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# SCALAR CURVATURES ON $S^2$

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ABSTRACT. A theorem for the existence of solutions of the nonlinear elliptic equation $-\Delta u + 2 = R(x)e^u$, $x \in S^2$, is proved by using a "mass center" analysis technique and by applying a continuous "flow" in $H^1(S^2)$ controlled by $\nabla R$.

**0. Introduction.** Given a function $R(x)$ on the two dimensional unit sphere $S^2$, one wishes to know when it can actually be the scalar curvature of some metric $g$ that is pointwise conformal to the standard metric $g_0$ on $S^2$. This is an interesting problem in geometry (cf. [**1**]). In order to find an answer, people usually consider the differential equation

$$(*) \qquad \Delta u - 2 + R(x)e^u = 0, \qquad x \in S^2.$$

It is well known that if $u$ is a solution of $(*)$, then $R(x)$ turns out to be the scalar curvature corresponding to the metric $g = e^u g_0$, which, obviously is pointwise conformal to $g_0$.

There are some necessary conditions for the solvability of $(*)$ pointed out by Kazdan and Warner (cf. [**2**]), which show that not all smooth functions $R(x)$ can be achieved as such a scalar curvature. Then for which $R$ can one solve $(*)$? This has been an open problem for many years (cf. [**3**]).

Moser [**4**] proved that if $R(x) = R(-x)$, for any $x \in S^2$, and $R$ is positive somewhere, then $(*)$ has a solution. Recently, Hong [**5**] considered the case where $R$ is rotationally symmetric and established some existence theorems. In our previous paper [**6**], we generalized the results of Moser and Hong to the case where $R$ possesses some kinds of generic symmetries, that is, $R$ is invariant under the action of some subgroups of the orthogonal transformation group in $\mathbf{R}^3$. Then it is natural for one to ask, "What happens when $R$ is not symmetric?" So far we know, there have not yet been any existence results in this situation. This is the motivation for this present paper.

In this paper, without any symmetry assumption on $R$, we find some sufficient conditions so that $(*)$ can be solved, which is independent of the results in [**6**]. To find a solution of $(*)$, we consider the functional

$$J(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2\, dA + 2\int_{S^2} u\, dA - 8\pi \ln \int_{S^2} Re^u\, dA$$

defined on

$$H_* = \left\{ u \in H^1(S^2)\colon \int_{S^2} Re^u\, dA > 0 \right\}$$

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and seek critical points of $J$. It is easily seen that a critical point of $J$ in $H_*$ plus a suitable constant makes a solution of $(*)$. Since $J$ is bounded from below on $H_*$, a natural idea is to seek a minimum of $J$. Unfortunately, it was shown (cf. [**5**]) that $\inf_{H_*} J$ can never be attained unless $R$ is a constant. So one is led to find saddle points of $J$. Under some appropriate assumptions on $R$, using a family of transformations on $H^1(S^2)$ and a continuous "flow" in $H_*$, by a careful "mass center" analysis, we prove the existence of a saddle point of $J$ in $H_*$ and establish the following

THEOREM. *Assume*
$(R_0)$ $R \in C^2(S^2)$.
$(R_1)$ *There exist two points, say $a$ and $b$, on $S^2$, such that*

$$R(a) = R(b) = m \equiv \max_{S^2} R > 0,$$

*and*

$$\sup_{h\in\Gamma} \min_{x\in h([0,1])} R(x) = \nu < m,$$

*where* $\Gamma = \{h\colon h \in C([0,1], S^2),\ h(0) = a,\ h(1) = b\}$.
$(R_2)$ *There is* $h_0 \in \Gamma$ *such that* $\min_{h_0([0,1])} R = \nu$, *and for any*

$$x \in K \equiv \{x \in h_0([0,1])\colon R(x) = \nu\}, \qquad \Delta R(x) > 0.$$

$(R_3)$ *There is no critical value of $R$ in the interval* $(\nu, m)$.
*Then problem* $(*)$ *possesses at least one solution.*

OUTLINE OF THE PROOF. Due to its complexity, we divide our proof into five sections.

In §1, we find two families of separated points, say $\{\varphi_{\lambda,a}\}$ and $\{\varphi_{\lambda,b}\}$ with $\lambda \in [0,1)$, satisfying

$$J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b}) \to \inf_{H_*} J, \quad \text{as } \lambda \to 1.$$

And under the condition $(R_1)$ prove that as $\lambda$ gets sufficiently close to 1, there exists a "mountain pass" between the two points $\varphi_{\lambda,a}$ and $\varphi_{\lambda,b}$, i.e.

$$\mu_\lambda = \inf_{l\in L_\lambda} \max_{u\in l([0,1])} J(u) > \max\{J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b})\}. \tag{0.1}$$

where $L_\lambda = \{l\colon l \in C([0,1], H_*),\ l(0) = \varphi_{\lambda,a}, l(1) = \varphi_{\lambda,b}\}$. Now, by Ekeland's variational principle (see [**7**]) there is a sequence $\{u_k\}$ in $H_*$ such that $J(u_k) \to \mu_\lambda$, $J'(u_k) \to 0$, as $k \to \infty$. If $\{u_k\}$ possesses a strongly convergent subsequence, then we have solved our problem.

When does the sequence $\{u_k\}$ converge strongly? In order to investigate this, we verify, in §2 a modified (P.S.) condition for the functional $J$, that is

PROPOSITION 2.1. *Assume* $\{v_k\} \subset H_*$, $J(v_k) \le \beta < +\infty$, $J'(v_k) \to 0$, *as* $k \to \infty$, *and* $|P(v_k)| \le 1 - \gamma < 1$. *Let* $\tilde v_k = v_k - (1/4\pi)\int_{S^2} v_k\, dA$. *Then* $\{\tilde v_k\}$ *possesses a strongly convergent subsequence in* $H_*$, *whose limit* $v_0$ *verifies* $J'(v_0) = 0$ *where* $P(u)$ *stands for the mass center of the function* $e^{u(x)}$ *defined on* $S^2$.

Due to Proposition 2.1, the key point to the solution of problem $(*)$ lies in controlling the behavior of the sequence $\{P(u_k)\}$. To do this, we divide, according

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to the value of $R$, the sphere $S^2$ into several areas, and try to find such a sequence $\{u_k\}$ that $\{P(u_k)\}$ can reach none of the areas on the sphere.

Thus, we introduce in §3 a family of transformations on $H^1(S^2)$ which leaves the functionals

$$F(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2\, dA + 2\int_{S^2} u\, dA \quad \text{and} \quad G(u) = \int_{S^2} e^u\, dA$$

invariant. With the help of this family of transformations, we obtain in §4 the following propositions providing some useful information when $P(u_k) \to S^2$.

PROPOSITION 4.2. *Suppose $\{v_k\} \subset H_*$, $\{J(v_k)\}$ is bounded; $J'(v_k) \to 0$, and $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$. Then there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$, and $\alpha_i$, $\varsigma_i$, with $\alpha_i \to 1$, $\varsigma_i \to \varsigma$ as $i \to \infty$ such that $\int_{S^2} |\nabla(v_{k_i} - \varphi_{\alpha_i,\varsigma_i})|^2 \to 0$ as $i \to \infty$, where $\varphi_{\lambda,\varsigma}(x) = \ln[(1-\lambda^2)/(1-\lambda\cos r(x,\varsigma))^2]$, with $r(x,\varsigma)$ the geodesic distance between the two points $x$ and $\varsigma$ on $S^2$.*

PROPOSITION 4.3. *Let $\{v_k\} \subset H_*$, $J(v_k)$ bounded, and $J'(v_k) \to 0$, $P(v_k) \to \varsigma \in S^2$ with $R(\varsigma) > 0$. Then there is a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that $J(v_{k_i}) \to 8\pi \ln 4\pi R(\varsigma)$.*

PROPOSITION 4.4. *Assume $\{u_k\}$, $\{v_k\}$ in $H_*$ satisfying*
*(1) $\{J(u_k)\}$, $\{J(v_k)\}$ are bounded; and $J'(v_k) \to 0$, as $k \to \infty$.*
*(2) $\int_{S^2} |\nabla(u_k - v_k)|^2\, dA \to 0$, as $k \to \infty$.*
*(3) $P(u_k) \to \eta \in S^2$, $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$.*
*Then $\eta = \varsigma$.*

Condition $(R_2)$ enables us to establish the estimate

$$\mu_\lambda < -8\pi \ln 4\pi\nu$$

for $\lambda$ sufficiently close to 1. Then for such $\lambda$ we prove

PROPOSITION 4.6. *There exist $\alpha_0$, $\delta_0 > 0$, such that for any $\{v_k\}$ in $H_*$, if $J(v_k) \le \mu_\lambda + \delta_0$ $(k = 1, 2, \dots)$ and $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$ then $R(\varsigma) \ge \nu + \alpha_0$.*

Finally, in §5, we utilize $\nabla R$ to construct a continuous "flow" in $H_*$. Based on the results in the proceding sections, mainly in §4, and applying the "flow", we are able to pick a sequence $\{u_k\}$ in $H_*$, such that as $k \to \infty$, $J(u_k) \to \mu_{\lambda_0}$ (for some $\lambda_0$ sufficiently close to 1), $J'(u_k) \to 0$, and $\{P(u_k)\}$ is bounded away from the sphere $S^2$. Therefore we arrive at the conclusion that $\mu_{\lambda_0}$ is a critical value of $J$, and complete the proof of our Theorem.

REMARK 0.1. In the Theorem, if $\nu \le 0$, then assumption $(R_2)$ can be omitted.

REMARK 0.2. Condition $(R_1)$ may be generalized as

$(\tilde{R}_1)$ Let $m = \max_{S^2} R$, $M = R^{-1}(m)$; $M$ is not contractible in itself, but is contractible on $S^2$. Let

$$\Gamma = \left\{ U = \bigcup_{t\in[0,1]} h_t(M) \;\middle|\; \begin{array}{l} h_t(\cdot) \text{ is a deformation of } M \text{ on } S^2; \\ h_0(M) = M \text{ and } h_1(M) \text{ is a point on } S^2 \end{array} \right\}$$

and suppose that $\nu = \sup_{U\in\Gamma} \min_{x\in U} R(x) < m$. Then by a similar argument as in the proof of our Theorem, one can show that condition $(R_0)$, $(\tilde{R}_1)$, $(R_2)$ and $(R_3)$ are sufficient for problem $(*)$ to have a solution.

We assume $(R_0)$–$(R_3)$ throughout the paper.

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**1. Mountain pass.** Consider the functional $J$ defined on $H_*$. For $x, \varsigma \in S^2$, $\lambda \in [0,1)$, define

$$\varphi_{\lambda,\varsigma}(x) = \ln\frac{1-\lambda^2}{(1-\lambda\cos r(x,\varsigma))^2}$$

where $r(x,\varsigma)$ is the geodesic distance between two points $x$ and $\varsigma$ on $S^2$. A direct computation shows that, as $\lambda \to 1$,

$$J(\varphi_{\lambda,a}) = -8\pi\ln\int_{S^2} Re^{\varphi_{\lambda,a}}\,dA \to -8\pi\ln 4\pi R(a) = -8\pi\ln 4\pi m.$$

Similarly,

$$J(\varphi_{\lambda,b}) \to -8\pi\ln 4\pi m. \tag{1.1}$$

On the other hand, the inequality (cf. [**5**])

$$\int_{S^2} e^u\,dA \le 4\pi\exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla u|^2\,dA + \frac{1}{4\pi}\int_{S^2} u\,dA\right) \quad \forall u \in H^1(S^2), \tag{1.2}$$

leads to

$$J(u) \ge -8\pi\ln 4\pi m \quad \forall u \in H_*.$$

Therefore,

$$\inf_{H_*} J = -8\pi\ln 4\pi m. \tag{1.3}$$

(1.1) and (1.3) inform us that there are two separated points, say $\varphi_{\lambda,a}$ and $\varphi_{\lambda,b}$, in $H_*$ at which the values of $J$ are as close to $\inf_{H_*} J$ as we wish. This phenomenon would naturally lead one to expect that there might be a "mountain pass" between the two separated points. In order to show this, we need the concept of mass center introduced in [**6**]. Now let us first recall it.

For $u \in H^1(S^2)$, regard $S^2$ as a rigid body with density $e^{u(x)}$ at point $x \in S^2$, denote the mass center of this rigid body by $P(u)$. Then in an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$,

$$P(u) = \left(\frac{\int_{S^2} x_1 e^u\,dA}{\int_{S^2} e^u\,dA}, \frac{\int_{S^2} x_2 e^u\,dA}{\int_{S^2} e^u\,dA}, \frac{\int_{S^2} x_3 e^u\,dA}{\int_{S^2} e^u\,dA}\right).$$

This concept and its analysis played an important role in [**6**] and will be a powerful tool in the following investigation.

*Define* $Q(u) = P(u)/|P(u)|$, and $d(u) = |Q(u) - P(u)|$, $u \in H^1(S^2)$.

LEMMA 1.1. *There exists constant* $C_0 = C(|R|_{C^1(S^2)})$, *such that*

$$\left|\int_{S^2}\{R(x) - R(Q(u))\}e^u\,dA\right| \le C_0\sqrt[3]{d(u)}\int_{S^2} e^u\,dA. \tag{1.4}$$

PROOF. For simplicity, write $Q = Q(u) = (Q_1, Q_2, Q_3)$, and $S_r = S_r(Q) = \{x \in S^2 \colon r(x,Q) < r\}$. Choose $r = \sqrt[3]{d(u)}$. Since

$$\int_{S^2\setminus S_r}\left(1 - \sum_i x_iQ_i\right)e^u\,dA \Big/ \int_{S^2} e^u\,dA \le 1 - |P(u)| = d(u)$$

and

$$1 - \sum_i x_iQ_i \ge r^2/4 \quad \forall x \in S^2\setminus S_r,$$

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we have

$$\int_{S^2\setminus S_r} e^u\,dA \Big/ \int_{S^2} e^u\,dA \le 4\sqrt[3]{d(u)}.$$

Therefore,

$$\left|\int_{S^2}\{R(x)-R(Q)\}e^u\,dA\right| \le \left|\int_{S_r}\{R(x)-R(Q)\}e^u\,dA\right| + \left|\int_{S^2\setminus S_r}\{R(x)-R(Q)\}e^u\right|$$
$$\le \left\{\max_{S^2}|\nabla R|\cdot r + 8\max_{S^2}|R|\sqrt[3]{d(u)}\right\}\int_{S^2} e^u\,dA \le C_0\sqrt[3]{d(u)}\int_{S^2} e^u\,dA.$$

LEMMA 1.2. *Let $u \in H_*$, $J(u) \le \beta$, $|P(u)| \le 1-\gamma < 1$. Then*

$$\int_{S^2}|\nabla u|^2\,dA \le C(\beta,\gamma).$$

PROOF. Analogous to the proof of Proposition 1.2 in [**6**].

*Define*

$$L_\lambda = \{l\colon l \in C([0,1],H_*), l(0) = \varphi_{\lambda,a}, l(1) = \varphi_{\lambda,b}\},$$
$$\mu_\lambda = \inf_{l\in L_\lambda}\max_{u\in l([0,1])} J(u).$$

For $\lambda$ sufficiently close to 1, $L_\lambda$ is nonempty. In fact, since $R(a) = R(b) > 0$, one has $\int_{S^2} Re^{\varphi_{\lambda,a}}\,dA$, $\int_{S^2} Re^{\varphi_{\lambda,b}}\,d\Lambda > 0$ for $\lambda$ close to 1. Take

$$u_\lambda^t = \ln[(1-t)e^{\varphi_{\lambda,a}} + te^{\varphi_{\lambda,b}}], \qquad t \in [0,1];$$

then $\int_{S^2} Re^{u_\lambda^t}\,dA > 0$; hence $l_\lambda = \{u_\lambda^t : t \in [0,1]\} \in L_\lambda$.

PROPOSITION 1.3 (MOUNTAIN PASS). *For $\lambda$ sufficiently close to* 1,

$$\mu_\lambda > \max\{J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b})\}. \tag{1.5}$$

PROOF. We argue indirectly. Suppose there exists $\{\lambda_k\}$, $\lambda_k \to 1$, such that

$$\mu_{\lambda_k} \le \max\{J(\varphi_{\lambda_k,a}), J(\varphi_{\lambda_k,b})\}, \qquad k = 1,2,\ldots.$$

Then by (1.1), one can find $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, and $\mu_{\lambda_k} < -8\pi\ln 4\pi m + \varepsilon_k$. By the definition of $\mu_k$, there is $l_k \in L_{\lambda_k}$ such that

$$\max_{l_k([0,1])} J < -8\pi\ln 4\pi m + \varepsilon_k. \tag{1.6}$$

Let $d_0 > 0$ be sufficiently small, so that if $d(u) \le d_0$, then

$$C_0\sqrt[3]{d(u)} < \tfrac{1}{2}(m-\nu). \tag{1.7}$$

By $(R_1)$, this is possible.

*Case* (1). There exists $k_0$ such that if $k \ge k_0$, then for any $u$ on $l_k$, $d(u) \le d_0$. Then by (1.6) and $(R_1)$, one can pick some $k \ge k_0$ so that

$$\max_{l_k([0,1])} J < -8\pi\ln 2\pi(m+\nu). \tag{1.8}$$

Fix such $k$ and set $h(t) = Q(l_k(t))$, $t \in [0,1]$. It is obvious that $h([0,1])$ is a continuous curve on $S^2$ joining $a$ and $b$. Let $t_0 \in (0,1)$ satisfy

$$R(h(t_0)) = \min_{h([0,1])} R.$$

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Then

$$R(h(t_0)) \le \nu. \tag{1.9}$$

Write $v = l_k(t_0)$, $Q = Q(v) = h(t_0)$. Then by (1.4) and (1.7)-(1.9),

$$\begin{aligned} -8\pi\ln 2\pi(m+\nu) &> J(v) \\ &\ge \frac{1}{2}\int_{S^2}|\nabla v|^2\,dA + 2\int_{S^2} v\,dA - 8\pi\ln\int_{S^2} e^v\,dA - 8\pi\ln(R(Q) + C_0\sqrt[3]{d(v)}) \\ &\ge -8\pi\ln 4\pi - 8\pi\ln\tfrac{1}{2}(\nu+m) = -8\pi\ln 2\pi(m+\nu), \end{aligned}$$

a contradiction.

*Case* (2). There exists a subsequence of $\{l_k\}$ (still denoted by $\{l_k\}$) such that for any $k$, one can pick out a $u_k \in l_k$ satisfying $d(u_k) > d_0$. Then by Lemma 1.2, we infer that $\int_{S^2}|\nabla u_k|^2\,dA$ are bounded, which implies that the sequence $\{\tilde{u}_k = u_k - (1/4\pi)\int_{S^2} u_k\,dA\}$ is bounded in $H^1(S^2)$ and hence possesses a subsequence coverging weakly to an element, say $u_0$, in $H^1(S^2)$. Since $u_k \in H_*$, it is easy to verify that $\tilde{u}_k$ and the weak limit $u_0$ are in $H_*$; consequently, $J(u_0) = \inf_{H_*} J$. This is impossible because by $(R_1)$ $R$ is not constant, so $\inf_{H_*} J$ can never be attained.

The above argument shows that our hypothesis at the beginning of the proof is false, so (1.5) must hold for $\lambda$ sufficiently close to 1.

## 2. A modified (P.S.) condition.

PROPOSITION 2.1. *Assume $\{u_k\} \subset H_*$, $J(u_k) \le \beta < +\infty$, $J'(u_k) \to 0$, as $k \to \infty$, and also assume $|P(u_k)| \le 1 - \gamma < 1$. Let $\tilde{u}_k = u_k - (1/4\pi)\int_{S^2} u_k\,dA$. Then $\{u_k\}$ possesses a strongly convergent subsequence in $H^1(S^2)$ whose limit $u_0$ verifies $J'(u_0) = 0$.*

PROOF. In the proof of Proposition 1.3, we have already seen that $\{\tilde{u}_k\} \subset H_*$ and there is a subsequence of $\{\tilde{u}_k\}$ (still denoted by $\{\tilde{u}_k\}$) converging weakly to $u_0$ in $H_*$.

Since for any constant $C$, $J(u + C) = J(u)$, one concludes, from the definition of $J'$, that

$$J'(\tilde{u}_k) = J'(u_k) \to 0, \quad \text{as } k \to \infty. \tag{2.1}$$

Hence

$$-\Delta\tilde{u}_k - \frac{8\pi}{\int_{S^2} Re^{\tilde{u}_k}\,dA} Re^{\tilde{u}_k} = 2 + o(1).$$

Consequently,

$$\begin{aligned} \int_{S^2}|\nabla(\tilde{u}_i - \tilde{u}_j)|^2 &= 8\pi\int_{S^2}\left\{\frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}}\right\}(\tilde{u}_i - \tilde{u}_j)\,dA + o(1) \\ &\le 8\pi\left(\int_{S^2}\left|\frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}}\right|^2 dA\right)^{1/2}\left(\int_{S^2}(\tilde{u}_i - \tilde{u}_j)^2\,dA\right)^{1/2} + o(1). \end{aligned} \tag{2.2}$$

The boundedness of $\{\tilde{u}_k\}$ (in $H^1(S^2)$) and of $J(\tilde{u}_k)$ leads to

$$\int_{S^2} Re^{\tilde{u}_k}\,dA \ge \alpha > 0, \qquad k = 1, 2, \ldots,$$

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which, together with (1.2), implies that, for any $i, j = 1, 2, \ldots$, the integrals

$$\int_{S^2} \left\{ \frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}} \right\}^2 dA$$

are bounded. By the compact embedding $H^1(S^2) \hookrightarrow L^2(S^2)$, we have

$$\int_{S^2} (\tilde{u}_i - \tilde{u}_j)^2 \, dA \to 0, \quad \text{as } i, j \to \infty.$$

Now, it follows from (2.2) that

$$\int_{S^2} |\nabla(\tilde{u}_i - \tilde{u}_j)|^2 \, dA \to 0, \quad \text{as } i, j \to \infty.$$

Consequently,

$$\tilde{u}_k \to u_0 \quad \text{in } H^1(S^2).$$

Therefore, by (2.1), $J'(u_0) = 0$. This completes the proof.

Let $\lambda_0$ be so close to 1 that (1.5) is valid. Then by Ekeland's variational principle (cf. [7], the proof for Mountain Pass Lemma), there is a sequence $\{u_k\} \subset H_*$ satisfying $J(u_k) \to \mu_{\lambda_0}$ and $J'(u_k) \to 0$ as $k \to \infty$. Set $\tilde{u}_k = u_k - (1/4\pi)\int_{S^2} u_k \, dA$; then from Proposition 2.1, we know that whether $\{\tilde{u}_k\}$ converges strongly in $H^1(S^2)$ depends on the behavior of the mass center $P(u_k)$. Does $\{P(u_k)\}$ remain bounded away from the sphere $S^2$? This has now become a key point to the solution of $(*)$. In order to analyze this, we introduce in the following section a family of transformations on $H^1(S^2)$ which possesses some important properties and is very useful in our later investigations.

**3. A family of transformations on $H^1(S^2)$.** Let $u \in H^1(S^2)$, $\varsigma \in S^2$. Select a spherical polar coordinate system $x = (\theta, \varphi)$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$, so that $\varsigma = (0, \varphi)$. Define a family of transformations $A_{\lambda,\varsigma}$ by

$$A_{\lambda,\varsigma} u(\theta, \varphi) = u \circ h_{\lambda,\varsigma}(\theta, \varphi) + \psi_{\lambda,\varsigma}(\theta)$$

where $0 < \lambda \le 1$, $h_{\lambda,\varsigma}(\theta, \varphi) = (2\tan^{-1}(\lambda \tan(\theta/2)), \varphi)$ is a conformal transformation on $S^2$, and

$$\psi_{\lambda,\varsigma}(\theta) = \ln \frac{\lambda^2}{(\cos^2(\theta/2) + \lambda^2 \sin^2(\theta/2))^2}.$$

The following propositions describe some important properties of $A_{\lambda,\varsigma}$.

PROPOSITION 3.1. *Define*

$$I(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 \, dA + 2\int_{S^2} u \, dA - 8\pi \ln \int_{S^2} e^u$$

*for $u \in H^1(S^2)$. Then*

$$(3.1) \qquad I(A_{\lambda,\varsigma} u) = I(u), \qquad \lambda \in (0, 1], \; \varsigma \in S^2;$$

*and consequently, if $u$ is a solution of the equation*

$$(**) \qquad -\Delta u + 2 = 2e^u$$

*then $A_{\lambda,\varsigma} u$ is also a solution.*

PROOF. (1)

$$\int_{S^2} e^{A_{\lambda,\varsigma} u} \, dA = \int_0^{2\pi} \int_0^{\pi} e^{u(2\tan^{-1}(\lambda \tan(\theta/2)), \varphi)} e^{\psi_{\lambda,\varsigma}(\theta)} \sin\theta \, d\theta \, d\varphi.$$

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Let $\theta' = 2\tan^{-1}(\lambda \tan(\theta/2))$, $\varphi' = \varphi$. Then $e^{\psi_{\lambda,\varsigma}(\theta)} \sin\theta \, d\theta = \sin\theta' \, d\theta'$. Hence

$$\int_{S^2} e^{A_{\lambda,\varsigma} u} \, dA = \int_0^{2\pi} \int_0^{\pi} e^{u(\theta',\varphi')} \sin\theta' \, d\theta' \, d\varphi' = \int_{S^2} e^u \, dA. \tag{3.2}$$

(2)

$$\begin{aligned} \int_{S^2} |\nabla(A_{\lambda,\varsigma} u)|^2 \, dA &= \int_{S^2} |\nabla(u \circ h_{\lambda,\varsigma})|^2 \, dA - 2\int_{S^2} u \circ h_{\lambda,\varsigma} \Delta\psi_{\lambda,\varsigma} + \int_{S^2} |\nabla\psi_{\lambda,\varsigma}|^2 \\ &= \int_{S^2} |\nabla u|^2 \, dA + 4\int_{S^2} u \circ h_{\lambda,\varsigma}(e^{\psi_{\lambda,\varsigma}} - 1) \, dA + \int_{S^2} |\nabla\psi_{\lambda,\varsigma}|^2 \\ &= \int_{S^2} |\nabla u|^2 \, dA + 4\int_{S^2} u \, dA - 4\int_{S^2} u \circ h_{\lambda,\varsigma} \, dA + \int_{S^2} |\nabla\psi_{\lambda,\varsigma}|^2. \end{aligned} \tag{3.3}$$

Here we have employed the fact that $\psi_{\lambda,\varsigma}$ satisfies (cf. [**5**])

$$-\Delta\psi_{\lambda,\varsigma} = 2e^{\psi_{\lambda,\varsigma}} - 2. \tag{3.4}$$

Meanwhile, a direct computation shows

$$\frac{1}{2}\int_{S^2} |\nabla\psi_{\lambda,\varsigma}|^2 \, dA + 2\int_{S^2} \psi_{\lambda,\varsigma} \, dA = 0.$$

Substitute this into (3.3) to get

$$\frac{1}{2}\int_{S^2} |\nabla(A_{\lambda,\varsigma} u)|^2 \, dA + 2\int_{S^2} A_{\lambda,\varsigma} u \, dA = \int_{S^2} \left(\frac{1}{2}|\nabla u|^2 + 2u\right) dA \tag{3.5}$$

which, in addition to (3.2), implies (3.1).

(3) Denote the dual pairing between $H^1(S^2)$ and its dual space by $\langle \cdot, \cdot \rangle$. Then by the definition of the Gâteaux derivative and (3.1), one has

$$\begin{aligned} \langle I'(A_{\lambda,\varsigma} u), v \rangle &= \lim_{t\to 0} \frac{1}{t}\{I(A_{\lambda,\varsigma} u + tv) - I(A_{\lambda,\varsigma} u)\} \\ &= \lim_{t\to 0} \frac{1}{t}\{I(A_{\lambda,\varsigma}(u + tv \circ h_{\lambda,\varsigma}^{-1})) - I(A_{\lambda,\varsigma} u)\} \\ &= \lim_{t\to 0} \frac{1}{t}\{I(u + tv \circ h_{\lambda,\varsigma}^{-1}) - I(u)\} = \langle I'(u), v \circ h_{\lambda,\varsigma}^{-1} \rangle \end{aligned} \tag{3.6}$$

where $h_{\lambda,\varsigma}^{-1}$ is the inverse of $h_{\lambda,\varsigma}$.

If $u$ is a solution of $(**)$, then $I'(u) = 0$ and $\int_{S^2} e^u = 4\pi$. Consequently, by (3.6) and (3.2), $I'(A_{\lambda,\varsigma} u) = 0$ and $\int_{S^2} e^{A_{\lambda,\varsigma} u} = 4\pi$, which implies $A_{\lambda,\varsigma} u$ is also a solution of $(**)$. This completes our proof.

PROPOSITION 3.2. *Define*

$$J_{\lambda,\varsigma}(u) = \int_{S^2} \left(\tfrac{1}{2}|\nabla u|^2 + 2u\right) - 8\pi \ln \int_{S^2} R \circ h_{\lambda,\varsigma} e^u.$$

*Then*

$$\forall v \in H^1(S^2), \quad \langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma} u), v \rangle = \langle J'(u), v \circ h_{\lambda,\varsigma}^{-1} \rangle. \tag{3.7}$$

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PROOF.

$$\langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma}u), v\rangle = \int_{S^2} \nabla(u\circ h_{\lambda,\varsigma})\nabla v + \int_{S^2}\nabla\psi_{\lambda,\varsigma}\cdot\nabla v + 2\int_{S^2} v - \frac{8\pi}{\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}}\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}v.$$

A direct computation leads to

$$\int_{S^2}\nabla(u\circ h_{\lambda,\varsigma})\nabla v = \int_{S^2}\nabla(u\circ h_{\lambda,\varsigma})\nabla(v\circ h_{\lambda,\varsigma}^{-1})\circ h_{\lambda,\varsigma} = \int_{S^2}\nabla u\nabla(v\circ h_{\lambda,\varsigma}^{-1}),$$

$$\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u} = \int_{S^2} Re^u,$$

$$\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}v = \int_{S^2} Re^u(v\circ h_{\lambda,\varsigma}^{-1}).$$

By (3.4),

$$\int_{S^2}\nabla\psi_{\lambda,\varsigma}\cdot\nabla v + 2\int_{S^2} v = 2\int_{S^2} e^{\psi_{\lambda,\varsigma}}v = 2\int_{S^2} v\circ h_{\lambda,\varsigma}^{-1}.$$

Therefore

$$\begin{aligned}\langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma}u), v\rangle &= \int_{S^2}\nabla u\nabla(v\circ h_{\lambda,\varsigma}^{-1}) + 2\int_{S^2} v\circ h_{\lambda,\varsigma}^{-1}\\ &\quad - \frac{8\pi}{\int_{S^2} Re^u}\int_{S^2} Re^u(v\circ h_{\lambda,\varsigma}^{-1})\\ &= \langle J'(u), v\circ h_{\lambda,\varsigma}^{-1}\rangle.\end{aligned}$$

This completes the proof.

PROPOSITION 3.3. *For any $u\in H_*$, there exist $\lambda\in(0,1]$ and $\varsigma\in S^2$ such that*

$$P(A_{\lambda,\varsigma}u) = 0. \tag{3.8}$$

PROOF. Use the spherical polar coordinate system with pole $\varsigma$ introduced at the beginning of this section, and denote the corresponding orthogonal coordinate system in $\mathbf{R}^3$ by $(x_1, x_1, x_3)_\varsigma$. That is

$$x_1 = r\sin\theta\cos\varphi,\quad x_2 = r\sin\theta\sin\varphi,\quad x_3 = r\cos\theta.$$

In this system, let $P(A_{\lambda,\varsigma}u) = (a_1(\lambda), a_2(\lambda), a_3(\lambda))_\varsigma$. We first show that for any fixed $\varsigma\in S^2$, as $\lambda\to 0$,

$$(a_1(\lambda),\ a_2(\lambda),\ a_3(\lambda))_\varsigma \to (0,0,-1)_\varsigma. \tag{3.9}$$

In fact,

$$a_i(\lambda) = \int_{S^2} x_i\circ h_{\lambda,\varsigma}^{-1}e^u \Big/ \int_{S^2} e^u,\qquad i=1,2,3.$$

It is easily seen that as $\lambda\to 0$

$$x_i\circ h_{\lambda,\varsigma}^{-1}\to 0,\quad \text{in } L^2(S^2),\qquad i=1,2,$$

$$x_3\circ h_{\lambda,\varsigma}^{-1}\to -1,\quad \text{in } L^2(S^2).$$

Hence (3.9) is valid.

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Set $F(x) = P(A_{1-t,\varsigma}u)$ with $x = t\varsigma$, $t \in [0,1)$, $\varsigma \in S^2$. Then $F$ is a continuous mapping from $B^3 = [0,1) \times S^2 \to B^3$. (3.9) enables us to extend $F$ continuously to $\overline{B}^3$, the closure of $B^3$, so that on $\partial B^3 = S^2$, $F(x) = -x$. By a well-known result on topological degree, we have

$$\deg(F, B^3, 0) = \deg(-x, B^3, 0) \neq 0.$$

So there exist $t \in [0,1)$, $\varsigma \in S^2$, such that $F(t\varsigma) = 0$. That is, (3.8) holds. This completes the proof.

PROPOSITION 3.4. *Assume* $\{u_k\} \subset H_*$, $P(u_k) \to \bar{\varsigma} \in S^2$. *Choose* $\lambda_k$, $\varsigma_k$, *such that* $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. *Then* $\lambda_k \to 0$ *and* $\varsigma_k \to \bar{\varsigma}$ *as* $k \to \infty$.

PROOF. (1) Suppose there exists a subsequence $\{\lambda_{k_i}\}$ of $\{\lambda_k\}$ such that $\lambda_{k_i} \to \lambda_0 > 0$. Then passing to a subsequence, we have $\lambda_k \to \lambda_0$ and $\varsigma_k \to \varsigma_0$, as $k \to \infty$ for some $\varsigma_0 \in S^2$.

Fix an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$. Then $\lambda_0 > 0$ implies that, as $k \to \infty$,

$$x_i \circ h^{-1}_{\lambda_k,\varsigma_k}(x) \to x_i \circ h^{-1}_{\lambda_0,\varsigma_0}(x), \quad \text{uniformly for } x \in S^2,\ i = 1,2,3. \tag{3.10}$$

Let $P(A_{\lambda_k,\varsigma_k}u_k) = (a_1^k, a_2^k, a_3^k)$. Since

$$\int_{S^2} x_i e^{A_{\lambda_k,\varsigma_k}u_k} = \int_{S^2} x_i \circ h^{-1}_{\lambda_k,\varsigma_k} e^{u_k} \quad \text{and} \quad \int_{S^2} e^{A_{\lambda_k,\varsigma_k}u_k} = \int_{S^2} e^{u_k},$$

(3.10) leads to

$$a_i^k = \frac{\int_{S^2} x_i \circ h^{-1}_{\lambda_0,\varsigma_0} e^{u_k}}{\int_{S^2} e^{u_k}} + o(1).$$

Applying Lemma 1.1 to the function $x_i \circ h^{-1}_{\lambda_0,\varsigma_0}$ instead of $R(x)$, we obtain, as $k \to \infty$, $a_i^k \to x_i \circ h^{-1}_{\lambda_0,\varsigma_0}(\bar{\varsigma})$; that is $P(A_{\lambda_k,\varsigma_k}u_k) \to h^{-1}_{\lambda_0,\varsigma_0}(\bar{\varsigma}) \in S^2$, obviously a contradiction with our assumption $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. Hence, one must have

$$\lambda_k \to 0, \quad \text{as } k \to \infty. \tag{3.11}$$

(2) Suppose there exists a subsequence of $\{\varsigma_k\}$ (still denoted by $\{\varsigma_k\}$) such that $\varsigma_k \to \varsigma_0 \neq \bar{\varsigma}$. Choose an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$, such that $\varsigma_0 = (0,0,1)$. Then it is easy to see that for $0 < \varepsilon < r(\bar{\varsigma}, \varsigma_0)$, as $k \to \infty$, $x_3 \circ h^{-1}_{\lambda_k,\varsigma_k} \to -1$, uniformly on $S^2 \backslash S_\varepsilon(\varsigma_0)$. Now by a similar argument as in step (1), we arrive at $a_3^k \to -1$, as $k \to \infty$, again a contradiction to $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. This completes the proof.

## 4. Mass center analysis.

LEMMA 4.1. *All solutions of*

$$-\Delta u + 2 - 2e^u = 0, \qquad x \in S^2, \tag{**}$$

*are rotationally symmetric with respect to some axis and hence assume the form*

$$\varphi_{\alpha,\varsigma}(x) = \ln \frac{1-\alpha^2}{(1-\alpha\cos r(\varsigma,x))^2} \quad \text{with } \alpha \in [0,1).$$

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PROOF. Suppose $u$ is a solution of (**). Then $e^u g_0$ is a metric on $S^2$ having constant Gaussian curvature 1, where $g_0$ is the standard metric of $S^2$. By well-known results in differential geometry, $(S^2, e^u g_0)$ is isometric to $(S^2, g_0)$; i.e., there exists a diffeomorphism $\varphi\colon S^2 \to S^2$ such that $\varphi^* g_0 = e^u g_0$. It follows that $\varphi$ is a conformal transformation of $(S^2, g_0)$. Since all conformal transformations of $(S^2, g_0)$ are explicitly known, we see easily that $u$ has to be rotationally symmetric with respect to some axis.

Moreover, by a result of Hong (cf. [**5**, Lemma 3.1]), $u$ must equal $\varphi_{\alpha,\varsigma}(x)$ with $\alpha \in [0,1)$ and $\varsigma \in S^2$.

PROPOSITION 4.2. *Suppose $\{u_k\} \subset H_*$, $\{J(u_k)\}$ bounded, $J'(u_k) \to 0$ and $P(u_k) \to \varsigma \in S^2$, as $k \to \infty$. Then there exists a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ and corresponding $\{\alpha_i\}$, $\{\varsigma_i\}$, with $\alpha_i \to 1$, $\varsigma_i \to \varsigma$ as $i \to \infty$ such that*

$$\int_{S^2} |\nabla(u_{k_i} - \varphi_{\alpha_i,\varsigma_i})|^2 \to 0, \quad \text{as } i \to \infty.$$

PROOF. (1) By Proposition 3.3, there exist $\lambda_k, \varsigma_k \in S^2$, such that $P(A_{\lambda_k,\varsigma_k} u_k) = 0$. Let $v_k = A_{\lambda_k,\varsigma_k} u_k$, $\tilde{v}_k = v_k - (1/4\pi)\int_{S^2} v_k$. By Lemma 1.1,

$$J(u_k) = I(u_k) - 8\pi \ln(R(\varsigma) + o(1)) \tag{4.2}$$

which implies the boundedness of $\{I(u_k)\}$. Taking (3.1) into account and by the obvious fact $I(u + C) = I(u)$ for any $u \in H^1(S^2)$ for any constant $C$, we see that

$$I(\tilde{v}_k) = I(u_k). \tag{4.3}$$

Hence $\{I(\tilde{v}_k)\}$ are bounded. And apparently $P(\tilde{v}_k) = P(v_k) = 0$. Now, due to Proposition 1.2 in [**6**], $\{\tilde{v}_k\}$ is bounded in $H^1(S^2)$, so there exists a subsequence of $\{\tilde{v}_k\}$ (still denoted by $\{\tilde{v}_k\}$) converging weakly to $v_0 \in H^1(S^2)$. Obviously

$$\int_{S^2} v_0 = 0. \tag{4.4}$$

(2) Applying (3.7), we have

$$\begin{aligned} \langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle &= \langle J'_{\lambda_k,\varsigma_k}(v_k), v\rangle = \langle J'(u_k), v \circ h^{-1}_{\lambda_k,\varsigma_k}\rangle \\ &= \langle J'(u_k), w_k\rangle \quad \forall v \in H^1(S^2) \end{aligned} \tag{4.5}$$

where

$$w_k = v \circ h^{-1}_{\lambda_k,\varsigma_k} - \frac{1}{4\pi}\int_{S^2} v \circ h^{-1}_{\lambda_k,\varsigma_k}.$$

Since $\int_{S^2} w_k = 0$ and $\int_{S^2} |\nabla w_k|^2 = \int_{S^2} |\nabla(v \circ h^{-1}_{\lambda_k,\varsigma_k})|^2 = \int_{S^2} |\nabla v|^2$ we infer from (4.5) that

$$|\langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle| \le C\|J'(u_k)\| \left(\int_{S^2} |\nabla v|^2\right)^{1/2} \tag{4.6}$$

with some constant $C$ independent of $u_k$ and $v$.

By Proposition 3.4, $P(u_k) \to \varsigma \in S^2$ implies, as $k \to \infty$, $\varsigma_k \to \varsigma$ and $\lambda_k \to 0$; hence

$$R \circ h_{\lambda_k,\varsigma_k} \to R(\varsigma) \quad \text{in } L^2(S^2). \tag{4.7}$$

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And it follows that

$$\langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle \to \langle I'(v_0), v\rangle, \quad \text{as } k\to\infty,\ \forall v\in H^1(S^2).$$

On the other hand, by (4.6)

$$\langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle \to 0, \quad \text{as } k\to\infty.$$

Therefore,

$$\langle I'(v_0), v\rangle = 0 \quad \forall v\in H^1(S^2). \tag{4.8}$$

By Lemma 4.1, $v_0(x) = \varphi_{\alpha,\eta}(x) + C$, for some constant $C$, $\alpha$ and $\eta\in S^2$. The weak convergence of $\{\tilde{v}_k\}$ to $v_0$ and $P(\tilde{v}_k)=0$ imply $P(v_0)=0$, and it follow that $\alpha=0$ and $v_0=$ const. Now, by (4.4), we obtain $v_0=0$.

(3) Due to (4.6), $\|J'(u_k)\|\to 0$ and $\tilde{v}_k \rightharpoonup 0$, we have

$$o(1) = \langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), \tilde{v}_k\rangle = \int_{S^2}|\nabla\tilde{v}_k|^2 - 8\pi\frac{\int_{S^2} R\circ h_{\lambda_k,\varsigma_k} e^{\tilde{v}_k}\tilde{v}_k}{\int_{S^2} R\circ h_{\lambda_k,\varsigma_k} e^{\tilde{v}_k}}.$$

It is not difficult to see from (4.7) that the last term in the above equality vanishes as $k\to\infty$ since the boundedness of $\{J(u_k)\}$ and $P(u_k)\to\varsigma$ imply $R(\varsigma)>0$ (cf. [**6**]). Hence

$$\int_{S^2}|\nabla\tilde{v}_k|^2 \to 0, \quad \text{as } k\to\infty, \tag{4.9}$$

which means $\tilde{v}_k\to 0$ strongly in $H^1(S^2)$.

(4) By (4.9),

$$\int_{S^2}|\nabla(u\circ h_{\lambda_k,\varsigma_k} + \psi_{\lambda,\varsigma_k})|^2 \to 0.$$

It follows that $\int_{S^2}|\nabla(u_k+\psi_{\lambda_k,\varsigma_k}\circ h^{-1}_{\lambda_k,\varsigma_k})|^2\to 0$, due to the conformal invariance of the integral $\int_{S^2}|\nabla u|^2$. Let $\alpha_k = (1-\lambda_k^2)/(1+\lambda_k^2)$. Then a straightforward computation shows

$$\psi_{\lambda_k,\varsigma_k}\circ h^{-1}_{\lambda_k,\varsigma_k}(x) = -\varphi_{\alpha_k,\varsigma_k}(x).$$

Obviously $\alpha_k\to 1$, since $\lambda_k\to 0$. This completes our proof.

PROPOSITION 4.3. *Let $\{u_k\}\subset H_*$ and assume $\{J(u_k)\}$ is bounded, $J'(u_k)\to 0$, and $P(u_k)\to\varsigma\in S^2$ as $k\to\infty$. Then there is a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $J(u_{k_i})\to -8\pi\ln 4\pi R(\varsigma)$.*

PROOF. By (4.2), (4.3) and (4.9), one can pick a subsequence $\{u_{k_i}\}$ of $u_k$ such that

$$\begin{aligned} J(u_{k_i}) &= I(u_{k_i}) - 8\pi\ln(R(\varsigma)+o(1)) = I(\tilde{v}_{k_i}) - 8\pi\ln(R(\varsigma)+o(1)) \\ &\to I(0) - 8\pi\ln R(\varsigma) = -8\pi\ln 4\pi R(\varsigma). \end{aligned}$$

This completes the proof.

PROPOSITION 4.4. *Assume $\{u_k\}$, $\{v_k\}\subset H_*$.*

*(1) $\{J(u_k)\}$, $\{J(v_k)\}$ bounded;*

*(2) $\int_{S^2}|\nabla(u_k-v_k)|^2\to 0$, as $k\to\infty$;*

*(3) $P(u_k)\to\eta\in S^2$, $P(v_k)\to\varsigma\in S^2$, as $k\to\infty$.*

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*Then* $\eta = \varsigma$.

PROOF. We argue indirectly. Suppose $\eta \neq \varsigma$.

By Proposition 4.2, one can pick a subsequence of $\{v_k\}$ (still denoted by $\{v_k\}$) and corresponding $\{\alpha_k\}$, $\{\varsigma_k\}$ with $\alpha_k \to 1$, $\varsigma_k \to \varsigma$ such that

$$\int_{S^2} |\nabla(v_k - \varphi_{\alpha_k,\varsigma_k})|^2 \to 0, \quad \text{as } k \to \infty.$$

From assumption (2)

$$\int_{S^2} |\nabla(\tilde{u}_k - \tilde{\varphi}_{\alpha_k,\varsigma_k})|^2 \to 0, \quad \text{as } k \to \infty. \tag{4.10}$$

Here we again used the notation $\tilde{u} = u - (1/4\pi)\int_{S^2} u$.

Noting that $P(\tilde{u}) = P(u)\ \forall u \in H^1(S^2)$, $\int_{S^2} |\nabla \tilde{\varphi}_{\alpha_k,\varsigma_k}|^2 \to \infty$ and $I(\tilde{\varphi}_{\alpha_k,\varsigma_k}) = -8\pi\ln 4\pi$; applying Lemma 2.2 in [**5**], we obtain, for any $x \in S^2$, $x \neq \varsigma$,

$$\tilde{\varphi}_{\alpha_k,\varsigma_k}(x) \to -\infty, \quad \text{as } k \to \infty. \tag{4.11}$$

Let $\varepsilon = \frac{1}{2} r(\eta, \varsigma)$, $S_k = \{x \in S_\varepsilon(\eta) \colon \tilde{u}_k(x) \geq 0\}$, $\tilde{u}_k^+ = \max\{\tilde{u}_k, 0\}$; then by (4.10),

$$\int_{S^k} \{|\nabla(\tilde{u}_k^+ - \tilde{\varphi}_{\alpha_k,\varsigma_k})|^2 + |\tilde{u}_k^+ - \tilde{\varphi}_{\alpha_k,\varsigma_k}|^2\} \to 0, \quad \text{as } k \to \infty.$$

Now, the boundedness of $\int_{S_k} |\nabla \tilde{\varphi}_{\alpha_k,\varsigma_k}|^2$ and (4.11) imply the boundedness of $\int_{S_k} (|\nabla \tilde{u}_k^+|^2 + \tilde{u}_k^{+2})$, hence of $\int_{S_\varepsilon(\eta)} (|\nabla \tilde{u}_k^+|^2 + |\tilde{u}_k^+|^2)$. Consequently, by Theorem 2.46 in [**8**], we infer that $\int_{S_\varepsilon(\eta)} e^{\tilde{u}_k^+}$ are bounded. It follows, from $P(u_k) \to \eta \in S^2$ and the proof of Lemma 1.1, that $\int_{S^2} e^{\tilde{u}_k}$ are bounded. Assumption (1) implies $\int_{S^2} |\nabla \tilde{u}_k|^2$ are bounded; hence there exists a subsequence $\{\tilde{u}_{k_i}\}$ of $\{\tilde{u}_k\}$ converging weakly to some element $u_0 \in H^1(S^2)$, which leads to $P(\tilde{u}_{k_i}) \to P(u_0)$. However, it is evidently that $P(u_0)$ can never lie on $S^2$. This contradicts $\eta \in S^2$, and the proof is completed.

LEMMA 4.5. *If $\lambda$ is sufficiently close to* 1, *then*

$$\mu_\lambda < -8\pi\ln 4\pi\nu. \tag{4.12}$$

*Here we assume $\nu > 0$.*

PROOF. Using $h_0$ in $(R_2)$, set

$$l_\lambda(t) = \varphi_{\lambda,h_0(t)}, \qquad t \in [0,1].$$

Apparently, for $\lambda$ sufficiently close to 1, $l_\lambda \in L_\lambda$. We want to show that

$$\max_{l_\lambda([0,1])} J < -8\pi\ln 4\pi\nu.$$

Since $J(\varphi_{\lambda,\varsigma}) = -8\pi\ln \int_{S^2} Re^{\varphi_{\lambda,\varsigma}}$, it suffices to verify the inequality

$$\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} > 4\pi\nu \quad \forall \varsigma \in h_0([0,1]). \tag{4.13}$$

Set $N_\delta = \{x \in h_0([0,1]) \colon \operatorname{dist}(x, K) < \delta\}$, where $K$ was defined in $(R_2)$ and $\operatorname{dist}(\cdot,\cdot)$ stands for the geodesic distance on standard $S^2$. By $(R_2)$, one can choose a small $\delta$ so that

$$\Delta R|_{N_\delta} > 0. \tag{4.14}$$

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(1) $\varsigma \in h_0([0,1])\backslash N_\delta$. In this case, there is an $\varepsilon > 0$, such that

$$R(\varsigma) \geq \nu + \varepsilon \quad \forall \varsigma \in h_0([0,1])\backslash N_\delta.$$

Let $\lambda$ be sufficiently close to 1 so that $d(\varphi_{\lambda,\varsigma}) \leq (\varepsilon/2C_0)^3$. (Note that here $d(\varphi_{\lambda,\varsigma})$ is independent of $\varsigma$.) Then by Lemma 1.1, we have

$$\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} \geq (R(\varsigma) - \varepsilon/2) \int_{S^2} e^{\varphi_{\lambda,\varsigma}} = 4\pi(R(\varsigma) - \varepsilon/2) > 4\pi\nu.$$

(2) $\varsigma \in N_\delta$. Due to the continuity of $J$ and the compactness of $\overline{N}_\delta$, the closure of $N_\delta$ on $S^2$, it suffices to show that for each $\varsigma \in N_\delta$ there is a $\lambda(\varsigma)$ such that, as $1 > \lambda \geq \lambda(\varsigma)$, (4.13) holds. For this aim, again choose a spherical polar coordinate system $x(\theta, \psi)$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$, so that $\varsigma = (0, \psi)$. Then for any $\varepsilon > 0$,

$$\begin{aligned}\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} &= (1-\lambda^2)\left\{\int_0^{2\pi}\int_0^{\varepsilon} \frac{[R(x) - R(\varsigma)]\sin\theta}{(1-\lambda\cos\theta)^2}\,d\theta\,d\psi + \int_0^{2\pi}\int_{\varepsilon}^{\pi}(\cdots)\right\} \\ &\quad + 4\pi R(\varsigma) \\ &\equiv (1-\lambda^2)\{(\mathrm{I}) + (\mathrm{II})\} + 4\pi R(\varsigma) \\ &\geq (1-\lambda^2)\{(\mathrm{I}) + (\mathrm{II})\} + 4\pi\nu.\end{aligned}$$

Thus one need only verify that

$$\text{(I)} + \text{(II)} > 0 \text{ as } \lambda \text{ sufficiently close to 1.} \tag{4.15}$$

In fact, for any fixed $\varepsilon$, integral (II) is bounded for all $\lambda \leq 1$. Using the second order Taylor expansion of $R$ at point $\varsigma$, taking into account that $\varphi_{\lambda,\varsigma}$ depends on $\theta$ only, and by a direct calculation, we arrive at

$$\text{(I)} = \pi \int_0^{\varepsilon} \frac{\{\Delta R(\varsigma)\sin^2\theta + o(\theta^2)\}}{(1-\lambda\cos\theta)^2}\sin\theta\,d\theta.$$

Let $\varepsilon$ be so small that

$$\text{(I)} \geq \frac{\pi}{2}\Delta R(\varsigma)\int_0^{\varepsilon}\frac{\sin^3\theta\,d\theta}{(1-\lambda\cos\theta)^2}.$$

Then an integration by parts shows

$$\int_0^{\varepsilon}\frac{\sin^3\theta\,d\theta}{(1-\lambda\cos\theta)^2} \to +\infty, \quad \text{as } \lambda \to 1.$$

Therefore (4.15) holds, and the proof is completed.

Let $\lambda_0$ be so close to 1 that both (1.5) and (4.12) hold. We write $\mu = \mu_{\lambda_0}$ and $L = L_{\lambda_0}$.

PROPOSITION 4.6. *There exist* $\alpha_0$, $\delta_0 > 0$, *such that for any* $\{v_k\}$ *in* $H_*$, *if* $J(v_k) \leq \mu + \delta_0$ $(k = 1, 2, \dots)$ *and* $P(v_k) \to \varsigma \in S^2$, *as* $k \to \infty$; *then*

$$R(\varsigma) \geq \nu + \alpha_0. \tag{4.16}$$

PROOF. Estimate (4.12) implies the existence of constants $\alpha_0$, $\delta_0 > 0$, such that

$$\mu + \delta_0 \leq -8\pi \ln 4\pi(\nu + \alpha_0). \tag{4.17}$$

By (1.2) and (1.4)

$$J(v_k) \geq -8\pi \ln 4\pi[R(Q(v_k)) + C_0\sqrt[3]{d(v_k)}].$$

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Clearly, as $k \to \infty$, both $d(v_k) \to 0$ and $R(Q(v_k)) \to R(\varsigma)$; hence

$$\mu + \delta_0 \geq \overline{\lim_k} J(v_k) \geq -8\pi \ln 4\pi R(\varsigma),$$

which, with (4.17), implies $R(\varsigma) \geq \nu + \alpha_0$. This completes the proof.

**5. Constructing a continuous "flow" in $H_*$ to complete the proof of the Theorem.** Let $M = R^{-1}(m) \equiv \{x \in S^2 \colon R(x) = m\}$. Choose $\varepsilon_1 > 0$, so that

$$\mu > -8\pi \ln 4\pi(m - \varepsilon_1). \tag{5.1}$$

Define $U_d = \{u \in H_* \colon R(Q(u)) \leq m - \varepsilon_1, d(u) \leq d\}$. Let $\mu_0 = \inf_{H_*} J = -8\pi \ln 4\pi m$.

LEMMA 5.1 (*A result of a continuous "flow"*). *There exist $\delta, d > 0$ and a continuous mapping $\mathcal{T} \colon H_* \to H_*$, such that*

(a) $J(\mathcal{T}(u)) \leq J(u)$, $\forall u \in H_*$;

(b) $\mathcal{T}(J^{-1}(\mu_0, \mu + \delta) \cap U_d) \subset J^{-1}(\mu_0, \mu - \delta)$;

(c) $\mathcal{T}|_{J^{-1}(\mu_0, \mu_0 + \delta)} = \mathrm{id}$, $\mu_0 + \delta < \mu - \delta$;

(d) $\mathcal{T}(H_* \backslash U_d) \subset H_* \backslash U_d$,

*where $J^{-1}(\alpha, \beta)$ stands for $\{u \in H_* \colon \alpha < J(u) < \beta\}$.*

PROOF. Analogous to the proof of Proposition 4.6 and by the definition of $U_d$, it is easy to show that there are constants $d_1, \delta_1, \varepsilon_2 > 0$, such that if $\delta < \delta_1$, $d < d_1$,

$$\forall u \in J^{-1}(\mu_0, \mu + \delta) \cap U_d, \qquad Q(u) \in R^{-1}[\nu + \varepsilon_2, m - \varepsilon_1]. \tag{5.2}$$

By $(R_3)$, there is a constant $\alpha_1 > 0$, such that

$$|\nabla R(x)| \geq \alpha_1, \quad \forall x \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]. \tag{5.3}$$

(1) Let $u \in H_*$ such that $Q(u) \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]$. Let $z(u)$ be the straight line passing through 0 which is perpendicular to the plane spanned by the vectors $Q(u)$ and $\nabla R(Q(u))$. Define $T(\theta, u)$ to be the rotation in $\mathbf{R}^3$ which takes $z(u)$ as its axis and which rotates along the direction $\nabla R(Q(u))$ by angle $\theta$.

Let $u$ be fixed for a moment, and write $x_0 = Q(u)$, $T_\theta = T(\theta, u)$. Consider

$$f(\theta) = \left(\int_{S^2} e^u\right)^{-1} \left(\int_{S^2} R(x) e^{u(T_\theta^{-1} x)} - \int_{S^2} R(x) e^{u(x)}\right)$$

where $T_\theta^{-1}$ is the inverse of $T_\theta$. Noting that $T_\theta$ is an orthogonal transformation in $\mathbf{R}^3$, we arrive immediately at

$$f(\theta) = \left(\int_{S^2} e^u\right)^{-1} \cdot \int_{S^2} [R(T_\theta x) - R(x)] e^{u(x)}.$$

The first order Taylor expansion of $R$ at $x_0$ leads to

$$R(T_\theta x_0) - R(x_0) \geq \tfrac{1}{2} |\nabla R(x_0)| \cdot |T_\theta x_0 - x_0|, \tag{5.4}$$

for $\theta$ sufficiently small. Let

$$\theta(x_0) = \max\{\alpha \colon \text{as } \theta \leq \alpha, (5.4) \text{ is valid}\},$$
$$\theta_0 = \inf\{\theta(x_0) \colon x_0 \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]\}.$$

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Then by the smoothness of $R$ and the compactness of $R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2]$, we have $\theta_0>0$. (5.4) and the continuity of $R$ imply, as $r$ sufficiently small,

$$R(T_{\theta_0}x)-R(x)\ge \tfrac{1}{4}|\nabla R(x_0)|\,|T_{\theta_0}x_0-x_0| \quad \forall x\in S_r(x_0). \tag{5.5}$$

Let $r(x_0)=\max\{s\colon \text{as } r\le s,\ (5.5)\text{ holds}\}$. Define

$$r_0=\inf\{r(x_0)\colon x_0\in R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2]\}.$$

Then similarly, one has $r_0>0$.

From the proof of Lemma 1.1, we see that there exists $d_2>0$ such that if $d(u)\le d_2$, then

$$\frac{\int_{S_{r_0}(x_0)}e^u}{\int_{S^2}e^u}\ge\frac{1}{2} \quad\text{and}\quad \max_{S^2}|\nabla R|\frac{\int_{S^2\setminus S_{r_0}(x_0)}e^u}{\int_{S^2}e^u}\le\frac{\alpha_1}{16}. \tag{5.6}$$

Apparently, $d_2$ is independent of $x_0$.

Now, by (5.3), (5.5) and (5.6), noting that for any $\theta, x$, $|T_\theta x_0-x_0|\ge|T_\theta x-x|$, we obtain, for all $\theta\le\theta_0$,

$$\begin{aligned} f(\theta)&=\left(\int_{S^2}e^u\right)^{-1}\left\{\int_{S_{r_0}(x_0)}[R(T_\theta x)-R(x)]e^u+\int_{S^2\setminus S_{r_0}(x_0)}(\cdots)\right\}\\ &\ge\frac{1}{4}|\nabla R(x_0)|\cdot|T_\theta x_0-x_0|\frac{\int_{S_{r_0}(x_0)}e^u}{\int_{S^2}e^u}\\ &\quad-\max_{S^2}|\nabla R|\frac{\int_{S^2\setminus S_{r_0}(x_0)}|T_\theta x-x|e^u}{\int_{S^2}e^u}\\ &\ge\frac{\alpha_1}{16}|T_\theta x_0-x_0|\equiv\alpha_2|T_\theta x_0-x_0|. \end{aligned} \tag{5.7}$$

(2) Choose $\delta_2, d_3>0$ so small that

$$m\cdot e^{-\delta_2/8\pi}-C_0d_3^{1/3}>m-\varepsilon_1/2$$

where $C_0$ was defined in Lemma 1.1. Then by the inequality

$$J(u)\ge-8\pi\ln4\pi[R(Q(u))+C_0\sqrt[3]{d(u)}]$$

derived from (1.2) and (1.4), it is easy to verify that for all $u\in H_*$

$$\text{if } J(u)\le\mu_0+\delta_2 \text{ and } d(u)\le d_3,\ \text{then } R(Q(u))>m-\varepsilon_1/2. \tag{5.8}$$

Let

$$G=\left\{x\in B^3\colon \frac{x}{|x|}\in R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2], 1-|x|\le\min\{d_2,d_3\}\right\},$$

$$G'=\left\{x\in G\colon \frac{x}{|x|}\in R^{-1}[\nu+\varepsilon_2, m-\varepsilon_1], 1-|x|\le\frac{1}{2}\min\{d_2,d_3\}\right\}.$$

Choose a $C^\infty$ function $g$ on $\overline{B}^3$, satisfying $0\le g(x)\le1$, $\forall x\in\overline{B}^3$; $g\equiv1$, $x\in G'$; $g\equiv0$, $x\in\overline{B}^3\setminus G$. Define

$$\mathcal{T}_tu(x)=u(T^{-1}(\tan(P(u)),u)x),\qquad t\in[0,\theta_0].$$

Note that $P(u)$, $Q(u)$ depend continuously on $u$ (in the $H^1(S^2)$ topology) and $R$ is smooth. We see that $T^{-1}(\theta,u)$ depends continuously on $u$, while the continuity

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of $T^{-1}(\theta,u)$ with respect to $\theta$ is obvious. Therefore $\mathcal{T}$ is a continuous map from $[0,\theta_0]\times H^1(S^2)$ to $H^1(S^2)$, and thus defines a continuous "flow" in $H^1(S^2)$ which is nonincreasing with respect to the functional $J$, that is

$$J(\mathcal{T}_t u)\le J(u)\quad \forall u\in H_*, t\in[0,\theta_0]. \tag{5.9}$$

In case $P(u)\notin G$, the above inequality holds obviously, since $g(P(u))=0$, $\mathcal{T}_t u=u$. And in case $P(u)\in G$, by (5.7)

$$\int_{S^2} R(x)e^{\mathcal{T}_t u}\ge\int_{S^2} R(x)e^u\quad \forall t\in[0,\theta_0].$$

Moreover, since $T^{-1}(\theta,u)$ is an orthogonal transformation we have

$$\frac{1}{2}\int_{S^2}|\nabla(\mathcal{T}_t u)|^2+2\int_{S^2}\mathcal{T}_t u=\frac{1}{2}\int_{S^2}|\nabla u|^2+2\int_{S^2}u,$$

for any $t\in[0,\theta_0]$, $u\in H^1(S^2)$. Therefore (5.9) also holds.

If $P(u)\in G'$, by the definition of $g$

$$\mathcal{T}_t u(x)=u(T^{-1}(t,u)x).$$

Write $J$ as

$$J(u)=I(u)-8\pi\ln\frac{\int_{S^2}Re^u}{\int_{S^2}e^u}.$$

It is easily see that $\int_{S^2}e^{\mathcal{T}_t u}=\int_{S^2}e^u$; hence $I(\mathcal{T}_t u)=I(u)$. By (5.7)

$$\frac{\int_{S^2}R(x)e^{\mathcal{T}_{\theta_0}u}}{\int_{S^2}e^{\mathcal{T}_{\theta_0}u}}-\frac{\int_{S^2}R(x)e^u}{\int_{S^2}e^u}\ge\alpha_2|T(\theta_0,u)Q(u)-Q(u)|\ge\alpha_3>0.$$

Consequently, there exists $\delta_3>0$ such that

$$J(\mathcal{T}_{\theta_0}u)\le J(u)-\delta_3,\quad \text{for all } u\in H_*,\ P(u)\in G'. \tag{5.10}$$

(3) Now define $\mathcal{T}(u)=\mathcal{T}_{\theta_0}u$. Let

$$\delta<\min\{\delta_1,\delta_2,\delta_3/2\},\qquad d<\min\{d_1,d_2/2,d_3/2\}.$$

Then equation (5.9) implies (a). (5.10) implies (b), since by (5.2), for any $u\in J^{-1}(\mu_0,\mu+\delta)\cap U_d$, $P(u)\in G'$. (5.8) and the definition of $G$ and of $g$ imply (c). Finally, noting that $d(\mathcal{T}_t u)=d(u)$, $R(Q(\mathcal{T}_t u))\ge R(Q(u))$ and by the definition of $U_d$, we see that the conclusion (d) of the lemma is true. This completes the proof.

PROOF OF THE THEOREM. For simplicity, we write $U_d$ in Lemma 5.1 by $U$, and $l([0,1])$ by $l$, for $l\in L$.

Choose $l_k\in L$, $k=1,2,\dots$, such that $\max_{l_k}J(u)<\mu+\delta$ and $\max_{l_k}J\to\mu$, as $k\to\infty$. By (a) and (c) in Lemma 5.1,

$$\mathcal{T}(l_k)\equiv\tilde{l}_k\in L\quad\text{and}\quad\max_{\tilde{l}_k}J\to\mu. \tag{5.11}$$

And by (b) and (d)

$$J|_{\tilde{l}_k\cap U}<\mu-\delta. \tag{5.12}$$

Now choose $u_k\in\tilde{l}_k$, so that $J(u_k)=\max_{\tilde{l}_k}J\to\mu$. It can be shown that (cf. e.g. [**7**], the proof for Mountain Pass Lemma by using Ekeland's variational principle) there exist $\{v_k\}\subset H_*$, such that

$$\|u_k-v_k\|_{H^1}\to 0,\quad J(v_k)\to\mu\quad\text{and}\quad J'(v_k)\to 0. \tag{5.13}$$

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By (5.12), $u_k \in H_* \backslash U$. Thus there are only two possibilities:
(1) $d(u_k) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$, or
(2) $P(u_k) \to \varsigma \in S^2$, with $R(\varsigma) > m - \varepsilon_1$.

In case (1), by (5.13), $\{P(v_k)\}$ is bounded away from the sphere $S^2$. Then by Proposition 2.1 and (5.13), $\{\tilde{v}_k \equiv v_k - (1/4\pi)\int_{S^2} v_k\}$ converges strongly in $H_*$ to some $v_0$, that $J'(v_0) = 0$ and $J(v_0) = \mu$. Hence $\mu$ is a critical value of $J$.

In case (2), by Proposition 4.4, $P(v_k) \to \varsigma$; then due to Proposition 4.3,

$$J(v_k) \to -8\pi \ln 4\pi R(\varsigma),$$

that is,

$$\mu = -8\pi \ln 4\pi R(\varsigma).$$

By (5.1), $-8\pi \ln 4\pi R(\varsigma) < -8\pi \ln 4\pi(m - \varepsilon_1) < \mu$, a contradiction. Therefore, $\mu$ is a critical value of $J$. This completes the proof of our Theorem.

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