Scalable noise estimation with random unitary operators

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Abstract
We describe a scalable stochastic method for the experimental measurement
of generalized fidelities characterizing the accuracy of the implementation
of a coherent quantum transformation. The method is based on the motion
reversal of random unitary operators. In the simplest case our method
enables direct estimation of the average gate fidelity. The more general
fidelities are characterized by a universal exponential rate of fidelity loss. In
all cases the measurable fidelity decrease is directly related to the strength of
the noise affecting the implementation, quantified by the trace of the
superoperator describing the non-unitary dynamics. While the scalability of
our stochastic protocol makes it most relevant in large Hilbert spaces (when
quantum process tomography is infeasible), our method should be
immediately useful for evaluating the degree of control that is achievable in
any prototype quantum processing device. By varying over different
experimental arrangements and error-correction strategies, additional
information about the noise can be determined.

Keywords: noise estimation, quantum computing, randomization, gate
fidelity

1. Introduction

The characterization and elimination of decoherence and
other noise sources has emerged as one of the major
challenges confronting the coherent experimental control of
increasingly large multi-body quantum systems. Decoherence
arising from undesired interactions with background (or
environment) systems and imprecision in the classical control
fields lead to severe limits on the observation of mesoscopic
and macroscopic quantum phenomena, such as interference
effects, and, in particular, the realization of quantum
communication and computation algorithms. Measurement
of the strength and other detailed properties of the noise
mechanisms affecting a physical implementation is a critical
part of optimizing, improving, and benchmarking the physical
device and experimental protocol [1, 2]. Moreover, in
the case of quantum devices capable of universal control,
knowledge of specific characteristics of the noise enables

the selection and optimization of passive and active error-
prevention strategies [3–8].

The exact method for characterizing the noise affecting
an implementation is quantum process tomography (QPT)
[8]. Let D denote the dimensionality of the Hilbert space
(constituted, e.g., from $n_q = \log_2(D)$ qubits). For QPT,
the desired transformation (usually a unitary operator) must
be applied to each member of a complete set of D^2 input
states (spanning the state space), followed by tomographic
measurement of the output state. This allows for a complete
reconstruction of the superoperator (completely positive linear
map) representing the imperfect implementation of the target
transformation. From this superoperator the cumulative noise
superoperator can be extracted from conventional analysis
of the matrix. The QPT approach to noise estimation
suffers from several practical deficiencies. First, often the
intrinsic properties of the noise operators are of interest, but
the noise superoperator determined from QPT will depend

on the symmetries between the noise mechanisms and the choice of target transformation. Second, the number of experiments that must be carried out grows exponentially in the number of qubits $D^4 = 2^{4n_q}$. Third, conventional numerical analysis of the tomographic data requires the manipulation of matrices of exponentially increasing dimension ($D^2 \times D^2$). For these last two reasons QPT becomes infeasible for processes involving more than about a dozen qubits, far fewer than the one thousand or so qubits required for the fault-tolerant implementation of quantum algorithms that outperform conventional computation. Hence the infeasibility of complete noise estimation via tomography prompts the question of whether there exist efficient methods by which specific features of the noise may be determined.

We show below that the overall noise strength and the associated accuracy of an implementation may be estimated by a scalable experimental method. Specifically we show that the average gate fidelity (1), and some more generalized fidelities described below, can be estimated directly with an accuracy $O(1/\sqrt{DN})$ where N is the number of independent experiments. This method provides a solution to the important problem of efficiently measuring which member of a set of experimental configurations and algorithmic techniques produces the most accurate implementation of an arbitrary target transformation. By varying over different experimental methods and noise-reduction algorithms and then directly measuring the variation in the associated fidelity this method enables estimation of more detailed characteristics of the noise.

2. Efficient estimation of the average gate fidelity

A convenient starting point for our analysis is the average gate fidelity

$$\overline{F_g}(\Lambda) \equiv \mathbb{E}_\psi(F_g(U, \Lambda, \psi))$$
$$\equiv \int d\psi \; \langle\psi|U^{-1}(\Lambda(U|\psi\rangle\langle\psi|U^{-1}))U|\psi\rangle \quad (1)$$

where

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger \quad (2)$$

is a completely positive (CP) map characterizing the noise. The gate fidelity F_g is the inner product of the state obtained from the actual implementation with the state that would be ideally obtained under the target unitary. The measure $d\psi$ denotes the natural, unitarily invariant (Fubini–Study) measure on the set of pure states and hence the average gate fidelity provides an indicator that is independent of the choice of initial state. If the implementation is perfect then $\overline{F_g} = 1$ and under increasing noise $\overline{F_g}$ decreases. Due to the invariance of the Fubini–Study measure the average fidelity depends only on the noise operator and can been expressed in the form [9–11]

$$\overline{F_g}(\Lambda) = \frac{\sum_k |\mathrm{Tr}(A_k)|^2 + D}{D^2 + D}. \quad (3)$$

Hence the average fidelity can be determined if the noise operator is known. The noise operator can be determined experimentally by measuring the CP map $\Lambda(U \cdot U^{-1})$ tomographically and then factoring out the inverse of the target map $U^{-1} \cdot U$. This procedure has been carried out

recently for three qubits in an implementation of the quantum Fourier transform using liquid state NMR techniques [2]. As noted above, this method requires $\mathcal{O}(D^4)$ experiments and the conventional manipulation of matrices also of dimension $D^2 \times D^2$. Recently Nielsen has proposed a method [11] for the direct measurement of $\overline{F_g}$ that requires D^2 (rather than D^4) experiments.

We now describe how the average gate fidelity (1) can be estimated accurately from a simple experimental protocol. Our method requires the physical implementation of the 'motion reversal' transformation $U^{-1}U|\psi\rangle\langle\psi|U^{-1}U$ on an arbitrary state $|\psi\rangle\langle\psi|$. Under this transformation, the CP map Λ in the gate fidelity (1) can be interpreted as the decoherence and experimental control errors arising under the imperfect implementation of the motion reversal experiment, i.e., $\Lambda = \Lambda_{U^{-1}U}$, rather than as the noise associated with only the forward transformation U, i.e., $\Lambda = \Lambda_U$. The key idea is to choose the target transformation U randomly from the Haar measure [12]. This will earn us the advantage of the concentration of measure in large Hilbert spaces, as described further below, and leads to a universal form of the gate fidelity depending only on the intrinsic strength of the cumulative noise. This universal form will allow us to evaluate the average fidelity for more generalized motion reversal protocols.

Our starting point is the gate fidelity uniformly averaged over all unitaries,

$$\mathbb{E}_U(F_g) = \int_{U(D)} dU \; \mathrm{Tr}[\rho U^{-1}\Lambda(U\rho U^{-1})U], \quad (4)$$

where in the above dU denotes the unitarily invariant Haar measure on $U(D)$ and $\rho = |\psi\rangle\langle\psi|$. In order to evaluate this integral we use the superoperator representation of the map (2),

$$\hat{\Lambda} = \sum_k A_k \otimes A_k^*, \quad (5)$$

and similarly $\hat{U} = U \otimes U^*$, where $*$ denotes complex conjugation. The Haar-averaged gate fidelity takes the form

$$\mathbb{E}_U(F_g) = \mathrm{Tr}\left(\rho\left[\int dU \; \hat{U}\hat{\Lambda}\hat{U}^{-1}\right]\rho\right) \quad (6)$$
$$= \mathrm{Tr}(\rho\hat{\Lambda}^{\mathrm{ave}}\rho) = F_g(\hat{\Lambda}^{\mathrm{ave}}) \quad (7)$$

where $\hat{\Lambda}^{\mathrm{ave}} \equiv \int dU \; \hat{U}\hat{\Lambda}\hat{U}^{-1}$. As shown in the appendix, the Haar-averaged superoperator $\hat{\Lambda}^{\mathrm{ave}}$ is $U(D)$ invariant and thus can be expressed as a depolarizing channel

$$\hat{\Lambda}^{\mathrm{ave}}\rho = p\rho + (1-p)\frac{\mathbb{I}}{D}, \quad (8)$$

(assuming $\mathrm{Tr}(\rho) = 1$) characterized by the single 'strength' parameter

$$p = \frac{\sum_k |\mathrm{Tr}(A_k)|^2 - 1}{D^2 - 1}, \quad (9)$$

where $p \in [0, 1]$ and we have made use of the fact that $\mathrm{Tr}(\hat{\Lambda}^{\mathrm{ave}}) = \mathrm{Tr}(\hat{\Lambda}) = \sum_k |\mathrm{Tr}(A_k)|^2$. Direct substitution leads to

$$\mathbb{E}_U(F_g) = F_g(\hat{\Lambda}^{\mathrm{ave}}) = p + \frac{(1-p)}{D}. \quad (10)$$

Hence the gate fidelity for the Haar-averaged operator resulting from a motion reversal experiment depends only on the single

parameter $\mathrm{Tr}(\hat{\Lambda})$ which represents the intrinsic strength of the cumulative noise. We remark that this result holds for general (possibly non-unital) noise. Furthermore, suppressing the arguments of F we note that the unitary invariance of the natural measure on pure states implies the equivalence

$$\overline{F_g}(\Lambda) = \mathbb{E}_\psi(F_g) = \mathbb{E}_U(F_g), \qquad (11)$$

and hence we recover equation (3).

We now describe why and how the intrinsic noise strength (characterized by p or $\mathrm{Tr}(\hat{\Lambda})$) can be estimated via an efficient experimental protocol. By implementing a single target transformation U that is randomly drawn from the Haar measure, we gain the advantage of the concentration of measure in large Hilbert spaces: the motion reversal (gate) fidelity for the single random U is exponentially close to the Haar-averaged motion reversal (gate) fidelity. From the unitary invariance of the Fubini–Study measure we know that

$$\mathbb{E}_\psi(F_g^2) = \mathbb{E}_U(F_g^2). \qquad (12)$$

As will be shown in [13], the typical fluctuation for a random initial state $|\psi\rangle$, given a fixed U and Λ, decreases exponentially with the number of qubits,

$$(\Delta_\psi F_g)^2 \equiv \overline{F_g^2} - \overline{F_g}^2 \leqslant \mathrm{O}(1/D). \qquad (13)$$

Therefore it follows that

$$(\Delta F)_U^2 \equiv \mathbb{E}_U(F_g^2) - \mathbb{E}_U(F_g)^2 \leqslant \mathrm{O}(1/D). \qquad (14)$$

Hence the fidelity under motion reversal of a single random U and arbitrary (non-random) initial state is exponentially close to the Haar-averaged fidelity

$$F_g(U, \Lambda, \psi) = F_g(\Lambda^{\mathrm{ave}}) + \mathrm{O}(1/\sqrt{D})$$
$$= p + \frac{(1-p)}{D} + \mathrm{O}(1/\sqrt{D}). \qquad (15)$$

The protocol is now clear: after the motion reversal sequence has been applied experimentally, the single parameter p characterizing the average gate fidelity appears as the residual population of the initial state. Due to the invariance of the Haar measure we may choose the initial state to be the computational basis state $(|0\rangle\langle 0|)^{\otimes n_q}$. Hence the gate fidelity can be determined directly from a standard read-out (projective measurement) of the final state in the computational basis. When the noise strength is actually non-negligible (e.g., the noise strength does not decrease as a polynomial function of $1/D$) an accurate estimate of p is possible with only a few experimental trials. If in each of N repetitions of the motion reversal experiment an independent random unitary is applied, then the observed average will approach the Haar average as $\mathcal{O}(1/\sqrt{DN})$.

3. Generalized fidelities in a discrete-time scenario

More generally we imagine the ability to implement a set of independent random unitary operators $\{U_j\}$ and their inverses. The entire sequence is subject to some unknown noise, consisting of the decoherence processes and control errors affecting the implementation. Such generalized motion

reversal sequences are relevant not only for noise estimation, but also have important applications in studies of fidelity decay [14] and decoherence rates [15] for quantum chaos and many-body complex systems.

We first consider the fidelity loss arising under an iterated motion reversal sequence of the form

$$\rho(n) = \hat{U}_n^{-1} \hat{\Lambda} \hat{U}_n \cdots \hat{U}_2^{-1} \hat{\Lambda} \hat{U}_2 \, \hat{U}_1^{-1} \hat{\Lambda} \hat{U}_1 \rho(0), \qquad (16)$$

where $\hat{\Lambda}_j = \hat{\Lambda}_{U_j^{-1} U_j}$ denotes the cumulative noise from the motion reversal of U_j and we now allow arbitrary (possibly mixed) initial states $\rho(0)$. The fidelity of this iterated transformation is

$$F_n(\psi, \{U_j\}) = \mathrm{Tr}(\rho(0) \hat{U}_n^{-1} \hat{\Lambda}_n \hat{U}_n \cdots \hat{U}_1^{-1} \hat{\Lambda}_1 \hat{U}_1 \rho(0)). \quad (17)$$

Averaging over the Haar measure for each U_j takes the form

$$\overline{F_n} \equiv \mathbb{E}_{\{U_j\}}(F_n(\psi, \{U_j\}))$$
$$\equiv \int_{U(D)^{\otimes n}} (\Pi_{j=1}^n \mathrm{d}U_j) F_n(\psi, \{U_j\}) \qquad (18)$$
$$= \mathrm{Tr}(\rho(0)[\Pi_{j=1}^n \hat{\Lambda}_j^{\mathrm{ave}}] \rho(0)), \qquad (19)$$

where $\mathrm{d}U_j$ denotes the Haar measure and we have defined the Haar-averaged noise operator,

$$\hat{\Lambda}_j^{\mathrm{ave}} \equiv \mathbb{E}_{U_j}(\hat{\Lambda}_j) \equiv \int_{U(D)} \mathrm{d}U \, \hat{U}^{-1} \hat{\Lambda}_j \hat{U}. \qquad (20)$$

As noted above and shown in the appendix, $\hat{\Lambda}^{\mathrm{ave}} \equiv \int \mathrm{d}U \hat{U} \hat{\Lambda} \hat{U}^{-1}$ is a depolarizing channel

$$\hat{\Lambda}_j^{\mathrm{ave}} \rho = p_j \rho + (1 - p_j) \frac{\mathbb{1}}{D}, \qquad (21)$$

with strength parameter

$$p_j = \frac{\mathrm{Tr}(\hat{\Lambda}_j) - 1}{D^2 - 1}. \qquad (22)$$

Because each U_j is random, we can further simplify this result by assuming that the cumulative noise for each U_j has the same strength $p_j = p$, in which case we obtain for arbitrary noise a universal exponential decay of the averaged fidelity

$$\overline{F_n} = p^n \mathrm{Tr}[\rho(0)^2] + \frac{(1 - p^n)}{D} \qquad (23)$$

depending only on the noise strength. In the limit of large n, we see that $\overline{F_n} \to D^{-1}$, as may be expected from the average fidelity between random states [16]. Most importantly, due to the concentration of measure (14), for large D the fidelity loss under iterated motion reversal of a single sequence of random unitary operators will be exponentially close to the Haar average, and hence the noise strength can be estimated with only a few experimental runs.

Another important generalized fidelity is the one obtained under the imperfect 'Loschmidt echo' sequence [14, 17]

$$\rho(n) = \hat{U}_1^{-1} \cdots \hat{U}_n^{-1} \hat{\Lambda}_n \hat{U}_n \cdots \hat{\Lambda}_1 \hat{U}_1 \rho(0), \qquad (24)$$

where the superoperator $\hat{\Lambda}_j$ represents the cumulative noise during the implementation of each U_j. The fidelity between

the initial state and final state in the Loschmidt echo experiment takes the form

$$
\begin{aligned}
&F_n^{\text{echo}}(\psi, \{\Lambda_j\}, \{U_j\}) \\
&= \text{Tr}(\rho(0)\hat{U}_1^{-1}\hat{U}_2^{-1}\cdots\hat{U}_n^{-1}\hat{\Lambda}_n\hat{U}_n\cdots\hat{\Lambda}_1\hat{U}_1\rho(0)).
\end{aligned} \tag{25}
$$

Moving to the interaction picture we define

$$
\hat{\Lambda}_j(j) = \hat{U}_1^{-1}\cdots\hat{U}_j^{-1}\hat{\Lambda}_j\hat{U}_j\cdots\hat{U}_1, \tag{26}
$$

so we have

$$
\begin{aligned}
&F_n^{\text{echo}}(\psi, \{\Lambda_j\}, \{U_j\}) \\
&= \text{Tr}(\rho(0)\hat{\Lambda}_n(n)\hat{\Lambda}_{n-1}(n-1)\cdots\hat{\Lambda}_1(1)\rho(0)).
\end{aligned} \tag{27}
$$

From the invariance of the Haar measure the average fidelity simplifies to

$$
\overline{F_n^{\text{echo}}} = \text{Tr}(\rho(0)\hat{\Lambda}_n^{\text{ave}}\hat{\Lambda}_{n-1}^{\text{ave}}\cdots\hat{\Lambda}_1^{\text{ave}}\rho(0)) \tag{28}
$$

with $\hat{\Lambda}_j^{\text{ave}}$ given by equation (21). As before, we can simplify this result by assuming that the cumulative noise for each step has the same strength ($p_j = p$), in which case we obtain for arbitrary noise a universal exponential form for the decay of fidelity

$$
\overline{F_n^{\text{echo}}}(p) = p^n + \frac{(1-p^n)}{D}. \tag{29}
$$

A generalized version of this Loschmidt echo that is more relevant to noise estimation is one for which noise appears in both the forward and backward sequences of the motion reversal. The associated fidelity is

$$
\begin{aligned}
&F_n^{\text{gen}}(\psi, \Lambda, \{U_j\}) \\
&= \text{Tr}(\rho(0)\hat{\Lambda}\hat{U}_1^{-1}\hat{\Lambda}\hat{U}_2^{-1}\cdots\hat{\Lambda}\hat{U}_n^{-1}\hat{\Lambda}\hat{U}_n\cdots\hat{\Lambda}\hat{U}_1\rho(0)).
\end{aligned} \tag{30}
$$

While we have not directly evaluated the average of this fidelity analytically in the general case, we have obtained analytic and numerical analysis for the special case of unitary noise which leads us to conjecture that the relation

$$
F_n^{\text{gen}} \simeq F_{2n}^{\text{echo}} \tag{31}
$$

should hold in general for large n.

4. Generalized fidelities for continuous-time weak noise

We describe our system by the Markovian master equation [18, 19]

$$
\frac{\text{d}}{\text{d}t}\rho = -\text{i}[H_C(t), \rho] + \epsilon\hat{L}(\rho) \tag{32}
$$

where $H_C(t)$ governs a controlled reversible part of the dynamics and the generator

$$
\hat{L}\rho \equiv L(\rho) = -\text{i}[H, \rho] + \frac{1}{2}\sum_\alpha([V_\alpha, \rho V_\alpha^\dagger] + [V_\alpha\rho, V_\alpha^\dagger]) \tag{33}
$$

with the condition $\text{Tr}\,H = \text{Tr}\,V_\alpha = 0$ (which fixes the decomposition of $\hat{L}$ into Hamiltonian and dissipative parts [19]) describes all sources of imperfections and noise. Here $0 < \epsilon \ll 1$ is a small parameter characterizing noise strength.

The time dependent fidelity of the initial state ϕ is given by

$$
F_\phi(t) = \langle\phi|\mathbf{T}\exp\left\{\epsilon\int_0^t\hat{L}(s)\,\text{d}s\right\}(|\phi\rangle\langle\phi|)|\phi\rangle \tag{34}
$$

where $\mathbf{T}$ denotes the chronological order, and

$$
\begin{aligned}
&\hat{L}(s) = \hat{U}^\dagger(s, 0)\hat{L}\hat{U}(s, 0), \\
&U(t, s) = \mathbf{T}\exp\left\{-\text{i}\int_s^t H_C(u)\,\text{d}u\right\}.
\end{aligned} \tag{35}
$$

Using the notation

$$
\hat{\Gamma}(t) = \mathbf{T}\exp\left\{\epsilon\int_0^t\hat{L}(s)\,\text{d}s\right\} \tag{36}
$$

we can write down the following 'cumulant expansion' of the dynamics with respect to the small parameter ϵ:

$$
\hat{\Gamma}(t) = \exp\{\epsilon\hat{K}_1(t) + \epsilon^2\hat{K}_2(t) + \cdots\}. \tag{37}
$$

Using the Wilcox formula for the matrix-valued functions

$$
\begin{aligned}
&\frac{\text{d}}{\text{d}x}\exp A(x) \\
&= \left(\int_0^1\exp(\lambda A(x))\frac{\text{d}}{\text{d}x}A(x)\exp(-\lambda A(x))\,\text{d}\lambda\right)\exp A(x)
\end{aligned} \tag{38}
$$

one obtains

$$
\begin{aligned}
\hat{K}_1(t) &= \int_0^t\hat{L}(s)\,\text{d}s, \\
\hat{K}_2(t) &= \frac{1}{2}\int_0^t\text{d}s\int_0^s\text{d}u[\hat{L}(s), \hat{L}(u)].
\end{aligned} \tag{39}
$$

We assume now the following *ergodic hypothesis*: (a) the ergodic mean exists and is equal to the Haar average

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T\hat{L}(t)\,\text{d}t = \hat{L}^{\text{ave}} = \int_{U(D)}\text{d}U\,\hat{U}\hat{L}\hat{U}^\dagger; \tag{40}
$$

(b) the fluctuations $\delta\hat{L}(t) \equiv \hat{L}(t) - \hat{L}^{\text{ave}}$ around the ergodic mean are *normal*, i.e. for long t

$$
\left\|\int_s^{s+t}\delta\hat{L}(u)\,\text{d}u\right\| \sim t^{1/2}. \tag{41}
$$

These conditions are satisfied, for instance if the time dependent dynamics $t \mapsto U(t)$ can be modelled by a random walk on the group $U(D)$ or by a trajectory on $U(D)$ given by a certain deterministic dynamics with strong enough ergodic properties. The norm of $\hat{K}_2(t)$ can be estimated using (41)

$$
\begin{aligned}
&\|\hat{K}_2(t)\| = \frac{1}{2}\left\|\int_0^t\text{d}s\int_0^s\text{d}u\right. \\
&\left.([\delta\hat{L}(s), \delta\hat{L}(u)] + [\delta\hat{L}(s), \hat{L}^{\text{ave}}] + [\hat{L}^{\text{ave}}, \delta\hat{L}(u)])\right\| \sim t^{3/2}.
\end{aligned} \tag{42}
$$

Therefore for small enough ϵ and long enough times t such that ϵt is fixed, the first term dominates and we can write

$$
\hat{\Gamma}(t) \simeq \exp\left(\epsilon\int_0^t\hat{L}(s)\,\text{d}s\right). \tag{43}
$$

Then replacing $\hat{\Gamma}(t)$ by $\exp(\epsilon \hat{L}_{\text{av}} t)$ and using the explicit expression (56), (57) we obtain the universal exponential decay of the fidelity

$$F_\phi(t) \simeq e^{-\gamma t} + \frac{1}{D}(1 - e^{-\gamma t}),$$

$$\gamma = \frac{D}{2(D^2 - 1)} \sum_\alpha \text{Tr}(|V_\alpha|^2). \qquad (44)$$

5. Discussion

We have described how generalized Haar-averaged fidelities may be directly estimated with only a few experimental measurements. By implementing a motion reversal sequence with a Haar-random unitary transformation, the observed fidelity decay provides a direct experimental estimate of the intrinsic strength of the noise. Moreover, because the target transformation is a Haar-random unitary, the cumulative noise measured by this method will not be biased by any special symmetries of the target transformation.

The only inefficiency of our protocol is the requirement of experimentally implementing a Haar-random unitary: the decomposition into elementary one- and two-qubit gates requires an exponentially long gate sequence [12]. However, the randomization provided by Haar-random unitary operators may be unnecessarily strong and this leads to the open question of whether efficient random circuits [20, 21] can provide an adequate degree of randomization for the above protocols. Indeed the experimental results of [2] suggest that even a structured transformation such as the quantum Fourier transform is sufficiently complex to approximately average the cumulative noise to an effective depolarizing channel, and from studies of quantum chaos it is known that efficient chaotic quantum maps are faithful to the universal Haar-averaged fidelity decay under imperfect motion reversal [14]. While more conclusive evidence is needed to answer this question, it appears likely that the inefficiency associated with implementing Haar-random unitary operators may be overcome.

An additional question is whether the implementation of random unitary operators (e.g., Haar-random unitary operators or even efficient random circuits) leads to an even stronger form of averaging. We have throughout our analysis made the usual assumption that the noise superoperator Λ is independent of the specific target transformation but depends only on the duration of the experiment. However, it is known that the actual noise in general depends sensitively on the choice of target transformation U. Moreover, the cumulative noise operator generally also depends on the particular sequence of elementary one- and two-qubit gates applied to generate U. For example, the implementation of the quantum Fourier transform [1, 2] will generate very different cumulative noise to the trivial implementation of the identity operator $U = \mathbb{I}$ for the same time τ. However, it appears likely that the cumulative noise operators, and in particular their intrinsic noise strength, under a specific but random gate sequence should become concentrated about an average value depending only on the length of the sequence. If this is the case, then the usual assumption that the noise is independent of the actual gate sequence becomes statistically well motivated,

and the measured fidelity under motion reversal can provide a benchmark of an intrinsic noise strength that is fully independent of the target unitary.

Appendix. Haar-averaged superoperators

We consider a linear superoperator $\hat{\Lambda}$ acting on the space $\mathbf{M}_D$ of $D \times D$ complex matrices treated as a Hilbert space with a scalar product $(X, Y) = \text{Tr}(X^\dagger Y)$. The superoperator $\hat{\Lambda}$ has a $D^2 \times D^2$-dimensional matrix representation and $\text{Tr}\,\hat{\Lambda}$ denotes the usual sum over the diagonal elements of the matrix. For clarity of notation we will sometimes express the linear operation $\hat{\Lambda}\rho$ in the form $\Lambda(\rho)$. By $\{|k\rangle\}$ we denote an orthonormal basis in $\mathbf{C}^D$ while $\{E_{kl} = |k\rangle\langle l|\}$ is a corresponding basis in $\mathbf{M}_D$. The group $U(D)$ of unitary $D \times D$ matrices has its natural unitary representation on $\mathbf{M}_D$ defined by

$$U(D) \ni U \mapsto \hat{U}, \qquad \hat{U}X = UXU^\dagger. \qquad (45)$$

This representation is reducible and implies the decomposition of $\mathbf{M}_D$ into two irreducible invariant subspaces:

$$\mathbf{M}_D = \mathbf{M}_D^c \oplus \mathbf{M}_D^0, \qquad \mathbf{M}_D^0 = \{X \in \mathbf{M}_D; \text{Tr}\,X = 0\}, \qquad (46)$$

$$\mathbf{M}_D^c = \{X = c\,\mathbb{I}\},$$

where c is an arbitrary complex number.

Any superoperator $\hat{\Lambda}$ possesses exactly two linear $U(D)$ invariants, i.e. the linear functionals on superoperator space which are invariant with respect to all transformations of the form $\hat{\Lambda} \mapsto \hat{U}\hat{\Lambda}\hat{U}^\dagger$:

$$\text{Tr}[\Lambda(\mathbb{I})] = \sum_{k=1}^D \langle k|\Lambda(\mathbb{I})|k\rangle \qquad (47)$$

and

$$\text{Tr}(\hat{\Lambda}) \equiv \sum_{k,l=1}^D (E_{kl}, \Lambda(E_{kl})) = \sum_{k,l=1}^D \langle k|\Lambda(E_{kl})|l\rangle. \qquad (48)$$

Example. Take $\Lambda(X) = AXB$; then $\text{Tr}[\Lambda(\mathbb{I})] = \text{Tr}(AB)$ and $\text{Tr}(\hat{\Lambda}) = \text{Tr}(A)\,\text{Tr}(B)$.

The following lemma completely characterizes $U(D)$-invariant trace-preserving superoperators.

Lemma 1. *Let $\hat{\Lambda}^{\text{inv}}$ be a $U(D)$ invariant, i.e. $\hat{\Lambda}^{\text{inv}} = \hat{U}\hat{\Lambda}^{\text{inv}}\hat{U}^\dagger$ for any $U \in U(D)$, trace-preserving operator. Then*

$$\hat{\Lambda}^{\text{inv}} X \equiv \Lambda^{\text{inv}}(X) = pX + (1 - p)\,\text{Tr}(X)\frac{\mathbb{I}}{D}, \qquad (49)$$

where

$$p = \frac{\text{Tr}(\hat{\Lambda}^{\text{inv}}) - 1}{D^2 - 1}. \qquad (50)$$

Proof. Schur's lemma implies the form (49) for $U(D)$-invariant trace-preserving operators. From the normalization $\mathrm{Tr}[\Lambda^{\mathrm{inv}}(\mathbb{I})] = D$ for the trace, the detailed expression (50) can be explicitly calculated by comparing $U(D)$ invariants for both sides of equation (49). $\qquad\square$

The Haar-averaged superoperator corresponding to the noise under the imperfect motion-reversal protocol, averaged over all possible unitary operators, is a $U(D)$-invariant superoperator

$$\hat{\Lambda}^{\mathrm{ave}} = \int_{U(D)} \mathrm{d}U \, \hat{U} \hat{\Lambda} \hat{U}^{\dagger}. \tag{51}$$

where $\mathrm{d}U$ is the normalized Haar measure on $U(D)$. Using lemma 1 we can easily compute the averaged form of the dynamical map for both the Schrödinger operator

$$\Lambda(\rho) = \sum_{\alpha} A_{\alpha} \rho A_{\alpha}^{\dagger}, \qquad \sum_{\alpha} A_{\alpha}^{\dagger} A_{\alpha} = \mathbb{I} \tag{52}$$

and the semigroup generator

$$\hat{L}\rho \equiv L(\rho) = -\mathrm{i}[H, \rho] + \tfrac{1}{2} \sum_{\alpha} ([V_{\alpha}, \rho V_{\alpha}^{\dagger}] + [V_{\alpha}\rho, V_{\alpha}^{\dagger}]) \tag{53}$$

with the condition $\mathrm{Tr}\, H = \mathrm{Tr}\, V_{\alpha} = 0$ which fixes the decomposition of $\hat{L}$ into Hamiltonian and dissipative parts. From the fact that $\mathrm{Tr}(\hat{\Lambda}^{\mathrm{ave}}) = \mathrm{Tr}(\hat{\Lambda})$ we obtain

$$\Lambda^{\mathrm{ave}}(\rho) = p\rho + (1-p)\,\mathrm{Tr}(\rho)\frac{\mathbb{I}}{D} \tag{54}$$

where

$$p = \frac{\mathrm{Tr}(\hat{\Lambda}) - 1}{D^2 - 1} = \frac{\sum_{\alpha} |\mathrm{Tr}(A_{\alpha})|^2 - 1}{D^2 - 1}. \tag{55}$$

Similarly for the generator we obtain

$$\hat{L}^{\mathrm{ave}}\rho \equiv L^{\mathrm{ave}}(\rho) = -\gamma\left(\rho - \mathrm{Tr}(\rho)\frac{\mathbb{I}}{D}\right) \tag{56}$$

where

$$\gamma = \frac{D}{2(D^2 - 1)} \sum_{\alpha} \mathrm{Tr}(|V_{\alpha}|^2). \tag{57}$$

Note added in proof. Additional evidence for the conjectured relation (31) can be found in [22].

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