

Scalar field equation in the presence of signature change

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We consider the (massless) scalar field on a two-dimensional manifold with metric that changes signature from Lorentzian to Euclidean. Requiring a conserved momentum in the spatially homogeneous case leads to a particular choice of propagation rule. The resulting mix of positive and negative frequencies depends only on the total (conformal) size of the spacelike regions and not on the detailed form of the metric. Reformulating the problem using junction conditions, we then show that the solutions obtained above are the unique ones which satisfy the natural distributional wave equation everywhere. We also give a variational approach, obtaining the same results from a natural Lagrangian.

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I. INTRODUCTION

In previous work [1] we argued that signature change of a spacetime metric should lead to particle production by determining the junction conditions on the scalar field. A detailed consideration of quite general propagation rules was given in [2], where the presence of symmetry was invoked to demand a conserved momentum, thus singling out the propagation rule proposed in [1]. In this paper we give a mathematically cleaner presentation of the result that a conserved momentum leads to a particular junction condition on the scalar field. We also propose a generalization using distributional language which could be applied in a more general setting.

In Sec. II we establish our notation and then introduce our homogeneous signature-changing model in Sec. III. In Sec. IV we show that the added physical requirement that momentum be conserved determines, using Stokes' theorem, the propagation of the scalar field across a surface of signature change. In Sec. V we reformulate the theory in terms of distributions, deriving the natural distributional wave equation without invoking any symmetry, and show that solutions of this wave equation automatically satisfy the propagation condition above. In Sec. VI we again reformulate the theory, this time using a variational approach, and show that a natural choice of action is equivalent to the distributional wave equation of the previous section. The results in Secs. V and VI do not require the assumption used in Sec. IV that the

momentum be conserved, but instead derive momentum conservation as a consequence of the theory. Finally, in Sec. VII we discuss our results, contrasting the various formulations.

II. NOTATION

We first review the usual theory of the massless scalar field equation in the absence of signature change using the language of differential forms. We set up our formalism on an n -dimensional manifold and then apply it to a particular two-dimensional model. Associated with any closed $(n-1)$ -form α there is an integral conserved quantity obtained from Stokes' theorem, namely

$$0 = \int_V d\alpha = \int_{\partial V} \alpha. \quad (1)$$

It is therefore useful to express the theory in terms of forms.

The Lagrangian \mathcal{L} for the massless scalar field with respect to an arbitrary metric g_{ab} is given by¹

$$\mathcal{L} * 1 = d\Phi \wedge *d\Phi \quad (2)$$

where $*$ denotes the Hodge dual, from which one derives the wave equation

$$d*d\Phi = 0 \quad (3)$$

or, in tensor language, $\square\Phi = 0$. By virtue of (3), the $(n-1)$ -form

$$K = \Phi *d\Psi - \Psi *d\Phi \quad (4)$$

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¹For simplicity we assume in this section that Φ is real.

is closed ($dK = 0$) for any two solutions Φ and Ψ of (3). In tensor language, the associated conserved quantity is just the symplectic product (from which the Klein-Gordon product is constructed), namely

$$0 = \int_{\partial V} K = \int_{\partial V} n^a (\Phi \partial_a \Psi - \Psi \partial_a \Phi) d\Sigma \quad (5)$$

where n^a is the unit normal and $d\Sigma$ the volume element on ∂V . Finally, associated with any Killing vector X there is a conserved current given by the closed form

$$J_X = i_X d\Phi \wedge *d\Phi + d\Phi \wedge i_X *d\Phi \quad (6)$$

where i_X denotes interior product, so that for example $i_X(d\Phi) = d\Phi(X) = X(\Phi)$. In tensor language, $J^a = T^{ab}X_b$ is conserved due to the conservation of the stress-energy tensor $\nabla_m T^{ma} = 0$ and Killing's equation $\nabla_{(a}X_{b)} = 0$.

III. SIGNATURE CHANGE

Consider the manifold $M = \mathbb{R} \times \mathbb{S}$ with metric

$$\begin{aligned} ds^2 &= f(t) dt^2 + g(t) dx^2 \\ &= g(h(t) dt^2 + dx^2) \end{aligned} \quad (7)$$

where x is periodic, $h = f/g$, g is everywhere positive, and we assume that f (and hence h) has at least one and at most countably many isolated roots $\{t_0, t_1, \dots\}$. For the remainder of this section, we will assume that f has only one root, which occurs at $t = 0$, and that $\text{sgn}(f) = \text{sgn}(t)$. Note that the vector $X = \partial_x$ is a Killing vector. The Hodge dual operator associated with this metric is given by

$$*1|_{U^\pm} = \epsilon^\pm \sqrt{|h|} dt \wedge dx \Big|_{U^\pm} \quad (8)$$

where $\epsilon^\pm = \pm 1$ according to the choice of orientation of the volume forms in $U^\pm := \{\pm t > 0\}$.

Introduce new “time” parameters τ for $t < 0$ and σ for $t > 0$ by

$$\tau = \int_0^t \sqrt{-h} dt, \quad \sigma = \int_0^t \sqrt{h} dt \quad (9)$$

so that, away from $t = 0$, the metric takes the form

$$ds^2 = g(\text{sgn}(h) dT^2 + dx^2) \quad (10)$$

where T is τ or σ as appropriate. Note that while the conformal “time” parameter T is continuous, it is not C^1 related to t (because t is not a C^1 function of T) and thus cannot be used as a coordinate in a region that includes $t = 0$.

Away from $t = 0$, it is easy to find a (complex) basis of solutions of (3) using conformal coordinates, namely

$$\begin{aligned} u_k &= e^{ikx} e^{-i|k|\tau} & (h < 0), \\ v_k &= e^{ikx} e^{-k\sigma} & (h > 0), \end{aligned} \quad (11)$$

and their complex conjugates, where k takes on suitable discrete values so that periodic boundary conditions (in x) obtain. Solutions of (3) are thus well-behaved functions of T even where $h = 0$, at least in the sense of one-sided limits. Note that these are just the usual positive and negative frequency solutions for $h < 0$ and (anti)analytic functions of $x + i\sigma$ for $h > 0$ as expected.

IV. STOKES' THEOREM

Now consider a manifold M with a preferred hypersurface Σ that partitions M into two manifolds (with boundary) M^\pm , and suppose that the n -forms α^\pm are defined on M^\pm . Let V be an arbitrary region of M , let V^\pm denote the intersections $V \cap M^\pm$, and let $(\partial V)^\pm = \partial V \cap (M^\pm - \Sigma)$. Then we may use Stokes' theorem (1) in M^\pm to write

$$\begin{aligned} \int_{V^+} d\alpha^+ + \int_{V^-} d\alpha^- &= \int_{(\partial V)^+} \alpha^+ + \int_{(\partial V)^-} \alpha^- \\ &\quad + \int_{\Sigma \cap V} (\alpha^+ - \alpha^-) \end{aligned} \quad (12)$$

where an assumption has been made about the relative orientation of Σ . Consequently, if α^\pm are closed on M^\pm and the pullbacks of α^\pm to Σ agree, then the standard arguments can be applied to generate conserved quantities associated with the form α defined to be α^\pm on M^\pm .²

We now assume that Σ corresponds to the surface of signature change at $t = 0$ in the model of Sec. III, and that the wave equation (3) [with $*$ as defined in (8)] is satisfied on M^\pm . In order for the (integral) momentum associated with the Killing vector ∂_x to be conserved, we need J_X to satisfy the additional condition above, namely that the pullbacks from M^\pm to Σ agree. But away from $t = 0$

$$\begin{aligned} J_X &= \epsilon^\pm \left(\frac{\Phi_{,t}^2}{h} - \Phi_{,x}^2 \right) \sqrt{|h|} dt \\ &\quad + 2\epsilon^\pm \text{sgn}(h) \frac{\Phi_{,x} \Phi_{,t}}{\sqrt{|h|}} dx \end{aligned} \quad (13)$$

whose pullback to Σ is $2\epsilon^\pm \text{sgn}(h) \Phi_{,x} \Phi_{,T} dx$. With the natural requirement that Φ be continuous at Σ , the condition at a surface of signature change thus becomes

$$\Phi_{,\sigma}|_\Sigma = -\epsilon \Phi_{,\tau}|_\Sigma \quad (14)$$

where $\epsilon = \epsilon^-/\epsilon^+$ is the relative orientation of M^\pm . As shown in [2], these requirements uniquely determine the propagation of a solution of (3) across Σ .

It is interesting to note that although (14) implies that the pullback condition is also satisfied for K , so that Klein-Gordon products are automatically conserved, the converse is not true.

²A related discussion of the divergence theorem in the presence of signature change appears in [3] which points out that conservation of matter does not then follow from the junction conditions imposed on the spacetime.

V. DISTRIBUTIONS

The massless wave equation for a 0-form Φ on a smooth manifold M with a non-degenerate metric may be written

$$dF = 0 \quad (15)$$

where

$$F = *d\Phi \quad (16)$$

in terms of the Hodge map $*$ defined by the metric. We wish to extend the theory of the massless scalar field to manifolds that admit a metric that changes signature. We sketch here how this may be achieved using a distributional language and refer to [4] for further mathematical details. We naturally require that (15) be satisfied on $U^\pm = M^\pm - \Sigma$, and we shall assume that Φ is continuous on M and denote $\Phi|_{U^\pm}$ by Φ^\pm .

We shall call a p -form F on M *regularly discontinuous* if the restrictions $F^\pm = F|_{U^\pm}$ are smooth and the (1-sided) limits $F^\pm|_\Sigma = \lim_{t \rightarrow 0^\pm} F$ exist. The *discontinuity* of F is the tensor distribution on Σ defined by

$$[F]_\Sigma = F^+|_\Sigma - F^-|_\Sigma. \quad (17)$$

Denote by Θ^\pm the Heaviside distributions with support in U^\pm and such that

$$d\Theta^\pm = \pm \delta \quad (18)$$

where δ is the hypersurface Dirac distribution with support on Σ .³ Now introduce, as a distribution on M ,

$$F = \Theta^+ F^+ + \Theta^- F^-. \quad (19)$$

It follows that

$$dF = \Theta^+ dF^+ + \Theta^- dF^- + \delta \wedge [F]_\Sigma. \quad (20)$$

We readily deduce the consequences of requiring dF to be the zero distribution. By evaluating dF on a set of test vectors with support in U^\pm we deduce

$$dF|_{U^\pm} = 0 \quad (21)$$

as expected. Similarly it follows that

$$\delta \wedge [F]_\Sigma = 0. \quad (22)$$

In order to derive the junction conditions for matching derivatives at Σ we shall only admit solutions such that $*d\Phi$ is regularly discontinuous at Σ so that $[*d\Phi]_\Sigma$ is well defined. We seek distributional solutions to

$$dF = 0 \quad (23)$$

where F is defined as above with $F|_{U^\pm} = *d\Phi|_{U^\pm} = *d\Phi^\pm$. Using (8) we see that

$$*d\Phi^\pm = \epsilon^\pm \left(\text{sgn}(h) \frac{\partial_t \Phi dx}{\sqrt{|h|}} - \partial_x \Phi \sqrt{|h|} dt \right) \Big|_{U^\pm}. \quad (24)$$

If $\partial_t \Phi / \sqrt{|h|}$ is bounded as $t \rightarrow 0^\pm$ then, since Φ is assumed continuous, $*d\Phi$ is regularly discontinuous at Σ and

$$[*d\Phi]_\Sigma \wedge dt = \epsilon^+ \left(\lim_{t \rightarrow 0^+} \frac{\partial_t \Phi}{\sqrt{|h|}} + \epsilon \lim_{t \rightarrow 0^-} \frac{\partial_t \Phi}{\sqrt{|h|}} \right) dx \wedge dt. \quad (25)$$

Then (22) implies

$$[*d\Phi]_\Sigma \wedge dt = 0, \quad (26)$$

which provides junction conditions for regularly discontinuous solutions of the equations

$$d*d\Phi|_{U^\pm} = 0. \quad (27)$$

Furthermore, these junction conditions are identical to (14).

VI. VARIATIONAL APPROACH

It is not mandatory to use distributions to generate junction conditions. We offer a variational approach that yields the same results for regularly discontinuous forms. Consider the action

$$S[\Phi] = \frac{1}{2} \int_{U^+} d\Phi^+ \wedge *d\Phi^+ + \frac{1}{2} \int_{U^-} d\Phi^- \wedge *d\Phi^- \quad (28)$$

where the fields and metric are piecewise continuous and the appropriate Hodge maps are understood in the regions U^\pm . Consider field variations $\Phi^\pm \rightarrow \Phi^\pm + \delta\Phi^\pm$

$$\begin{aligned} \delta S &= \int_{U^+} d(\delta\Phi^+) \wedge *d\Phi^+ + \int_{U^-} d(\delta\Phi^-) \wedge *d\Phi^- \\ &= \int_{\partial U^+} \delta\Phi^+ *d\Phi^+ + \int_{\partial U^-} \delta\Phi^- *d\Phi^- \\ &\quad + \int_{U^+} \delta\Phi^+ d*d\Phi^+ + \int_{U^-} \delta\Phi^- d*d\Phi^- \end{aligned} \quad (29)$$

using Stokes' theorem in U^\pm . Now postulate that $\delta S = 0$ for all field variations of compact support. Choosing support entirely in U^\pm yields (27). Now let the variation have compact support on any domain that includes Σ and assume that $*d\Phi$ is regularly discontinuous with respect to Σ . Then the continuity of Φ allows us to write $\delta\Phi = \delta\Phi^+ \equiv \delta\Phi^-$ so that

$$\int_\Sigma \delta\Phi (*d\Phi^+ - *d\Phi^-) = 0. \quad (30)$$

Since $\delta\Phi$ is arbitrary we conclude that the pullback of the form $[*d\Phi]_\Sigma$ to the hypersurface Σ must vanish. If Σ is given by $\psi = 0$ then this may be expressed in terms of a restriction

³The properties of these distributions will be discussed more fully in [4].

$$[*d\Phi]_{\Sigma} \wedge d\psi = 0 \quad (31)$$

as before.

The above argument can be generalized to other field theories. We plan to discuss the junction conditions for spinor fields in a discontinuous or degenerate metric elsewhere; see also the recent work of Romano [5]. We also observe that hypersurface sources are readily accommodated within this language by considering actions of the form

$$S = \int_{U^+} \Lambda_+ + \int_{U^-} \Lambda_- + \int_{\Sigma} \Lambda \quad (32)$$

where the source is described by the hypersurface action density Λ .

VII. DISCUSSION

It is important to distinguish between the assumptions made in our three derivations of the matching condition (14). The first derivation based on Stokes' theorem used conservation of momentum in the presence of a Killing vector but did *not* make use of a particular form of the wave equation at the surface of signature change. On the other hand, the second derivation assumes a particular distributional form for the wave equation on the whole manifold, while the third assumes a particular form for the action; neither makes any assumptions about symmetries. The latter two derivations can therefore be applied in more general spacetimes provided one is willing to accept either (15) as being the correct wave equation or (28) as being the correct action. We plan to apply this approach to an explicit imbedding of the trousers spacetime in three-dimensional Minkowski space.

If we assume that f in (7) has precisely 2 (simple) roots corresponding to the T values T_i and T_f , and that $f < 0$ as $|t| \rightarrow \infty$, so that our spacetime is asymptotically Lorentzian, then, as claimed in [1] and shown in detail in [2], the above solutions, satisfying the condition (14) and continuity, correspond to the relationship

$$u_k^{\text{out}} e^{+i|k|T_f} = u_k^{\text{in}} e^{+i|k|T_i} \cosh(k\Delta T) + \bar{u}_{-k}^{\text{in}} e^{-i|k|T_i} i \sinh(|k|\Delta T) \quad (33)$$

between basis solutions at early and at late times. The mixing of positive and negative frequencies, and hence particle production, is controlled by the last term.

We note an interesting freedom in the derivations of the junction conditions presented above. In all three

derivations, the choice of Hodge map in the regions separated by Σ is fixed only up to a relative sign. Physically, this corresponds to different choices of time orientation in one or more regions. For the example just considered with two surfaces of signature change, there are 8 different choices of orientations. Since our (classical) theory is invariant under a global change of time orientation, this number is immediately reduced to 4. (However, one might want on physical grounds to use different boundary conditions depending on the global choice of time orientation.) Furthermore, it can easily be shown [2] that changing the "time" orientation of the middle, Euclidean region results only in an unimportant phase factor in (33), so that there is no need to worry about which "time" orientation to pick in this region. (Specifically, the second term picks up a minus sign.) Equation (33) corresponds physically to a model with asymptotic "in" and "out" regions. The only remaining distinct choice corresponds to both Lorentzian regions being to the future (or past) of the Euclidean region, corresponding to two universes sharing a common big bang or big crunch.

A related case of interest is a paraboloid, e.g., with the induced metric obtained from embedding it in Minkowski three-space with the rotation axis being the time axis. Deleting the point on the axis yields a manifold with topology $\mathbb{R} \times S$ and a metric of the form (7), but with only one signature change: from an initial Euclidean region to a final Minkowskian region. This picture is reminiscent of quantum cosmology, and is related to the models recently considered by Ellis *et al.* [6] and Hayward [7]. Note that there will be now be an extra regularity condition at the axis which will affect the observed particle spectrum at late times.

Finally, we note that our junction condition (14) holds regardless of which way the signature changes, although our derivation [and in particular (25)] made use of specific assumptions about $\text{sgn}(h)$. The calculation is easily generalized to a degenerate metric of the form (7) which does not change signature; this merely changes the relative sign in (14).

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