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SCALAR CURVATURE OF CONTACT THREE *CR*-SUBMANIFOLDS IN A UNIT $(4m+3)$-SPHERE

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Abstract. In this paper we derive an integral formula on an $(n+3)$-dimensional, compact, minimal contact three CR-submanifold M of $(p-1)$ contact three CR-dimension immersed in a unit $(4m+3)$-sphere S^{4m+3}. Using this integral formula, we give a sufficient condition concerning the scalar curvature of M in order that such a submanifold M is to be a generalized Clifford torus.

1. Introduction

Let S^{4m+3} be a $(4m+3)$-dimensional unit sphere, that is,

$$S^{4m+3} = \{q \in Q^{m+1} \ : \ \|q\| = 1\},$$

where Q^{m+1} is the real $4(m+1)$-dimensional quaternionic number space. For any point $q \in S^{4m+3}$, we put

$$\xi = Jq, \quad \eta = Kq, \quad \zeta = Lq,$$

where $\{J, K, L\}$ denotes the canonical quaternionic Kähler structure of Q^{m+1}. Then $\{\xi, \eta, \zeta\}$ becomes a Sasakian three structure, that is, ξ, η and ζ are mutually orthogonal unit Killing vector fields which satisfy

$$\begin{aligned} \bar\nabla_Y \bar\nabla_X \xi &= g(X,\xi)Y - g(Y,X)\xi, \\ \bar\nabla_Y \bar\nabla_X \eta &= g(X,\eta)Y - g(Y,X)\eta, \\ \bar\nabla_Y \bar\nabla_X \zeta &= g(X,\zeta)Y - g(Y,X)\zeta \end{aligned} \tag{1.1}$$

for any vector fields X, Y tangent to S^{4m+3}, where g denotes the canonical metric on S^{4m+3} induced from that of Q^{m+1} and $\bar\nabla$ the Riemannian connection

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with respect to g. In this case, putting

$$\phi X = \bar{\nabla}_X \xi, \quad \psi X = \bar{\nabla}_X \eta, \quad \theta X = \bar{\nabla}_X \zeta, \tag{1.2}$$

it follows that

$$\begin{aligned} &\phi\xi = 0,\ \psi\eta = 0,\ \theta\zeta = 0,\\ &\psi\zeta = -\theta\eta = \xi,\ \theta\xi = -\phi\zeta = \eta,\ \phi\eta = -\psi\xi = \zeta,\\ &[\eta,\zeta] = -2\xi,\ [\zeta,\xi] = -2\eta,\ [\xi,\eta] = -2\zeta, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\phi^2 = -I + f_\xi \otimes \xi, \quad \psi^2 = -I + f_\eta \otimes \eta, \quad \theta^2 = -I + f_\zeta \otimes \zeta,\\ &\psi\theta = \phi + f_\zeta \otimes \eta, \quad \theta\phi = \psi + f_\xi \otimes \zeta, \quad \phi\psi = \theta + f_\eta \otimes \xi,\\ &\theta\psi = -\phi + f_\eta \otimes \zeta, \quad \phi\theta = -\psi + f_\zeta \otimes \xi, \quad \psi\phi = -\theta + f_\xi \otimes \eta, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} g(\phi X, Y) &= -g(X, \phi Y),\\ g(\psi X, Y) &= -g(X, \psi Y),\\ g(\theta X, Y) &= -g(X, \theta Y), \end{aligned} \tag{1.5}$$

where I denotes the identity transformation and

$$f_\xi(X) = g(X, \xi), \quad f_\eta(X) = g(X, \eta), \quad f_\zeta(X) = g(X, \zeta). \tag{1.6}$$

Moreover, from (1.1) and (1.2), we have

$$\begin{aligned} (\bar{\nabla}_Y \phi)X &= g(X, \xi)Y - g(Y, X)\xi,\\ (\bar{\nabla}_Y \psi)X &= g(X, \eta)Y - g(Y, X)\eta,\\ (\bar{\nabla}_Y \theta)X &= g(X, \zeta)Y - g(Y, X)\zeta \end{aligned} \tag{1.7}$$

for any vector fields X, Y tangent to S^{4m+3} (cf. [4, 5, 6, 7, 8]).

Let M be an $(n+3)$-dimensional submanifold tangent to the structure vectors ξ, η and ζ of S^{4m+3}. If there exists a subbundle ν of the normal bundle $TM^\perp$ such that

$$\begin{aligned} &\phi\nu_x \subset \nu_x, \quad \psi\nu_x \subset \nu_x, \quad \theta\nu_x \subset \nu_x,\\ &\phi\nu_x^\perp \subset T_xM, \quad \psi\nu_x^\perp \subset T_xM, \quad \theta\nu_x^\perp \subset T_xM \end{aligned} \tag{1.8}$$

at any point $x \in M$, where TM denotes the tangent bundle of M and $\nu^\perp$ is the complementary orthogonal subbundle to ν in $TM^\perp$, then the submanifold is called a *contact three CR-submanifold* of S^{4m+3} and the dimension of ν *contact three CR-dimension*. In particular we can easily see that real hypersurfaces tangent to ξ, η and ζ of S^{4m+3} are typical examples of such submanifolds.

In this paper we shall study $(n+3)$-dimensional contact three CR-submanifolds with $(p-1)$ contact three CR-dimension of S^{4m+3}, where p is $4m-n$ the codimension. In this case the maximal $\{\phi, \psi, \theta\}$-invariant subspace

$$\mathcal{D}_x = T_xM \cap \phi T_xM \cap \psi T_xM \cap \theta T_xM$$

of T_xM has constant dimension $n-3$ because the orthogonal complement $\mathcal{D}_x^\perp$ to $\mathcal{D}_x$ in T_xM has constant dimension 6 at any point $x \in M$ (cf. See §2 and [7]).

Moreover we shall investigate some geometric characterizations of

$$S^{4r+3}(a) \times S^{4s+3}(b) \quad (a^2+b^2=1,\ r+s=(n-3)/4)$$

as a contact three CR-submanifold of S^{4m+3}.

2. Preliminaries

Let M be an $(n+3)$-dimensional contact three CR-submanifold with $(p-1)$ contact three CR-dimension of S^{4m+3}. Then we may set $\nu^\perp = \mathrm{Span}\{N\}$ for a unit normal vector field N to M since dim $\nu_x = p-1$ at every $x \in M$. From now on we put

$$\phi N = -U, \qquad \psi N = -V, \qquad \theta N = -W. \tag{2.1}$$

Then it follows from (1.3)-(1.6) and (1.8) that U, V, W are mutually orthogonal unit tangent vector fields to M and satisfy

$$\begin{aligned} &g(\xi,U)=0, \quad g(\xi,V)=0, \quad g(\xi,W)=0,\\ &g(\eta,U)=0, \quad g(\eta,V)=0, \quad g(\eta,W)=0,\\ &g(\zeta,U)=0, \quad g(\zeta,V)=0, \quad g(\zeta,W)=0. \end{aligned} \tag{2.2}$$

Moreover ξ, η, ζ, U, V and W are all contained in $\mathcal{D}_x^\perp$ and consequently dim $\mathcal{D}_x^\perp = 6$, or equivalently dim $\mathcal{D}_x = n-3$ at any point $x \in M$ (cf. [7]). It is clear that

$$\phi\mathcal{D}_x^\perp \subset \mathrm{Span}\{N\}, \quad \psi\mathcal{D}_x^\perp \subset \mathrm{Span}\{N\}, \quad \theta\mathcal{D}_x^\perp \subset \mathrm{Span}\{N\} \tag{2.3}$$

at any point $x \in M$. Hence for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\dots,p}$ $(N_1 := N)$ of normal vectors to M, we have the following decomposition in tangential and normal components:

$$\begin{aligned} &\text{(i) } \phi X = FX + u(X)N,\\ &\text{(ii) } \psi X = GX + v(X)N,\\ &\text{(iii) } \theta X = HX + w(X)N, \end{aligned} \tag{2.4}$$

$$\phi N_\alpha = \sum_{\beta=2}^{p} P^\phi_{\alpha\beta} N_\beta,\ \psi N_\alpha = \sum_{\beta=2}^{p} P^\psi_{\alpha\beta} N_\beta,\ \theta N_\alpha = \sum_{\beta=2}^{p} P^\theta_{\alpha\beta} N_\beta,\ \alpha = 2,\dots,p, \tag{2.5}$$

where $\{F, G, H\}$ define skew-symmetric linear endomorphisms acting on T_xM and $\{u, v, w\}$ are local 1-forms on M. Since the structure vector fields $\{\xi, \eta, \zeta\}$ are tangent to M, it follows from (1.3), (1.5), (2.1) and (2.4) that

$$\begin{aligned} &F\xi = 0, \quad F\eta = \zeta, \quad F\zeta = -\eta,\\ &G\xi = -\zeta, \quad G\eta = 0, \quad G\zeta = \xi\\ &H\xi = \eta, \quad H\eta = -\xi, \quad H\zeta = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &FU = 0, \quad FV = W, \quad FW = -V, \\ &GU = -W, \quad GV = 0, \quad GW = U \\ &HU = V, \quad HV = -U, \quad HW = 0, \end{aligned} \tag{2.7}$$

$$g(U,X) = u(X), \quad g(V,X) = v(X), \quad g(W,X) = w(X). \tag{2.8}$$

Next, applying ϕ to both side of $(2.4)_{(\mathrm{i})}$ and using (1.4), (1.6), (2.1) and $(2.4)_{(\mathrm{i})}$, we have

$$F^2X = -X + u(X)U + g(\xi,X)\xi, \quad u(FX) = g(U,FX) = 0. \tag{2.9}$$

Similarly, from $(2.4)_{(\mathrm{ii})}$ and $(2.4)_{(\mathrm{iii})}$ it follows that

$$G^2X = -X + v(X)V + g(\eta,X)\eta, \quad v(GX) = g(V,GX) = 0, \tag{2.10}$$

$$H^2X = -X + w(X)W + g(\zeta,X)\zeta, \quad w(HX) = g(W,HX) = 0. \tag{2.11}$$

Also applying ψ and θ to both side of $(2.4)_{(\mathrm{i})}$, respectively, and using (1.4)-(1.6), (2.1) and (2.4), we get

$$GFX = -HX + u(X)V + g(\xi,X)\eta, \quad v(FX) = -w(X), \tag{2.12}$$

$$HFX = GX + u(X)W + g(\xi,X)\zeta, \quad w(FX) = v(X). \tag{2.13}$$

Similarly, it follows from $(2.4)_{(\mathrm{ii})}$ and $(2.4)_{(\mathrm{iii})}$ that

$$HGX = -FX + v(X)W + g(\eta,X)\zeta, \quad w(GX) = -u(X), \tag{2.14}$$

$$FGX = HX + v(X)U + g(\eta,X)\xi, \quad u(GX) = w(X), \tag{2.15}$$

$$FHX = -GX + w(X)U + g(\zeta,X)\xi, \quad u(HX) = -v(X), \tag{2.16}$$

$$GHX = FX + w(X)V + g(\zeta,X)\eta, \quad v(HX) = u(X). \tag{2.17}$$

3. Fundamental equations for the contact three CR-submanifold

Let M be as in §2. Then, by means of (1.4), (1.6) and (2.5), we can take a local orthonormal basis $\{N, N_a, N_{a^*}, N_{a^{**}}, N_{a^{***}}\}_{a=1,\dots,q:=(p-1)/4}$ of normal vectors to M in such a way that

$$N_{a^*} := \phi N_a, \quad N_{a^{**}} := \psi N_a, \quad N_{a^{***}} := \theta N_a. \tag{3.1}$$

Let ∇ and $\nabla^\perp$ denote the covariant differentiation in M and the normal connection induced from $\bar\nabla$ on the normal bundle $TM^\perp$, respectively. Then Gauss and Weingarten formulae are given by

$$\bar\nabla_X Y = \nabla_X Y + h(X,Y), \tag{3.2}$$

(3.3)

(i) $\bar\nabla_X N = -AX + \nabla^\perp_X N$

$$= -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*} + s_{a^{**}}(X)N_{a^{**}} + s_{a^{***}}(X)N_{a^{***}}\},$$

(ii) $\bar{\nabla}_X N_a = -A_a X - s_a(X)N$

$$+\sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*} + s_{ab^{**}}(X)N_{b^{**}} + s_{ab^{***}}(X)N_{b^{***}}\},$$

(iii) $\bar{\nabla}_X N_{a^*} = -A_{a^*} X - s_{a^*}(X)N$

$$+\sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*} + s_{a^*b^{**}}(X)N_{b^{**}} + s_{a^*b^{***}}(X)N_{b^{***}}\},$$

(iv) $\bar{\nabla}_X N_{a^{**}} = -A_{a^{**}} X - s_{a^{**}}(X)N$

$$+\sum_{b=1}^{q}\{s_{a^{**}b}(X)N_b + s_{a^{**}b^*}(X)N_{b^*} + s_{a^{**}b^{**}}(X)N_{b^{**}} + s_{a^{**}b^{***}}(X)N_{b^{***}}\},$$

(v) $\bar{\nabla}_X N_{a^{***}} = - A_{a^{***}} X - s_{a^{***}}(X)N$

$$+\sum_{b=1}^{q}\{s_{a^{***}b}(X)N_b + s_{a^{***}b^*}(X)N_{b^*} + s_{a^{***}b^{**}}(X)N_{b^{**}} + s_{a^{***}b^{***}}(X)N_{b^{***}}\}$$

for any vector fields X, Y tangent to M, where $s's$ are coefficients of the normal connection $\nabla^{\perp}$. Here and in the sequel h denotes the second fundamental form and A, A_a, A_{a^*}, $A_{a^{**}}$ and $A_{a^{***}}$ the shape operators corresponding to the normals N, N_a, N_{a^*}, $N_{a^{**}}$ and $N_{a^{***}}$, respectively. They are related by

$$\begin{aligned} h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{&g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*} \\ &+ g(A_{a^{**}}X,Y)N_{a^{**}} + g(A_{a^{***}}X,Y)N_{a^{***}}\}. \end{aligned} \tag{3.4}$$

On the other hand, since the ambient manifold S^{4m+3} is a space form of the constant curvature 1, its curvature tensor $\bar{R}$ satisfies

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$

Hence, by means of the equation of Gauss, we can easily see that the Ricci tensor $\mathrm{Ric}(Y,Z)$ turns out to be

(3.5)

$$\begin{aligned} &\mathrm{Ric}(Y,Z) \\ &= (n+2)g(Y,Z) + (\mathrm{tr}A)g(AY,Z) - g(A^2Y,Z) \\ &\quad + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)g(A_aY,Z) - g(A_a^2Y,Z) + (\mathrm{tr}A_{a^*})g(A_{a^*}Y,Z) - g(A_{a^*}^2Y,Z) \\ &\quad + (\mathrm{tr}A_{a^{**}})g(A_{a^{**}}Y,Z) - g(A_{a^{**}}^2Y,Z) + (\mathrm{tr}A_{a^{***}})g(A_{a^{***}}Y,Z) - g(A_{a^{***}}^2Y,Z)\} \end{aligned}$$

and consequently the scalar curvature ρ is given by

$$\begin{aligned} \rho =&(n+2)(n+3) + (\mathrm{tr}A)^2 - \mathrm{tr}A^2 + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)^2 - \mathrm{tr}A_a^2 \\ &+ (\mathrm{tr}A_{a^*})^2 - \mathrm{tr}A_{a^*}^2 + (\mathrm{tr}A_{a^{**}})^2 - \mathrm{tr}A_{a^{**}}^2 + (\mathrm{tr}A_{a^{***}})^2 - \mathrm{tr}A_{a^{***}}^2\}. \end{aligned} \tag{3.6}$$

Moreover, by means of the equation of Codazzi, we also have

(3.7)
$$\begin{aligned}&(\nabla_X A)Y-(\nabla_Y A)X\\ =&\sum_{a=1}^{q}\{s_a(X)A_aY-s_a(Y)A_aX+s_{a^*}(X)A_{a^*}Y-s_{a^*}(Y)A_{a^*}X\\ &+s_{a^{**}}(X)A_{a^{**}}Y-s_{a^{**}}(Y)A_{a^{**}}X+s_{a^{***}}(X)A_{a^{***}}Y-s_{a^{***}}(Y)A_{a^{***}}X\}.\end{aligned}$$

Now differentiating $(2.4)_{(\mathrm{i})}$ covariantly and using (1.7), (3.2), $(3.3)_{(\mathrm{i})}$ and (3.4), we have

$$\begin{aligned}&(\nabla_Y F)X=g(X,\xi)Y-g(X,Y)\xi-g(AX,Y)U+u(X)AY,\\ &(\nabla_Y u)X=-g(AFX,Y).\end{aligned}\tag{3.8}$$

Similarly, from $(2.4)_{(\mathrm{ii})}$ and $(2.4)_{(\mathrm{iii})}$, we also obtain

$$\begin{aligned}&(\nabla_Y G)X=g(X,\eta)Y-g(X,Y)\eta-g(AX,Y)V+v(X)AY,\\ &(\nabla_Y v)X=-g(AGX,Y),\end{aligned}\tag{3.9}$$

$$\begin{aligned}&(\nabla_Y H)X=g(X,\zeta)Y-g(X,Y)\zeta-g(AX,Y)W+w(X)AY,\\ &(\nabla_Y w)X=-g(AHX,Y).\end{aligned}\tag{3.10}$$

Differentiating (2.1) covariantly and using (1.7), (2.4), (3.2), $(3.3)_{(\mathrm{i})}$ and (3.4), we have

$$\nabla_X U=FAX,\qquad \nabla_X V=GAX,\qquad \nabla_X W=HAX.\tag{3.11}$$

Moreover, it is clear from (1.2), (3.2) and (3.4) that

$$\nabla_X\xi=FX,\quad \nabla_X\eta=GX,\quad \nabla_X\zeta=HX,\tag{3.12}$$

$$A\xi=U,\quad A\eta=V,\quad A\zeta=W,\tag{3.13}$$

$$\begin{aligned}&A_a\xi=0,\quad A_{a^*}\xi=0,\quad A_{a^{**}}\xi=0,\quad A_{a^{***}}\xi=0,\\ &A_a\eta=0,\quad A_{a^*}\eta=0,\quad A_{a^{**}}\eta=0,\quad A_{a^{***}}\eta=0,\\ &A_a\zeta=0,\quad A_{a^*}\zeta=0,\quad A_{a^{**}}\zeta=0,\quad A_{a^{***}}\zeta=0,\quad a=1,\dots,q.\end{aligned}\tag{3.14}$$

On the other hand, since the structure vector fields $\{\xi,\eta,\zeta\}$ are tangent to M, it follows from (3.1) and $(3.3)_{(\mathrm{iii})}$ that

(3.15)
$$\begin{aligned}\phi\bar\nabla_X N_a=&-A_{a^*}X-s_{a^*}(X)N\\ &+\sum_{b=1}^{q}\{s_{a^*b}(X)N_b+s_{a^*b^*}(X)N_{b^*}+s_{a^*b^{**}}(X)N_{b^{**}}+s_{a^*b^{***}}(X)N_{b^{***}}\}.\end{aligned}$$

Applying ϕ to (3.15) and using (1.4), (1.6), (2.1), $(2.4)_{(\mathrm{i})}$, (3.1) and (3.14), we get

$$\bar{\nabla}_X N_a = FA_{a^*}X - s_{a^*}(X)U + g(A_{a^*}X, U)N - \sum_{b=1}^{q}\{s_{a^*b}(X)N_{b^*} - s_{a^*b^*}(X)N_b + s_{a^*b^{**}}(X)N_{b^{***}} - s_{a^*b^{***}}(X)N_{b^{**}}\},$$

which together with $(3.3)_{(\mathrm{ii})}$ implies

$$A_a X = -FA_{a^*}X + s_{a^*}(X)U, \quad s_a(X) = -g(A_{a^*}X, U) = -u(A_{a^*}X).$$

Applying ψ and θ to (3.15), respectively and using (1.4), (1.6), (2.1), $(2.4)_{(\mathrm{ii})}$, $(2.4)_{(\mathrm{iii})}$, (3.1) and (3.14), we also have

$$\bar{\nabla}_X N_{a^{***}} = GA_{a^*}X - s_{a^*}(X)V + g(A_{a^*}X, V)N - \sum_{b=1}^{q}\{s_{a^*b}(X)N_{b^{**}} - s_{a^*b^*}(X)N_{b^{***}} - s_{a^*b^{**}}(X)N_b + s_{a^*b^{***}}(X)N_{b^*}\},$$

$$\bar{\nabla}_X N_{a^{**}} = -HA_{a^*}X + s_{a^*}(X)W - g(A_{a^*}X, W)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_{b^{***}} + s_{a^*b^*}(X)N_{b^{**}} - s_{a^*b^{**}}(X)N_{b^*} - s_{a^*b^{***}}(X)N_b\},$$

thus comparing the above two equations with $(3.3)_{(\mathrm{iv})}$ and $(3.3)_{(\mathrm{v})}$, we obtain

$$A_{a^{**}}X = HA_{a^*}X - s_{a^*}(X)W, \qquad s_{a^{**}}(X) = g(A_{a^*}X, W) = w(A_{a^*}X),$$
$$A_{a^{***}}X = -GA_{a^*}X + s_{a^*}(X)V, \quad s_{a^{***}}(X) = -g(A_{a^*}X, V) = -v(A_{a^*}X).$$

Similarly, from $(3.3)_{(\mathrm{iv})}$ it follows that

$$\begin{aligned}&\psi\bar{\nabla}_X N_a\\ &= -A_{a^{**}}X - s_{a^{**}}(X)N\\ &\quad + \sum_{b=1}^{q}\{s_{a^{**}b}(X)N_b + s_{a^{**}b^*}(X)N_{b^*} + s_{a^{**}b^{**}}(X)N_{b^{**}} + s_{a^{**}b^{***}}(X)N_{b^{***}}\},\end{aligned}$$

which implies

$$\begin{aligned}&\bar{\nabla}_X N_a\\ &= GA_{a^{**}}X - s_{a^{**}}(X)V + g(A_{a^{**}}X, V)N\\ &\quad - \sum_{b=1}^{q}\{s_{a^{**}b}(X)N_{b^{**}} - s_{a^{**}b^*}(X)N_{b^{***}} - s_{a^{**}b^{**}}(X)N_b + s_{a^{**}b^{***}}(X)N_{b^*}\},\end{aligned}$$

$$\begin{aligned}&\bar{\nabla}_X N_{a^{***}}\\ &= -FA_{a^{**}}X + s_{a^{**}}(X)U - g(A_{a^{**}}X, U)N\\ &\quad + \sum_{b=1}^{q}\{s_{a^{**}b}(X)N_{b^*} - s_{a^{**}b^*}(X)N_b + s_{a^{**}b^{**}}(X)N_{b^{***}} - s_{a^{**}b^{***}}(X)N_{b^{**}}\},\end{aligned}$$

$$\begin{aligned}&\bar{\nabla}_X N_{a^*}\\&= HA_{a^{**}}X - s_{a^{**}}(X)W + g(A_{a^{**}}X, W)N\\&\quad - \sum_{b=1}^{q}\{s_{a^{**}b}(X)N_{b^{***}} + s_{a^{**}b^*}(X)N_{b^{**}} - s_{a^{**}b^{**}}(X)N_{b^*} - s_{a^{**}b^{***}}(X)N_b\}.\end{aligned}$$

Hence we have

$$\begin{aligned}&A_a X = -GA_{a^{**}}X + s_{a^{**}}(X)V, \quad s_a(X) = -g(A_{a^{**}}X, V) = -v(A_{a^{**}}X),\\&A_{a^*}X = -HA_{a^{**}}X + s_{a^{**}}(X)W, \quad s_{a^*}(X) = -g(A_{a^{**}}X, W) = -w(A_{a^{**}}X),\\&A_{a^{***}}X = FA_{a^{**}}X - s_{a^{**}}(X)U, \quad s_{a^{***}}(X) = g(A_{a^{**}}X, U) = u(A_{a^{**}}X).\end{aligned}$$

Also, by means of $(3.3)_{(\mathrm{v})}$ we have

$$\begin{aligned}\theta\bar{\nabla}_X N_a = &- A_{a^{***}}X - s_{a^{***}}(X)N\\&+ \sum_{b=1}^{q}\{s_{a^{***}b}(X)N_b + s_{a^{***}b^*}(X)N_{b^*} + s_{a^{***}b^{**}}(X)N_{b^{**}}\\&\qquad + s_{a^{***}b^{***}}(X)N_{b^{***}}\},\end{aligned}$$

which yields

$$\begin{aligned}\bar{\nabla}_X N_a = &HA_{a^{***}}X - s_{a^{***}}(X)W + g(A_{a^{***}}X, W)N\\&- \sum_{b=1}^{q}\{s_{a^{***}b}(X)N_{b^{***}} + s_{a^{***}b^*}(X)N_{b^{**}} - s_{a^{***}b^{**}}(X)N_{b^*}\\&\qquad - s_{a^{***}b^{***}}(X)N_b\},\end{aligned}$$

$$\begin{aligned}\bar{\nabla}_X N_{a^{**}} = &FA_{a^{***}}X - s_{a^{***}}(X)U + g(A_{a^{***}}X, U)N\\&- \sum_{b=1}^{q}\{s_{a^{***}b}(X)N_{b^*} - s_{a^{***}b^*}(X)N_b + s_{a^{***}b^{**}}(X)N_{b^{***}}\\&\qquad - s_{a^{***}b^{***}}(X)N_{b^{**}}\},\end{aligned}$$

$$\begin{aligned}\bar{\nabla}_X N_{a^*} = &- GA_{a^{***}}X + s_{a^{***}}(X)V - g(A_{a^{***}}X, V)N\\&+ \sum_{b=1}^{q}\{s_{a^{***}b}(X)N_{b^{**}} - s_{a^{***}b^*}(X)N_{b^{***}} - s_{a^{***}b^{**}}(X)N_b\\&\qquad + s_{a^{***}b^{***}}(X)N_{b^*}\}.\end{aligned}$$

Thus we have

$$\begin{aligned}&A_a X = -HA_{a^{***}}X + s_{a^{***}}(X)W,\ s_a(X) = -g(A_{a^{***}}X, W) = -w(A_{a^{***}}X),\\&A_{a^*}X = GA_{a^{***}}X - s_{a^{***}}(X)V,\ s_{a^*}(X) = g(A_{a^{***}}X, V) = v(A_{a^{***}}X),\\&A_{a^{**}}X = -FA_{a^{***}}X + s_{a^{***}}(X)U,\ s_{a^{**}}(X) = -g(A_{a^{***}}X, U) = -u(A_{a^{***}}X).\end{aligned}$$

Finally, applying ϕ, ψ and θ to $(3.3)_{(\mathrm{ii})}$, respectively and using (1.4), (1.6), (1.7), (2.1), (2.4) and (3.1), we have

$$\begin{aligned}\bar{\nabla}_X N_{a^*} =& - FA_aX + s_a(X)U - g(A_aX, U)N\\ &+ \sum_{b=1}^{q}\{s_{ab}(X)N_{b^*} - s_{ab^*}(X)N_b + s_{ab^{**}}(X)N_{b^{***}} - s_{ab^{***}}(X)N_{b^{**}}\},\\ \bar{\nabla}_X N_{a^{**}} =& - GA_aX + s_a(X)V - g(A_aX, V)N\\ &+ \sum_{b=1}^{q}\{s_{ab}(X)N_{b^{**}} - s_{ab^*}(X)N_{b^{***}} - s_{ab^{**}}(X)N_b + s_{ab^{***}}(X)N_{b^*}\},\\ \bar{\nabla}_X N_{a^{***}} =& - HA_aX + s_a(X)W - g(A_aX, W)N\\ &+ \sum_{b=1}^{q}\{s_{ab}(X)N_{b^{***}} + s_{ab^*}(X)N_{b^{**}} - s_{ab^{**}}(X)N_{b^*} - s_{ab^{***}}(X)N_b\},\end{aligned}$$

thus comparing the above three equations with $(3.3)_{(\mathrm{iii})}$, $(3.3)_{(\mathrm{iv})}$ and $(3.3)_{(\mathrm{v})}$, we obtain

$$\begin{aligned}&A_{a^*}X = FA_aX - s_a(X)U, \quad s_{a^*}(X) = g(A_aX, U) = u(A_aX),\\ &A_{a^{**}}X = GA_aX - s_a(X)V, \quad s_{a^{**}}(X) = g(A_aX, V) = v(A_aX),\\ &A_{a^{***}}X = HA_aX - s_a(X)W, \quad s_{a^{***}}(X) = g(A_aX, W) = w(A_aX).\end{aligned}$$

Summing up, we have:

Lemma 3.1. *Let M be an $(n+3)$-dimensional contact three CR-submanifold of S^{4m+3} with contact three CR-dimension $(p-1)$. Then the following relationships* (3.16) *and* (3.17) *are established on M, where $p = 4m - n$.*

$$\begin{aligned}A_aX &= -FA_{a^*}X + s_{a^*}(X)U = -GA_{a^{**}}X + s_{a^{**}}(X)V\\ &= -HA_{a^{***}}X + s_{a^{***}}(X)W,\\ A_{a^*}X &= FA_aX - s_a(X)U = GA_{a^{***}}X - s_{a^{***}}(X)V\\ &= -HA_{a^{**}}X + s_{a^{**}}(X)W,\\ A_{a^{**}}X &= -FA_{a^{***}}X + s_{a^{***}}(X)U = GA_aX - s_a(X)V\\ &= HA_{a^*}X - s_{a^*}(X)W,\\ A_{a^{***}}X &= FA_{a^{**}}X - s_{a^{**}}(X)U = -GA_{a^*}X + s_{a^*}(X)V\\ &= HA_aX - s_a(X)W,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\text{(i)} \quad s_a(X) = -u(A_{a^*}X) = -v(A_{a^{**}}X) = -w(A_{a^{***}}X),\\ &\text{(ii)} \quad s_{a^*}(X) = u(A_aX) = v(A_{a^{***}}X) = -w(A_{a^{**}}X),\\ &\text{(iii)} \quad s_{a^{**}}(X) = -u(A_{a^{***}}X) = v(A_aX) = w(A_{a^*}X),\\ &\text{(iv)} \quad s_{a^{***}}(X) = u(A_{a^{**}}X) = -v(A_{a^*}X) = w(A_aX).\end{aligned} \tag{3.17}$$

Because of Lemma 3.1 and the facts that F, G, H are skew-symmetric and A_a, A_{a^*}, $A_{a^{**}}$, $A_{a^{***}}$ are symmetric, (3.16) yields

$$\begin{aligned} g((A_aF + FA_a)X, Y) &= s_a(X)u(Y) - s_a(Y)u(X),\\ g((A_aG + GA_a)X, Y) &= s_a(X)v(Y) - s_a(Y)v(X),\\ g((A_aH + HA_a)X, Y) &= s_a(X)w(Y) - s_a(Y)w(X), \end{aligned} \tag{3.18}$$

$$\begin{aligned} g((A_{a^*}F + FA_{a^*})X, Y) &= s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X),\\ g((A_{a^*}G + GA_{a^*})X, Y) &= s_{a^*}(X)v(Y) - s_{a^*}(Y)v(X),\\ g((A_{a^*}H + HA_{a^*})X, Y) &= s_{a^*}(X)w(Y) - s_{a^*}(Y)w(X), \end{aligned} \tag{3.19}$$

$$\begin{aligned} g((A_{a^{**}}F + FA_{a^{**}})X, Y) &= s_{a^{**}}(X)u(Y) - s_{a^{**}}(Y)u(X),\\ g((A_{a^{**}}G + GA_{a^{**}})X, Y) &= s_{a^{**}}(X)v(Y) - s_{a^{**}}(Y)v(X),\\ g((A_{a^{**}}H + HA_{a^{**}})X, Y) &= s_{a^{**}}(X)w(Y) - s_{a^{**}}(Y)w(X), \end{aligned} \tag{3.20}$$

$$\begin{aligned} g((A_{a^{***}}F + FA_{a^{***}})X, Y) &= s_{a^{***}}(X)u(Y) - s_{a^{***}}(Y)u(X),\\ g((A_{a^{***}}G + GA_{a^{***}})X, Y) &= s_{a^{***}}(X)v(Y) - s_{a^{***}}(Y)v(X),\\ g((A_{a^{***}}H + HA_{a^{***}})X, Y) &= s_{a^{***}}(X)w(Y) - s_{a^{***}}(Y)w(X). \end{aligned} \tag{3.21}$$

It is also clear from (3.14) and (3.17) that

$$\begin{aligned} s_a(\xi) = s_{a^*}(\xi) = s_{a^{**}}(\xi) = s_{a^{***}}(\xi) = 0,\\ s_a(\eta) = s_{a^*}(\eta) = s_{a^{**}}(\eta) = s_{a^{***}}(\eta) = 0,\\ s_a(\zeta) = s_{a^*}(\zeta) = s_{a^{**}}(\zeta) = s_{a^{***}}(\zeta) = 0. \end{aligned} \tag{3.22}$$

On the other hand, we can take an orthonormal basis

$$\{e_i\}_{i=1,\dots,4l+6}, \quad l := (n-3)/4$$

of tangent vectors to M in such a way that

$$\begin{aligned} e_{l+1} := Fe_1,\ \dots\ ,\ e_{2l} := Fe_l, \quad e_{2l+1} := Ge_1,\ \dots\ ,\ e_{3l} := Ge_l,\\ e_{3l+1} := He_1,\ \dots\ ,\ e_{4l} := He_l, \end{aligned} \tag{3.23}$$

$$e_{4l+1} := U,\ e_{4l+2} := V,\ e_{4l+3} := W,\ e_{4l+4} := \xi,\ e_{4l+5} := \eta,\ e_{4l+6} := \zeta. \tag{3.24}$$

Replacing X by Fe_i in $(3.17)_{(\mathrm{i})}$, we have

$$s_a(Fe_i) = -g(A_{a^*}Fe_i, U) = -g(A_{a^{**}}Fe_i, V) = -g(A_{a^{***}}Fe_i, W),$$

which together with (2.7), (3.19), (3.20) and (3.21) implies

$$s_a(Fe_i) = -s_{a^*}(e_i) = -w(A_{a^{**}}e_i) = v(A_{a^{***}}e_i).$$

But it follows from $(3.17)_{(\mathrm{ii})}$ that

$$s_{a^*}(e_i) = -w(A_{a^{**}}e_i) = v(A_{a^{***}}e_i),$$

which and the above equation imply

$$s_a(Fe_i) = 0, \quad s_{a^*}(e_i) = 0, \quad i = 1, \dots, l. \tag{3.25}$$

Similarly, replacing X by Ge_i and He_i in $(3.17)_{(\mathrm{i})}$, respectively, we also have

$$s_a(Ge_i) = -g(A_{a^*}Ge_i, U) = -g(A_{a^{**}}Ge_i, V) = -g(A_{a^{***}}Ge_i, W),$$
$$s_a(He_i) = -g(A_{a^*}He_i, U) = -g(A_{a^{**}}He_i, V) = -g(A_{a^{***}}He_i, W),$$

which together with (2.7), (3.19), (3.20) and (3.21) yields

$$s_a(Ge_i) = w(A_{a^*}e_i) = -s_{a^{**}}(e_i) = -u(A_{a^{***}}e_i),$$
$$s_a(He_i) = -v(A_{a^*}e_i) = u(A_{a^{**}}e_i) = -s_{a^{***}}(e_i).$$

But it follows from $(3.17)_{(\mathrm{iii})}$ and $(3.17)_{(\mathrm{iv})}$ that

$$s_{a^{**}}(e_i) = w(A_{a^*}e_i) = -u(A_{a^{***}}e_i), \quad s_{a^{***}}(e_i) = -v(A_{a^*}e_i) = u(A_{a^{**}}e_i),$$

which and the above equation give

$$(3.26) \quad s_a(Ge_i) = 0, \quad s_a(He_i) = 0, \quad s_{a^{**}}(e_i) = 0, \quad s_{a^{***}}(e_i) = 0 \quad i = 1, \ldots, l.$$

Next, replacing X by Fe_i, Ge_i and He_i in $(3.17)_{(\mathrm{ii})}$, respectively, we have

$$s_{a^*}(Fe_i) = u(A_aFe_i) = v(A_{a^{***}}Fe_i) = -w(A_{a^{**}}Fe_i),$$
$$s_{a^*}(Ge_i) = u(A_aGe_i) = v(A_{a^{***}}Ge_i) = -w(A_{a^{**}}Ge_i),$$
$$s_{a^*}(He_i) = u(A_aHe_i) = v(A_{a^{***}}He_i) = -w(A_{a^{**}}He_i)$$

from which together with (2.7), (3.18), (3.20) and (3.21),

$$s_{a^*}(Fe_i) = s_a(e_i) = w(A_{a^{***}}e_i) = v(A_{a^{**}}e_i),$$
$$s_{a^*}(Ge_i) = -w(A_ae_i) = s_{a^{***}}(e_i) = -u(A_{a^{**}}e_i),$$
$$s_{a^*}(He_i) = v(A_ae_i) = -u(A_{a^{***}}e_i) = -s_{a^{**}}(e_i).$$

But $(3.17)_{(\mathrm{i})}$, $(3.17)_{(\mathrm{iii})}$ and $(3.17)_{(\mathrm{iv})}$ yield

$$s_a(e_i) = -v(A_{a^{**}}e_i) = -w(A_{a^{***}}e_i),$$

which together with the above equation and (3.26) gives

$$(3.27) \quad s_a(e_i) = 0, \quad s_{a^*}(Fe_i) = 0, \quad s_{a^*}(Ge_i) = 0, \quad s_{a^*}(He_i) = 0, \quad i = 1, \ldots, l.$$

Replacing X by Fe_i, Ge_i and He_i in $(3.17)_{(\mathrm{iii})}$, respectively, we have

$$s_{a^{**}}(Fe_i) = -u(A_{a^{***}}Fe_i) = v(A_aFe_i) = w(A_{a^*}Fe_i),$$
$$s_{a^{**}}(Ge_i) = -u(A_{a^{***}}Ge_i) = v(A_aGe_i) = w(A_{a^*}Ge_i),$$
$$s_{a^{**}}(He_i) = -u(A_{a^{***}}He_i) = v(A_aHe_i) = w(A_{a^*}He_i),$$

from which together with (2.7), (3.18), (3.19) and (3.21),

$$s_{a^{**}}(Fe_i) = -s_{a^{***}}(e_i) = w(A_ae_i) = -v(A_{a^*}e_i),$$
$$s_{a^{**}}(Ge_i) = w(A_{a^{***}}e_i) = s_a(e_i) = u(A_{a^*}e_i),$$
$$s_{a^{**}}(He_i) = -v(A_{a^{***}}e_i) = -u(A_ae_i) = s_{a^*}(e_i).$$

Thus (3.25), (3.26) and (3.27) give

$$(3.28) \qquad s_{a^{**}}(Fe_i) = 0, \quad s_{a^{**}}(Ge_i) = 0, \quad s_{a^{**}}(He_i) = 0, \quad i = 1, \ldots, l.$$

Finally, replacing X by Fe_i, Ge_i and He_i in $(3.17)_{(\text{iv})}$, respectively, we have

$$s_{a^{***}}(Fe_i) = u(A_{a^{**}}Fe_i) = -v(A_{a^*}Fe_i) = w(A_aFe_i),$$
$$s_{a^{***}}(Ge_i) = u(A_{a^{**}}Ge_i) = -v(A_{a^*}Ge_i) = w(A_aGe_i),$$
$$s_{a^{***}}(He_i) = u(A_{a^{**}}He_i) = -v(A_{a^*}He_i) = w(A_aHe_i),$$

from which together with (2.7), (3.18), (3.19), and (3.20),

$$s_{a^{***}}(Fe_i) = s_{a^{**}}(e_i), \quad s_{a^{***}}(Ge_i) = -s_{a^*}(e_i), \quad s_{a^{**}}(He_i) = s_a(e_i).$$

Hence (3.25), (3.26) and (3.27) imply

$$s_{a^{***}}(Fe_i) = 0, \quad s_{a^{***}}(Ge_i) = 0, \quad s_{a^{***}}(He_i) = 0, \quad i = 1, \dots, l. \tag{3.29}$$

4. An integral formula on the compact contact three *CR*-submanifold

Let M be as in §2 and put

$$T := \nabla_U U + \nabla_V V + \nabla_W W + (\text{div } U)U + (\text{div } V)V + (\text{div } W)W$$

and take the same orthonormal basis $\{e_i\}_{i=1,\dots,4l+6}$ $(l = (n-3)/4)$ of tangent vectors to M as given in (3.23) and (3.24), where div $U = \sum_{i=1}^{4l+6} g(e_i, \nabla_{e_i}U)$. Since F is skew-symmetric and A is symmetric, (3.11) implies

$$T = FAU + GAV + HAW. \tag{4.1}$$

We note that T is a global function on M. Now, for later use we shall compute div $T = \sum_{i=1}^{4l+6} g(e_i, \nabla_{e_i}T)$.

First of all, differentiating both side of (4.1) covariantly and using (3.8)-(3.11), we have

$$\begin{aligned}\nabla_X T = &(\nabla_X F)AU + F(\nabla_X A)U + FA\nabla_X U\\ &+ (\nabla_X G)AV + G(\nabla_X A)V + GA\nabla_X V\\ &+ (\nabla_X H)AW + H(\nabla_X A)W + HA\nabla_X W,\end{aligned}$$

that is,

$$\begin{aligned}\nabla_X T = &g(AU,\xi)X - g(AU,X)\xi - g(A^2U,X)U + u(AU)AX + F(\nabla_X A)U\\ &+ FAFAX + g(AV,\eta)X - g(AV,X)\eta - g(A^2V,X)V + v(AV)AX\\ &+ G(\nabla_X A)V + GAGAX + g(AW,\zeta)X - g(AW,X)\zeta - g(A^2W,X)W\\ &+ w(AW)AX + H(\nabla_X A)W + HAHAX,\end{aligned}$$

which and (3.13) give

(4.2)

$$\begin{aligned}\nabla_X T = &3X - g(AU,X)\xi - g(AV,X)\eta - g(AW,X)\zeta\\ &+ \{u(AU) + v(AV) + w(AW)\}AX - g(A^2U,X)U - g(A^2V,X)V\\ &- g(A^2W,X)W + FAFAX + GAGAX + HAHAX + F(\nabla_X A)U\\ &+ G(\nabla_X A)V + H(\nabla_X A)W.\end{aligned}$$

Thus, from (2.6), (2.7) and (2.9)-(2.17), we have

$$\begin{aligned}
&\operatorname{div}T\\
&= 3(n+2) + \operatorname{tr}A\{u(AU) + v(AV) + w(AW)\}\\
&\quad - g(A^2U,U) - g(A^2V,V) - g(A^2W,W)\\
&\quad + \sum_{i=1}^{n+3} g(FAFAe_i + GAGAe_i + HAHAe_i, e_i)\\
&\quad + \sum_{i=1}^{l}\{g((\nabla_{Fe_i}A)e_i - (\nabla_{e_i}A)Fe_i, U) + g((\nabla_{Ge_i}A)e_i - (\nabla_{e_i}A)Ge_i, V)\\
&\quad + g((\nabla_{He_i}A)e_i - (\nabla_{e_i}A)He_i, W) + g((\nabla_{He_i}A)Ge_i - (\nabla_{Ge_i}A)He_i, U)\\
&\quad + g((\nabla_{Fe_i}A)He_i - (\nabla_{He_i}A)Fe_i, V) + g((\nabla_{Ge_i}A)Fe_i - (\nabla_{Fe_i}A)Ge_i, W)\}\\
&\quad + g((\nabla_W A)V - (\nabla_V A)W, U) + g((\nabla_U A)W - (\nabla_W A)U, V)\\
&\quad + g((\nabla_V A)U - (\nabla_U A)V), W) + g((\nabla_\zeta A)\eta - (\nabla_\eta A)\zeta, U)\\
&\quad + g((\nabla_\xi A)\zeta - (\nabla_\zeta A)\xi, V) + g((\nabla_\eta A)\xi - (\nabla_\xi A)\eta), W),
\end{aligned}$$

which together with (3.7), (3.22) and (3.25)-(3.29) implies

$$\begin{aligned}
&\operatorname{div}T\\
&= 3(n+2) + \operatorname{tr}A\{u(AU) + v(AV) + w(AW)\}\\
&\quad - g(A^2U,U) - g(A^2V,V) - g(A^2W,W)\\
&\quad + \sum_{i=1}^{n+3} g(FAFAe_i + GAGAe_i + HAHAe_i, e_i)\\
&\quad + \sum_{a=1}^{q}\{s_a(W)u(A_aV) - s_a(V)u(A_aW) + s_{a^*}(W)u(A_{a^*}V)\\
&\qquad - s_{a^*}(V)u(A_{a^*}W) + s_{a^{**}}(W)u(A_{a^{**}}V) - s_{a^{**}}(V)u(A_{a^{**}}W)\\
&\qquad + s_{a^{***}}(W)u(A_{a^{***}}V) - s_{a^{***}}(V)u(A_{a^{***}}W)\}\\
&\quad + \sum_{a=1}^{q}\{s_a(U)v(A_aW) - s_a(W)v(A_aU) + s_{a^*}(U)v(A_{a^*}W) - s_{a^*}(W)v(A_{a^*}U)\\
&\qquad + s_{a^{**}}(U)v(A_{a^{**}}W) - s_{a^{**}}(W)v(A_{a^{**}}U)\\
&\qquad + s_{a^{***}}(U)v(A_{a^{***}}W) - s_{a^{***}}(W)v(A_{a^{***}}U)\}\\
&\quad + \sum_{a=1}^{q}\{s_a(V)w(A_aU) - s_a(U)w(A_aV) + s_{a^*}(V)w(A_{a^*}U) - s_{a^*}(U)w(A_{a^*}V)\\
&\qquad + s_{a^{**}}(V)w(A_{a^{**}}U) - s_{a^{**}}(U)w(A_{a^{**}}V)\\
&\qquad + s_{a^{***}}(V)w(A_{a^{***}}U) - s_{a^{***}}(U)w(A_{a^{***}}V)\},
\end{aligned}$$

that is,

$$\begin{aligned}\operatorname{div}T &= 3(n+2) + \operatorname{tr}A\{u(AU) + v(AV) + w(AW)\} \\ &\quad - \|AU\|^2 - \|AV\|^2 - \|AW\|^2 \\ &\quad + \sum_{i=1}^{n+3} g(FAFAe_i + GAGAe_i + HAHAe_i, e_i).\end{aligned} \tag{4.3}$$

On the other hand, using (2.9)-(2.17) and (3.13), we can easily verify that

$$\sum_{i=1}^{n+3} g(FAFAe_i, e_i) = \frac{1}{2}\|FA - AF\|^2 - \operatorname{tr}A^2 + \|AU\|^2 + 1,$$

$$\sum_{i=1}^{n+3} g(GAGAe_i, e_i) = \frac{1}{2}\|GA - AG\|^2 - \operatorname{tr}A^2 + \|AV\|^2 + 1,$$

$$\sum_{i=1}^{n+3} g(HAHAe_i, e_i) = \frac{1}{2}\|HA - AH\|^2 - \operatorname{tr}A^2 + \|AW\|^2 + 1,$$

that is,

$$\begin{aligned}&\sum_{i=1}^{n+3} g(FAFAe_i + GAGAe_i + HAHAe_i, e_i) \\ &= \frac{1}{2}(\|FA - AF\|^2 + \|GA - AG\|^2 + \|HA - AH\|^2) \\ &\quad - 3\operatorname{tr}A^2 + \|AU\|^2 + \|AV\|^2 + \|AW\|^2 + 3,\end{aligned}$$

which and (4.3) yield

$$\begin{aligned}\operatorname{div}T &= 3\{\rho - (n+1)(n+3)\} + \operatorname{tr}A\{u(AU) + v(AV) + w(AW)\} \\ &\quad + \frac{1}{2}(\|FA - AF\|^2 + \|GA - AG\|^2 + \|HA - AH\|^2) \\ &\quad - 3(\operatorname{tr}A)^2 - 3\sum_{a=1}^{q}\{(\operatorname{tr}A_a)^2 - \operatorname{tr}A_a^2 + (\operatorname{tr}A_{a^*})^2 - \operatorname{tr}A_{a^*}^2 \\ &\quad + (\operatorname{tr}A_{a^{**}})^2 - \operatorname{tr}A_{a^{**}}^2 + (\operatorname{tr}A_{a^{***}})^2 - \operatorname{tr}A_{a^{***}}^2\}.\end{aligned} \tag{4.4}$$

Thus we have:

Lemma 4.1. *Let M be an $(n+3)$-dimensional compact, minimal contact three CR-submanifold of S^{4m+3} with contact three CR-dimension $(p-1)$. If the scalar curvature ρ is greater or equal to $(n+1)(n+3)$, then*

$$FA = AF, \quad GA = AG, \quad HA = AH, \tag{4.5}$$

$$A_a = A_{a^*} = A_{a^{**}} = A_{a^{***}} = 0, \quad a = 1, \ldots, q. \tag{4.6}$$

5. The proof of main theorem

For the submanifold M given in Lemma 4.1, it is clear from (4.6) that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection $\nabla^{\perp}$ with the aid of $(3.3)_{(\mathrm{i})}$ and (3.17). Thus we may apply Erbacher's reduction theorem ([3]) and this yields that there is an $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} such that $M \subset S^{n+4}$. Here we note that $n+4 = \dim S^{n+4}$ is of the type $4(l+1)+3$. Moreover, since the tangent space T_xS^{n+4} of the totally geodesic submanifold S^{n+4} at $x \in M$ is $T_xM \oplus \mathrm{Span}\{N\}$, S^{n+4} is an invariant submanifold of S^{4m+3} with respect to the Sasakian three structure $\{\xi, \eta, \zeta\}$ (that is, ξ, η and ζ are all tangent to S^{n+4}, and $\phi(T_xS^{n+4}) \subset T_xS^{n+4}$, $\psi(T_xS^{n+4}) \subset T_xS^{n+4}$ and $\theta(T_xS^{n+4}) \subset T_xS^{n+4}$ for any $x \in S^{n+4}$) because of (2.1) and (2.4). Hence the submanifold M given in Lemma 4.1 can be regarded as a real hypersurface of S^{n+4} which is a totally geodesic invariant submanifold of S^{4m+3}.

Tentatively we denote S^{n+4} by M' and by i_1 the immersion of M into M' and i_2 the totally geodesic immersion of M' into S^{4m+3}. Then, from the Gauss equation (3.1), it follows that

$$\nabla'_{i_1X} i_1Y = i_1\nabla_X Y + h'(X,Y) = i_1\nabla_X Y + g(A'X,Y)N', \tag{5.1}$$

where h' denotes the second fundamental form of M in M', A' the corresponding shape operator and N' a unit normal vector field to M in M'. Since $i = i_2 \circ i_1$, we have

$$\begin{aligned} \bar{\nabla}_{i_2\circ i_1X} i_2 \circ i_1Y &= i_2\nabla'_{i_1X} i_1Y + \bar{h}(i_1X, i_1Y) \\ &= i_2(i_1\nabla_X Y + g(A'X,Y)N'), \end{aligned} \tag{5.2}$$

because M' is totally geodesic in S^{4m+3}. Comparing (5.2) with (3.2), we can easily see that

$$N = i_2N', \quad A = A'. \tag{5.3}$$

Since M' is an invariant submanifold of S^{4m+3}, for any $X' \in TM'$,

$$\phi i_2X' = i_2\phi'X', \quad \psi i_2X' = i_2\psi'X', \quad \theta i_2X' = i_2\theta'X' \tag{5.4}$$

are valid, where $\{\phi', \psi', \theta'\}$ is the induced Sasakian three structure of M'. Thus it follows from (2.4) that

$$\begin{aligned} \phi iX &= \phi i_2 \circ i_1X = i_2\phi' i_1X = i_2(i_1F'X + u'(X)N') \\ &= iF'X + u'(X)i_2N' = iF'X + u'(X)N, \\ \psi iX &= \psi i_2 \circ i_1X = i_2\psi' i_1X = i_2(i_1G'X + v'(X)N') \\ &= iG'X + v'(X)i_2N' = iG'X + v'(X)N, \\ \theta iX &= \theta i_2 \circ i_1X = i_2\theta' i_1X = i_2(i_1H'X + w'(X)N') \\ &= iH'X + w'(X)i_2N' = iH'X + w'(X)N. \end{aligned}$$

Comparing those equations with (2.4), we have $F = F'$, $u' = u$; $G = G'$, $v' = v$ and $H = H'$, $w' = w$. Hence M is a real hypersurface of S^{n+4} which satisfies $F'A' = A'F'$, $G'A' = A'G'$ and $H'A' = A'H'$. Now applying the theorem(cf. Theorem 10 in [8]) due to the second author, we can conclude:

Theorem. *Let M be an $(n+3)$-dimensional compact, minimal, contact three CR-submanifold of $(p-1)$ contact three CR-dimension in S^{4m+3}. If the scalar curvature is greater or equal to $(n+1)(n+3)$, then*

$$M = S^{4r+3}(a) \times S^{4s+3}(b), \quad a^2 + b^2 = 1, \quad r + s = (n-3)/4.$$

References

[1] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1986.

[2] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker Inc., New York, 1973.

[3] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Diff. Geom. **5** (1971), 333–340.

[4] S. Ishihara and M. Konish, *Differential geometry of fibred spaces*, Study Group of Differential Geometry, Tokyo, 1973.

[5] T. Kashiwada, *A note on a Riemannian space with Sasakian* 3*-structure*,Natur. Sci. Rep. Ochanomizu Univ. **22** (1971), 1–2.

[6] Y. Y. Kuo *On almost contact* 3*-structure*, Tohoku Math. J. **22** (1970), 325–332.

[7] J.-H. Kwon and J. S. Pak, *On contact three CR-submanifolds of a $(4m+3)$-dimensional unit sphere*, Comm. Korean Math Soc. **13** (1998), no. 3, 561–577.

[8] J. S. Pak, *Real hypersurfaces in quaternionic Kaehlerian manifolds with constant Q-sectional curvature*, Kodai Math. Sem. Rep. **29** (1977), no. 1-2, 22–61.

[9] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker Inc., New York, 1970.

[10] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser, Boston-Basel-Stuttgart, 1983.

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