PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 3, May 1973

# SCALAR CURVATURE, INEQUALITY AND SUBMANIFOLD

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ABSTRACT. Using an inequality relation between scalar curvature and length of second fundamental form, we may conclude that a submanifold must have nonnegative (or positive) sectional curvatures. An application to compact submanifolds in obtained.

1. **Statement of results.**[1] Let $M$ be an $n$-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $N$ of constant sectional curvature $c$, and let $h$ and $H$ be the second fundamental form and the mean curvature vector field respectively. Let $h_{ij}^{\alpha}$, $i, j=1, \cdots, n$, $\alpha= n+1, \cdots, n+p$, be the coefficients of the second fundamental form $h$ with respect to a local field of orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$. Then the square of length of second fundamental form, $S$, and the scalar curvature, $R$, of $M$ are given respectively by

$$S = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2, \tag{1}$$

$$R = n^2 H \cdot H - S + n(n-1)c, \tag{2}$$

where dot "·" denotes the scalar product of vectors. A normal vector field $\eta$ is said to be parallel if $D\eta=0$ identically, where $D$ denotes the connection of the normal bundle. The purpose of this paper is to show the following

THEOREM 1. *Let $M$ be an $n$-dimensional submanifold of a Riemannian manifold $N$ of constant curvature $c$. If the scalar curvature $R$ satisfies*

$$R \geqq (n-2)S + (n-2)(n-1)c \qquad (\textit{resp. } R > (n-2)S - (n-2)(n-1)c) \tag{3}$$

*at a point $p \in M$, then the sectional curvatures of $M$ are nonnegative (resp. positive) at $p$.*

Received by the editors May 2, 1972 and, in revised form, July 31, 1972.
*AMS (MOS) subject classifications* (1970). Primary 53B25, 53C40.
*Key words and phrases.* Scalar curvature, length of second fundamental form, sectional curvature, parallel normal vector field, minimal submanifold.

[1] For notations and formulas we refer to [2].

THEOREM 2. *Let $M$ be an $n$-dimensional compact submanifold of euclidean $(n+p)$-space $E^{n+p}$. Then the mean curvature vector $H$ is parallel and we have $R>(n-2)S$ if and only if $M$ is a hypersphere of a linear $(n+1)$-subspace of $E^{n+p}$ when $n\geqq 3$, and $M$ is a minimal surface of a hypersphere of $E^{n+p}$ with positive Gaussian curvature when $n=2$.*

REMARK 1. If the connection of the normal bundle is flat, $n>2$, or if the submanifold is a hypersurface, Theorem 2 was proved by one of the present authors ([**3**], [**4**]).

2. **Proof of Theorem 1.** First we state the following lemma which is a slight generalization of a lemma given in [**4**]. The method of the proofs are quite the same.

LEMMA. *Let $a_1, \cdots, a_n$, $b$ be $n+1$ $(n\geqq 2)$ real numbers satisfying the following inequality*:

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqq (n-1)\sum_{i=1}^{n} a_i^2 + b \quad (resp. >); \tag{4}$$

*then, for any distinct $i, j$; $1\leqq i<j\leqq n$, we have*

$$2a_i a_j \geqq b/(n-1) \quad (resp. >). \tag{5}$$

This lemma is proved in the following way: (4) can be rewritten as

$$(n-2)a_n^2 - 2\left(\sum_{i=1}^{n-1} a_i\right)a_n + \left[(n-2)\sum_{i=1}^{n-1} a_i^2 - 2\sum_{i<j<n} a_i a_j + b\right] \leqq 0,$$

(resp. $<$). Denote the left-hand side by $-r$. Since $a_n$ is real,

$$\begin{aligned}\left(\sum_{i=1}^{n-1} a_i\right)^2 &\geqq (n-2)\left[(n-2)\sum_{i=1}^{n-1} a_i^2 - 2\sum_{i<j<n} a_i a_j + b + r\right]\\ &\geqq (n-2)\left[(n-1)\sum_{i=1}^{n-1} a_i^2 - \left(\sum_{i=1}^{n-1} a_i\right)^2 + b\right].\end{aligned}$$

Hence we obtain

$$\left(\sum_{i=1}^{n-1} a_i\right)^2 \geqq (n-2)\sum_{i=1}^{n-1} a_i^2 + \left(\frac{n-2}{n-1}\right)b \quad \text{(resp. >)}.$$

Continuing the same process $(n-2)$ times, we obtain (5).

Substituting (2) into (3), we obtain

$$n^2 H\cdot H \geqq (n-1)S - 2(n-1)c \quad \text{(resp. >)} \quad \text{at } p. \tag{6}$$

For simplicity we may choose a local field of orthonormal frame $e_1, \cdots$, $e_n$, $e_{n+1}, \cdots, e_{n+p}$ around $p$ such that $e_{n+1}$ is parallel to the mean curvature vector $H$ and $e_1, \cdots, e_n$ are in the principal directions of $e_{n+1}$ at

$p \in M$. (If $H=0$ at $p$, we may choose an arbitrary $e_{n+1}$.) Then we have

$$(7)\qquad (h_{ij}^{n+1}) = \begin{pmatrix} h_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & \\ & & & h_n \end{pmatrix}, \qquad n^2 H \cdot H = \left(\sum_{i=1}^{n} h_i\right)^2 \text{ at } p.$$

Thus we obtain from (6):

$$(8)\qquad \left(\sum_{i=1}^{n} h_i\right)^2 \geqq (n-1)\sum_{i=1}^{n} h_i^2 + (n-1)\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 - 2(n-1)c$$

(resp. $>$).

Applying the lemma to (8), we get

$$\begin{aligned}(9)\qquad 2h_ih_j &\geqq \sum_{\alpha=n+2}^{n+p}\sum_{k,m=1}^{n}(h_{km}^{\alpha})^2 - 2c \\ &\geqq \sum_{\alpha=n+2}^{n+p}[(h_{ii}^{\alpha})^2 + (h_{jj}^{\alpha})^2 + 2(h_{ij}^{\alpha})^2] - 2c \\ &\geqq 2\sum_{\alpha=n+2}^{n+p}[h_{ii}^{\alpha}h_{jj}^{\alpha} + (h_{ij}^{\alpha})^2] - 2c,\end{aligned}$$

for any $1 \leqq i < j \leqq n$ at $p$. Thus the sectional curvature at $p$,

$$K_{ij} = \sum_{\alpha=n+1}^{n+p}[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] + c,$$

for the plane section spanned by $e_i$ and $e_j$ is nonnegative (resp. positive). This proves the theorem.

3. **Proof of Theorem 2.** Let $M$ be an $n$-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\bar{N}$ of constant sectional curvature $c$ and $\eta$ be a parallel unit normal vector over $M$. If we choose the local fields of orthonormal frame in such a way that $e_{n+1}=\eta$ and $e_1, \cdots, e_n$ are in the principal directions of $e_{n+1}$, then we have

$$H_{n+1} = \begin{pmatrix} h_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & h_n \end{pmatrix}.$$

We assume that Tr $H_{n+1}$ is constant. Then a recent paper of Smyth [5] gives the following formula:

$$(10)\qquad \sum_{i,j=1}^{n} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} = \sum_{i<j}\left[K_{ij} + \sum_{\beta}(h_{ij}^{\beta})^2\right](h_i - h_j)^2,$$

where $\Delta h_{ij}^{n+1}$ denotes the Laplacian of the second fundamental form $h_{ij}^{n+1}$ in the direction of $e_{n+1}$. Now, suppose that $M$ is an $n$-dimensional compact submanifold of $E^{n+p}$ such that the mean curvature vector $H$ is parallel and $R>(n-2)S$. Then, by Theorem 1, we see that the sectional curvatures of $M$ are all positive, that is, $K_{ij}>0$ for $1\leqq i<j\leqq n$. Therefore, we see that $\sum_{i,j} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1}\geqq 0$. Hence we get

$$\frac{1}{2}\Delta(\mathrm{Tr}\, H_{n+1}^2) = \sum_{i,j} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} + \sum_{i,j,k} (h_{ijk}^{n+1})^2 \geqq 0. \tag{11}$$

By Hopf's lemma we see that $h_{ijk}^{n+1}=0$ and $\sum_{i,j} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1}=0$. Hence, from (10) we have

$$h_1 = \cdots = h_n \neq 0. \tag{12}$$

This shows that $M$ is pseudo-umbilical in $E^{n+p}$ and $H$ is parallel. Hence, we see that $M$ is contained in a hypersphere $S^{n+p-1}$ of $E^{n+p}$ as a minimal submanifold (see, for instance, [**1**]). Without loss of generality, we may assume that $S^{n+p-1}$ is of radius 1. Then, by the assumption, $R>(n-2)S$, we see that the square of the length of second fundamental form of $M$ in $S^{n+p-1}$, say $\bar{S}$, satisfies

$$\bar{S} < n/(n-1). \tag{13}$$

Therefore, by a result of Chern-do Carmo-Kobayashi [**2**], we find that if $n\geqq 3$, then $M$ must be totally geodesic in $S^{n+p-1}$. Hence $M$ is a hypersphere of a linear $(n+1)$-subspace of $E^{n+p}$. If $n=2$, then the condition $R>(n-2)S$ implies that the Gaussian curvature of $M$ is positive. This proves a part of the theorem. The remaining part is obvious.

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