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CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATIONS

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Abstract: In this paper, we define the classes $T_j^\lambda(m, l, A, B)$, using Janowski class and the multiplier transformations $I(m, \lambda, l)f(z)$ and we study distortion bounds, extreme points and many more properties. We also establish some results concerning partial sums for functions belonging to these classes.

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1. Introduction

Let $\mathcal{A}(j)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \ (a_k \geq 0; \, j \in \ \mathbf{N})$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{T}(j)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \ (a_k \geq 0; \, j \in \ \mathbf{N}) \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.

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For $-1 \leq A < B \leq 1$, let $\mathcal{P}(A, B)$ [3] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where ω is a bounded analytic function satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$.

For $f \in \mathcal{A}(j)$, $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\lambda, l \geq 0$, the operator [1] $I(m, \lambda, l)f(z)$ is defined by

$$I(m, \lambda, l)f(z) = z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1) + l + 1}{l+1} \right)^m a_k z^k.$$

We say that a function $f \in \mathcal{T}(j)$ is in the class $\mathcal{T}_j^{\lambda}(m, l, A, B)$ if and only if

$$\frac{z[I(m, \lambda, l)f(z)]'}{I(m, \lambda, l)f(z)} \prec \frac{1 + Az}{1 + Bz}, \ (m \in \mathbb{N}_0),$$

for $-1 \leq A < B \leq 1$, $l, \lambda > 0$ and for all $z \in \mathcal{U}$.

Note that $T_j^0(1, 0, 2\alpha - 1, 1) = S_j^*(\alpha)$ introduced by Chatterjea [2] and $T_j^1(1, 0, 2\alpha - 1, 1) = C_j(\alpha)$ studied by Srivastava [7].

We shall need the following Theorem of Ravikumar and Latha [4] to prove our results.

2. Main Results

Theorem 2.1. *A function $f \in \mathcal{T}_j$ is in the class $\mathcal{T}_j^{\lambda}(m, l, A, B)$ if and only if*

$$\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1) + l + 1}{l+1} \right)^m [k(B+1) - (A+1)]a_k \leq B - A \qquad (2.1)$$

for $m \in \mathbb{N}_0$, $-1 \leq A < B \leq 1$, $l, \lambda > 0$ and $z \in \mathcal{U}$.

Corollary 2.2. *If function $f(z) \in \mathcal{T}_j$ is in the class $\mathcal{T}_j^{\lambda}(m, l, A, B)$ then*

$$|a_k| \leq \frac{(B-A)}{\left(\frac{\lambda(k-1) + l + 1}{l+1} \right)^m [k(B+1) - (A+1)]},$$

for some $-1 \leq A < B \leq 1, m \in \mathbb{N}_0$, $l, \lambda > 0$ and $z \in \mathcal{U}$.

Now we determine extreme points for the class $\mathcal{T}_j^{\lambda}(m,l,A,B)$.

Theorem 2.3. *Let $f(z) \in \mathcal{T}_j^{\lambda}(m,l,A,B)$. Define $f_1(z) = z$ and*

$$f_k(z) = z - \frac{B-A}{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)]} z^k, \quad k \geq j+1,$$

for some $-1 \leq A < B \leq 1, m \in \mathbb{N}_0$, $l, \lambda > 0$ and $z \in \mathcal{U}$. Then $f \in \mathcal{T}_j^{\lambda}(m,l,A,B)$ if and only if f can be expressed as $f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum_{k=j}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z)$ with $\sum_{k=j}^{\infty} \mu_k = 1, \quad \mu_k \geq 0$, then

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)]}{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)]} \mu_k (B-A) \\
&= \sum_{k=j+1}^{\infty} \mu_k (B-A) = (1-\mu_j)(B-A) \\
&\leq (B-A).
\end{aligned}$$

Hence $f(z) \in \mathcal{T}_j^{\lambda}(m,l,A,B)$.

Conversely, let $f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \in \mathcal{T}_j^{\lambda}(m,l,A,B)$, define

$$\mu_k = \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)]|a_k|}{(B-A)}, \quad k \geq j+1,$$

and define $\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k$. From Theorem 2.1, $\sum_{k=j+1}^{\infty} \mu_k \leq 1$ and hence $\mu_j \geq 0$.

Since $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$,

$\sum_{k=j+1}^{\infty} \mu_k f_k(z) = z - \sum_{k=j+1}^{\infty} a_k z^k = f(z).$ □

Theorem 2.4. *The class $\mathcal{T}_j^{\lambda}(m,l,A,B)$ is closed under convex linear combination.*

Proof. Let $f(z),\ g(z)\in\mathcal{T}_j^{\lambda}(m,l,A,B)$ and let

$$f(z)=z-\sum_{k=j+1}^{\infty}a_kz^k,\quad g(z)=z-\sum_{k=j+1}^{\infty}b_kz^k.$$

For η such that $0\le\eta\le1$, it suffices to show that the function defined by $h(z)=(1-\eta)f(z)+\eta g(z),\ z\in\mathcal{U}$ belongs to $\mathcal{T}_j^{\lambda}(m,l,A,B)$. Now

$$h(z)=z-\sum_{k=j+1}^{\infty}[(1-\eta)a_k+\eta b_k]z^k.$$

Applying Theorem 2.1, to $f(z),\ g(z)\in\mathcal{T}_j^{\lambda}(m,l,A,B)$, we have

$$\begin{aligned}&\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m[k(B+1)-(A+1)][(1-\eta)a_k+\eta b_k]\\&=(1-\eta)\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m[k(B+1)-(A+1)]a_k\\&+\eta\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m[k(B+1)-(A+1)]b_k\\&\le(1-\eta)(B-A)+\eta(B-A)=(B-A).\end{aligned}$$

This implies that $h(z)\in\mathcal{T}_j^{\lambda}(m,l,A,B)$. ☐

Theorem 2.5. *Let for $i=1,2,\cdots,k$, $f_i(z)=z-\sum_{k=j+1}^{\infty}a_{k,i}z^k\in\mathcal{T}_j^{\lambda}(m,l,A,B)$ and $0<\beta_i<1$ such that $\sum_{i=1}^{k}\beta_i=1$, then the function $F(z)$ defined by*
$F(z)=\sum_{i=1}^{k}\beta_if_i(z)$ is also in $\mathcal{T}_j^{\lambda}(m,l,A,B)$.

Proof. For each $i\in\{1,2,3,\cdots,k\}$ we obtain

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m[k(B+1)-(A+1)]|a_{k,i}|<(B-A).$$

Since

$$F(z) = \sum_{i=1}^{k} \beta_i (z - \sum_{k=j+1}^{\infty} a_{k,i} z^k)$$
$$= z - \sum_{k=j+1}^{\infty} (\sum_{i=1}^{k} \beta_i a_{k,i}) z^k.$$

$$\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)] \left[\sum_{i=1}^{k} \beta_i a_{k,i}\right]$$
$$= \sum_{i=1}^{k} \beta_i \left[\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1)-(A+1)]\right]$$
$$< \sum_{i=1}^{k} \beta_i (B-A) < (B-A).$$

Therefore $F(z) \in \mathcal{T}_j^{\lambda}(m,l,A,B)$. □

3. Partial Sums

Consider partial sums of functions in the class $\mathcal{T}_j^{\lambda}(m,l,A,B)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f_n'(z)$. In this paper, applying methods used by Silverman [5] and Silvia [6] , we will investigate the ratio of a function $f(z)$ of the form (1.1) to its sequence of partial sums

$$f_n(z) = z - \sum_{k=j+1}^{n} a_k z^k$$

when the coefficients are sufficiently small to satisfy conditions (2.1). More precisely, we will determine sharp lower bounds for

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\}, \Re\left\{\frac{f_n(z)}{f(z)}\right\}, \Re\left\{\frac{f'(z)}{f_n'(z)}\right\}, \, and \, \Re\left\{\frac{f_n'(z)}{f'(z)}\right\}.$$

In the sequel, we will make use of the result that

$$\Re\frac{(1+\omega(z))}{(1-\omega(z))} > 0 \quad (z \in \mathcal{U})$$

if and only if $\omega(z) = \sum_{k=j+1}^{\infty} c_k z^k$ satisfies the inequality $|\omega(z)| < |z|$.

Theorem 3.1. *If $f(z) \in \mathcal{T}_j^{\lambda}(m, l, A, B)$, then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{n+1}}, \ (z \in \mathcal{U}) \tag{3.1}$$

where

$$c_k = \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1) - (A+1)]}{B - A}. \tag{3.2}$$

The estimates in (3.1) are sharp.

Proof. The function $f(z) \in \mathcal{T}_j^{\lambda}(m, l, A, B)$, if and only if $\sum_{k=j+1}^{\infty} c_k a_k \leq 1$.

It is easy to verify that $c_{k+1} > c_k > 1$.
Therefore we have

$$\sum_{k=j+1}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=j+1}^{\infty} c_k a_k \leq 1. \tag{3.3}$$

By Setting

$$\omega(z) = c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\}$$

$$= 1 - \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^{n} a_k z^{k-1}}$$

and applying (3.3), we find that

$$\left|\frac{1 - \omega(z)}{1 + \omega(z)}\right| \leq \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=j+1}^{n} a_k - c_{n+1} \sum_{k=n+1}^{\infty} a_k}$$

$$\leq 1, \ \ z \in \mathcal{U},$$

which readily yields the assertion (3.3).

To see that

$$f(z) = z - \frac{z^{k+1}}{c_{k+1}}$$

gives sharp results, we observe that

$$\frac{f(z)}{f_n(z)} = 1 - \frac{z^k}{c_{k+1}}.$$

Letting $z \to 1^-$, we have

$$\frac{f(z)}{f_n(z)} = 1 - \frac{1}{c_{k+1}},$$

which shows that the bounds in is the best possible for each $n \in \mathcal{N}$.

□

Theorem 3.2. *If $f(z) \in \mathcal{T}_j^{\lambda}(m, l, A, B)$, then*

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1 + c_{n+1}}, \ (z \in \mathcal{U}) \tag{3.4}$$

where c_n is defined in (3.2).The result is sharp for every n with extremal function given by (3.4).

Proof. By Setting

$$\omega(z) = (1 + c_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}}\right\}$$

$$= \frac{1 - \sum_{k=j+1}^{n} a_k z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k z^{k-1}},$$

and using (3.3) we find that

$$\left|\frac{1 - \omega(z)}{1 + \omega(z)}\right| \leq \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=j+1}^{n} a_k - (1 - c_{n+1}) \sum_{k=n+1}^{\infty} a_k}$$

$$\leq 1, \ \ z \in \mathcal{N}.$$

which gives (3.4), The bound in (3.4) is sharp for all $n \in \mathcal{N}$ with the external function given by (3.2). □

Theorem 3.3. *If $f(z) \in \mathcal{T}_j^{\lambda}(m, l, A, B)$, then*

$$\Re\left\{\frac{f'(z)}{f'_n(z)}\right\} > 1 - \frac{n+1}{c_{n+1}}, \ (z \in \mathcal{U}) \tag{3.5}$$

where c_n is defined in (3.2).

Proof. By Setting

$$\omega(z) = c_{n+1}\left\{\frac{f'(z)}{f'_n(z)} - \left(1 - \frac{n+1}{c_{n+1}}\right)\right\}$$

$$= \frac{1 - \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k z^{k-1} - \sum_{k=j+1}^{\infty} ka_k z^{k-1}}{1 - \sum_{k=j+1}^{n} ka_k z^{k-1}}$$

$$= 1 - \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k z^{k-1}}{1 - \sum_{k=j+1}^{n} ka_k z^{k-1}}.$$

Now

$$\left|\frac{1-\omega(z)}{1+\omega(z)}\right| \leq \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k}{2 - 2\sum_{k=j+1}^{n} ka_k - \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k} \leq 1$$

if and only if

$$2\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k \leq 2 - 2\sum_{k=j+1}^{n} ka_k,$$

which is equivalent to

$$\sum_{k=j+1}^{n} ka_k + \frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k \leq 1$$

thus we obtain (3.5). The result is sharp with functions given by (3.2). □

Theorem 3.4. *If $f(z) \in \mathcal{T}_j^{\lambda}(m, l, A, B)$, then*

$$\Re\left\{\frac{f_n'(z)}{f'(z)}\right\} > \frac{c_{n+1}}{nm + 1 + c_{n+1}}, \ (z \in \mathcal{U}) \tag{3.6}$$

where c_n is defined in (3.2).The result is sharp for every n with extremal function given by (3.6).

Proof. By Setting

$$\omega(z) = [(n+1) + c_{n+1}]\left\{\frac{f_n'(z)}{f'(z)} - \frac{c_{n+1}}{n+1+c_{n+1}}\right\}$$

$$= 1 + \frac{(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 - \sum_{k=j+1}^{m} k a_k z^{k-1}}.$$

We deduce that that

$$\left|\frac{1-\omega(z)}{1+\omega(z)}\right| \le \frac{(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k}{2 - 2\sum_{k=j+1}^{n} k a_k - (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k} \le 1,$$

if and only if

$$2(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k \le 2 - 2\sum_{k=j+1}^{n} k a_k,$$

which is equivalent to

$$\sum_{k=j+1}^{n} k a_k + (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k \le 1.$$

which gives (3.6). The bound in (3.3) is sharp for all $n \in \mathcal{N}$ with the external function given by (3.2). □

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