

SCALAR CURVATURE OF COMPLEX SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

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1. Statement of results

Let $P_{n+p}(\mathbf{C})$ be a complex projective space of complex dimension $n + p$ with the Fubini-Study metric of constant holomorphic sectional curvature 1, and X be an n -dimensional compact complex submanifold of $P_{n+p}(\mathbf{C})$ with the induced Kaehler structure. Then X is algebraic by a well known theorem of Chow. Throughout this paper, we assume that X is a complete intersection of p hypersurfaces in general position in $P_{n+p}(\mathbf{C})$, i.e., that there exist p hypersurfaces X_1, \dots, X_p of degree a_1, \dots, a_p in $P_{n+p}(\mathbf{C})$ such that $X = X_1 \cap \dots \cap X_p$. As a matter of course, every compact complex hypersurface in $P_{n+p}(\mathbf{C})$ is under consideration. The purpose of the present paper is to prove the following results:

Theorem. *Let X be a complete intersection of p hypersurfaces of degree a_1, \dots, a_p in general position in $P_{n+p}(\mathbf{C})$, and ρ be the scalar curvature of X . Then*

$$\int_X \rho * 1 = n\{n + p + 1 - (a_1 + \dots + a_p)\} \int_X * 1,$$

where $* 1$ denotes the volume element of X .

This theorem implies that the average of the scalar curvature depends only on the degree of X , while the scalar curvature itself on the equations defining X .

Corollary 1. *If $\rho > n^2$ everywhere on X , then $X = P_n(\mathbf{C})$.*

Corollary 2. *Let X be a hypersurface of $P_{n+1}(\mathbf{C})$. If $n(n - \nu + 1) < \rho \leq n(n - \nu + 2)$ everywhere on X , then X is an algebraic manifold of degree ν .*

Let S be the square of the length of the second fundamental form of the imbedding so that $S = n(n + 1) - \rho$. The following corollaries are equivalent to Corollary 1 and Corollary 2 respectively.

Corollary 1'. *If $S < n$ everywhere on X , then $X = P_n(\mathbf{C})$.*

Corollary 2'. *Let X be a hypersurface of $P_{n+1}(\mathbf{C})$. If $n(\nu - 1) \leq S < n\nu$ everywhere on X , then X is an algebraic manifold of degree ν .*

In a previous paper [3], we have proved that if $S < (n + 2)/(4 - 1/p)$ everywhere on X , then $X = P_n(\mathbf{C})$. Corollary 1' is an improvement of this result and is best possible for the following reason: Let $Q_n(\mathbf{C}) = \{(z_0, \dots, z_{n+1}) \in P_{n+1}(\mathbf{C}) \mid \sum z_i^2 = 0\}$, where z_0, \dots, z_{n+1} be the homogeneous coordinates

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in $P_{n+1}(\mathbf{C})$. Then $Q_n(\mathbf{C})$ is an Einstein-Kaehler manifold and $S = n$ everywhere on it.

2. Proof of results

Let $g = 2 \sum g_{\alpha\beta} dz_\alpha d\bar{z}_\beta$ and $\Phi = \frac{2}{\sqrt{-1}} \sum g_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$ be the Kaehler metric and the fundamental 2-form of X respectively, and let $\text{Ric} = 2 \sum R_{\alpha\beta} dz_\alpha d\bar{z}_\beta$ be the Ricci tensor of X . Then the first Chern class $c_1(X)$ of X is represented by the closed 2-form

$$\gamma = \frac{1}{2\pi\sqrt{-1}} \sum R_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta .$$

We designate $[\Phi]$ and $[\gamma]$ to be the cohomology classes represented by Φ and γ respectively, so that $c_1(X) = [\gamma]$.

Let h be the generator of $H^2(P_{n+p}(\mathbf{C}), \mathbf{Z})$ corresponding to the divisor class of a hyperplane $P_{n+p-1}(\mathbf{C})$. Then the first Chern class $c_1(P_{n+p}(\mathbf{C}))$ of $P_{n+p}(\mathbf{C})$ is given by

$$c_1(P_{n+p}(\mathbf{C})) = (n + p + 1)h .$$

Let $j: X \rightarrow P_{n+p}(\mathbf{C})$ be the imbedding, and \tilde{h} the image of h under the homomorphism $j^*: H^2(P_{n+p}(\mathbf{C}), \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z})$. Then we have

$$c_1(X) = \{n + p + 1 - (a_1 + \dots + a_p)\}\tilde{h} .$$

Let Ψ be the fundamental 2-form of $P_{n+p}(\mathbf{C})$ so that

$$c_1(P_{n+p}(\mathbf{C})) = \frac{n + p + 1}{8\pi} [\Psi] .$$

These, together with the fact that $\Phi = j^*\Psi$, imply

$$[\Phi] = 8\pi\tilde{h}$$

so that

$$c_1(X) = \frac{1}{8\pi} \{(n + p + 1 - (a_1 + \dots + a_p))[\Phi]\} .$$

Thus there exists a 1-form η such that

$$(1) \quad \gamma = \frac{1}{8\pi} \{(n + p + 1 - (a_1 + \dots + a_p))[\Phi] + d\eta\} .$$

Let δ, Λ and M be the usual operators in harmonic integral theory (cf. [2]). Operating Λ on both sides of (1) we have

$$(2) \quad -\rho/(2\pi) = -n\{n + p + 1 - (a_1 + \dots + a_p)\}/(2\pi) + \Lambda d\eta,$$

since $\Lambda\Phi = *(\Phi \wedge *\Phi) = -4n$ and $\Lambda\gamma = *(\Phi \wedge *\gamma) = -\rho/(2\pi)$. On the other hand, using the identity $d\Lambda - \Lambda d = \delta M - M\delta$ and the relation $d\Lambda\eta = M\delta\eta = 0$ we obtain

$$\Lambda d\eta = -\delta M\eta,$$

and therefore, by (2),

$$(3) \quad \rho/(2\pi) = n\{n + p + 1 - (a_1 + \dots + a_p)\}/(2\pi) + \delta M\eta.$$

Integration of both sides of (3) on X thus gives

$$(4) \quad \frac{1}{2\pi} \int_X \rho * 1 = \frac{n}{2\pi} \{n + p + 1 - (a_1 + \dots + a_p)\} \int_X * 1 + \int_X \delta M\eta * 1.$$

The second term of the right hand side of (4) vanishes since $\int_X \delta M\eta * 1 = (\delta M\eta, 1) = (M\eta, d1) = 0$, where $(,)$ denotes the global scalar product. Hence we have

$$\int_X \rho * 1 = n\{n + p + 1 - (a_1 + \dots + a_p)\} \int_X * 1,$$

which proves our theorem.

If $n^2 < \rho$ everywhere on X , then

$$n^2 \int_X * 1 < n\{n + p + 1 - (a_1 + \dots + a_p)\} \int_X * 1,$$

which implies $a_1 + \dots + a_p < p + 1$, that is, $a_1 = \dots = a_p = 1$, proving Corollary 1.

To prove Corollary 2, we put $p = 1$ and $a_1 = a$. If $n(n - \nu + 1) < \rho \leq n(n - \nu + 2)$ everywhere on X , then

$$n(n - \nu + 1) \int_X * 1 < n(n - a + 2) \int_X * 1 \leq n(n - \nu + 2) \int_X * 1,$$

which implies $\nu \leq a < \nu + 1$, that is, $a = \nu$. Hence Corollary 2 is proved.

Bibliography

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