J. DIFFERENTIAL GEOMETRY
59 (2001) 479-522

SCALAR CURVATURE AND PROJECTIVE EMBEDDINGS, I

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Abstract

We prove that a metric of constant scalar curvature on a polarised Kähler manifold is the limit of metrics induced from a specific sequence of projective embeddings; satisfying a condition introduced by H. Luo. This gives, as a Corollary, the uniqueness of constant scalar curvature Kähler metrics in a given rational cohomology class. The proof uses results in the literature on the asymptotics of the Bergman kernel. The arguments are presented in a general framework involving moment maps for two different group actions.

1. Introduction

Two classical results are:

(1) A compact Riemann surface can be embedded in complex projective space, as a complex algebraic curve;

(2) A compact Riemann surface admits a metric of constant curvature, unique up to the action of the holomorphic automorphisms of the surface.

It is natural to ask if there are connections between these two renowned results, linking the algebro-geometric and differential-geometric points of view. Specifically, a complex algebraic curve in $\mathbf{CP}^N$ has a metric induced from the Fubini-Study metric on projective space and one can ask how metrics of this kind are related to the constant-curvature metric. This is the question we study here, not just for a Riemann surface but for a polarised manifold of any dimension, i.e., a compact complex manifold X of complex dimension n with a given positive line bundle $L \to X$.

Received November 13, 2001.

The main motivation for the work comes from the questions about the existence and uniqueness of Kähler metrics of constant *scalar curvature* on X—generalising the constant curvature metrics in the Riemann surface case. (In the case when the line bundle is a power of the canonical bundle of the manifold the metrics involved are Kähler-Einstein and the issues we investigate here fall very much into the mould of ideas which have been propounded by S.-T. Yau over many years [11].) The thrust of our results is to establish a precise correspondence between constant scalar curvature Kähler metrics and the asymptotics of *sequences* of projective embeddings, where one replaces L by L^k and lets $k \to \infty$.

A central role in this paper is a played by an idea due to Luo [9]. Let $[Z_0, \ldots, Z_N]$ be standard homogeneous coordinates on $\mathbf{CP}^N$ and define

$$b_{\alpha\beta} = \frac{Z_\alpha \overline{Z_\beta}}{\|Z\|^2},$$

where $\|Z\|^2 = \sum |Z_\alpha|^2$. Clearly the $b_{\alpha\beta}$ define functions on $\mathbf{CP}^N$. If $V \subset \mathbf{CP}^N$ is any projective variety we define $M(V)$ to be the skew-adjoint $(N+1) \times (N+1)$ matrix with entries

$$M(V)_{\alpha\beta} = i \int_V b_{\alpha\beta}\, d\mu_V, \tag{1}$$

where $d\mu_V$ is the standard measure on V induced by the Fubini-Study metric. Another way of expressing this is that we take the standard embedding of $\mathbf{CP}^N$ in the Lie algebra $\mathfrak{u}(N+1)$, as a co-adjoint orbit: then $M(V) \in \mathfrak{u}(N+1)$ is the centre of mass of $(V, d\mu_V)$ in the Euclidean space $\mathfrak{u}(N+1)$. Following Luo, we study projective varieties V such that $M(V)$ is a multiple of the identity matrix—or in other words the projection to $\mathfrak{su}(N+1)$ is zero. We will call such a variety a "balanced variety" in $\mathbf{CP}^N$. The main result of Luo is that a balanced variety is *stable* in the sense of Hilbert-Mumford.

The hypothesis we will make throughout this paper on the polarised manifold (X, L) is that the group $\mathrm{Aut}\,(X, L)$ of holomorphic automorphisms of the pair (X, L) (modulo the trivial automorphisms $\mathbf{C}^*$) is *discrete*; i.e., the Lie algebra of this group is trivial. (In the sequel to this paper we will remove this condition.) We consider the powers L^k of the positive line bundle $L \to X$ for large k. The Kodaira embedding theorem asserts that for large enough k the sections $H^0(L^k)$ define a projective embedding

$$\iota_k : X \to \mathbf{P}(H^0(L^k)^*),$$

and of course a choice of basis in $H^0(L^k)$ identifies the target space with the standard $\mathbf{CP}^N$, where $N = N_k$ is (for large enough k) given by the Riemann-Roch formula

(2)
$$N_k + 1 = \chi(L^k) = \int_X \exp(kc_1(L))\mathrm{Td}\ (X) = a_0 k^n + a_1 k^{n-1} + \cdots + a_n.$$

Here the coefficients a_i are topological invariants of (X, L), the salient ones for this paper being the leading coefficient

$$a_0 = \frac{1}{n!}\int_X c_1(L)^n, \tag{3}$$

and the subsequent term

$$a_1 = \frac{1}{2(n-1)!}\int_X c_1(L)^{n-1}c_1(X). \tag{4}$$

We say that (X, L^k) is balanced if one can choose a basis in $H^0(L^k)$ such that $\iota_k(X)$ is a balanced variety in $\mathbf{CP}^N$. Our first result is:

Theorem 1. *Suppose* $\mathrm{Aut}\,(X, L)$ *is discrete. If* (X, L^k) *is balanced then the choice of basis in* $H^0(L^k)$ *such that* $\iota_k(X)$ *is balanced is unique up to the action of* $\mathrm{U}(N+1) \times \mathbf{R}^*$.

Another way of expressing this is that, if (X, L^k) is balanced, there is a unique Hermitian metric, up to scale, on $H^0(L^k)$ such that $\iota_k(X)$ is balanced if one chooses an orthonormal basis with respect to this metric. In turn, there is a well-defined Kähler metric $\iota_k^*(\omega_{FS})$ on X obtained by the restriction of the Fubini-Study metric on $\mathbf{CP}^N$. We normalise this by setting

$$\omega_k = \frac{2\pi}{k}\iota_k^*(\omega_{FS}), \tag{5}$$

so the cohomology class $[\omega_k] = 2\pi c_1(L)$ in $H^2(X)$ is independent of k. We will prove:

Theorem 2. *Suppose that* $\mathrm{Aut}\,(X, L)$ *is discrete and* (X, L^k) *is balanced for all sufficiently large* k*. Suppose that the metrics* ω_k *converge in* C^∞ *to some limit* ω_∞ *as* $k \to \infty$*. Then* ω_∞ *has constant scalar curvature.*

The main result of this paper is a converse:

Theorem 3. *Suppose that* $\mathrm{Aut}\,(X, L)$ *is discrete and that* ω_∞ *is a Kähler metric in the class* $2\pi c_1(L)$ *with constant scalar curvature. Then* (X, L^k) *is balanced for large enough* k *and the sequence of metrics* ω_k *converges in* C^∞ *to* ω_∞ *as* $k \to \infty$.

To put these results in context, following the work of Yau, Tian and others in the Kalher-Einstein case (when L is the canonical bundle K_X) and the general formal picture described in [3], [4], one hopes that the existence of a constant scalar curvature metric should be related to some appropriate algebro-geometric notion of stability. The theorems stated above show that this question can be reduced to, on the one hand, the finite-dimensional issue of the relation between the balanced condition and stability and, on the other hand, to the question of the convergence of the metrics ω_k as $k \to \infty$. In principle one might be able to prove the existence of constant scalar curvature metrics by showing directly that the ω_k converge—avoiding PDE theory—but it is hard to see how one might go about this. Even in the classical case of Riemann surfaces it is hard to see how one could obtain this convergence without knowing the existence of the constant curvature metric.

We can obtain two new results as Corollaries of Theorem 3. First, using Luo's work, we have:

Corollary 4. *If* $\mathrm{Aut}\,(X, L)$ *is discrete and* X *admits a Kähler metric of constant scalar curvature in the class* $2\pi c_1(L)$ *then the embedding* $\iota_k(X)$ *is stable in the sense of Hilbert-Mumford for large enough* k.

Second, we have:

Corollary 5. *Suppose* $\mathrm{Aut}\,(X, L)$ *is discrete. Then there is at most one Kähler metric of constant scalar curvature in the cohomology class* $2\pi c_1(L)$.

This follows immediately from Theorems 1 and 3.

We will begin the body of this paper, in Section 2 below, by setting up a formal context for the work. Following the philosophy of [3], [4] we express the equations we want to solve as the vanishing of "moment maps" for appropriate symmetry groups. A key point is that there are two groups involved in the discussion: a finite dimensional unitary group and an infinite dimensional symplectomorphism group. These two groups tie up with the algebro-geometric and differential-geometric points of view respectively, and the interaction between the two is connected to the classical limit in geometric quantisation (with k^{-1} playing the role of Planck's constant). This interaction, which we will study further in the sequel, forms the main theme of the paper and raises a variety of further questions. For example, in the case when $c_1(X) < 0$, X. Chen has proved the uniqueness result of Corollary 5 [2]. His proof involves the construction of a "geodesic path" in the space of Kähler metrics joining any two given points, and the restriction to $c_1(X) < 0$ is due to technical complications arising from the fact that this path is not known to be smooth. Spelling out our proof of Corollary 5 our argument involves joining two points by a geodesic in the symmetric space $\mathrm{SL}(N+1;\mathbf{C})/\mathrm{SU}(N+1)$ for large $N = N_k$. One might expect that these two approaches are fundamentally the same in that the geodesics in the space of Kalher metrics are approximated by geodesics in the symmetric space as $k \to \infty$; or, in other words, that our argument is the "quantisation" of Chen's.

The two symmetry groups of the problem alluded to above lead to two ways of thinking about the balanced condition. On the one hand it appears as a condition involving finite-dimensional matrix groups. On the other hand it can be expressed in terms of Kähler geometry, which brings out the connection with scalar curvature. Suppose we have any fibre metric h on L such that the corresponding curvature form is $-i\omega$, where $\omega > 0$. The fibre metric h is uniquely determined by ω, up to a constant scalar multiple. Using ω as a Kähler metric on X, we get a standard L^2-inner product on $H^0(L^k)$. Let s_α be an orthonormal basis of this vector space and define a function $\rho_k(\omega)$ on X by

$$\rho_k(\omega) = \sum_\alpha |s_\alpha|^2. \tag{6}$$

A moment's thought shows that this function does not depend on the choice of orthonormal basis. It is also unchanged if we replace h by a constant scalar multiple. Thus, as the notation suggests, it is an invariant of the Kähler form ω. The balanced condition for (X, L^k) is equivalent to the existence of a metric ω_k such that the function $\rho_k(\omega_k)$ is *constant* on X. Now the asymptotic behaviour of the "density of states" function $\rho_k(\omega)$ as $k \to \infty$, with ω fixed, has been studied by Catlin [1], Tian [10], Zelditch [12] and Lu [8]. Notice that, for any metric ω

$$\int_X \rho_k(\omega)\, d\mu = \dim\ H^0(L^k) = a_0 k^n + a_1 k^{n-1} + \dots,$$

and from the Chern-Weil theory the coefficients a_i in this Riemann-Roch formula can be expressed as integrals involving the curvature of ω. The first two coefficients are given by (3) and (4): the first coefficient a_0 is just the Riemannian volume of X and

$$a_1 = \frac{1}{2\pi}\int_X S(\omega)d\mu, \tag{7}$$

where $S(\omega)$ is the scalar curvature of ω. The result we need is:

Proposition 6.

(1) *For fixed ω, there is an asymptotic expansion as $k \to \infty$*

$$\rho_k(\omega) \sim A_0(\omega)k^n + A_1(\omega)k^{n-1} + \dots,$$

where $A_i(\omega)$ are smooth functions on X defined locally by ω.

(2) *In particular*

$$A_0(\omega) = 1,\ A_1(\omega) = \frac{1}{2\pi}S(\omega).$$

(3) *The expansion holds in C^∞ in that for any $r, N \geq 0$*

$$\left\| \rho_k(\omega) - \sum_{i=0}^{N} A_i(\omega)k^{n-i} \right\|_{C^r(X)} \leq K_{r,N,\omega}k^{n-N-1}$$

for some constants $K_{r,N,\omega}$. Moreover the expansion is uniform in that for any r, N there is an integer s such that if ω runs over a set of metrics which are bounded in C^s, and with ω bounded below, the constants $K_{r,N,\omega}$ are bounded by some $K_{r,N}$ independent of ω.

Here part (1) is the result of Catlin [1] and Zelditch [12] and part (2) is proved by Lu in [8]. (It is also clear from Lu's work that all the functions $A_{\omega,i}$ are polynomials in the curvature of ω and its covariant derivatives.) The uniformity statement in part (3) is not given explicitly in those references but can be obtained by tracing through the same arguments (the author is grateful to Professor Zelditch for advice on this point). The sequel to this paper will contain a further discussion of these asymptotic expansions.

The existence of this asymptotic expansion strongly suggests that results like Theorems 2 and 3 should be true. In fact Theorem 2 is a straightforward consequence of Proposition 6. To prove Theorem 3 we have to obtain a solution of the equation $\rho_{\omega,k} =$ Constant, for large k, starting from solution of the equation $A_1(\omega) =$ Constant, this being the first effective term in the asymptotic expansion. The proof involves two main parts; in each of which a certain linear differential operator

$$\mathcal{D}^*\mathcal{D} : C^\infty(X) \to C^\infty(X) \tag{8}$$

plays a vital role. In one part, we construct a formal power series solution to the problem. This is straightforward (given the results of Catlin, Lu and Zelditch), and is done in Section 4.1 below. However, there is no reason to suppose that this formal power series converges, so in the other, more substantial, part of the proof we pass to the other point of view and solve an equation in a large but finite-dimensional matrix group, essentially by means of an implicit function theorem. This is done is Section 3 below, which is the heart of the paper.

2. Preliminaries

2.1 Formal set-up

In this Section we will explain how the problems we address can be put in the framework of "moment map geometry", in the spirit of [3], [5]. To begin with we consider a general compact symplectic manifold (M,ω), of dimension $2n$, and suppose $L \to M$ is a Hermitian line bundle with connection, having curvature $-i\omega$. As usual, the space of functions $C^\infty(M)$ is a Lie algebra under the Poisson bracket. The Hamiltonian construction maps a function f to the symplectic vector field ξ_f with $i_{\xi_f}(\omega) = df$, and this is a Lie algebra homomorphism. For each integer k the space $\Gamma(L^k)$ of sections of L^k has a standard L^2 norm

$$\|s\|^2 = \int_M |s|^2 d\mu,$$

where $d\mu$ is the volume form $\frac{\omega^n}{n!}$. For any function f on M and section s of $L^k \to M$ we set

$$R_f(s) = \nabla_{\xi_f}(s) - ikfs. \tag{9}$$

Proposition 7. *The map $f \mapsto R_f$ defines a unitary action of the Lie algebra $C^\infty(M)$ on the complex vector space $\Gamma(L^k)$.*

This is a standard result and we omit the proof, which is a straightforward exercise (see [6], Sect. (6.5.1)). We define the group $\mathcal{G}$ to be the group of Hermitian bundle maps from L to L which preserve the connection. Then the Lie algebra of $\mathcal{G}$ is $C^\infty(M)$ and the action of $\mathcal{G}$ on $\Gamma(L^k)$ induces the Lie algebra action of Proposition 7.

The space of sections $\Gamma(L^k)$ has a natural symplectic form Ω associated to the Hermitian metric:

$$\Omega(s_1, s_2) = \mathrm{Re} \int_M (is_1, s_2)\ d\mu. \tag{10}$$

Thus it makes sense to ask for a *moment map* for the action of $\mathcal{G}$ on $\Gamma(L^k)$. Given a section s of L^k we write

$$\mu(s) = \frac{1}{n!}\left(-\frac{i}{2}\nabla s \wedge \nabla \overline{s} \wedge \omega^{n-1} + k|s|^2\omega^n\right). \tag{11}$$

Here $\overline{s}$ is the section of the dual bundle L^{-k} defined using the standard anti-linear isomorphism between L^k and L^{-k} and in (11) we tacitly use the dual pairing between L^k, L^{-k}. Thus we have a map μ from $\Gamma(L^k)$ to the vector space of $2n$-forms on M, which in turn has a natural pairing with the Lie algebra $C^\infty(M)$.

Proposition 8. *The map μ is an equivariant moment map for the action of $\mathcal{G}$ on $\Gamma(L^k)$.*

First, it is clear that μ is $\mathcal{G}$-equivariant. The identity which has to be established to show that it is a moment map is

$$\delta\ \langle \mu(s), f\rangle = \Omega(R_f s, \delta s), \tag{12}$$

where δ denotes the functional derivative with respect to s. To verify this, let us write $\delta s \sigma$ so the left hand side of (12) is a sum, $A+B$ say, where

$$A = -\frac{i}{2n!}\int_M \nabla s \wedge \nabla\overline{\sigma} \wedge \omega^{n-1} + \nabla\sigma \wedge \nabla\overline{s} \wedge \omega^{n-1},$$

and

$$B = \frac{k}{n!}\int_M (s\ \overline{\sigma} + \overline{s}\ \sigma)\ \omega^n.$$

Then we can apply Stokes Theorem to write

$$A = \frac{-i}{2n!}\int_M (df \wedge \nabla s\ \overline{\sigma} - df \wedge \nabla\overline{s}\ \sigma)\omega^{n-1} + f(d_\nabla(\nabla s)\ \overline{\sigma} - d_\nabla(\nabla \overline{s})\ \sigma)\omega^{n-1}.$$

We have the curvature identities on $L^{\pm k}$

$$d_\nabla(\nabla s) = -ik\omega s, \quad d_\nabla(\nabla \overline{s}) = ik\omega \overline{s},$$

so

$$A = \frac{-i}{2n!}\int_M df \wedge (\overline{\sigma}\nabla s - \sigma \nabla \overline{s})\omega^{n-1} - ikf(s\overline{\sigma} + \overline{s}\sigma)\omega^n.$$

Thus the sum $A + B$ is

$$\frac{-i}{2n!}\int_M df \wedge (\nabla s\overline{\sigma} - \sigma \nabla \overline{s})\omega^{n-1} + ikf(s\overline{\sigma} + \overline{s}\sigma)\omega^n.$$

Now it is easy to check that one has an identity

$$\nabla_{\xi_f} s\ \omega^n = -df \wedge \nabla s \wedge \omega^{n-1}, \tag{13}$$

so we can write

$$A + B = \frac{i}{2}\int_M \left[\left(\nabla_{\xi_f} s - ikfs\right)\overline{\sigma} - \overline{\left(\nabla_{\xi_f} s - ikfs\right)}\sigma\right] \frac{\omega^n}{n!},$$

and this is precisely $\Omega(R_f s, \sigma)$, as required.

Now suppose that there is a compatible complex struture on the manifold M, making it into a Kähler manifold with Kähler form ω. The line bundle L has curvature of type $(1,1)$ and is thus endowed, by the connection, with a holomorphic structure. Suppose that the section s of L^k is also holomorphic. In this situation we can write the moment map $\mu(s)$ in a different way.

Lemma 9. *If s is a holomorphic section over a Kähler manifold then*

$$\mu(s) = (\tfrac{1}{2}\Delta|s|^2 + k|s|^2)\tfrac{\omega^n}{n!},$$

where $\Delta = d^*d$ is the usual Laplace operator on functions.

To see this we observe that, in this situation, we can write

$$\nabla s \wedge \nabla \overline{s} \wedge \omega^{n-1} = \frac{i}{2}|\nabla s|^2 \omega^n.$$

Now

$$\Delta|s|^2 = -2|\nabla s|^2 + 2(s, \nabla^*\nabla s)$$

and for a holomorphic section s the Weitzenböck formula gives

$$\nabla^*\nabla s = 2\overline{\partial}^*\overline{\partial}s + kns = kns.$$

So

$$\tfrac{1}{2}\Delta|s|^2 = kn|s|^2 - |\nabla s|^2$$

and the result follows.

The action of the group $\mathcal{G}$ on $\Gamma(L^k)$ does not preserve the set of holomorphic sections, for a fixed holomorphic structure. To get around this we consider, as in [3], the set $\mathcal{J}_{\mathrm{int}}$ of all compatible complex structures on the manifold M. This is a subset of the set $\mathcal{J}$ of all compatible *almost-complex* structures—the cross-sections of a fibre bundle over M with fibre $\mathrm{Sp}(2n, \mathbf{R})/\mathrm{U}(n)$. (Thus $\mathcal{J}_{\mathrm{int}} \subset \mathcal{J}$ is the set where the Nijenhuis tensor vanishes.) The group $\mathcal{G}$ acts on $\mathcal{J}$, preserving $\mathcal{J}_{\mathrm{int}}$. At the Lie algebra level this action is given as follows. We identify the tangent space to $\mathcal{J}$ at a given almost-complex structure I with the subspace of $\Omega^{0,1}(T)$ given by the sections of the bundle of symmetric tensors $s^2(\overline{T}^*)$ (embedded in $\Omega^{0,1}(T)$ by the ismorphism of T with $\overline{T}^*$ furnished by the metric). Then the infinitesimal action of a function f in $C^\infty(M)$ on $\mathcal{J}$ is given, at a point of $\mathcal{J}_{\mathrm{int}}$, by

$$\mathcal{D}f = \overline{\partial}(\xi_f), \tag{14}$$

where

$$\overline{\partial} : \Gamma(T) \to \Omega^{0,1}(T)$$

is the $\overline{\partial}$-operator on the tangent bundle defined by the given complex structure (see [3], Lemma 10). To display the nature of the operator $\mathcal{D}$ more clearly, we can choose osculating complex coordinates z_λ at a given point on M. Then, at this point,

$$\mathcal{D}f = \sum_{\lambda\mu} \frac{\partial^2 f}{\partial \overline{z}_\lambda \partial \overline{z}_\mu} d\overline{z}_\lambda d\overline{z}_\mu.$$

With this background in place, we consider the diagonal action of $\mathcal{G}$ on the product $\Gamma(L^k) \times \mathcal{J}_{\mathrm{int}}$. Let $\mathcal{H} \subset \Gamma(L^k) \times \mathcal{J}_{\mathrm{int}}$ be the subset consisting of pairs (s, I) where s is a nontrivial holomorphic section with respect to the complex structure I.

Lemma 10. *The diagonal action of $\mathcal{G}$ preserves $\mathcal{H}$.*

On the one hand, this is obviously true on general grounds since the group $\mathcal{G}$ acts as automorphisms of the line bundle L^k, covering symplectomorphisms of M and the action of $\mathcal{G}$ on $\mathcal{J}$ is just the natural action of these symplectomorphisms of M. Alternatively, the statement of the Lemma follows from a differential-geometric identity

$$(\mathcal{D}f)\partial_{L^k}s + \overline{\partial}_{L^k}(R_f s) = 0, \tag{15}$$

for any function f on M and I-holomorphic section s. Here $\partial_{L^k}, \overline{\partial}_{L^k}$ are defined in the usual way using the connection and and the pairing $(\mathcal{D}f)\partial_{L^k}s$ is the algebraic pairing

$$\Omega^{0,1}(T) \otimes \Omega^{1,0}(L^k) \to \Omega^{0,1}(T)$$

defined by the duality between T and $\Omega^{1,0}$.

We introduce an equivalence relation on $\mathcal{H}$ by decreeing that $(s, I) \sim (s', I')$ if there is a bundle-map $\hat{F}$ of L^k (regarded as a $\mathbf{C}^*$-bundle, and forgetting the connection) covering a diffeomorphism F of M, taking s to s' and I to I'. This is just saying that the two pairs are equivalent in the usual sense as pairs "complex manifold plus holomorphic section of a holomorphic line bundle". We notice that the action of $\mathcal{G}$ preserves the equivalence classes, since it is defined by an action on the line bundle L. We also notice that $\Gamma(L^k)$ and $\mathcal{J}_{\text{int}}$ both have natural *complex structures* (that in the latter case being discussed in [3]). Now, in the spirit of [4], we have:

Proposition 11. *The equivalence classes in $\mathcal{H}$ are formal orbits of the action of the complexified group $\mathcal{G}^c$ on $\mathcal{H}$.*

Just as in [4], the point here that while there is in reality no complexified group $\mathcal{G}^c$ the equivalence classes play the role of the orbits of the action, if such a group did exist. More precisely, we mean that each equivalence class $\mathfrak{o}$ in $\mathcal{H}$ is an infinite-dimensional manifold and the tangent space of $\mathfrak{o}$ at a point (s, I) is the complexification $\mathcal{T} + J\mathcal{T}$ where $\mathcal{T}$ is the tangent space to the $\mathcal{G}$-orbit and J is the complex structure defined by those on $\Gamma(L^k)$ and $\mathcal{J}$.

Having thus clarified the meaning of Proposition 11 we can now proceed to the proof. We will be content with a somewhat formal argument, as in the similar statements in [3], [4], [5], since we will make use of the statement at this level in the proofs of our main results. Thus we pass to the corresponding discussion at the level of tangent vectors where we suppose that we have a $\mathbf{C}^*$-invariant vector field $\hat{v}$ on the total space of L^k covering a vector field v on M. The infinitesimal action of $\hat{v}$ on a section s of L^k is a section $\sigma = L_{\hat{v}}s$ and the action of v on the almost-complex structure I is $\mu = L_v I$. We ask when (σ, μ) is a tangent vector in $\mathcal{H}$ at the point (s, I). The condition for this, analogous to (15), is

$$\mu \partial_{L^k} s + \overline{\partial}_{L^k} \sigma = 0. \tag{16}$$

We write

$$\hat{v} = \widetilde{v} + \psi \underline{t},$$

where $\widetilde{v}$ is the horizontal lift of v defined by the connection on L^k, $\underline{t}$ is the canonical vertical vector field on L^k and ψ is a complex-valued function on M. Then we have

$$\sigma = -i_v(\partial_{L^k} s) + \psi s,$$

and $\mu = \overline{\partial} v$. Then

$$\overline{\partial}_{L^k}(\sigma) = -(\overline{\partial} v)\partial_{L^k} s - i_v^{0,1}(\overline{\partial}_{L^k} \partial_{L^k} s) + \overline{\partial}\psi \ s,$$

where $i_v^{0,1}$ denotes the $(0,1)$ part of the contraction. Then (16) gives

$$i_v^{0,1}(\overline{\partial}_{L^k} \partial_{L^k} s) = \overline{\partial}\psi s.$$

Now $\overline{\partial}_{L^k} \partial_{L^k} s = -ik\omega s$ so we obtain

$$\overline{\partial}\psi = k i_v^{0,1}(\omega). \tag{17}$$

If we write $\psi = f + ig$, in real and imaginary parts, then (17) is equivalent to

$$v = k^{-1}(\xi_f + I\xi_g)$$

where ξ_f, ξ_g are the Hamiltonian vector fields of f and g and I denotes the action of complex multiplication on tangent vectors. In turn it follows that

$$k(\sigma, \mu) = (R_f s, \mathcal{D} f) + (iR_g s, I\mathcal{D} g), \tag{18}$$

so (σ, μ) lies in $\mathcal{T} + J\mathcal{T}$. Conversely, for any f, g in $C^\infty(M)$ we can define (σ, μ) by (18) and the formulae above show that (σ, μ) is a tangent vector to the equivalence class, where we define the vector field $\hat{v}_\psi$, for $\psi = f + ig$, on L^k by

$$\hat{v}_\psi = \widetilde{v} + \psi \underline{t}, \tag{19}$$

with $\widetilde{v}$ the horizontal lift of $\xi_f + I\xi_g$.

There is a more concrete way of representing these equivalence classes in $\mathcal{H}$. Suppose $\hat{F}, F$ are diffeomorphisms of the kind considered in the definition of the equivalence relation above. Then the pull-back of the Hermitian metric h_0 on L^k by $\hat{F}$ can be viewed as another metric on the fixed holomorphic bundle L^k. Conversely, if we take any function ϕ on M such that $\omega + i\overline{\partial}\partial\phi$ is a positive form (working with respect to a fixed complex stucture), then we have a triple $(s, I, e^{-\phi}h_0)$ consisting of a holomorphic Hermitian line with a holomorphic section over the fixed complex manifold (M, I). We can choose a diffeomorphism F of M so that $F^*(\omega + i\overline{\partial}\partial\phi) = \omega$, and this lifts to a bundle map which defines a point in the equivalence class of (s, I). The ambiguity in the choice of F and $\hat{F}$ precisely corresponds to the action of $\mathcal{G}$ on $\mathcal{H}$. The conclusion is that, up to the action of $\mathcal{G}$, the points in an equivalence class can be identified with the Hermitian metrics of positive curvature on a fixed holomorphic line bundle.

A result which is related to Proposition 11 and which is central to this paper is:

Lemma 12. *Let I be a point of $\mathcal{J}_{\rm int}$. The map $\psi \mapsto \hat{v}_\psi$ of (19) yields an isomorphism from the kernel of $\mathcal{D}$ acting on the complex-valued functions over M, to the Lie algebra of the group of automorphisms of (L, M, I).*

To prove this, we write an element of the Lie algebra of the automorphisms of (L, M, I) as a vector field $\hat{v}$ on L^k, covering a vector field v on M. We can apply the formulae above, with $\sigma = L_{\hat{v}} s$, but in this case $\mu = 0$ since v preserves the complex structure. It follows as before that there is a function $\psi = f + ig$ on M such that $\sigma = k^{-1}(R_f s + iR_g s)$ but now we have $\mathcal{D}g + i\mathcal{D}g = \mathcal{D}\psi = 0$. Conversely, for any $\psi = f + ig$ with $\mathcal{D}\psi = 0$ the vector field $\widetilde{v} + \psi\underline{t}$ lies in the Lie algebra of the automorphisms of (L, M, I), where $\widetilde{v}$ is the horizontal lift of $k^{-1}(\xi_f + I\xi_g)$.

We can extend the discussion above to obtain an action of $\mathcal{G}$ on

$$\Gamma(L^k) \times \cdots \times \Gamma(L^k) \times \mathcal{J}_{\rm int}\,,$$

taking $N + 1$ copies of $\Gamma(L^k)$. We let $\mathcal{H}_0$ be the subset of this product consisting of those $(s_0, \ldots, s_N; I)$ such that the s_α are holomorphic with respect to the complex structure I and are linearly independent, as elements of $\Gamma(L^k)$. We let N be defined by the Riemann-Roch formula (2), and suppose that k is sufficiently large: so in other words we are considering *bases* for the set of holomorphic sections of L^k defined using the complex structure I. Just as before we obtain a $\mathcal{G}$-action on $\mathcal{H}_0$ and the complexified orbits are equivalence classes under the standard equivalence of pairs "complex manifold with basis for holomorphic sections of a line bundle". From now on we assume that k is so large that the holomorphic sections define a projective embedding of the manifold M, for all complex structures I. (In reality we will be restricting attention presently to a single complex orbit $\mathfrak{o}$ so this assumption will be permissible.) Then the complex orbits in $\mathcal{H}_0$ are labelled by certain projective varieties in $\mathbf{CP}^N$.

Now consider the projection π_1 from $\mathcal{H}_0$ to $\Gamma(L^k)^{N+1}$. We have:

Proposition 12′. *The projection π_1 is an injective immersion of $\mathcal{H}_0$ into $\Gamma(L^k)^{N+1}$.*

(There may be difficulties in interpreting this statement because $\mathcal{H}$ may not be a manifold: again however we can ignore these because we will be presently concerned with a single complex orbit where these difficulties disappear.) The fact that the projection is an injection on $\mathcal{H}_0$ is rather obvious once one unwinds the statement. It just asserts that the complex structure on M is determined by a projective embedding. Similarly for the immersive property: a tangent vector to $\mathcal{H}_0$ has the form $(\sigma_0, \dots, \sigma_N; \mu)$ where $\sigma_\alpha \in \Gamma(L^k)$ and $\mu \in \Omega^{0,1}(T)$ satisfy

$$\overline{\partial}\mu = 0;\ \overline{\partial}_{L^k}\sigma_\alpha + \mu\partial_{L^k}s_\alpha = 0.$$

If this tangent vector projects to zero then all the σ_α vanish, but this imples that μ is also zero, since the derivatives $\partial_{L^k}s_\alpha$ generate $T^*M \otimes L^k$ at each point.

We use Proposition 12′ to define a symplectic (in fact Kähler) structure on $\mathcal{H}_0$. We start with the standard form Ω on $\Gamma(L^k)$, take the sum of this over the $N+1$ copies of $\Gamma(L^k)$ in the usual way and then lift this form up to $\mathcal{H}_0$ using the projection π_1. We write Ω again for the resulting form on $\mathcal{H}_0$.

Proposition 13. *The group $\mathcal{G}$ acts on $\mathcal{H}_0$ preserving the complex structure and Kähler form Ω. The moment map for the action is given by*

$$\mu_{\mathcal{G}}(s_0, \dots, s_N; I) = \left(\tfrac{1}{2}\Delta + k\right)\rho,$$

where $\rho(s_0, \dots, s_N; I) = \sum |s_\alpha|^2$.

Here the calculation of the moment map follows imediately from Proposition 8, since the projection map π_1 is equivariant. (In Proposition 13 we have dropped the distinction between functions on M and $2n$-forms, using the standard volume form to identify the two.)

There is another natural symmetry group acting on $\mathcal{H}_0$. This is the finite-dimensional unitary group $\mathrm{U}(N+1)$ acting on the bases $(s_0, \dots s_N)$. The moment map for this action is a map from $\mathcal{H}_0$ to the skew-adjoint matrices $\mathfrak{u}(N+1)$, and one easily verifies that this is given by

$$\mu_{\mathrm{U}}(s_0, \dots, s_N; I) = i\left(\langle s_\alpha, s_\beta\rangle\right),$$

where $\langle\ ,\ \rangle$ is the usual L^2 inner product. (Here we are identifying the Lie algebra $\mathfrak{u}(N+1)$ with its dual, using the form $(A, B) \mapsto -\mathrm{Tr}\,(AB)$.) Of course $\mathrm{U}(N+1)$ has a genuine complexification $\mathrm{GL}(N+1, \mathbf{C})$ and the action obviously extends to a holomorphic action of this group on $\mathcal{H}_0$. The actions of the groups $\mathcal{G}$ and $\mathrm{U}(N+1)$ on $\mathcal{H}_0$ commute, so we have an action of the product $\mathcal{G} \times \mathrm{U}(N+1)$. The orbits of $\mathcal{G}^c \times \mathrm{GL}(N+1, \mathbf{C})$ are labelled by equivalence classes of pairs (X, L) where L is a positive line bundle over a complex manifold X.

We now turn to symplectic quotients. Each of the groups $\mathcal{G}, \mathrm{U}(N+1)$ has a one-dimensional centre, given by the constant functions and the multiples of the identity matrix respectively, and these act in the same way on $\mathcal{H}_0$. To avoid this duplication, we restrict to the subgroup $\mathrm{SU}(N+1)$ of $\mathrm{U}(N+1)$. The moment map μ_{SU} for this is the projection of μ_{U} to the trace-free matrices:

$$\mu_{\mathrm{SU}}(s_0, \dots, s_N; I) = i\left(\langle s_\alpha, s_\beta\rangle - \frac{1}{N+1}\left(\sum_\gamma \|s_\gamma\|^2\right)\delta_{\alpha\beta}\right). \tag{21}$$

The moment map for the action of the product is just the direct sum $\mu_{\mathcal{G}} \oplus \mu_{\mathrm{SU}}$ of the individual moment maps. For fixed $a > 0$ we may consider the symplectic quotient

$$\mathcal{H}_0//(\mathcal{G} \times \mathrm{SU}(N+1)) = \frac{\mu_{\mathcal{G}}^{-1}(a) \cap \mu_{\mathrm{SU}}^{-1}(0)}{\mathcal{G} \times \mathrm{SU}(N+1)}. \tag{22}$$

We can think about this in two ways, taking the group actions one at a time (this is a general pheomenon for the symplectic quotients of commuting actions). On the one hand we can consider first the symplectic quotient by $\mathcal{G}$:

$$\mathcal{H}_0//\mathcal{G} = \mu_{\mathcal{G}}^{-1}(a)/\mathcal{G}. \tag{23}$$

The action of $\mathrm{SU}(N+1)$ on $\mathcal{H}_0$ induces an action on $\mathcal{H}_0//\mathcal{G}$ and we can take the symplectic quotient for this latter action: this is precisely the same as the quotient (22) for the action of the product. On the other hand, we can interchange the role of the two groups, taking first the $\mathrm{SU}(N+1)$ quotient. The utility of these two points of view stems from the fact that the two partial symplectic quotients are easy to understand. We need a preliminary observation.

Lemma 14. *Suppose $L^k \to X$ is a Hermitian holomorphic line bundle and $s_0, \ldots, s_N$ are holomorphic sections of L^k such that the function $\sum |s_\alpha|^2$ is a nonzero constant on X. Then the curvature of the compatible unitary connection on L^k is $-i\iota^*(\omega_{FS})$ where ω_{FS} is the Fubini-Study form on $\mathbf{C}P^N$ and $\iota : X \to \mathbf{CP}^N$ is the map defined by the sections s_α.*

We leave the proof to the reader.

Proposition 15.

(1) *Any $\mathcal{G}^c$-orbit in $\mathcal{H}_0$ contains a point in $\mu_{\mathcal{G}}^{-1}(a)$, unique up to the action of $\mathcal{G}$.*

(2) *Any $\mathrm{SL}(N+1, \mathbf{C})$ orbit in $\mathcal{H}_0$ contains a point in $\mu_{\mathrm{SU}}^{-1}(0)$, unique up to the action of $\mathrm{SU}(N+1)$.*

For the first part, recall that $\mu_{\mathcal{G}}(s_0, \ldots, s_N; I) \left(\frac{1}{2}\Delta + k\right)\rho$. If $\left(\frac{1}{2}\Delta + k\right)\rho = a$ we take the L^2 inner product with the eigenfunctions of the Laplacian to see that ρ itself must be constant, in fact $\rho = a/k$. So the moment map equation $\mu_{\mathcal{G}} = a$ is equivalent to

$$\sum |s_\alpha|^2 = a/k.$$

Now consider the $\mathcal{G}^c$ orbit of some point $(\widetilde{s}_0, \ldots, \widetilde{s}_N; I)$. We have seen that the points of this, up to the action of $\mathcal{G}$, correspond to fixing the complex structure, holomorphic line bundle and sections and varying the Hermitian metric on the bundle by a factor e^ϕ. Thus the moment map equation becomes an equation for ϕ

$$e^\phi \sum |\widetilde{s}_\alpha|^2 = a/k,$$

and the unique solution is

$$\phi = -\log\left(\frac{k\sum|s_\alpha|^2}{a}\right). \tag{24}$$

Here we use Lemma 14 to see that the metric defined by this formula has positive curvature.

The second part is completely straightforward: a point $(s_0, ..., s_N; I)$ lies in $\mu_{\mathrm{SU}}^{-1}(0)$ if and only if there is a positive scalar b such that the rescaled sections bs_α form an orthonormal basis for the holomorphic sections.

To see the relevance of this we can finally go back to fit Luo's condition into our formalism.

Proposition 16. *A pair (X, L^k) is balanced if and only if, for any $a > 0$, the corresponding complexified orbit $\mathfrak{o} \subset \mathcal{H}_0$ contains a point in $\mu_{\mathcal{G}}^{-1}(a) \cap \mu_{\mathrm{SU}}^{-1}(0)$, or equivalently if and only if the complexified orbit is represented by a point in the symplectic quotient (22).*

To see this we take the $\mathcal{G}$-quotient first. We have seen that points in $\mathcal{H}_0//\mathcal{G}$ correspond to sections s_α with $\sum |s_\alpha|^2 = a/k$. Thus, by Lemma 14, the pull-back of the Fubini-Study form is a multiple of the given symplectic form ω on M. The moment map for the induced $\mathrm{SU}(N+1)$ action on $\mathcal{H}_0//\mathcal{G}$ is given by the same formula (21) and hence, identifying our manifold with a projective variety, this is exactly the trace-free part of Luo's map. Thus a point in $\mu_{\mathrm{SU}}^{-1}(0) \subset \mathcal{H}_0//\mathcal{G}$ is exactly the same as a balanced variety in the sense of Section 1.

2.2 Generalities on moment maps and complex orbits

We have now seen how to fit the problem of constructing balanced varieties into a familar form and we will now recall some standar results about this set-up. Thus suppose a compact Lie group G acts on a Kähler manifold Z and that $\nu : Z \to \mathfrak{g}$ is a moment map for the action. Here we have identified the Lie algebra $\mathfrak{g}$ with its dual using an invariant inner product. At each point $z \in Z$ we have the infinitesimal action

$$\sigma_z : \mathfrak{g} \to TZ_z.$$

We define an endomorphism of $\mathfrak{g}$ by

$$Q_z = \sigma_z^* \sigma_z, \tag{25}$$

where the adjoint is formed using the metrics on $\mathfrak{g}$ and TZ. By the definition of the moment map, this is also given by

$$Q_z = d\nu_z \circ I \circ \sigma_z \tag{26}$$

The maps Q_z are important in understanding the existence and uniqueness of zeros of the moment map inside an orbit of the complexified group G^c. We recall first the standard proof of uniqueness. Suppose that $\nu(z) = \nu(gz) = 0$, where $g \in G^c$. After acting by an element of G, we may assume that $g = e^{i\xi}$ for some $\xi \in \mathfrak{g}$. Then define

$$f(t) = \langle \xi, \nu(e^{it\xi} z)\rangle. \tag{27}$$

Differentiating with respect to t gives

$$f'(t) = \langle \xi, Q_{e^{it\xi}z}\xi\rangle, \tag{28}$$

and this is nonnegative, vanishing if and only if $\sigma_z(\xi) = 0$, which occurs only when $z = z'$. This is the well-known fact that the zeros of ν in a G^c-orbit are unique up to the action of G. The argument above shows more: let Γ_G be the stabiliser in G of the point z, and let Γ_{G^c} be the stabiliser in G^c. Write $\Gamma^0_G, \Gamma^0_{G^c}$ for the identity components of these two groups. Then the inclusion of Γ_G in Γ_{G^c} induces an isomorphism between the discrete groups Γ_G/Γ^0_G and $\Gamma_{G^c}/\Gamma^0_{G^c}$, and $\Gamma^0_{G^c}$ is the complexification of Γ^0_G.

Our main concern is with the question of existence of zeros of the moment map. Suppose now that the stabilisers of all points under the G-action are discrete, so σ_z is injective and Q_z is invertible for all z in Z. Let Λ_z be the operator norm of $Q_z^{-1} : \mathfrak{g} \to \mathfrak{g}$, defined using the invariant Euclidean metric on $\mathfrak{g}$.

Proposition 17. *Suppose given $z_0 \in Z$ and real numbers λ, δ such that $\Lambda_z \leq \lambda$ for all $z = e^{i\xi} z_0$ with $|\xi| \leq \delta$. Suppose that $\lambda|\nu(z_0)| < \delta$. Then there is point $w = e^{i\eta} z_0$ with $\nu(w) = 0$, where $|\eta| \leq \lambda|\nu(z_0)|$.*

To prove this Proposition, we consider the flow on Z defined by

$$\frac{dz}{dt} = -I\sigma_z(a(z)),$$

where

$$a(z) = Q_z^{-1}(\nu(z)).$$

Taking the initial point z_0, this defines a path z_t in Z, at least for parameters t is some interval $[0, T)$, and we write $a_t = a(z_t)$ for the corresponding path in the Lie algebra $\mathfrak{g}$. By construction, we have

$$\frac{d\nu(z_t)}{dt} = -\nu(z_t),$$

(using (26)), so $\nu(z_t) = e^{-t}\nu(z_0)$. We also have

$$|a(t)| \leq \Lambda_{z_t}|\nu(z_t)| = \Lambda_{z_t}|\nu(z_0)|e^{-t}.$$

Now the path z_t lies in a single G^c-orbit: indeed we have $z_t = g_t(z_0)$ where the path g_t in G^c is defined by the ODE

$$\frac{dg}{dt} g^{-1} = i a_t,$$

with $g_0 = 1$. We consider the projection $p : G^c \to G^c/G$ and the resulting path $h_t = p(g_t)$ in the symmetric space G^c/G. To make the notation more transparent, we will suppose here that the Lie group G is $\mathrm{SU}(N+1)$ (the case we will need in our application), although the arguments go through easily to the general situation. In this case the quotient space G^c/G can be identified with the space $\mathcal{S}$ of positive definite Hermitian matrices with determinant 1, with the map p given by $p(g) = g^* g$. The invariant Riemannian metric on G^c/G can be represented by

$$\frac{1}{4} \mathrm{Tr}\ (\ (h^{-1}\delta h)^2).$$

Thus we have $h_t = g_t^* g_t$ and $\frac{dh_t}{dt} = 2 g_t^* a_t g_t$ and so the square of the length of the velocity vector $\frac{dh_t}{dt}$, computed using the natural Riemannian metric, is just

$$\frac{1}{4} \mathrm{Tr} \left(\left(\frac{dh_t}{dt} h_t^{-1} \right)^2 \right) = -\mathrm{Tr}\,(a_t^2) = |a_t|^2.$$

Now the paths $e^{i\xi t}$ in G^c, for $\xi \in \mathfrak{g}$, project to geodesics in $\mathcal{S}$ and the Riemannian distance between $p(e^{i\xi})$ and $p(1)$ is just $|\xi|$. So the hypothesis in the Proposition, and the G-invariance of the whole set-up, asserts that so long as h_t lies inside the δ-ball B_δ about $p(1)$ in G^c/G we have $\Lambda_{z_t} \leq \lambda$, and in this case we have

$$|a_t| \leq \lambda |\nu(z_0)| e^{-t}.$$

But now the assumption $\lambda |\nu(z_0)| < \delta$ means that the total length of the path h_t (in the natural Riemannian metric) is less than

$$\delta \int_0^\infty e^{-t} dt = \delta,$$

so the path h_t must indeed lie inside B_δ for all $t < T$. It follows easily that the flow, with this initial condition, is defined for all positive time and converges to a limit $p(e^{i\eta})$ where $|\eta|$ is bounded by the total path length, and hence by $\lambda|\nu(z_0)|$. Clearly $e^{i\eta}z_0$ is the desired zero of the moment map ν.

The last general point we need to recall concerns the situation where the space Z is itself a Kähler quotient $Z = W//H$ say, and the action of G on Z is induced by an action of $G \times H$ on the Kähler manifold W. For each point $w \in W$ the derivative of the action gives linear maps

$$\sigma_{G,w} : \mathfrak{g} \to TW_w, \ \ \sigma_{H,w} : \mathfrak{h} \to TW_w.$$

Lemma 18. *Let $z \in W//H$ be represented by a point $w \in W$. Then for $\xi \in \mathfrak{g}$ the endomorphism Q_z of $\mathfrak{g}$ associated to the G action on $W//H$ satisfies*

$$\langle Q_z \xi, \xi \rangle = |\pi(\sigma_{G,w}\xi)|^2$$

where $\pi : TW_w \to TW_w$ is the projection onto the orhogonal complement of the image of $\sigma_{H,w}$. In particular

$$\Lambda_z = \left(\min_{\xi \in \mathfrak{g}} \frac{|\pi(\sigma_{G,w}\xi)|}{|\xi|} \right)^{-2}.$$

The proof of this is just a matter of unwinding the definitions. (By the definition of the symplectic quotient the tangent space of $W//H$ at z can be isometrically identified with the orthogonal complement of the complexification of the image of $\sigma_{H,w}$.)

3. Estimates for the linearised problem

3.1 Explicit formulae

This Section 3 is the heart of the paper. We consider the action of the group $\mathrm{SU}(N+1)$ on the symplectic quotient

$$\mathcal{Z} = \mathcal{H}_0 // \mathcal{G}.$$

We fix attention on a single orbit of the complexification $\mathrm{SL}(N+1, \mathbf{C})$; that is, we fix attention on a given polarised variety $L^k \to X$. Our main goal is to prove the existence of points in $\mu_{\mathrm{SU}}^{-1}(0)$ in the given complex orbit, under appropriate conditions, and to this end we need to estimate the quantity Λ_z, for points $z \in \mathcal{Z}$, using the formula of Lemma 18 (with $G = \mathrm{SU}(N+1)$ and $Z = \mathcal{Z}$). We need to hold in mind two points of view: an element of the orbit is represented by a pair $(s_0, \dots, s_N; I)$ with

$\sum_\alpha |s_\alpha|^2 = 1$, or equivalently by an embedding of X in $\mathbf{CP}^N$. Given a matrix $iA = i(a_{\alpha\beta})$ of $\mathfrak{su}(N+1)$ we write

$$\sigma_\alpha = \sum_\beta a_{\alpha\beta} s_\beta.$$

To apply Lemma 18, we need to find the orthogonal projection in the Hilbert space $\Gamma(L^k)^{N+1}$ of

$$\underline{\sigma} = (\sigma_0, \dots, \sigma_N)$$

to the orthogonal complement of the subspace

$$P = \{(R_f s_0, \dots, R_f s_N) : f \in C^\infty(M)\}.$$

Proposition 19. *Given s_α and $A = (a_{\alpha\beta})$ as above, define a function $H = H_A \in C^\infty(M)$ by*

$$H = \sum a_{\alpha\beta}(s_\alpha, s_\beta).$$

Then the orthogonal projection of $\underline{\sigma}$ to the subspace P is

$$\underline{p} = (k^{-1} R_H s_0, \dots, k^{-1} R_H s_N).$$

(Recall, in the definition of the function H, that (s_α, s_β) is the pointwise inner product of the sections, defined by the Hermitian metric on L^k.)

To prove Proposition 19, write

$$\psi_\alpha = ik^{-1} R_H(s_\alpha) - \sigma_\alpha = ik^{-1} R_H(s_\alpha) - \sum_\beta a_{\alpha\beta} s_\beta. \tag{29}$$

We have to show that, for any function $f \in C^\infty(M)$,

$$\sum \langle R_f s_\alpha, \psi_\alpha \rangle = 0.$$

We write

$$\langle R_f(s_\alpha), \psi_\alpha \rangle = \Omega(R_f(s_\alpha), i\psi_\alpha) + i\Omega(R_f(s_\alpha, \psi_\alpha)$$

and apply the moment map identity of Proposition 13. Thus we have

$$\langle R_f(s_\alpha), \psi_\alpha \rangle = \int_M f\ [\delta_{i\psi_\alpha} \mu(s_\alpha) + i\delta_{\psi_\alpha} \mu(s_\alpha)],$$

where δ_{ψ_α} denotes the derivative of μ in the direction ψ_α and μ is the moment map

$$\mu(s) = \frac{1}{2}\Delta|s|^2 + k|s|^2.$$

Thus, evaluated at s_α,

$$\delta_{\psi_\alpha}\mu + i\delta_{i\psi_\alpha}\mu = \left[\Delta(s_\alpha, \psi_\alpha) + 2k(s_\alpha, \psi_\alpha)\right].$$

So we need to show that

$$\sum_\alpha \left(\Delta(\psi_\alpha, s_\alpha) + 2k(\psi_\alpha, s_\alpha)\right) = 0,$$

which is certainly true if

$$\sum_\alpha (\psi_\alpha, s_\alpha) = 0.$$

Now recall that $H = \sum_{\alpha\beta} a_{\alpha\beta}(s_\alpha, s_\beta)$, and that

$$R_H s_\alpha = \nabla_{\xi_H} s_\alpha - ikHs_\alpha.$$

Thus

$$(iR_H s_\alpha, s_\alpha) = (i\nabla_{\xi_H} s_\alpha, s_\alpha) + kH|s_\alpha|^2,$$

and

$$\sum_\alpha (\psi_\alpha, s_\alpha) = \sum_\alpha (ik^{-1}\nabla_{\xi_H} s_\alpha, s_\alpha) + H\sum_\alpha |s_\alpha|^2 - \sum_{\alpha\beta} a_{\alpha\beta}(s_\alpha, s_\beta).$$

Using the definition of H, and the fact that $\sum |s_\alpha|^2 = 1$, we see that the last two terms cancel and we are left with

$$\sum_\alpha (\psi_\alpha, s_\alpha) = ik^{-1}\sum_\alpha (\nabla_{\xi_H} s_\alpha, s_\alpha).$$

Now for any vector field v on M we have

$$\sum_\alpha \nabla_v |s_\alpha|^2 = 2\mathrm{Re}\left(\sum \nabla_v s_\alpha, s_\alpha\right) = 0$$

since $\sum |s_\alpha|^2$ is constant. Taking $v = \xi_H$, this shows that imaginary part of $\sum(\psi_\alpha, s_\alpha)$ vanishes. On the other hand, taking $v = I\xi_H$ and using the fact that

$$\nabla_{I\xi_H} s_\alpha = i\nabla_{\xi_H} s_\alpha$$

(since the sections s_α are holomorphic), we see that the real part of $\sum(\psi_\alpha, s_\alpha)$ also vanishes.

We can now apply Lemma 18 to see that the quantity Λ_z associated to our problem is given by

$$\Lambda_z^{-1} = \min \sum \|\psi_\alpha\|^2 = \min \sum_\alpha \left\| ik^{-1} R_H(s_\alpha) - \sum_\beta a_{\alpha\beta} s_\beta \right\|^2, \tag{30}$$

where the minimum runs over the trace-free Hermitian matrices $(a_{\alpha\beta})$ with $\|(a_{\alpha\beta})\|^2 \sum_{\alpha\beta} |a_{\alpha\beta}|^2 = 1$, and H is defined, in terms of the $a_{\alpha\beta}$ and s_α, as above. Our problem is to find a lower bound for the sum appearing on the right hand side of (30).

Proposition 20. *Continuing the notation above, we have*

$$\sum \|\overline{\partial}_{L^k} \psi_\alpha\|^2 k^{-1} \|\mathcal{D}H\|^2,$$

where $\overline{\partial}_{L^k}$ is the $\overline{\partial}$-operator on sections of L^k defined by the given complex structure I on M and $\|\ \|$ is the standard L^2 norm defined by the metric ω.

In fact, more is true: we have a pointwise equality

$$\sum_\alpha |\overline{\partial}_{L^k} \psi_\alpha|^2 = k|\mathcal{D}H|^2, \tag{31}$$

at each point of X. To see this, observe first that

$$\overline{\partial}_{L^k} \psi_\alpha = k^{-1} \overline{\partial}_{L^k}(R_H s_\alpha),$$

since the sections σ_α are holomorphic. Now we apply the identity (15) to see that

$$\overline{\partial}_{L^k} \psi_\alpha = k^{-1} (\mathcal{D}H) \partial_\nabla s_\alpha.$$

We fix a point x on X and simplify the calculations by using a basis adapted to this point. Let $U = (u_{\alpha\beta})$ be any unitary matrix and define $\widetilde{s}_\alpha = \sum_\beta u_{\alpha\beta} s_\beta$ and $\widetilde{A} = U^* A U$ then the functions $H = H_A$ is the same as $\sum \widetilde{a}_{\alpha\beta}(\widetilde{s}_\alpha, \widetilde{s}_\beta)$ and the pointwise sums, $\sum |(\mathcal{D}H)\partial_\nabla s_\alpha|^2$ and $\sum |(\mathcal{D}H)\partial_\nabla \widetilde{s}_\alpha|^2$ are equal. Thus it is equivalent to prove the equality (31) for the original basis s_α or the new basis $\widetilde{s}_\alpha$. By elementary linear algebra we may as well, therefore, suppose that we are in the position where the sections $s_1, \ldots, s_N$ vanish at x, as do the derivatives

$\partial s_{n+1}, \ldots \partial s_N$. Then the section s_0 trivialises the fibre of L^k over x and the derivatives $\partial s_1, \ldots \partial s_n$ can be viewed as a basis for the cotangent space $(T^*X)_x$. We claim that this is an orthonormal basis. Indeed, if we write $s_\alpha = f_\alpha s_0$ near x, for local holomorphic functions f_α, the condition that $\sum |s_\alpha|^2 = 1$ shows that

$$|s_0|^2 = \left(1 + \sum_{\alpha=1}^{N} |f_\alpha|^2\right)^{-1},$$

so the form $-ik\omega$, which is the curvature of L^k, is

$$\overline{\partial}\partial \log \left(1 + \sum |f_\alpha|^2\right),$$

which, evaluated at x, is just $\sum_{\alpha=1}^N \overline{\partial f_\alpha} \partial f_\alpha$. On the other hand, evaluated at x,

$$\partial s_\alpha = \partial f_\alpha s_0,$$

so only the first n terms in the sum contribute, and we have

$$-ik\omega = \sum_{\alpha=1}^{n} \overline{\partial f_\alpha} \partial f_\alpha,$$

and this precisely asserts that $k^{-1/2}\partial f_1, \ldots, k^{-1/2}\partial f_n$ form an orthonormal basis for the cotangent space. Note also that the condition $\sum |s_\alpha|^2 = 1$ implies that ∂s_0 vanishes at x. Now it is clear that

$$k|\mathcal{D}H|^2 = \sum_{\alpha=1}^{n} |\mathcal{D}H\ \partial s_\alpha|^2 = \sum_{\alpha=0}^{N} |\mathcal{D}H\ \partial s_\alpha|^2 = \sum_\alpha |\overline{\partial}_{L^k} \psi_\alpha|^2,$$

at x, as required.

3.2 Analytical estimates

In this subsection we will obtain an explicit estimate on the quantity Λ, using the formulae from (3.1) and some simple analytical arguments. The estimates we obtain are not sharp, we discuss the scope for improvements in Section 4.

The crucial thing is to keep track of the parameter k. In the course of our arguments we need to estimate the norms of various tensor fields; one can do this either with respect to metrics in the fixed cohomology class $2\pi c_1(L)$, or in the rescaled class $k2\pi c_1(L)$. Of course it is an entirely elementary matter to transform between the two points of view, introducing appropriate powers of k.

Fix any reference metric ω_0 in the Kähler class $c_1(L)$. We also fix an integer $r \geq 4$. Given k, let $\widetilde{\omega}_0$ be the rescaled metric $k\omega_0$. For $R > 0$ we say that another metric $\widetilde{\omega}$ in the cohomology class $kc_1(L)$ has *R-bounded geometry* if $\widetilde{\omega} > R^{-1}\widetilde{\omega}_0$ and

$$\|\widetilde{\omega} - \widetilde{\omega}_0\|_{C^r} < R,$$

where the norm $\| \ \|_{C^r}$ is the standard C^r norm determined by the metric $\widetilde{\omega}_0$. These conditions scale to the conditions $\omega > R^{-1}\omega_0$ and

$$\|\omega - \omega_0\|_{C^r(\omega_0)} < k^{r/2}R,$$

where $\| \ \|_{C^r(\omega_0)}$ is the norm determined by the metric ω_0. Clearly, at the cost of a change in R, this notion is independent of the choice of ω_0. Now, as in (3.1), we consider a basis s_α for $H^0(L^k)$ which determines a unique metric on L^k such that with $\sum |s_\alpha|^2 = 1$ at each point. We say that the basis(s_α) has R-bounded geometry if the Kähler metric $\widetilde{\omega}$ induced from the Fubini-Study metric by the embedding of X in $\mathbf{CP}^N$ does.

Throughout this section we work with the "large" metric $\widetilde{\omega}$ on X . Thus the volume of X in this metric is Vk^n, where V is the volume of X in the metric ω_0. We use the L^2 inner product on sections of L^k defined by this metric. To fit the preceding discussion into this framework we need to replace the symplectic form ω by $k\omega$ and L by L^k, while removing the explicit k dependence in formulae such as (9), (29), (30) and Proposition 20. Notice that, working with this large metric,

$$\sum_\alpha \|s_\alpha\|^2 = Vk^n.$$

We write

$$\langle s_\alpha, s_\beta \rangle = \frac{Vk^n}{N+1}\delta_{\alpha\beta} + \eta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Thus the matrix $E(\eta_{\alpha\beta})$ is a trace-free Hermitian matrix, and $E = 0$ if and only if the projective embedding is balanced. (Notice that the factor $Vk^n/N + 1$ tends to 1 as $k \to \infty$.)

We continue with the notation from (3.1), so for any matrix $A = (a_{\alpha\beta}) \in i\mathfrak{su}(N+1)$ we define a function H and sections ψ_α by (29) and the statement of Proposition 19, but setting $k = 1$ in the formulae as explained above. We write $\underline{\psi}$ for the "vector" of sections ψ_α so

$$\|\underline{\psi}\|^2 = \sum_\alpha \|\psi_\alpha\|^2.$$

We will make use of two standard norms on the Hermitian matrices: the Hilbert-Schmidt norm

$$\|F\|^2 = \sum_{\alpha\beta} |F_{\alpha\beta}|^2$$

and the operator norm

$$\|F\|_{\mathrm{op}} = \max \frac{|F(\xi)|}{|\xi|}.$$

Alternatively, $\|F\|_{\mathrm{op}}$ is the maximum of the moduli of the eigenvalues of F. We will use the elementary inequalities, for $(N+1) \times (N+1)$ Hermitian matrices F, G

$$\begin{aligned} |\mathrm{Tr}(FGF)| &\leq \|F\|^2 \|G\|_{\mathrm{op}} \\ |\mathrm{Tr}(FG)| &\leq \sqrt{N+1}\|F\| \; \|G\|_{\mathrm{op}}\,. \end{aligned} \tag{32}$$

The goal of this subsection is to prove:

Theorem 21. *Suppose* Aut (X, L) *is discrete. For any R there are constants $C = C(R, \omega_0)$ and $\epsilon = \epsilon(R, \omega_0) < 1/10$ such that, for any k, if the basis s_α of $H^0(L^k)$ has R-bounded geometry and $\|E\|_{\mathrm{op}} < \epsilon$ then for any trace-free Hermitian matrix A,*

$$\|A\| \leq Ck^2 \|\underline{\psi}\|.$$

By Lemma 18, this yields:

Corollary 22. *If z is the point in $\mathcal{Z}$ determined by a basis (s_α) of satisfying the hypotheses of Theorem* 21 *we have*

$$\Lambda_z \leq C^2 k^4.$$

(Here Λ is the operator norm defined by (30), but using the large metric $\widetilde{\omega}$ and consequently with the factor of k^{-1} removed.)

The analytical estimates required to prove Theorem 21 are summed up in the following:

Proposition 23. *Suppose* $\mathrm{Aut}\,(X, L)$ *is discrete. If the basis* s_α *has* R*-bounded geometry and* $\|E\|_{\mathrm{op}} \leq \frac{1}{10}$ *then there are constants* $C_1, \ldots, C_4$*, depending only on* R *and* ω_0*, such that for sufficiently large* k *we have:*

(i) $$\|\mathcal{D}H\|^2 \leq C_1 \|\underline{\psi}\| \|A\|;$$

(ii) $$\frac{9}{10}\|A\|^2 \leq \frac{11}{10}\|H\|^2 + \|\underline{\psi}\|^2 + C_2\|\nabla H\|^2;$$

(iii) $$\|\nabla H\|^2 \leq \max\left(\|\mathcal{D}H\|^2, C_3\|H\|^2\right);$$

(iv) $$\|H\|^2 \leq C_4 k^2 \|\mathcal{D}H\|^2 + 2\|E\|_{\mathrm{op}}^2 \|A\|^2.$$

Here the operator $\mathcal{D}$ *is that defined by the large metric* $\widetilde{\omega}$ *and the norms* $\|\mathcal{D}H\|, \|\nabla H\|, \|H\|$ *are the standard* L^2*-norms defined by* $\widetilde{\omega}$*.*

The proof of Theorem 21 given Proposition 23 is completely elementary and we give it now. First, if $\|\underline{\psi}\|^2 \geq \frac{1}{10}\|A\|^2$ then the estimate asserted in Theorem 21 holds rather trivially, with $C = \sqrt{10}$. So we may as well suppose from now on that $\|\underline{\psi}\|^2 \leq \frac{1}{10}\|A\|^2$ so that (ii) of Proposition 23 gives

$$\frac{8}{10}\|A\|^2 \leq \frac{11}{10}\|H\|^2 + C_2\|\nabla H\|^2. \tag{33}$$

Now consider two cases depending on whether $\|\mathcal{D}H\|^2 \geq C_3\|H\|^2$ or not. If $\|\mathcal{D}H\|^2 \geq C_3\|H\|^2$ then (iii) gives $\|\nabla H\| \leq \|\mathcal{D}H\|$ and then (33) implies

$$\frac{8}{10}\|A\|^2 \leq \left(\frac{11}{10C_3} + 1\right)\|\mathcal{D}H\|^2.$$

Combining this with (i) we get

$$\frac{8}{10}\|A\|^2 \leq C_1\left(\frac{11}{10C_3} + 1\right)\|\underline{\psi}\|\|A\|,$$

so

$$\|A\| \leq \frac{10}{8}C_1\left(\frac{11}{10C_3} + 1\right)\|\underline{\psi}\|,$$

and the estimate asserted in Theorem 21 holds with

$$C = \frac{10}{8}C_1\left(\frac{11}{10C_3} + 1\right).$$

Now we go to the other case (which we expect to be the one that actually occurs), when $\|\mathcal{D}H\|^2 \leq C_3\|H\|^2$. In this case (iii) gives

$$\|\nabla H\|^2 \leq C_3\|H\|^2$$

so (ii) implies

$$\frac{8}{10}\|A\|^2 \leq \frac{11}{10}\|H\|^2 + C_2C_3\|H\|^2,$$

and (iv) implies

$$\frac{8}{10}\|A\|^2 \leq \left(\frac{11}{10} + C_2C_3\right)(C_4k^2\|\mathcal{D}H\|^2 + 2\|E\|^2_{\mathrm{op}}\|A\|^2).$$

Hence if we define ϵ by $2\epsilon^2\frac{1}{11+C_2C_3}$ then if $\|E\|_{\mathrm{op}} \leq \epsilon$ we have $(\frac{11}{10} + C_2C_3)\|\eta\|_{\mathrm{op}} \leq \frac{1}{10}$ and we get, using (i),

$$\frac{7}{10}\|A\|^2 \leq \frac{C_4}{10}k^2\|\mathcal{D}H\|^2 \leq \frac{C_1C_4}{10}k^2\|\underline{\psi}\|\|A\|.$$

This is the desired estimate with $C = \frac{C_1C_4}{7}$.

We now begin the proof of Proposition 23. The most important of the four inequalities is (i), so we start with that. We begin with a Lemma which expresses the fact that we can control the size of the derivatives of the holomorphic sections s_α.

Lemma 24. *Under the hypotheses of Theorem 21, there is a constant C_5 such that for any integer $j \leq 4$*

(1)
$$\sum_\alpha |\nabla^j s_\alpha|^2 \leq C_5$$

at each point of X;

(2)
$$\|\nabla^j H\|_{L^2} \leq C_5\|A\|.$$

(In the second item we have written $\| \ \|_{L^2}$ to emphasise the distinction with the pointwise estimate in the first item. We empasise again that in this Lemma we use the large metric $\widetilde{\omega}$ and the given connection and metric on the line bundle L^k.)

To prove (1) of Lemma 24, fix a point $x \in X$ and an embedded geodesic ball $B \subset X$ centred at x. It is standard that there is a constant K such that for any holomorphic section s of L^k

$$\left|\left(\nabla^j s\right)_x\right|^2 \leq K \int_B |s|^2 d\mu. \tag{34}$$

It is clear that, under our hypothesis of R-bounded geometry, we can choose a fixed constant K, depending only on R and ω_0, and using a ball of a some fixed radius, also depending only on R and ω_0. Now apply this to the sections s_α, and sum over α to get

$$\sum_\alpha |(\nabla^j s_\alpha)_x|^2 \leq K \int_B \sum |s_\alpha|^2 d\mu = K \text{ Vol } (B) \leq \text{Constant}.$$

To prove (2) of Lemma 24 we begin with a short digression. Let Z be a compact complex Hermitian manifold, let E be a Hermitian holomorphic vector bundle over Z and let $P \subset Z$ be a differentiable (real) submanifold. For each point p of P we can fix a small ball B_p in Z, of radius ρ say, centred on p so that for any holomorphic section σ of E the L^2 norm of σ over B_p controls the size of the covariant derivative $\nabla^j \sigma$ at p, just as in (34) above. If we now integrate over p in P we obtain an estimate

$$\|\nabla^j \sigma\|^2_{L^2(P)} \leq K' \|\sigma\|^2_{L^2(N)}, \tag{35}$$

where N is the ρ-neighbourhood of P in Z. It is clear that, provided the data Z, P, E has bounded local geometry in a suitable sense, and if the radius ρ is fixed, then the constant K' can be taken to be independent of the particular manifolds and bundles involved.

To apply this in our situation, consider the manifold $Z = X \times \overline{X}$, where $\overline{X}$ is X with the opposite complex structure. The connection defined by the Hermitian metric on $L^k \to X$ makes $\overline{L}^k \to \overline{X}$ into a holomorphic line bundle. Let $E \to Z$ be the tensor product of the lift of L^k from the first factor and the dual of $\overline{L}^k$ on the second factor: thus E is a holomorphic line bundle over Z. A holomorphic section s of $L^k \to X$ defines a holomorphic section $\widetilde{s}$ of the dual of $\overline{L}^k$ via the C^∞ bundle isomorphism defined by the fibre metric. (This is essentially the definition of the connection defined on a Hermitian holomorphic bundle.) Thus for any Hermitian matrix $A = (a_{\alpha\beta})$ we get a holomorphic section

$$\sigma_A = \sum a_{\alpha\beta} s_\alpha \otimes \widetilde{s}_\beta$$

of E over Z. We have

$$\|\sigma_A\|^2_{L^2(Z)} = \sum_{\alpha\alpha'\beta\beta'} a_{\alpha\beta}\overline{a_{\alpha'\beta'}}\langle s_\alpha, s_{\alpha'}\rangle\langle s_{\beta'}, s_\beta\rangle,$$

or in matrix notation

$$\|\sigma_A\|^2_{L^2(Z)} = \mathrm{Tr}\ (A(I+E)(I+E^*)A^*).$$

Since $\|E\|_{\mathrm{op}} \leq 1/10$ we deduce, using (32), that

$$\|\sigma_A\|_{L^2(Z)} \leq 11/10\|A\|. \tag{35$'$}$$

Now let P be the diagonal in $X \times \overline{X}$. The metric on L^k defines a C^∞ trivialisation of the bundle E over P and our function $H = \sum_{\alpha\beta} a_{\alpha\beta}(s_\alpha, s_\beta)$ is just the restriction of the section σ_A to the diagonal, in this trivialisation. Taking a suitable neighbourhood N of the diagonal, we can apply (34) and (35) to obtain

$$\|\nabla^r H\|^2_{L^2(X)} \leq K'\|\sigma_A\|^2_{L^2(N)} \leq K'\|\sigma_A\|^2_{L^2(Z)} \leq (11/10)^2 K'\|A\|^2.$$

Again it is clear that, in our situation, the constant K' can be chosen to depend only on R and ω_0. This completes the proof of Lemma 24.

We now give the key argument of this subsection, which establishes the first inequality of Proposition 23. Recall from Proposition 20 that

$$\sum \|\overline{\partial}\psi_\alpha\|^2 = \|\mathcal{D}H\|^2.$$

Now we write

$$\|\overline{\partial}\psi_\alpha\|^2 = \langle \psi_\alpha, \Delta_{L^k}\psi_\alpha\rangle,$$

where Δ_{L^k} is the Laplace-type operator $\overline{\partial}^*_{L^k}\overline{\partial}_{L^k}$ on sections of L^k. Thus we have

$$\|\overline{\partial}_{L^k}\psi_\alpha\|^2 \leq \|\Delta_{L^k}\psi_\alpha\|\|\psi_\alpha\|,$$

and so

$$\begin{aligned} \|\mathcal{D}H\|^2 = \sum_\alpha \|\overline{\partial}\psi_\alpha\|^2 &\leq \left(\sum_\alpha \|\Delta_{L^k}\psi_\alpha\|^2 \sum_\alpha \|\psi_\alpha\|^2\right)^{1/2} \\ &= \left(\sum_\alpha \|\Delta_{L^k}\psi_\alpha\|^2\right)^{1/2} \|\underline{\psi}\|. \end{aligned} \tag{36}$$

Now, for each α, we have

$$\Delta_{L^k}\psi_\alpha = i\overline{\partial}^*\overline{\partial}R_H(s_\alpha) = \overline{\partial}^*((\mathcal{D}H).\partial s_\alpha),$$

by (15). Using a schematic notation, we can write

$$\Delta_{L^k}\psi_\alpha = \nabla^3 H * \nabla s_\alpha + \nabla^2 H * \nabla^2 s_\alpha,$$

where $*$ denotes certain natural bilinear algebraic bundle maps. Thus, pointwise on X,

$$|\Delta_{L^k}\psi_\alpha|^2 \leq c(|\nabla^3 H|^2 |\nabla s_\alpha|^2 + |\nabla^2 H|^2 |\nabla^2 s_\alpha|^2),$$

for a suitable universal constant c. Summing over α we get, using the first part of Lemma 24,

$$\begin{aligned}\sum |\Delta_{L^k}\psi_\alpha|^2 &\leq c\left(|\nabla^3 H|\sum_\alpha |\nabla s_\alpha|^2 + |\nabla^2 H|^2 \sum_\alpha |\nabla^2 s_\alpha|^2\right)\\ &\leq cC_5(|\nabla^3 H|^2 + |\nabla^2 H|^2).\end{aligned}$$

Finally, integrating over X, we get

$$\sum_\alpha \|\Delta\psi_\alpha\|^2 \leq cC_5\|\nabla^3 H\|^2 + \|\nabla^2 H\|^2 \leq 2cC_5^2\|A\|^2, \tag{37}$$

using the second part of Lemma 24. Now (36) and (37) give the first inequality of Proposition 23 , with $C_1 = \sqrt{2c}C_5$.

We now go on to the second inequality of Proposition 23. Recall that we have an equality in the Hilbert space $\Gamma(L^k)^{N+1}$

$$\underline{\sigma} = \underline{\psi} + \underline{p}$$

where $\sigma_\alpha = \sum_\beta a_{\alpha\beta}s_\beta$ and $p_\alpha = R_H s_\alpha$. By Proposition 19, $\underline{\psi}$ and $\underline{p}$ are orthogonal vectors in the Hilbert space so

$$\|\underline{\sigma}\|^2 = \|\underline{\psi}\|^2 + \|\underline{p}\|^2. \tag{38}$$

Now

$$\|\underline{\sigma}\|^2 = \sum_{\alpha\beta\gamma} a_{\alpha\beta}\overline{a}_{\alpha\gamma}\langle s_\beta, s_\gamma\rangle = \sum_{\alpha\beta}|a_{\alpha\beta}|^2 + \sum_{\alpha\beta\gamma} a_{\alpha\beta}\eta_{\beta\gamma}a_{\gamma\alpha}.$$

By (32), the last term is bounded by $\|E\|_{\rm op}\|A\|^2$ so, since $\|E\|_{\rm op} \leq \frac{1}{10}$ by hypothesis, we deduce that

$$\|\underline{\sigma}\|^2 \geq \frac{9}{10}\|A\|^2. \tag{39}$$

Hence

$$\frac{9}{10}\|A\|^2 \leq \|\underline{\psi}\|^2 + \|\underline{p}\|^2.$$

Now $\|\underline{p}\|^2 = \sum_\alpha \|R_H s_\alpha\|^2$. Pointwise on X we have

$$|R_H s_\alpha|^2 = |H s_\alpha - i\nabla H.\nabla s_\alpha|^2 \leq (1+c)|Hs_\alpha|^2 + (1+c^{-1})|\nabla H.\nabla s_\alpha|^2,$$

for any $c>0$. We choose $c = 1/10$ so

$$|R_H s_\alpha|^2 \leq \frac{11}{10}|H|^2|s_\alpha|^2 + 11|\nabla H|^2|\nabla s_\alpha|^2.$$

Now sum over α and use Lemma 24 to get

$$\sum_\alpha |R_H s_\alpha|^2 \leq \frac{11}{10}|H|^2 + 11K|\nabla H|^2.$$

Integrating over X we get

$$\|\underline{p}\|^2 \leq \frac{11}{10}\|H\|^2 + 11K\|\nabla H\|^2,$$

and, combined with (38) and (39), this gives the second inequality of Proposition 23, with $C_2 = 11K$.

To obtain the third inequality of Proposition 23 we use a Weitzenböck formula. Recall that $\mathcal{D}H = \overline{\partial}\xi_H$ where ξ_H is the Hamiltonian vector field generated by H. For any vector field v on X we have

$$\|\nabla v\|^2 = 2\|\overline{\partial} v\|^2 + \int_X \mathrm{Ric}\ (v,v),$$

where Ric denotes the Ricci tensor of the metric $\widetilde{\omega}$. On the other hand $|\nabla \xi_H| = |\nabla dH|$, and the Bochner formula for closed 1-forms gives

$$\|\nabla dH\|^2 = \|\Delta H\|^2 - \int_X \mathrm{Ric}\ (\xi_H, \xi_H).$$

Combining these we get

$$\|\Delta H\|^2 = 2\|\mathcal{D}H\|^2 + 2\int_X \mathrm{Ric}\ (\xi_H,\xi_H) \leq 2\|\mathcal{D}H\|^2 + c\|\nabla H\|^2,$$

for a suitable constant c depending on R. Now we have

$$\|\nabla H\|^2 = \langle H, \Delta H\rangle \leq \|H\| \sqrt{2\|\mathcal{D}H\|^2 + c\|\nabla H\|^2}.$$

So if $\|\mathcal{D}H\|^2 \leq \|\nabla H\|^2$ we have

$$\|\nabla H\|^2 \leq \|H\| \sqrt{(2+c)\|\nabla H\|^2},$$

and hence $\|\nabla H\| \leq \sqrt{2+c}\|H\|$. In other words

$$\|\nabla H\|^2 \leq \max\ \left(\|\mathcal{D}H\|^2, (2+c)\|H\|^2\right),$$

and the third item of Proposition 24 holds with $C_3 = 2 + c$.

We come now to the fourth inequality of Proposition 23.

Lemma 25. *Suppose* $\mathrm{Aut}\,(X, L)$ *is discrete. Then for any* $L > 1$ *there is a constant* C*, depending only on* ω_0 *and* L*, such that if* ω *is any metric in the same cohomology class as* ω_0 *with* $L\omega_0 > \omega > L^{-1}\omega_0$ *and if* f *is any real-valued function on* X

$$\|f\|^2 \leq C\|\mathcal{D}_\omega f\|^2 + \frac{1}{V}\left(\int_X f d\mu_\omega\right)^2,$$

where $\mathcal{D}_\omega$ *is the operator defined by the metric* ω *and the norms are the* L^2 *norms defined by* ω*.*

By our assumption on the automorphism group of (X, L), the kernels of the operators $\mathcal{D}_\omega, \mathcal{D}_{\omega_0}$, acting on the complex-valued functions on X, consist of the constant functions. Thus an inequality of the kind stated in the Lemma with some constant, depending on ω, is standard. The point of the Lemma is that this constant can be controlled in terms of ω_0 and L. In the proof we repeatedly use the fact that the L^2 norms of any tensor field on X defined by the two metrics ω, ω_0 are equivalent.

Let U be the finite-dimensional space of holomorphic vector fields on X. For any metric ω there is a constant c_ω such that

$$\|\xi\|^2 \leq c_\omega\|\overline{\partial}\xi\|^2 + \|\pi_\omega\xi\|^2, \tag{40}$$

for any vector field ξ on X, where π_ω is the L^2- projection to U. The constant c_ω can be given by

$$c_\omega = \max_\xi \left(\min_{v\in U} \frac{\|\xi - v\|^2}{\|\overline{\partial}\xi\|^2}\right).$$

Since the L^2 norms defined by the two metrics ω, ω_0 are equivalent we see from this characterisation that we can replace c_ω in (40) by a constant c_1 depending only on L and ω_0. By the same argument, there is a constant c_2, depending only on L and ω_0, such that

$$\|f\|^2 \leq c_2 \|\nabla f\|^2 + \frac{1}{V}\left(\int_X f \, d\mu_\omega\right)^2, \tag{41}$$

for any real-valued function f on X. In the case when $U = 0$ the inequalities (40), (41) immediately give the desired result, since

$$\|\xi_f\|^2 = \|\nabla f\|^2.$$

To handle the case when $U \neq 0$ we observe that for any v in U the contraction $i_v^{0,1}(\omega)$ is a $\overline{\partial}$-closed $(0,1)$-form, defining a class in $H^{0,1}(X)$, and this class depends only on v and the cohomology class of ω. So we can write

$$i_v^{0,1}(\omega) = h_{\omega,v} + \overline{\partial}\psi_{\omega,v} \ , \ i_v^{0,1}(\omega_0) = h_{\omega_0,v} + \overline{\partial}\psi_{\omega_0,v}$$

where $h_{\omega,v}, h_{\omega_0,v}$ are the harmonic representatives of the same cohomology class with respect to the metrics ω, ω_0. The hypothesis that $\mathrm{Aut}\,(X, L)$ is discrete implies that $h_{\omega_0,v}$ is never zero, for nonzero v, since if it were $\psi_{\omega_0,v}$ would lie in the kernel of $\mathcal{D}_{\omega_0}$. Since U is finite-dimensional we have an inequality

$$\|v\|_{L^2(\omega_0)} \leq c\|h_{\omega_0,v}\|_{L^2(\omega_0)},$$

for all $v \in U$. Now the harmonic representative $h_{\omega_0,v}$ minimises the $L^2(\omega_0)$ norm over the cohomology class so

$$\|h_{\omega_0,v}\|_{L^2(\omega_0)} \leq \|h_\omega\|_{L^2(\omega_0)}.$$

Using again the fact that the L^2 norms are equivalent, we see that there is a constant c_3, depending only on ω_0 and L, such that

$$\|v\|_{L^2(\omega)} \leq c_3\|h_{\omega,v}\|_{L^2(\omega)}.$$

Thus

$$\|\overline{\partial}\psi_{\omega,v}\|^2 = \|v\|^2 - \|h_{\omega,v}\|^2 \leq (1 - c_3^{-2})\|v\|^2,$$

where from here on all norms are those defined by ω. Now the harmonic representative is L^2-orthogonal to $\overline{\partial}f$. for any function f on X. Thus

$$\langle \xi_f, u\rangle = \langle \overline{\partial}f, \overline{\partial}\psi_{\omega,v}\rangle \leq (1 - c_3^{-2})^{1/2}\|\xi_f\|\|v\|.$$

It follows that
$$\|\pi_\omega \xi_f\| \leq (1 - c_3^2)^{1/2} \|\xi_f\|,$$
and so
$$\|\xi_f\|^2 \leq c_1 \|\overline{\partial}\xi_f\|^2 + (1 - c_3^{-2}) \|\xi_f\|^2,$$
and thus, by (40),
$$\|\xi_f\|^2 \leq c_1 c_3^2 \|\overline{\partial}\xi_f\|^2 = c_1 c_3^2 \|\mathcal{D} f\|^2.$$
Then (41) gives the desired result, with $C = c_1 c_2 c_3^2$.

We proceed now to the proof of item (iv) of Proposition 23. Consideration of the scaling behaviour of the various terms of the result of Lemma 25 shows that if $\widetilde{\omega}$ is R-bounded
$$\|f\|^2 \leq Ck^2 \|\mathcal{D}_{\widetilde{\omega}} f\|^2 + \frac{1}{k^n V} \left(\int_X f \ d\mu_{\widetilde{\omega}} \right)^2,$$
where the constant C depends only on R and ω_0, and all norms are the L^2 norms defined by the large metric $\widetilde{\omega}$. In particular, this holds for the function H;
$$\|H\|^2 \leq Ck^2 \|\mathcal{D}_{\widetilde{\omega}} H\|^2 + \frac{1}{k^n V} \left(\int_X H \ d\mu_{\widetilde{\omega}} \right)^2. \tag{43}$$
The final step is to control the term involving the integral of H. Here we use the fact that the matrix $A = (a_{\alpha\beta})$ is trace-free. We have
$$\begin{aligned}\int_X H \ d\mu_{\widetilde{\omega}} &= \sum a_{\alpha\beta} \langle s_\alpha, s_\beta \rangle = \sum a_{\alpha\beta} \left(\frac{V}{N+1} \delta_{\alpha\beta} + \eta_{\alpha\beta} \right) \\ &= \sum a_{\alpha\beta} \eta_{\alpha\beta}.\end{aligned}$$
Thus, using (32),
$$\left| \int_X H \ d\mu_{\widetilde{\omega}} \right|^2 \leq (N+1) \|E\|_{\mathrm{op}}^2 \|A\|^2. \tag{44}$$
Then (43) and (44) give
$$\|H\|^2 \leq Ck^2 \|\mathcal{D}_{\widetilde{\omega}} H\|^2 + \frac{N+1}{Vk^n} \|E\|_{\mathrm{op}}^2 \|A\|^2,$$
which implies the desired result, since
$$\frac{N+1}{Vk^n} \to 1$$
as $k \to \infty$.

4. Synthesis

4.1 Construction of approximate solutions

We now use the asymptotic results of Catlin, Lu and Zelditch to constuct "nearly balanced" projective embeddings. Let $\chi(k) = \chi(X, L^k)$ be the Hilbert polynomial of (X, L). Recall that for any metric ω in the Kähler class $2\pi c_1(L)$, $\rho_k(\omega)$ is the function on X given by $\sum_\alpha |s_\alpha|^2$ where s_α is an orthonormal basis of $H^0(L^k)$.

Theorem 26. *Suppose that* $\mathrm{Aut}\,(X, L)$ *is discrete and that* ω_∞ *is a metric of constant scalar curvature in the Kähler class* $c_1(L)$*. There are functions* $\eta_1, \eta_2 \ldots,$ *on* X *such that for any* $q > 0$ *there is a constant* C_q *with the property that if* $\omega(k)$ *is the form*

$$\omega(k) = \omega_\infty + i\overline{\partial}\partial \left(\sum_{j=1}^{q} \eta_j k^{-j} \right)$$

(*which is a Kähler form for large enough* k) *then*

$$\rho_k(\omega(k)) = V^{-1}\chi(k) + \sigma_q(k)$$

where

$$\|\sigma_q(k)\|_{C^{r+2}} \leq C_q k^{n-q-1}$$

for all large enough k.

This is a straightforward consequence of the Catlin-Lu-Zelditch results. Recall that

$$\rho_k(\omega) = k^n + A_1(\omega)k^{n-1} + \cdots + A_q(\omega)k^{n-q} + O(k^{n-q-1}),$$

where the A_p are polynomials in the curvature tensor of ω and its covariant derivatives and the error term is uniformly bounded in C^{r+2} for all metrics ω in a bounded family. We can plainly make a Taylor expansion of the coefficients

$$A_p(\omega + i\overline{\partial}\partial\eta) = A_p(\omega) + \sum_{l=1}^{q} A_{p,l}(\eta) + O\left(\|\eta\|_{C^s}^{q+1}\right) \tag{45}$$

where $A_{p,l}(\eta)$ is a polynomial of degree l, depending on ω, in η and its covariant derivatives and s is sufficiently large (depending on r and q).

Thus, for any $\eta_1, \dots, \eta_q$, we can write

$$A_p\left(\omega + i\overline{\partial}\partial\left(\sum_{j=1}^{q}\eta_j k^{-j}\right)\right) = A_p(\omega) + \sum_{l=1}^{q} b_{p,l}k^{-l} + O(k^{-q-1}), \tag{46}$$

where the $b_{p,l}$ are certain multilinear expressions in the η_j, and their covariant derivatives, beginning with

$$b_{p,1} = A_{p,1}(\eta_1).$$

Thus we get

$$\rho_k\left(\omega_\infty + i\overline{\partial}\partial\left(\sum_{j=1}^{q}\eta_j k^{-j}\right)\right) = \sum_{p=0}^{q} k^{n-p}a_p(\omega_\infty) + \sum_{p,l=1}^{r} b_{p,l}k^{n-p-l} + O(k^{n-q-1}). \tag{47}$$

We now simply choose the η_j so that the terms in the right hand side of (47) are constant on X. Suppose, inductively, that we have chosen the η_j for $j \leq p$ so that the coefficients of k^{n-j} are constants for $j \leq p$.The new term η_{p+1} appears only once in the coefficient of k^{n-p-1}, in the form $A_{1,1}(\eta_{p+1})$. So we have to solve a linear equation for a function η_{p+1} and a constant c_{p+1},

$$A_{1,1}(\eta_{p+1}) - c_{p+1} = P_p \tag{48}$$

where P_p is determined by the previous terms η_j for $j \leq p$. The crucial fact we need now is that

$$A_{1,1}(\eta) = \mathcal{D}^*\mathcal{D}(\eta). \tag{49}$$

That is, the operator $\mathcal{D}^*\mathcal{D}$ gives the first variation of the scalar curvature. This is an old result of Lichnerowicz [7]. In the symplectic framework of [3], where one fixes the symplectic form on X and varies the complex structure, it appears as a consequence of the fact that the scalar curvature is a moment map for the action of the symplectomorphism group. In our current framework, with a fixed complex structure and varying Kähler form, the formula in the general case appears more complicated: for any Kähler metric ω'

$$S(\omega' + i\overline{\partial}\partial\eta) = S(\omega') + \mathcal{D}^*\mathcal{D}(\eta) + \nabla\eta.\nabla S + O(\eta^2).$$

The term $\nabla\eta.\nabla S$ appears from the infinitesimal diffeomorphism one needs to apply to pass between the two frameworks. But in our situation the scalar curvature of ω is constant by hypothesis, so this term drops out and we just arrive at (49). Now our hypothesis on the automorphisms of (X, L) means that the kernel of the self-adjoint operator $\mathcal{D}^*\mathcal{D}$ consists only of the constants (Lemma 12) so, by the Fredholm alternative, we can solve Equation (48) and thus complete the proof of Theorem 26.

Now fix a positive integer q and switch attention to the rescaled metrics $\widetilde{\omega}(k) = k\omega(k)$. The conclusion of Theorem 26 rescales to the condition

$$\rho_k(\widetilde{\omega}(k)) = \frac{\chi(k)}{Vk^n}(1 + \epsilon_k),$$

where $\epsilon_k = O(k^{-q-1})$ in C^{r+2}. Let h be the metric on L^k corresponding to $\widetilde{\omega}(k)$. Following the discussion in Section 2, we define a new metric h' on L^k by $h' = e^{u_k}h$ where $e^{u_k} = (1+\epsilon_k)^{-1}$: this gives a Kähler metric $\widetilde{\omega}'(k) = \widetilde{\omega}(k) + i\overline{\partial}\partial u_k$ on X. If we pick any orthonormal basis $\sqrt{\frac{\chi(k)}{Vk^n}}s_\alpha$ for $H^0(L^k)$ with respect to the metrics $h, \widetilde{\omega}(k)$ we have

$$\sum_\alpha |s_\alpha|^2_{h'} = 1,$$

and the metric $\widetilde{\omega}'(k)$ is induced from the Fubini-Study metric by the embedding of X in $\mathbf{CP}^N$ given by the sections s_α. Thus we are in the situation considered in Section 3, with a point, z say, in the symplectic quotient $\mathcal{Z} = \mathcal{H}_0//\mathcal{G}$. For any trace-free Hermitian matrix $B \in i\mathfrak{su}(N+1)$ we can use the action of $\mathrm{SL}(N+1, \mathbf{C})$ on $\mathcal{Z}$ to get another point $e^B z$ of the symplectic quotient, which gives another Kähler metric $\widetilde{\omega}_B$ on X. We let E_B be the matrix $(\eta_{\alpha\beta})$ considered in Section 3, for this Kähler metric. We take the reference metric ω_0 to be ω_∞.

Proposition 27. *If* $\|B\|_{\mathrm{op}} \le 1/10$ *then:*

(1) *There is a constant* c *such that if*

$$\|B\|_{\mathrm{op}} + \|\epsilon\|_{C^{r+2}} + k^{-1} \le cR$$

then the metrics $\widetilde{\omega}_B$ *are* R*-bounded.*

(2) *There is a constant* c' *such that*

$$\|E_B\|_{\mathrm{op}} \le c'(\|B\|_{\mathrm{op}} + \|\epsilon\|_{C^2}).$$

We now prove Proposition 27. The whole construction is invariant under the action of $\mathrm{SU}(N+1)$ (the choice of our original orthonormal basis s_α), so we may assume that B is a diagonal matrix $B = \mathrm{diag}(\lambda_\alpha)$. By definition the metric $\widetilde{\omega}_B$ is $\widetilde{\omega}'(k) + i\overline{\partial}\partial v$ where

$$e^{-v} = \sum e^{2\lambda_\alpha} |s_\alpha|^2_{h'} = 1 + \sum (e^{2\lambda_\alpha} - 1)|s_\alpha|^2 + \epsilon_k. \tag{50}$$

Clearly the metric $\widetilde{\omega}(k)$ differs from $\widetilde{\omega}$ in C^{r+2} norm by $O(k^{-1})$. It is also clear, using (50), that the C^{r+2} norm of v is controlled by that of ϵ_k and by $\max|\lambda_\alpha| \|B\|_{\mathrm{op}}$. More precisely, assuming $\|B\|_{\mathrm{op}} \leq 1/10$ (say), we have an inequality of the form

$$\|v\|_{C^{r+2}} \leq c'(\|B\|_{\mathrm{op}} + \|\epsilon_k\|_{C^{r+2}}), \tag{51}$$

and this gives part (1) of the Proposition. For part (2), recall that $E = (\eta_{\alpha\beta})$ where

$$\eta_{\alpha\beta} = \int_X F\ (s_\alpha, s_\beta)\ \widetilde{\omega}(k)^n \tag{52}$$

and where F is the function on X

$$F = \left(e^{\lambda_\alpha + \lambda_\beta} \frac{(\widetilde{\omega}_k + i\overline{\partial}\partial v)^n}{\widetilde{\omega}_k^n} e^{-v} - 1 \right). \tag{53}$$

As in part (1), the C^0 norm of F is bounded by a multiple of $\|B\|_{\mathrm{op}} + \|\epsilon\|_{C^2}$. We need to recall an elementary fact.

Lemma 28. *Let $V \to Y$ be a Hermitian vector bundle over a measure space Y and let $s_\alpha, \alpha = 0, \ldots N$ be an orthornormal set of sections of V with respect to the usual L^2 inner product. If F is a bounded function on Y and we define a matrix $E = (\eta_{\alpha\beta})$ by*

$$\eta_{\alpha\beta} = \int_Y F\ (s_\alpha, s_\beta)\ d\mu,$$

then $\|E\|_{\mathrm{op}} \leq \|F\|_{L^\infty}$.

The proof of the Lemma is simply to observe that E is the matrix of the operator $\pi \circ M_F \circ \iota$ where $M_F : L^2 \to L^2$ is multiplication by F, π is the projection to the span of the s_α and ι is the inclusion of this span in L^2.

Applying this Lemma to our situation we see that

$$\|E\|_{\mathrm{op}} \leq \|F\|_{C^0} \leq c'(\|B\|_{\mathrm{op}} + \|\epsilon\|_{C^2}),$$

and the proof of Proposition 27 is complete.

4.2 Proofs of the main theorems

The proofs of our main Theorems are now just a matter of putting together the results of the previous sections.

Proof of Theorem 1. This follows from the uniqueness property discussed in Section 2.2. Suppose that s_α and s'_α are two bases of $H^0(L^k)$ such that the two embeddings ι_k, ι'_k of X in $\mathbf{CP}^N$ are balanced. After suitable rescaling, we may suppose that

$$s_0 \wedge \cdots \wedge s_N = s'_0 \wedge \cdots \wedge s'_N \in \Lambda^{N+1} H^0(L^k).$$

We have seen that the data (X, L^k, s_α) defines a point z in $\mu_{\mathrm{SU}}^{-1}(0) \subset \mathcal{Z}$, and likewise the data (X, L^k, s'_α) define a point z'. By the uniquess result the points z, z' lie in the same $\mathrm{SU}(N+1)$ orbit in $\mathcal{Z}$. After changing the s'_α by an element of $\mathrm{SU}(N+1)$ we may suppose that $z = z'$. This means that there is a bundle map $\hat{F}$, covering a diffeomeorphism F of X, which takes the sections s_α to the s'_α. Now, since the s_α and s'_α are holomorphic sections, $(\hat{F}, F)$ defines a holomorphic automorphism of (X, L^k). Let Γ_k be the group of holomorphic automorphisms of (X, L^k) which act with determinant 1 on $H^0(L^k)$. This is the same as the stabiliser of the point z under the $\mathrm{SL}(N+1)$ action. Then Γ_k maps onto $\mathrm{Aut}\,(X, L^k)$ with finite kernel while $\mathrm{Aut}\,(X, L)$ maps to $\mathrm{Aut}\,(X, L^k)$ with finite kernel and cokernel. Thus the hypothesis that $\mathrm{Aut}\,(X, L)$ is discrete implies that Γ_k is also discrete. By the discussion in Section 2.2, Γ_k is also the stabiliser of z under the $\mathrm{SU}(N+1)$ action and it follows that there is a unitary automorphism of $\mathbf{C}^{N+1}$ taking the s_α to the s'_α, as required.

Proof of Theorem 2. This is an immediate consequence of the Catlin-Lu-Zelditch result, Proposition 6. Suppose that the balanced metrics ω_k converge to ω_∞. Proposition 6 gives

$$\left\| \rho_k(\omega_k) - k^n - \frac{S(\omega_k)}{2\pi} k^{n-1} \right\|_{C^0} \leq c k^{n-2}, \tag{54}$$

for a fixed constant c (since the metrics ω_k converge and are *a fortiori* bounded). By hypothesis,

$$\rho_k(\omega_k) = \frac{\chi(k)}{V}.$$

Let S_0 be the average value of $S(\omega_k)$, so $S(\omega_k) - S_0$ has integral zero. Then (54) implies

$$\|S(\omega_k) - S_0\|_{C^0} = O(k^{-1}),$$

and hence $S(\omega_\infty) = \lim S(\omega_k) = S_0$.

Proof of Theorem 3. This is, of course, the main result of the paper and it may be helpful to the reader if we pause to give a summary of the main line of the argument, before going to the details. Starting with a constant scalar curvature metric, we use the formal power series solution, truncated after a suitable number of terms, to get an approximately balanced metric. We then shift to the finite-dimensional point of view, so that the search for a balanced metric close to this approximate solution is the search for a solution to an equation for a matrix. This equation fits into the general moment map set-up of Section 2.2 and we need to show that we can find a solution by using the gradient flow method of that section. The crux of the argument is to show that the approximate solution is a sufficiently good approximation for us to be able to control this gradient flow, and this is underpinned by the estimates on the linearised problem that we have obtained.

Now for the details: fix any integer $r > 0$—we will prove that there is a sequence of balanced metrics converging in C^r. (It follows, by the uniqueness, that the sequence actually converges in C^∞.) Fix an arbitrary $R > 0$. We choose a value of q such that $q > \frac{n}{2} + 3 + r$. We first apply Proposition 27, which in turn hinges on the previous Theorem 26. This gives us metrics $\widetilde{\omega}'(k)$ with $\rho_k(\widetilde{\omega}'(k)) = \frac{\chi(k)}{Vk^n}(1 + \epsilon_k)$, with $\|\epsilon\|_{C^{r+2}} = O(k^{-q-1})$. We choose k so large that $\|\epsilon_k\|_{C^{r+2}} + k^{-1} \leq \frac{cR}{2}$, where c is the constant of Proposition 27. Then Proposition 27 tells us that if $\|B\|_{\mathrm{op}} \leq \min(cR/2, 1/10)$ then the metric $\widetilde{\omega}_B$ is R-bounded. Under this hypothesis we can apply Theorem 21, which tells us that $\Lambda(\widetilde{\omega}_B) \leq \lambda$ where we take $\lambda = C^2k^4$ for the constant C of Theorem 21. Now we apply Proposition 17. We take the constant δ of Proposition 17 to be $\min(cR/2, 1/10)$. Since we obviously have

$$\|B\|_{\mathrm{op}} \leq \|B\|,$$

we know that $\|B\| \leq \delta$ implies $\Lambda(\widetilde{\omega}_B) \leq \lambda$. Now the other ingredient entering into Proposition 17 is $|\nu(z_0)|$, which in our context is $\|E\|$. We have

$$\|E\| \leq \sqrt{N+1}\|E\|_{\mathrm{op}} \tag{55}$$

and since N is $O(k^n)$ we see that $\|E\| \leq C'k^{n/2-q-1}$. Thus we can apply Proposition 17 so long as

$$\lambda C'k^{n/2-q-1} = C^2C'k^{n/2-q+3} \leq \delta. \tag{56}$$

Our choice of r with $q > \frac{n}{2} + 3$ implies that this holds for large enough k and we obtain a solution to our problem with

$$\|B\|_{\mathrm{op}} \leq \|B\| \leq C^2 C' k^{n/2-q+3}. \tag{57}$$

The inequality (51) then shows that the metric $\widetilde{\omega}_k$ which corresponds to this solution differs from $\widetilde{\omega}_\infty$ in C^r by $O(k^{n/2-q+3})$. Finally we rescale back to metrics $\omega_k = k^{-1}\widetilde{\omega}_k$. From the scaling behaviour of the norms, ω_k differs from ω_∞ by $O(k^{n/2+3+r-q})$ in the C^r norm defined by the fixed metric ω_∞. Since $q > n/2 + 3 + r$ the metrics ω_k converge to ω_∞ in C^r as $k \to \infty$.

4.3 Discussion

The success of our proof of Theorem 3 depends crucially on the fact that we have, thanks to Catlin, Lu and Zelditch, a full asymptotic expansion of the "density of states function" ρ_k. This means that it does not matter precisely what power of k we have in, for example, the crucial inequality of Corollary 22. Whatever power we have we can always increase q—the number of terms in the formal series solution that we use—to compensate. If we only knew the weaker result that

$$\rho_k = k^n + \frac{S}{2\pi} k^{n-1} + o(k^{n-1}), \tag{58}$$

our proof, as it stands, would fail. To push through the proof one would need to improve the estimates. There are two main issues here. First, the power of k appearing in Corollary 22, and second the comparison between the Hilbert-Schmidt and operator norms. For each point in our space $\mathcal{Z}$ we can define a number Λ_{op} in the same manner as Λ but using the norm $\| \; \|_{\mathrm{op}}$ on the Lie algebra $\mathfrak{su}(N+1)$ in place of the Hilbert-Schmidt norm;

$$\Lambda_{\mathrm{op}} = \min \frac{\|Q\xi\|_{\mathrm{op}}}{\|\xi\|_{\mathrm{op}}}.$$

Then, with some modification, our scheme of proof would allow to deduce the existence of balanced metrics from the weaker result (58) provided we knew that Λ_{op} is $O(k)$. What we have in fact shown is merely that Λ is $O(k^4)$, and the only way we have of getting information about Λ_{op} is the rather trivial bound $\Lambda_{\mathrm{op}} \leq ck^{n/2}\Lambda \leq c'k^{n/2+4}$. It would be interesting to know what is the real asymptotic behaviour of Λ and

$\Lambda_{\rm op}$. The author guesses that in fact $\Lambda_{\rm op} \sim \Lambda = O(k)$. In this direction we point out that one can refine the argument we used to prove Theorem 21, to obtain a modest improvement, as follows. For any $p > 0$ we can write

$$\|\overline{\partial}_{L^k}\psi_\alpha\|^2 \leq \|\psi_\alpha\|^{2-1/p}\|\Delta^p_{L^k}\psi_\alpha\|^{1/p}.$$

We can estimate the norm of the $\Delta^p\psi_\alpha$ in terms of H and then A, much as before. Summing over α, using Proposition 20 (extended to cover arbitrarily high derivatives) and Holders inequality, we get

$$\|\mathcal{D}H\|^2 \leq \text{const. } \|\underline{\psi}\|^{2-1/p}\|A\|^{1/p}.$$

And from this we deduce, in place of Theorem 21, that

$$\|A\| \leq \text{const.} k^{2p/2p-1}\|\underline{\psi}\|,$$

or in other words $\Lambda \leq \text{const.} k^{4p/2p-1}$. Thus, taking p sufficiently large, we have $\Lambda = O(k^{2+\epsilon})$ for arbitrarily small $\epsilon > 0$.

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