 + \tau, a(t) \leq x \leq b(t) \right\},$$

and its time-section by $I(t) := [a(t), b(t)]$. Let $J := I_1 \cup I_2 \cup \dots \cup I_M$ be the union of the disjoint closed intervals $\{I_i\}_{i=1}^M$, and set

$$J(t) := I_1(t) \cup \dots \cup I_M(t), \quad A_J^{\nu,(t_0,\tau)} := \bigcup_{m=1}^M A_{I_m}^{\nu,(t_0,\tau)}.$$

We will now obtain wave balances in regions of the form $A_J^{\nu,(t_0,\tau)}$. Due to the genuinely non-linearity of the i -th family, the corresponding proof in [8] works, we will repeat it for completeness.

The balance on the region $A_J^{\nu,(t_0,\tau)}$ has to take into account also the contribution of the flux Φ_i^ν across boundaries of the segments $I_m(t)$: due to the definition of generalized characteristic and the wave-front approximation, it follows that Φ_i^ν is an atomic measure on the characteristics forming the border of $A_J^{\nu,(t_0,\tau)}$, and moreover a positive wave may enter the domain $A_J^{\nu,(t_0,\tau)}$ only if an interaction occurs at the boundary point (\hat{t}, \hat{x}) , which gives the estimate

$$\Phi_i^\nu(\{\hat{t}, \hat{x}\}) \leq \mathcal{O}(1)\mu_i^{\text{IC}}(\{\hat{t}, \hat{x}\}). \tag{42}$$

One thus obtains that

$$[v_i^\nu(\tau)](J(\tau)) - [v_i^\nu(t_0)](J) = \mu_i^\nu(A_J^{\nu,(t_0,\tau)}) + \Phi_i^\nu(A_J^{\nu,(t_0,\tau)}) + \mathcal{O}(1)\epsilon_\nu, \tag{43}$$

where the last term depends on the errors due to the wave-front approximation (a single rarefaction front may exit the interval I_m at t_0).

The same computation can be done for the jump part $v_i^{\nu,\text{jump}}$, obtaining

$$\begin{aligned} & [v_i^{\nu,\text{jump}}(\tau)](J(t)) - [v_i^{\nu,\text{jump}}(t_0)](J) \\ &= \mu_i^{\nu,\text{jump}}(A_J^{\nu,(t_0,\tau)}) + \Phi_i^{\nu,\text{jump}}(A_J^{\nu,(t_0,\tau)}). \end{aligned} \tag{44}$$

Since the flux $\Phi_i^{\nu,\text{jump}}$ only involves the contribution of (ϵ^0, ϵ^1) -shocks, it is clearly non-positive.

Subtracting (44) to (43), one finds the following equation for $v_i^{\nu,\text{cont}}$:

$$\begin{aligned} & [v_i^{\nu,\text{cont}}(\tau)](J(\tau)) - [v_i^{\nu,\text{cont}}(t_0)](J) \\ &= (\mu_i^\nu - \mu_i^{\nu,\text{jump}})(A_J^{\nu,\tau}) + (\Phi_i^\nu - \Phi_i^{\nu,\text{jump}})(A_J^{\nu,(t_0,\tau)}) + \mathcal{O}(1)\epsilon_\nu. \end{aligned}$$

Denote the difference between the two fluxes by

$$\Phi_i^{\nu, \text{cont}} := \Phi_i^\nu - \Phi_i^{\nu, \text{jump}}.$$

Since $\Phi_i^{\nu, \text{jump}}$ removes only some terms in the negative part of Φ_i^ν , one concludes that

$$\Phi_i^\nu - \Phi_i^{\nu, \text{jump}} \leq [\Phi_i^\nu]^+ \leq \mu_\nu^{\text{IC}}. \tag{45}$$

Setting

$$\mu_{i, \nu}^{\text{ICJ}} := \mu_\nu^{\text{IC}} + |\mu_i^{\nu, \text{jump}}|,$$

and using the estimate $|\mu_i^\nu| \leq \mathcal{O}(1)\mu_\nu^{\text{IC}}$, one has

$$\mu_i^\nu - \mu_i^{\nu, \text{jump}} \leq \mathcal{O}(1)\mu_{i, \nu}^{\text{ICJ}}. \tag{46}$$

6.2.4. Decay estimate

Due to the semigroup property of solutions, it is sufficient to prove the estimate for the measure $[v_i^{\nu, \text{cont}}(t = 0)]^-$. Consider thus a closed interval $I = [a, b]$ and let $z(t) := b(t) - a(t)$ where

$$a(t) := x^\nu(t; 0, a), \quad b(t) := x^\nu(t; 0, b)$$

and the minimal forward characteristics starting at $t = 0$ from a and b . For \mathcal{L}^1 -a.e. t one has

$$\dot{z}(t) = \tilde{\lambda}_i(t, b(t)) - \tilde{\lambda}_i(t, a(t)).$$

By introducing a piecewise Lipschitz continuous non-decreasing potential Φ to control the waves on the other families [9], with $\Phi(0) = 1$, one obtains

$$\left| \dot{z}(t) + \xi(t) - [v_i^\nu(t)](I(t)) \right| \leq \mathcal{O}(1)\epsilon_\nu + \dot{\Phi}(t)z(t), \tag{47}$$

where

$$\xi(t) := (\tilde{\lambda}_i(t, a(t)+) - \tilde{\lambda}_i(t, a(t)-)) + (\tilde{\lambda}_i(t, b(t)+) - \tilde{\lambda}_i(t, b(t)-)).$$

We consider two cases.

Case 1. If

$$\dot{z}(t) - \dot{\Phi}(t)z(t) < \frac{1}{4}[v_i^{\nu, \text{cont}}(0)](I)$$

for all $t > 0$, then

$$\begin{aligned} \frac{d}{dt} \left[e^{-\int_0^t \dot{\Phi}(s) ds} z(t) \right] &= e^{-\int_0^t \dot{\Phi}(s) ds} \{ \dot{z}(t) - \dot{\Phi}(t)z(t) \} \\ &< \frac{e^{-\int_0^t \dot{\Phi}(s) ds}}{4} [v_i^{\nu, \text{cont}}(0)](I). \end{aligned}$$

Integrating the above inequality from 0 to τ and remembering that $\Phi(0) = 1$ and $v_i^{\nu, \text{jump}}(0)$ is non-positive, one has

$$\begin{aligned} -\mathcal{L}^1(I) &= -z(0) \leq e^{-\int_0^\tau \dot{\Phi}(s) ds} z(\tau) - z(0) \\ &\leq \frac{1}{4} \int_0^\tau e^{-\int_0^t \dot{\Phi}(s) ds} d\tau [v_i^{\nu, \text{cont}}(0)](I) \\ &\leq \frac{1}{4} \tau [v_i^{\nu, \text{cont}}(0)](I), \end{aligned}$$

which reads as

$$-[v_i^{\nu, \text{cont}}(0)](I) \leq 4 \frac{\mathcal{L}^1(I)}{\tau}.$$

Case 2. Assume instead that

$$\dot{z}(\bar{t}) - \dot{\Phi}(\bar{t})z(\bar{t}) \geq \frac{1}{4} [v_i^{\nu, \text{cont}}(0)](I) \tag{48}$$

at some time $\bar{t} > 0$. From (29) and the fact that the i -th family is genuinely nonlinear and the fronts in $\mathcal{F}_{(\epsilon^0, \epsilon^1)}^{\nu, i}$ satisfy Rankine-Hugoniot conditions (up to a negligible error), we have

$$v_i^{\nu, \text{jump}}(t, a(t)) = \lambda_i(t, a(t)+) - \lambda_i(t, a(t)-),$$

Then by the assumption of genuine nonlinearity, we conclude that

$$\begin{aligned} \xi(t) &\geq \frac{3}{4} [[v_i^{\nu, \text{jump}}(t)](a(t)) + [v_i^{\nu, \text{jump}}(t)](b(t)) - 2\epsilon^1] \\ &\geq \frac{3}{4} [[v_i^{\nu, \text{jump}}(t)](I(t)) - 2\epsilon^1]. \end{aligned} \tag{49}$$

As $v_i^{\nu, \text{jump}}$ is non-positive, (47) and (49) yield that

$$\begin{aligned} \dot{z}(t) - \dot{\Phi}z(t) &\leq [v_i^{\nu, \text{cont}}(t)](I(t)) + [v_i^{\nu, \text{jump}}(t)](I(t)) - \xi(t) + \mathcal{O}(1)\epsilon_\nu \\ &\leq [v_i^{\nu, \text{cont}}(t)](I(t)) + \mathcal{O}(1)\epsilon_\nu + 2\epsilon^1. \end{aligned}$$

Recall the assumption (48), at time \bar{t} , we get

$$[v_i^{\nu, \text{cont}}(0)](I)/4 \leq [v_i^{\nu, \text{cont}}(\bar{t})](I(\bar{t})) + \mathcal{O}(1)\epsilon_\nu + 2\epsilon^1.$$

By the balance for $v_i^{\nu, \text{cont}}$ we get in Section 6.2.3, one obtains

$$[v_i^{\nu, \text{cont}}(0)](I)/4 \leq [v_i^{\nu, \text{cont}}(0)](I) + \mu_\nu^{\text{ICJ}}(A_I^{\nu, (0, \bar{t})}) + \mathcal{O}(1)\epsilon_\nu + 2\epsilon^1.$$

Combining the conclusion for the two cases one gets the uniform bound r.w.t ν

$$-[v_i^{\nu, \text{cont}}(0)](I) \leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}^1(I)}{t} + \mu_\nu^{\text{ICJ}}(A_I^{\nu, (0, t)}) + \epsilon^1 + \epsilon_\nu \right\}.$$

This gives the estimate (18) for the case of a single interval for the approximate solution.

By analogous computation for the region which is a finite union of intervals, as we have done in Section 6.2.3, one obtains the same bound as above, and since $v_i^{\nu, \text{cont}}$ is a Radon measure, the same result holds for any Borel sets, i.e.

$$-[v_i^{\nu, \text{cont}}(0)](B) \leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}^1(B)}{t} + \mu_\nu^{\text{ICJ}}(\overline{A_B^{\nu, (0, t)}}) + \epsilon^1 + \epsilon_\nu \right\},$$

where B is any Borel set in \mathbb{R} and

$$A_B^{\nu, (0, t)} := \left\{ (\tau, x^\nu(\tau; 0, x_0)) : x \in B, 0 < \tau \leq t \right\}.$$

As the solution is independent on the choice of the approximation, we can consider a particular converging sequence $\{u^\nu\}_{\nu \geq 1}$ of ϵ_ν -approximate solutions with the following additional properties:

$$\mathcal{Q}(u^\nu(0, \cdot)) \rightarrow \mathcal{Q}(u_0).$$

By lower semi-continuity of $[v_i(0)]^- + C_0 \mathcal{Q}(u(0))$ (35), one gets

$$[v_i(0)]^- + C_0 \mathcal{Q}(u(0)) \leq \text{weak}^* - \liminf_{\nu \rightarrow \infty} \{ [v_i^\nu(0)]^- + C_0 \mathcal{Q}(u^\nu(0)) \}. \quad (50)$$

Since $v_i^{\text{jump}}(0)$ has only negative part, from (50) and (31), up to a subsequence, one obtains for any open set $U \subset \mathbb{R}$,

$$\begin{aligned} & [v_i^{\text{cont}}(0)]^-(U) \\ &= [v_i(0)]^-(U) + [v_i^{\text{jump}}(0)](U) \\ &\leq \liminf_{\nu \rightarrow \infty} \{ [v_i^\nu(0)]^-(U) + C_0 \mathcal{Q}(u^\nu(0)) \} - C_0 \mathcal{Q}(u(0)) + \lim_{\nu \rightarrow \infty} [v_i^{\nu, \text{jump}}(0)](U) \\ &= \liminf_{\nu \rightarrow \infty} \{ [v_i^{\nu, \text{cont}}(0)]^-(U) + C_0 \mathcal{Q}(u^\nu(0)) \} - C_0 \mathcal{Q}(u(0)) \\ &\leq \liminf_{\nu \rightarrow \infty} \mathcal{O}(1) \left\{ \frac{\mathcal{L}^1(U)}{t} + \mu_i^{\nu, \text{ICJ}}(\overline{A_U^{\nu, (0, t)}}) + \epsilon^1 + \epsilon_\nu + \mathcal{Q}(u^\nu(0)) - \mathcal{Q}(u(0)) \right\} \\ &\leq \mathcal{O}(1) \left\{ \frac{\mathcal{L}^1(U)}{t} + \mu_i^{\text{ICJ}}([0, t] \times \mathbb{R}) \right\}, \end{aligned}$$

where μ_i^{ICJ} is defined as weak*-limit of measure $\mu_i^{\nu, \text{ICJ}}$ (up to a subsequence). Then the outer regularity of Radon measure yields the inequality for any Borel set.

The above estimate together with Theorem 6.1 gives (18).

7. SBV regularity for the i -th component of the i -th eigenvalue

This last section concerns the proof of Theorem 1.2, adapting the strategy of Section 2.

Proof of Theorem 1.2. As in the scalar case, we define the sets

$$\begin{aligned} J_\tau &:= \{x \in \mathbb{R} : u^L(\tau, x) \neq u^R(\tau, x)\}, \\ F_\tau &:= \{x \in \mathbb{R} : \nabla \lambda_i(u(\tau, x)) \cdot r_i(u(\tau, x)) = 0\}, \\ C &:= \{(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R} : \xi \in J_\tau \cup F_\tau\}, \quad C_\tau := J_\tau \cup F_\tau. \end{aligned}$$

By definition of continuous part

$$|v_i^{\text{cont}}(\tau)|(J_\tau) = 0,$$

and since

$$\nabla \lambda_i(u(\tau, F_\tau \setminus J_\tau)) \cdot r_i(u(\tau, F_\tau \setminus J_\tau)) = 0,$$

we conclude that

$$\begin{aligned} &|\nabla \lambda_i(u) \cdot r_i(u)v_i^{\text{cont}}(\tau)|(C_\tau) \\ &= |\nabla \lambda_i(u) \cdot r_i(u)v_i^{\text{cont}}(\tau)|(J_\tau) + |\nabla \lambda_i(u) \cdot r_i(u)v_i^{\text{cont}}(\tau)|(F_\tau \setminus J_\tau) \\ &= 0. \end{aligned}$$

For any $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \setminus C$, there exist strictly positive $b_0 = b_0(x_0, t_0)$, $c_0 = c_0(x_0, t_0)$ such that

$$|\nabla \lambda_i \cdot r_i(u(t_0, x))| \geq c_0 > 0$$

for every x in the open interval $I_0 :=]-b_0 + x_0, x_0 + b_0[$, because $u(t_0, x)$ is continuous at $x_0 \notin C_{t_0}$. Hence by Theorem 4.3, we know that there is a triangle

$$T_0 := \left\{ (t, x) : |x - x_0| < b'_0 - \bar{\eta}(t - t_0), \quad 0 < t - t_0 < b'_0/\bar{\eta} \right\}$$

with the basis $I'_0 :=]-b'_0 + x_0, x_0 + b'_0[\subset I_0$, such that

$$|\nabla \lambda_i \cdot r_i(u(t_0, x))| \geq \frac{c_0}{2} > 0, \tag{51}$$

by taking $b'_0 \ll 1$ in order to have that the total variation remains sufficiently small.

Since $u_{\perp T_0}$ coincides with the solution to

$$\begin{cases} \partial_t w + f(w)_x = 0, \\ w(x, t_0) = \begin{cases} u_{t_0}(x) & |x - x_0| < b'_0, \\ \frac{1}{2b'_0} \int_{x_0 - b'_0}^{x_0 + b'_0} u_{t_0}(y) dy & |x - x_0| \geq b_0, \end{cases} \end{cases} \quad (52)$$

and by taking b'_0 sufficiently small, we still have that (51) holds for the range of w . In particular w is SBV outside a countable number of times, and the same happens for u in T_0 .

As in the scalar case, one thus verifies that there is a countable family of triangles $\{T_i\}_{i=1}^\infty$ covering the complement of C outside a set whose projection on the t -axis is countable. The same computation of the scalar case concludes the proof: for any τ chosen as in (5)

$$\begin{aligned} |(\nabla \lambda_i \cdot r_i) v_i^c|(\mathbb{R}) &\leq |(\nabla \lambda_i \cdot r_i) v_i^c|(C_\tau) \\ &\quad + |(\nabla \lambda_i \cdot r_i) v_i^c|\left(\bigcup_i T_i \cap \{t = \tau\}\right) = 0. \end{aligned}$$

Recall the definition (17), we can finally conclude that the i -th component of $D_x \lambda_i(u(t, \cdot))$ has no Cantor part for every $t \in \mathbb{R}^+ \setminus S$ and $i \in \{1, 2, \dots, N\}$. \square

Similar to the scalar case, it is easy to get the following corollary from the Theorem 1.2 and (16).

COROLLARY 7.1. *Suppose u be a vanishing viscosity solution of the Cauchy problem for the strictly hyperbolic system (1)-(2). Let u be the vanishing viscosity solution of the problem (1), (2). Then the scalar measure $[D_x \lambda_i(u)]_i$ has no Cantor part in $\mathbb{R}^+ \times \mathbb{R}$.*

REMARK 7.2. *As we mentioned in the introduction, it no longer holds the SBV regularity of admissible solution to the general strictly hyperbolic system of conservation laws.*

Consider the following equations

$$\begin{cases} u_t = 0, \\ v_t + ((1 + v + u)v)_x = 0. \end{cases}$$

Since $D_x \lambda_2((u, v)) = D_x u + 2D_x v$, then it is clear that $D_x \lambda_2$ can have a Cantor part since the first equation is just trivial which means that the component u is not SBV regular if the initial data is not.

While from Theorem 4.1 we know that the Cantor part of the second component of $D_x \lambda_2(u)$,

$$\begin{aligned} [D_x^c \lambda_2(u)]_2 &= (D_u \lambda_2 \cdot r_2)(l_2 \cdot D_x^c(u, v)) \\ &= \frac{2}{1+u+2v} (v D_x^c u_x + (1+u+2v) D_x^c v) \end{aligned}$$

vanishes. (Notice that since the Cantor part of $(D_x u, D_x v)$ concentrates on the set of continuous points of (u, v) , we do not need to specify the coefficients at the jump points of (u, v) .)

REFERENCES

- [1] L. AMBROSIO AND C. DE LELLIS, *A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton-Jacobi equations*, J. Hyperbolic Differ. Equ. **1** (2004), 813–826.
- [2] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Clarendon Press, 2000.
- [3] F. ANCONA AND A. MARSON, *Existence theory by front tracking for general nonlinear hyperbolic systems*, Arch. Ration. Mech. Anal. **185** (2007), 287–340.
- [4] S. BIANCHINI, *Interaction estimates and Glimm functional for general hyperbolic systems*, Discrete Contin. Dyn. Syst. **9** (2003), 133–166.
- [5] S. BIANCHINI, *On the Riemann problem for non-conservative hyperbolic systems*, Arch. Ration. Mech. Anal. **166** (2003), 1–26.
- [6] S. BIANCHINI, *SBV regularity of genuinely nonlinear hyperbolic systems of conservation laws in one space dimension*, Acta Math. Sci. **32** (2012), 380–388.
- [7] S. BIANCHINI AND A. BRESSAN, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. **161** (2005), 223–342.
- [8] S. BIANCHINI AND L. CARAVENNA, *SBV regularity for genuinely nonlinear, strictly hyperbolic systems of conservation laws in one space dimension*, arXiv:1111.6246v1, November 2011.
- [9] A. BRESSAN, *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem*, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, USA, 2000.
- [10] A. BRESSAN AND R.M. COLOMBO, *Decay of positive waves in nonlinear systems of conservation laws*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 133–160.
- [11] A. BRESSAN AND P.G. LEFLOCH, *Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws*, Indiana Univ. Math. J. **48** (1999), 43–84.
- [12] A. BRESSAN AND T. YANG, *A sharp decay estimate for positive nonlinear waves*, SIAM J. Math. Anal. **36** (2004), 659–677.
- [13] C. M. DAFERMOS, *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. **26** (1977), 1097–1119.
- [14] C. M. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, Berlin, 2009.

- [15] C. DE LELLIS, *Hyperbolic equations and SBV functions*, Journées équations aux dérivées partielles **6** (2010), 1–10.
- [16] J. GLIMM, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 697–715.
- [17] R. ROBYR, *SBV regularity of entropy solutions for a class of genuinely nonlinear scalar balance laws with non-convex flux function*, J. Hyperbolic Differ. Equ. **5** (2008), 449–475.

Authors' addresses:

Stefano Bianchini
SISSA - International School for Advanced Studies
via Bonomea 265, 34136 Trieste, ITALY
E-mail: bianchin@sissa.it

Lei Yu
SISSA - International School for Advanced Studies
via Bonomea 265, 34136 Trieste, ITALY
E-mail: yulei@sissa.it

Received March 1, 2012
Revised September 7, 2012