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Scalar perturbations in scalar field quantum cosmology

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In this paper it is shown how to obtain the simplest equations for the Mukhanov-Sasaki variables describing quantum linear scalar perturbations in the case of scalar fields without potential term. This was done through the implementation of canonical transformations at the classical level, and unitary transformations at the quantum level, without ever using any classical background equation, and it completes the simplification initiated in investigations by Langlois [D. Langlois, Classical Quantum Gravity **11**, 389 (1994).], and Pinho and Pinto-Neto [E. J. C. Pinho and N. Pinto-Neto, Phys. Rev. D **76**, 023506 (2007).] for this case. These equations were then used to calculate the spectrum index n_s of quantum scalar perturbations of a nonsingular inflationary quantum background model, which starts at infinity past from flat space-time with Planckian size spacelike hypersurfaces, and inflates due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase. The result is $n_s = 3$, incompatible with observations.

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I. INTRODUCTION

The usual theory of cosmological perturbations, with their simple equations [1], relies essentially on the assumptions that the background is described by pure classical general relativity (GR), while the perturbations thereof stem from quantum fluctuations. It is a semiclassical approach, where the background is classical and the perturbations are quantized, and the fact that the background satisfies Einstein's equations is heavily used in the simplification of the equations. In Refs. [2–4], which assume the validity of the Einstein-Hilbert action, it was shown that such simple equations for quantum linear cosmological perturbations can also be obtained without ever using any equations for the background. This can be accomplished through a series of canonical transformations and redefinitions of the lapse function. These results open the way to also quantize the background, and use these simple equations to evaluate the evolution of the quantum linear perturbations on it. Indeed, such results were applied to quantum bouncing backgrounds, and spectral indices for tensor and scalar perturbations were calculated in Refs. [5,6].

The matter content used in these papers were assumed to be either a single perfect fluid or a single scalar field. In the case of perfect fluids, the equations were simplified up to their simplest possible form, both for tensor and scalar perturbations. For the case of scalar fields, this simplest form was achieved for tensor perturbations but not for scalar perturbations. One ended in an intermediate stage that needed further simplifications in order to be applied to quantum backgrounds [3,7].

Meanwhile, a nonsingular inflationary model was found [8] containing a single scalar field without potential term, which starts at infinity past from flat space-time with Planckian size spacelike hypersurfaces, and inflates, due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase. It should be interesting to investigate if this model could generate an almost scale invariant spectrum of scalar perturbations, as observed [9]. However, without simple equations governing the evolution of the perturbations, the investigation becomes rather cumbersome.

The aim of this paper is twofold: complete the simplification initiated in Refs. [3,7], and apply it to the background described in Ref. [8]. In fact, after performing some canonical transformations at the classical level, and unitary transformations at the quantum level, we were able to obtain the simple equations for linear scalar perturbations of Ref. [1] for the case of scalar fields without potential, without ever using any classical background equation. These perturbation equations were then used to calculate the spectrum index n_s of the background model of Ref. [8] yielding $n_s = 3$, incompatible with observations [9] ($n_s \approx 1$). Hence, even though the quantum background model has some attractive features, the model should be discarded.

The paper is organized as follows: in the next section, we briefly summarize the results of Ref. [8]. In Sec. III, the simplification of the second order Hamiltonian for the scalar perturbations is implemented, and the full quantization of the system, background and perturbations, is performed. The quantum background trajectories are then used to induce a time evolution for the Heisenberg operators describing the perturbations, yielding simple dynamical equations for the quantum perturbations. In Sec. IV, we calculate the spectral index of scalar perturbations in the

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background presented in Sec. II, using the equations obtained in Sec. III. Section V presents our conclusions.

II. BOHM-DE BROGLIE INTERPRETATION OF A QUANTUM NON-SINGULAR INFLATIONARY BACKGROUND MODEL

In this section, we first briefly highlight the main characteristics of the Bohm-de Broglie quantization scheme, restricting our discussion to the homogeneous minisuperspace models which have a finite number of degrees of freedom. We then apply it to the quantisation of the background geometry with a massless scalar field without potential term.

The Wheeler-DeWitt equation of a minisuperspace model is obtained through the Dirac quantization procedure, where the wave function must be annihilated by the operator version of the Hamiltonian constraint

$$\mathcal{H}(\hat{p}^{\mu}, \hat{q}_{\mu})\Psi(q) = 0. \tag{1}$$

The quantities $\hat{p}^{\mu}$, $\hat{q}_{\mu}$ are the phase space operators related to the homogeneous degrees of freedom of the model. Usually this equation can be written as

$$-\frac{1}{2} f_{\rho\sigma}(q_{\mu}) \frac{\partial\Psi(q)}{\partial q_{\rho}\partial q_{\sigma}} + U(q_{\mu})\Psi(q) = 0, \tag{2}$$

where $f_{\rho\sigma}(q_{\mu})$ is the minisuperspace DeWitt metric of the model, whose inverse is denoted by $f^{\rho\sigma}(q_{\mu})$.

Writing Ψ in polar form, $\Psi = R\exp(iS)$, and substituting it into (2), we obtain the following equations:

$$\frac{1}{2} f_{\rho\sigma}(q_{\mu}) \frac{\partial S}{\partial q_{\rho}} \frac{\partial S}{\partial q_{\sigma}} + U(q_{\mu}) + Q(q_{\mu}) = 0, \tag{3}$$

$$f_{\rho\sigma}(q_{\mu}) \frac{\partial}{\partial q_{\rho}}\left(R^2 \frac{\partial S}{\partial q_{\sigma}}\right) = 0, \tag{4}$$

where

$$Q(q_{\mu}) \equiv -\frac{1}{2R} f_{\rho\sigma} \frac{\partial^2 R}{\partial q_{\rho}\partial q_{\sigma}} \tag{5}$$

is called the quantum potential.

The Bohm-de Broglie interpretation applied to quantum cosmology states that the trajectories $q_{\mu}(t)$ are real, independently of any observations. Equation (3) represents their Hamilton-Jacobi equation, which is the classical one added with a quantum potential term Eq. (5) responsible for the quantum effects. This suggests to define

$$p^{\rho} = \frac{\partial S}{\partial q_{\rho}}, \tag{6}$$

where the momenta are related to the velocities in the usual way:

$$p^{\rho} = f^{\rho\sigma} \frac{1}{N} \frac{\partial q_{\sigma}}{\partial t}. \tag{7}$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations, called the guidance relations:

$$\frac{\partial S(q_{\rho})}{\partial q_{\rho}} = f^{\rho\sigma} \frac{1}{N} \dot{q}_{\sigma}. \tag{8}$$

Equations (8) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same space-time geometry for a given nonclassical solution $q_{\alpha}(t)$. There is no problem of time in the Bohm-de Broglie interpretation of minisuperspace quantum cosmology [10]. We will return to this point in the next section.

We now apply this interpretation to the situation where $\mathcal{H}$ in Eq. (1) is given by

$$H_0^{(0)} = \frac{\sqrt{2V}}{2\ell_{Pl} e^{3\alpha}}(-P_{\alpha}^2 + P_{\varphi}^2), \tag{9}$$

which was worked out in Ref. [8]. The variables are dimensionless with φ describing the scalar field degree of freedom and α associated to the scale factor through $\alpha \equiv \log(a)$. The main feature of this model is the possibility to obtain a nonsingular inflationary model similar to the pre-big bang model [11–14], with a minimum volume spatial section in the infinity past, or the emergent model [15] for flat spatial sections, without any graceful exit problem.

We take as solution of the background Wheeler-DeWitt equation, $\hat{H}_0^{(0)}\Psi(a, \varphi) = 0$, a Gaussian superposition of WKB solutions. The resulting wave function is (see Ref. [8] for details)

$$\Psi(\alpha, \varphi) = 2\sqrt{\pi|h|}\left[\exp i\left(-\frac{h}{2}(\alpha+\varphi)^2 + d(\alpha+\varphi) + \frac{\pi}{4}\right) + \exp i\left(-\frac{h}{2}(\alpha-\varphi)^2 + d(\alpha-\varphi) + \frac{\pi}{4}\right)\right], \tag{10}$$

where h and d are two positive free parameters associated to the variance and the displacement of the Gaussian superposition, respectively.

The norm of the wave function is given by $R = 4\sqrt{\pi|h|}\cos[\varphi(h\alpha - d)]$, yielding the quantum potential, Eq. (5),

$$Q = (h\alpha - d)^2 - h^2\varphi^2. \tag{11}$$

The guidance relations, given by Eq. (8) with the choice $N = \frac{\ell_{Pl}}{\sqrt{2V}} e^{3\alpha}$, reduce to

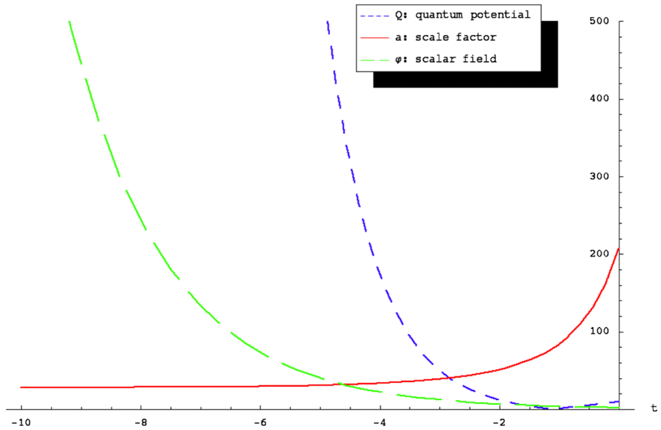

FIG. 1 (color online). Time evolution of the background variables. The solid line describe the accelerated expansion of the scale factor from a finite minimum size $a_0 = e^{d/h}$. The long-dashed line pictures the exponential decrease of the scalar field and the short-dashed line gives the decrease of the quantum potential until arriving in the classical region. The parameters were chosen to be $h = 3/5$, $d = 2$, and $\alpha_0 = 2$.

$$\dot{\alpha} = -\frac{\partial S}{\partial \alpha}, \qquad \dot{\varphi} = \frac{\partial S}{\partial \varphi}, \tag{12}$$

yielding

$$\dot{\alpha} = h\alpha - d, \qquad \dot{\varphi} = -h\varphi, \tag{13}$$

which can be directly integrated to give

$$a = e^{\alpha} = e^{d/h}\exp(\alpha_0 e^{ht}) \quad \text{and} \quad \varphi = \alpha_0 e^{-ht}, \tag{14}$$

where α_0 is an integration constant. Recall that the time parameter t is related to cosmic time τ through $\tau = \int dte^{3\alpha(t)} \Rightarrow \tau - \tau_0 = \mathrm{Ei}(3\alpha_0 e^{ht})/h$, where $\mathrm{Ei}(x)$ is the exponential-integral function.

These solutions represent ever expanding nonsingular models (see Fig. 1). For $t \ll 0$ the Universe expands accelerately from its minimum size $a_0 = e^{d/h}$ (remember that for the physical scale factor one has $a_0^{\mathrm{phys}} = \frac{\ell_{Pl}}{\sqrt{2V}} e^{d/h}$), which occurs in the infinity past $t \to -\infty$. The scalar field is very large in that phase. If $|ht| \leq \alpha_0$ is not very large, one has

$$a \approx e^{\alpha_0 + d/h}[1 + \alpha_0 ht + (1 + 1/\alpha_0)(\alpha_0 ht)^2/2! + (1 + 3/\alpha_0 + 1/\alpha_0^2)(\alpha_0 ht)^3/3! \ldots]. \tag{15}$$

Taking $\alpha_0 \gg 1$, one can write $a \approx e^{\alpha_0 + d/h}\exp(\alpha_0 ht)$. In that case, from $\tau = \int dta^3(t)$, one obtains that $a \propto (\tau - \tau_0)^{1/3}$ and $\varphi \propto \ln(\tau - \tau_0)$, as in the classical regime. Figure 1 exhibits the Bohmian trajectories and quantum potential for the parameters $h = 3/5$, $d = 2$, and $\alpha_0 = 2$.

III. SIMPLIFICATION OF THE SECOND ORDER HAMILTONIAN AND CANONICAL QUANTIZATION

The conventional approach to deal with quantum cosmological perturbations is to consider a semiclassical treatment that quantizes only the first order perturbations while the background is treated classically. Once the background dynamics has a classical evolution, one can use these equations to significantly simplify the second order Lagrangian before quantizing the system [1]. In this case, the background evolution induces a potential term that modifies the quantum dynamics of the perturbations.

One step further is to consider quantum corrections to the background evolution itself, as in minisuperspace models, Refs. [16–19]. In this case, the simplifications in the equations for the linear perturbations using the classical background cannot be implemented. It is worth to remind that the original lagrangian is quite involved, and the use of the background equation is a key step to rewrite the system in a treatable form.

Recent works using technics for Hamiltonian's systems [2–4] showed that it is also possible to simplify the full Hamiltonian system by a series of canonical transformations. Their main results focus in the scalar and tensor perturbations considering the matter content of the Universe described by a perfect fluid. Even though in Ref. [7] and in the Appendix A of Ref. [3] it is shown a long development that significantly simplifies the Hamiltonian for a scalar field with a generic potential $U(\varphi)$, there were still some delicate issues to be addressed to consistently quantize the scalar field case.

We will not reproduce the development made in these references but we will continue the development of the above mentioned Appendix. The main point to acquaint from this reference is that their simplification procedure use only canonical transformations, that guarantees the equivalence between the original and the simplified Hamiltonians, independently of the background equations of motion.

In the present work we will focus in the case of a vanishing potential $U(\varphi)$ and show how it is possible to consistently quantize simultaneously both the background and the perturbations. The background system is composed of a free massless scalar field in a spatially flat Friedmann-Lemaître-Robertson-Walker metric (FLRW). Since we are only interested in scalar perturbations, the perturbed metric can be written as

$$ds^2 = N^2(1 + 2\phi)dt^2 - NaB_{|i}dtdx^i - a^2[(1 + 2\psi)\delta_{ij} - 2E_{|i|j}]dx^i dx^j. \tag{16}$$

The matter content is defined by a free massless scalar field $\varphi(t, x) = \varphi_0(t) + \delta\varphi(t, x)$, where φ_0 is the background homogeneous scalar field. Using these definitions in the Lagrangian density for the scalar field, namely

$\mathcal{L}_m = \frac{1}{2}\varphi_{;\mu}\varphi^{;\mu}$, we find

$$\mathcal{L}_m = \frac{(1-2\phi)}{N^2}\left(\frac{\dot{\varphi}_0^2}{2} + \dot{\varphi}\delta\dot{\varphi}\right) + \frac{\dot{\varphi}_0^2}{N^2}\left(2\phi^2 - \frac{B^{|i}B_{|i}}{2}\right) - \frac{\dot{\varphi}_0}{Na}B^{|i}\delta\varphi_{|i} + \frac{\delta\dot{\varphi}^2}{2N^2} - \frac{1}{2a^2}\delta\varphi^{|i}\delta\varphi_{|i}. \tag{17}$$

As our starting point, let us consider the Hamiltonian (A39) of Ref. [3] with the scalar field potential $U(\varphi)$ taken to be null,

$$H = NH_0 + \int d^3x\left(-\frac{\ell_{Pl}^2 P_a^2}{2a^2V}\phi + \frac{3P_\varphi^2}{a^4 P_a V}\psi + \frac{3\ell_{Pl}^2 P_\varphi}{2a^4V}v\right)\tilde{\phi}_6 + \Lambda_N P_N + \int d^3x \Lambda_\phi \pi_\phi, \tag{18}$$

where $\tilde{\phi}_6 = \pi_\psi$, P_N and π_ϕ are first class constrains, and v is the Mukhanov-Sasaki variable. The quantity H_0 is defined as

$$H_0 = -\frac{\ell_{Pl}^2 P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3V} + \frac{1}{2a}\int d^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}v^{,i}v_{,i}\right) + \left[\frac{15\ell_{Pl}^2 P_\varphi^2}{4a^5V^2} - \frac{\ell_{Pl}^4 P_a^2}{16a^3V^2} - \frac{27P_\varphi^4}{4a^7V^2P_a^2}\right]\int d^3x\sqrt{\gamma}v^2, \tag{19}$$

where P_a, P_φ and π are the momenta canonically conjugate to a, φ_0 and v, respectively, $\ell_{Pl}^2 = \frac{8\pi G}{3}$, and V is the comoving volume of the compact spatial sections, i.e. $V < \infty$. The zero order Hamiltonian,

$$H_0^{(0)} \equiv -\frac{\ell_{Pl}^2 P_a^2}{4aV} + \frac{P_\varphi^2}{2a^3V}, \tag{20}$$

can be used to simplify further the masslike term for the perturbations, i.e. the function inside brackets multiplying the v^2 term. To do so, we rewrite P_φ as

$$P_\varphi^2 = 2a^3V\left(H_0^{(0)} + \frac{\ell_{Pl}^2 P_a^2}{4aV}\right).$$

Redefining the lapse function as

$$\tilde{N} = N\left\{1 + \left[\frac{15\ell_{Pl}^2}{2a^2V} - \frac{27}{aP_a^2}\left(H_0^{(0)} + \frac{\ell_{Pl}^2 P_a^2}{2aV}\right)\right] \times \int d^3x\sqrt{\gamma}v^2\right\},$$

and keeping only second order terms in NH_0, we can rewrite it as

$$NH_0 = \tilde{N}\left[H_0^{(0)} + \frac{1}{2a}\int d^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}v^{,i}v_{,i}\right) + \frac{\ell_{Pl}^4 P_a^2}{8a^3V^2}\int d^3x\sqrt{\gamma}v^2\right] + \mathcal{O}(v^4, v^2\pi^2) \tag{21}$$

Thus, by a simple redefinition of the lapse function, the masslike term simplifies significantly. Nonetheless, it is still tricky to quantize this term due to the momentum P_a. Furthermore, the scale factor is defined on the half-line which requires additional care in specifying the Hilbert space. To deal with these two points, it is convenient to define dimensionless variables $\alpha \equiv \log(\sqrt{2V}\ell_{Pl}^{-1}a)$ and $\varphi \to \frac{\ell_{Pl}}{\sqrt{2}}\varphi$ which give us the following relations:

$$P_\alpha = -\frac{\ell_{Pl}}{\sqrt{2V}}\frac{e^{3\alpha}}{N}\dot{\alpha}, \qquad \frac{\ell_{Pl}^2}{4V}\frac{P_a^2}{a} = \frac{\sqrt{2V}}{\ell_{Pl}}\frac{P_\alpha^2}{2e^{3\alpha}},$$

$$\frac{P_\varphi^2}{2a^3V} \to \frac{\sqrt{2V}}{\ell_{Pl}}\frac{P_\varphi^2}{2e^{3\alpha}}, \qquad H_0^{(0)} = \frac{\sqrt{2V}}{2\ell_{Pl}e^{3\alpha}}(-P_\alpha^2 + P_\varphi^2).$$

With these new variables we find,

$$H_0 = H_0^{(0)} + \frac{N\sqrt{2V}}{2\ell_{Pl}e^\alpha}\int d^3x\sqrt{\gamma}\left(\frac{\pi^2}{\gamma} + v^{,i}v_{,i} + \frac{P_\alpha^2}{e^{4\alpha}}v^2\right).$$

To eliminate the momentum in the masslike term we perform a canonical transformation generated by

$$\mathcal{F} = \mathcal{I} + \frac{P_\alpha}{2}\int d^3x\sqrt{\gamma}\tilde{v}^2 + e^{\tilde{\alpha}}\int d^3x\pi\tilde{v}, \tag{22}$$

which implies

$$\alpha = \tilde{\alpha} + \frac{1}{2}\int d^3x\sqrt{\gamma}\tilde{v}^2, \qquad v = e^{\tilde{\alpha}}\tilde{v},$$

$$\tilde{P}_\alpha = P_\alpha + e^{\tilde{\alpha}}\int d^3x\pi\tilde{v}, \qquad \tilde{\pi} = \sqrt{\gamma}\tilde{P}_\alpha\tilde{v} + e^{\tilde{\alpha}}\pi,$$

$$e^{3\alpha} = e^{3\tilde{\alpha}}\left(1 + \frac{3}{2}\int d^3x\sqrt{\gamma}\tilde{v}^2\right) + \mathcal{O}(\tilde{v}^3).$$

Once more, redefining the lapse function as

$$\tilde{N} = N\left[1 - \frac{3}{2}\int d^3x\sqrt{\gamma}\tilde{v}^2\right],$$

and omitting the tilde in the new variables, the Hamiltonian transforms into

$$H = H_0 + \int d^3x\left(-\frac{2V}{\ell_{Pl}^2}\frac{P_\alpha^2}{e^{4\alpha}}\phi + \frac{3\sqrt{2V}}{\ell_{Pl}}\frac{P_\varphi^2}{e^{3\alpha}P_\alpha}\psi + \frac{3\sqrt{2V}}{\ell_{Pl}}\frac{\sqrt{V}P_\varphi}{e^{4\alpha}}v\right)\pi_\psi + \Lambda_N P_N + \int d^3x\Lambda_\phi\pi_\phi \tag{23}$$

with,

$$H_0 = \frac{\sqrt{2V}}{2\ell_{Pl}e^{3\alpha}}\left[-P_\alpha^2 + P_\varphi^2 + \int d^3x\left(\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right)\right]. \tag{24}$$

The system described by this Hamiltonian can be immediately quantized. The Dirac's quantization procedure for constrained Hamiltonian systems requires that the first class constraints must annihilate the wave function

$$\frac{\partial}{\partial N}\Psi(\alpha,\varphi,v,N,\phi,\psi)=0,$$
$$\frac{\delta}{\delta\psi}\Psi(\alpha,\varphi,v,N,\phi,\psi)=0,$$
$$\frac{\delta}{\delta\phi}\Psi(\alpha,\varphi,v,N,\phi,\psi)=0.$$

Thus, the wave function must be independent of N, ϕ and ψ, i.e. $\Psi=\Psi(\alpha,\varphi,v)$ where v encode the perturbed degrees of freedom. Note that, due to the transformation (22), v is now the Mukhanov-Sasaki variable divided by a. The remaining equation is

$$\hat{H}_0\Psi(\alpha,\varphi,v)=0, \qquad (25)$$

which has only quadratic terms in the momenta.

A well-known feature of the quantization of time reparametrization invariant theories is that the state is not explicitly time dependent, hence one should find among intrinsic degrees of freedom a variable that can play the role of time. In the perfect fluid case, the Wheeler-DeWitt's equation assumes a Schrödinger-like form, due to a linear term in the momenta connected with the fluid degree of freedom. However, the Hamiltonian (24) does not possess such linear term, rendering ambiguous the choice of an intrinsic time variable. Notwithstanding, we still can define an evolutionary time for the perturbations if we use the Bohm-de Broglie interpretation. The procedure is similar to what is done in a semiclassical approach, where a time evolution for the quantum perturbations is induced from the classical background trajectory (see, e.g., Ref. [20] for details). Let us summarize it in the following paragraphs.

First of all, take the Hamiltonian NH_0, with H_0 given in Eq. (24) satisfying the Hamiltonian constraint $H_0\approx 0$, and let us solve it classically using the Hamilton-Jacobi theory. The respective Hamilton-Jacobi equation reads

$$-\frac{1}{2}\left(\frac{\partial S_T}{\partial\alpha}\right)^2+\frac{1}{2}\left(\frac{\partial S_T}{\partial\varphi}\right)^2+\frac{1}{2}\int d^3x\left[\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_T}{\delta v}\right)^2+\sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right], \qquad (26)$$

where the classical trajectories can be obtained from a solution S_T of Eq. (26) through

$$\dot{\alpha}=-P_\alpha=-\frac{\partial S_T}{\partial\alpha},\qquad \dot{\varphi}=P_\varphi=\frac{\partial S_T}{\partial\varphi},$$
$$\dot{v}=\frac{1}{\sqrt{\gamma}}\pi=\frac{1}{\sqrt{\gamma}}\frac{\delta S_T}{\delta v}, \qquad (27)$$

where we have chosen $N=l_{Pl}e^{3\alpha}/\sqrt{2V}$, and hence a time parameter t (a dot means derivative with respect to this parameter), related to conformal time through $dt\propto a^2 d\eta$.

We will now use the fact that the v variable is a small perturbation over the background variables α and φ, and that its backreaction in the dynamics of the background is negligible. In this case, one can write $S_T(\alpha,\varphi,v)$ as

$$S_T(\alpha,\varphi,v)=S_0(\alpha,\varphi)+S_2(\alpha,\varphi,v), \qquad (28)$$

where it is assumed that $S_2(\alpha,\varphi,v)$ cannot be splitted again into a sum involving a function of the background variables alone (which would just impose a redefinition of S_0). Noting that, in order to be a solution of the Hamilton-Jacobi Eq. (26), S_2 must be at least a second order functional of v (see Ref. [21]), then $S_2\ll S_0$ as well as their partial derivatives with respect to the background variables. Hence one obtains for the background that

$$\dot{\alpha}\approx-\frac{\partial S_0}{\partial\alpha},\qquad \dot{\varphi}\approx\frac{\partial S_0}{\partial\varphi}. \qquad (29)$$

Inserting the splitting given in Eq. (28) into Eq. (26), one obtains, order by order:

$$-\frac{1}{2}\left(\frac{\partial S_0}{\partial\alpha}\right)^2+\frac{1}{2}\left(\frac{\partial S_0}{\partial\varphi}\right)^2=0, \qquad (30)$$

$$-\left(\frac{\partial S_0}{\partial\alpha}\right)\left(\frac{\partial S_2}{\partial\alpha}\right)+\left(\frac{\partial S_0}{\partial\varphi}\right)\left(\frac{\partial S_2}{\partial\varphi}\right)+\frac{1}{2}\int d^3x\left[\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_2}{\delta v}\right)^2+\sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right]=0, \qquad (31)$$

$$-\frac{1}{2}\left(\frac{\partial S_2}{\partial\alpha}\right)^2+\frac{1}{2}\left(\frac{\partial S_2}{\partial\varphi}\right)^2+O(4)=0. \qquad (32)$$

In Eq. (32), the symbol $O(4)$ represents terms coming from high order corrections to the Hamiltonian (24). As we are interested only on linear perturbations, this equation will not be relevant. The first Eq. (30) is the Hamilton-Jacobi equation of the background which solution yields, together with Eqs. (29), the background classical trajectories. Once one obtains the classical trajectories $\alpha(t)$, $\varphi(t)$, the functional $S_2(\alpha,\varphi,v)$ becomes a functional of v and a function of t, $S_2(\alpha,\varphi,v)\rightarrow S_2(\alpha(t),\varphi(t),v)=\bar{S}_2(t,v)$. Hence Eq. (31), using Eqs. (29), can be written as

$$\frac{\partial S_2}{\partial t}+\frac{1}{2}\int d^3x\left(\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_2}{\delta v}\right)^2+\sqrt{\gamma}e^{4\alpha(t)}v^{,i}v_{,i}\right)=0. \qquad (33)$$

Equation (33) can now be understood as the Hamilton-Jacobi equation coming from the Hamiltonian

$$H_2=\frac{1}{2}\int d^3x\left(\frac{\pi^2}{\sqrt{\gamma}}+\sqrt{\gamma}e^{4\alpha(t)}v^{,i}v_{,i}\right), \qquad (34)$$

which is the generator of time t translations (and not anymore constrained to be null).

If one wants to quantize the perturbations, the corresponding Schrödinger equation should be

$$i\frac{\partial\chi}{\partial t}=\hat{H}_2\chi, \qquad (35)$$

where χ is a wave functional depending on v and t, and the dependences of $\hat{H}_2$ on the background variables are understood as a dependence on t.

Let us now go one step further and quantize both the background and perturbations. When the background is also quantized, this procedure can also be implemented in the framework of the Bohm-de Broglie interpretation of quantum theory, where there is a definite notion of trajectories as well, the Bohmian trajectories. In order to do that, we first note that Eqs. (24) and (25) imply that

$$(\hat{H}_0^{(0)} + \hat{H}_2)\Psi = 0, \tag{36}$$

where

$$\hat{H}_0^{(0)} = -\frac{\hat{P}_\alpha^2}{2} + \frac{\hat{P}_\varphi^2}{2}, \tag{37}$$

$$\hat{H}_2 = \frac{1}{2}\int d^3x\left(\frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma}e^{4\hat{\alpha}}\hat{v}^{,i}\hat{v}_{,i}\right). \tag{38}$$

We write the wave functional Ψ as $\Psi = \exp(A_T + iS_T) \equiv R_T\exp(iS_T)$, where both A_T and S_T are real functionals. Inserting it in the Wheeler-DeWitt Eq. (36), the two real equations we obtain are

$$-\frac{\partial}{\partial\alpha}\left(R_T^2\frac{\partial S_T}{\partial\alpha}\right) + \frac{\partial}{\partial\varphi}\left(R_T^2\frac{\partial S_T}{\partial\varphi}\right) + \int\frac{d^3x}{\sqrt{\gamma}}\frac{\delta}{\delta v}\left(R_T^2\frac{\delta S_T}{\delta v}\right) = 0, \tag{39}$$

$$\begin{aligned}&-\frac{1}{2}\left(\frac{\partial S_T}{\partial\alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_T}{\partial\varphi}\right)^2 + \frac{1}{2}\int d^3x\left(\frac{1}{\sqrt{\gamma}}\left(\frac{\delta S_T}{\delta v}\right)^2 + \sqrt{\gamma}e^{4\alpha}v^{,i}v_{,i}\right)\\ &+\frac{1}{2R_T}\left(\frac{\partial^2 R_T}{\partial\alpha^2} - \frac{\partial^2 R_T}{\partial\varphi^2}\right) - \frac{1}{2}\int\frac{d^3x}{\sqrt{\gamma}}\frac{1}{R_T}\frac{\delta^2 R_T}{\delta v^2} = 0.\end{aligned} \tag{40}$$

These two equations correspond to Eqs. (4) and (5), respectively.

The Bohmian guidance relations are the same as in the classical case,

$$\dot{\alpha} = -P_\alpha = -\frac{\partial S_T}{\partial\alpha}, \qquad \dot{\varphi} = P_\varphi = \frac{\partial S_T}{\partial\varphi}, \qquad \dot{v} = \frac{1}{\sqrt{\gamma}}\pi = \frac{1}{\sqrt{\gamma}}\frac{\delta S_T}{\delta v}, \tag{41}$$

with the difference that the new S_T satisfies a Hamilton-Jacobi equation different from the classical one due to the presence of the quantum potential terms (the two last terms in Eq. (40)), which are responsible for the quantum effects.

We have again made the choice $N \propto e^{3\alpha}$. Whether this procedure is unambiguously independent on the choice of the lapse function is a delicate point. Indeed, in a general framework (the full superspace), the Bohmian evolution of three-geometries may not even form a four-geometry (a space-time) in the sense described in Refs. [22–25], although the theory remains consistent ([23,24]), and its geometrical properties depends on the choice of the lapse function. However, in the case of homogeneous spacelike hypersurfaces, a preferred foliation of space-time is selected, the one where the time direction is perpendicular to the Killing vectors of these hypersurfaces. In this case, once one has chosen this preferred foliation, one can prove that the residual ambiguity in the lapse function (which is now independent of space coordinates) is geometrically irrelevant for the Bohmian trajectories (see Ref. [10]). This is also true when linear perturbations are present, where the Hamiltonian constraints reduce to a single one, and the supermomentum constraint can be solved, as it was shown in Ref. [6]. Again, the lapse function is just a time function. In this case, the Bohmian quantum background trajectories can be obtained without geometrical ambiguities [10], and they can be used to induce a time dependence on the perturbation quantum state, as we will see.

Let us assume, as in the classical case, that we can split $A_T(\alpha, \varphi, v) = A_0(\alpha, \varphi) + A_2(\alpha, \varphi, v)$ implying that $R_T(\alpha, \varphi, v) = R_0(\alpha, \varphi)R_2(\alpha, \varphi, v)$ and $S_T(\alpha, \varphi, v) = S_0(\alpha, \varphi) + S_2(\alpha, \varphi, v)$, and that $A_2 \ll A_0$, $S_2 \ll S_0$, together with their derivatives with respect to the background variables. The approximate guidance relations are

$$\dot{\alpha} \approx -\frac{\partial S_0}{\partial\alpha}, \qquad \dot{\varphi} \approx \frac{\partial S_0}{\partial\varphi}, \tag{42}$$

and the zeroth order terms of Eqs. (39) and (40) read

$$-\frac{\partial}{\partial\alpha}\left(R_0^2\frac{\partial S_0}{\partial\alpha}\right) + \frac{\partial}{\partial\varphi}\left(R_0^2\frac{\partial S_0}{\partial\varphi}\right) \approx 0, \tag{43}$$

$$-\frac{1}{2}\left(\frac{\partial S_0}{\partial\alpha}\right)^2 + \frac{1}{2}\left(\frac{\partial S_0}{\partial\varphi}\right)^2 + \frac{1}{2R_0}\left(\frac{\partial^2 R_0}{\partial\alpha^2} - \frac{\partial^2 R_0}{\partial\varphi^2}\right) \approx 0. \tag{44}$$

which, again, correspond to Eqs. (4) and (5) for the background, respectively.

A solution (S_0, R_0) of Eqs. (43) and (44) yield a Bohmian quantum trajectory for the background through Eq. (42). If S_0 and R_0 are obtained from Eq. (10), then the Bohmian trajectories will be given by Eq. (14).

As in the classical case, once one obtains the Bohmian quantum trajectories $\alpha(t)$, $\varphi(t)$, the functionals $S_2(\alpha, \varphi, v)$, $A_2(\alpha, \varphi, v)$ become functionals of v and functions of t, $S_2(\alpha, \varphi, v) \to S_2(\alpha(t), \varphi(t), v) = \bar{S}_2(t, v)$, $A_2(\alpha, \varphi, v) \to A_2(\alpha(t), \varphi(t), v) = \bar{A}_2(t, v)$.

Defining $\chi(\alpha, \varphi, v) \equiv R_2(\alpha, \varphi, v)\exp(iS_2(\alpha, \varphi, v))$, writing it as

$$\chi(\alpha, \varphi, v) = \int d\lambda G(\lambda, v)F(\lambda, \alpha, \phi), \tag{45}$$

where F satisfies

$$\frac{1}{2}\left(\frac{\partial^2 F}{\partial\alpha^2} - \frac{\partial^2 F}{\partial\varphi^2}\right) + \frac{1}{R_0}\left(\frac{\partial R_0}{\partial\alpha}\frac{\partial F}{\partial\alpha} - \frac{\partial R_0}{\partial\varphi}\frac{\partial F}{\partial\varphi}\right) = 0, \tag{46}$$

and G is an arbitrary functional of v, which also depends on an integration constant λ, then the next-to-leading-order terms of Eqs. (39) and (40) read

$$\frac{\partial \bar{R}_2^2}{\partial t} + \int \frac{d^3x}{\sqrt{\gamma}} \frac{\delta}{\delta v}\left(\bar{R}_2^2 \frac{\delta \bar{S}_2}{\delta v} d^3x\right) = 0, \tag{47}$$

$$\frac{\partial \bar{S}_2}{\partial t} + \frac{1}{2}\int d^3x\left(\frac{1}{\sqrt{\gamma}}\left(\frac{\delta \bar{S}_2}{\delta v}\right)^2 + \sqrt{\gamma} e^{4\alpha(t)} v^{,i} v_{,i}\right) - \frac{1}{2}\int \frac{d^3x}{\bar{R}_2\sqrt{\gamma}} \frac{\delta^2 \bar{R}_2}{\delta v^2} = 0, \tag{48}$$

where $\bar{R}_2(t, v) \equiv \exp(\bar{A}_2(t, v))$. In order to obtain these equations we used that

$$-\left(\frac{\partial S_0}{\partial \alpha}\right)\left(\frac{\partial S_2}{\partial \alpha}\right) + \left(\frac{\partial S_0}{\partial \varphi}\right)\left(\frac{\partial S_2}{\partial \varphi}\right) = \frac{\partial \bar{S}_2}{\partial t}, \tag{49}$$

and the same for R_2 and $\bar{R}_2$.

These two equations can be grouped into a single Schrödinger equation

$$i\frac{\partial \bar{\chi}}{\partial t} = \hat{H}_2 \bar{\chi}, \tag{50}$$

where $\bar{\chi}(t, v) = \chi(\alpha(t), \varphi(t), v)$ is a wave functional depending on v and t, and, as before, the dependences of $\hat{H}_2$ on the background variables are understood as a dependence on t.

For the specific example of Sec. II, Eq. (10), one possible solution of Eq. (46) yields for χ through Eq. (45)

$$\chi(\alpha, \varphi, v) = \frac{1}{R(\alpha, \varphi)} \int d\lambda G(\lambda, v) \times \exp\left\{\frac{(\alpha + \varphi - d/h)^2}{2\lambda} + \frac{\lambda h^2(\alpha - \varphi - d/h)^2}{8}\right\}. \tag{51}$$

From solution (51), we can construct $\bar{\chi}(t, v) \equiv \chi(\alpha(t), \varphi(t), v)$ solution of Eq. (50). Note that, as G is an arbitrary functional of v and the real parameter λ, the functional $\bar{\chi}(t, v)$ constructed from (51) via $\bar{\chi}(t, v) \equiv \chi(\alpha(t), \varphi(t), v)$ is also an arbitrary functional of t and v (even though $\chi(\alpha, \varphi, v)$ in (51) is not arbitrary in α and φ).

During our procedure, we have supposed that the evolution of the background is independent of the perturbations. This no backreaction assumption is based on the fact that terms induced by the linear perturbations in the zeroth order Hamiltonian are negligible, which should be the case when one assumes that quantum perturbations are initially in a vacuum quantum state, as it is argued in Ref. [26]. We will come back to this point in the conclusion.

Once one obtains the quantum trajectories for the background variables, they can be used to define a time dependent unitary transformation for the perturbative sector. This unitary transformation takes the vector $|\chi\rangle$ into $|\xi\rangle = U|\chi\rangle$, i.e. $|\chi\rangle = U^{-1}|\xi\rangle$. With respect to this transformation the Hamiltonian is taken into $\hat{H}_2 \longrightarrow \hat{H}_{2U}$ with

$$i\frac{d}{dt}|\xi\rangle = \hat{H}_{2U}|\xi\rangle = \left(U\hat{H}_2 U^{-1} - iU\frac{d}{dt}U^{-1}\right)|\xi\rangle. \tag{52}$$

Let us define this unitary transformation by

$$U = e^{iA}e^{-iB} \tag{53}$$

with,

$$A = \frac{1}{2}\int d^3x \sqrt{\gamma} \frac{\dot{a}}{a^3} \hat{v}^2, \tag{54}$$

$$B = \frac{1}{2}\int d^3x (\hat{\pi}\,\hat{v} + \hat{v}\,\hat{\pi}) \log(a). \tag{55}$$

Remember that the time derivative, $\dot{a} = \frac{da}{dt}$, is taken with respect to the parametric time t related to the cosmic time τ by $d\tau = N dt \propto a^3 dt$. In these expressions, the scale factor $a = a(t)$ should be understood as a function of time, instead of an operator, since we suppose that the background quantum equations have already been solved. Thus, $a = a(t)$ should be taken as the Bohmian trajectory associated with equations $\hat{H}_0^{(0)}|\phi\rangle = 0$.

Naturally, the $\hat{\pi}$ and $\hat{v}$ operators do not commute with the unitary transformation. Using the following relations

$$e^{iA}\hat{v}e^{-iA} = \hat{v}, \qquad e^{iA}\hat{\pi}e^{-iA} = \hat{\pi} - \frac{\dot{a}}{a^3}\sqrt{\gamma}\hat{v}$$
$$e^{-iB}\hat{v}e^{iB} = a^{-1}\hat{v}, \qquad e^{-iB}\hat{\pi}e^{iB} = a\hat{\pi}.$$

We can calculate the transformed Hamiltonian as

$$\hat{H}_{2U} = \frac{a^2}{2}\int d^3x\left[\frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma}\hat{v}^{,i}\hat{v}_{,i} - \left(\frac{\ddot{a}}{a^5} - 2\frac{\dot{a}^2}{a^6}\right)\sqrt{\gamma}\hat{v}^2\right]. \tag{56}$$

Note that the unitary transformation U takes us back to the Mukhanov-Sasaki variable.

Recalling that $dt = a^{-2}d\eta$, where η is the conformal time, we have $\dot{a} = a^2 a'$ and $\ddot{a} = a^4 a'' + 2a^3 a'^2$, and the Hamiltonian can be recast as

$$\hat{H}_{2U} = \frac{a^2}{2}\int d^3x\left[\frac{\hat{\pi}^2}{\sqrt{\gamma}} + \sqrt{\gamma}\hat{v}^{,i}\hat{v}_{,i} - \frac{a''}{a}\sqrt{\gamma}\hat{v}^2\right]. \tag{57}$$

So far our analysis has been made in the Schrödinger picture but now it is convenient to describe the dynamics using the Heisenberg representation. The equations of motion for the Heisenberg operators are written as

$$\dot{\hat{v}} = -i[\hat{v}, \hat{H}_{2U}] = a^2\frac{\hat{\pi}}{\sqrt{\gamma}},$$
$$\dot{\hat{\pi}} = -i[\hat{\pi}, \hat{H}_{2U}] = a^2\sqrt{\gamma}\left(\hat{v}^{,i}_{,i} + \frac{a''}{a}\hat{v}\right).$$

Combining these two equations and changing to conformal time, we find the following equations for the operator modes of wave number k, v_k:

$$v_k'' + \left(k^2 - \frac{a''}{a}\right)v_k = 0. \tag{58}$$

This is the same equation of motion for the perturbations known in the literature in the absence of a scalar field potential [1]. The crucial point is that we have not used the background equations of motion. Thus we have shown that Eq. (58) is well defined, independently of the background dynamics, and it is correct even if we consider quantum background trajectories.

Note, however, that this result was obtained using a specific subclass of wave functionals which satisfies the extra condition Eq. (46). What are the physical assumptions behind this choice?

When one approaches the classical limit, where R_0 is a slowly varying function of α and φ, condition (46) reduces to

$$\frac{\partial^2 F}{\partial \alpha^2} - \frac{\partial^2 F}{\partial \varphi^2} \approx 0. \tag{59}$$

If Eq. (59) were not satisfied, one would not obtain anymore the usual Schrödinger equation for quantum perturbations in a classical background (which arises when R_0 is a slowly varying function of α and φ), due to extra terms in Eqs. (47) and (48): there would be corrections originated from some quantum entanglement between the background and the perturbations, even when the background is already classical, which would spoil the usual semiclassical approximation. This could be a viable possibility driven by a different type of wave functional than the one considered here, but it seems that our Universe is not so complicated. In fact, the observation that the simple semiclassical model without this sort of entanglement works well in the real Universe indicates something about the wave functional of the Universe [27].[1] In other words, the validity of the usual semiclassical approximation imposes Eq. (59).

When R_0 is not slowly varying and quantum effects on the background become important causing the bounce, the two last terms of condition (46) cannot be neglected. They would also induce extra terms in Eqs. (47) and (48), again originated from some quantum entanglement between the background and the perturbations, but now in the background quantum domain, and the final quantum Eq. (58) for the perturbations we obtained would not be valid around the bounce. In this case, there is no observation indicating which class of wave functionals one should take and our choice in this no man's land resides only on assumptions of simplicity: there is no quantum entanglement between the background and the perturbations in the entire history of the Universe. This is the physical hypothesis behind the choice of the specific class os wave functionals satisfying condition (46).

In the next section we will apply the above formalism implying Eq. (58) to the specific example described in Sec. II.

IV. APPLICATION OF THE FORMALISM

We will now use Eq. (58) to evaluate the spectral index of scalar perturbations in the quantum background described by Eq. (14). The potential $V \equiv a''/a$ reads

$$V \equiv \frac{a''}{a} = \frac{1}{a^4}\left[\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2\right] = \frac{\alpha_0 h^2 \exp(ht)[1 - \alpha_0 \exp(ht)]}{a^4}. \tag{60}$$

Defining $u_k \equiv v_k/a$, Eq. (58) in terms of the t variable can be written as (from now on we will omit the index k),

$$\ddot u + k^2 a^4 u = 0. \tag{61}$$

When $ht \ll 0$, we can approximate $a \approx \exp(d/h) \times [1 + \alpha_0 \exp(ht)]$, and the general solution reads

$$u = A_+(k) J_\nu(z) - A_-(k) J_{-\nu}(z), \tag{62}$$

where J is the Bessel function of the first type, $\nu = i2k\exp(2d/h)/h$ and $z = 4\alpha_0^{1/2} k \exp(2d/h + ht/2)/h$. At $t \to -\infty$, when the scale factor becomes constant and space-time is flat, one can impose vacuum initial conditions

$$v_{\rm ini} = \frac{e^{ik\eta}}{\sqrt{k}}, \tag{63}$$

which implies that $A_+(k) = 0$, and $A_-(k) \propto k^{-1/2}\exp[i2k\ln(k)\exp(2d/h)/h]$. Hence, v in this region reads

$$v_I = aA_-(k) J_{-\nu}(z). \tag{64}$$

The solution can also be expanded in powers of k^2 according to the formal solution (see Ref. [1])

$$\frac{v}{a} \simeq A_1(k)\left[1 - k^2 \int^t \frac{d\bar\eta}{a^2(\bar\eta)} \int^{\bar\eta} a^2(\bar{\bar\eta}) d\bar{\bar\eta}\right] + A_2(k) \times \left[\int^\eta \frac{d\bar\eta}{a^2} - k^2 \int^\eta \frac{d\bar\eta}{a^2} \int^{\bar\eta} a^2 d\bar{\bar\eta} \int^{\bar{\bar\eta}} \frac{d\bar{\bar{\bar\eta}}}{a^2}\right] + \ldots, \tag{65}$$

When the mode is deep inside the potential, $k^2 \ll V$, we can neglect the k^2 terms yielding

[1]In these references, it is pointed out how the features of our Universe we take for granted (classicality, separability) impose severe restrictions on the initial wave-function of the Universe. In fact, our Universe could have been highly nonclassical, completely entangled, even when it is large, depending on the features of this initial wave solution.

$$v_{II} \approx a\left[A_1(k) + A_2(k)\int^{\eta}\frac{d\bar{\eta}}{a^2}\right] = a[A_1(k) + A_2(k)t]. \tag{66}$$

We can now perform the matching of v_I with v_{II} in order to calculate $A_1(k)$ and $A_2(k)$. As we are interested on large scales, $k \ll 1$, this matching can still be made when $ht \ll 0$. In this region one has $V \approx \alpha_0 h^2 \exp(ht - 4d/h)$, yielding the matching time

$$ht_M = \ln\left(\frac{k^2\exp(4d/h)}{\alpha_0 h^2}\right). \tag{67}$$

Note that the potential crossing condition relating the wave number k and the time t_M of the crossing is logarithmic. In fact, since in this region the scale factor is almost constant, the wave number is also logarithmically related to the conformal time. This dependence is drastically different from the slow roll scenario, where the conformal time of potential crossing is inversely proportional to the wave number, $k \propto 1/\eta_M$.

Performing the matching at this time and taking the leading order term in k, one obtains that

$$A_1(k) = k^{-1}A_2(k) \propto k^{-1/2}\exp[i6k\ln(k)\exp(2d/h)/h]. \tag{68}$$

Note that solution (65) is valid everywhere, hence we can use it in the period when the scale factor evolution becomes classical. During this period, unless for some fine tuning, the mode is also deep inside the potential and one can use Eq. (66) to calculate the Bardeen potential Φ through the classical equation [1]

$$\Phi = -\frac{(\epsilon + p)^{1/2}z}{k^2}\left(\frac{v}{z}\right)', \tag{69}$$

where $z \equiv a^2(\epsilon + p)^{1/2}/\mathcal{H}$. For the case of a scalar field without potential (stiff matter), $z \propto a$, yielding

$$\Phi \propto A_1(k) + \frac{A_2(k)}{k^2a^4}, \tag{70}$$

one constant and one decaying mode, as usual. The transition to radiation dominated and matter dominated phases may alter the amplitudes but not the spectrum. The power spectrum

$$\mathcal{P}_\Phi \equiv \frac{2k^3}{\pi^2}|\Phi|^2 \propto k^{n_s-1}, \tag{71}$$

yields for the spectral index, from the value of $A_1(k)$ in the constant mode given in Eq. (68), the value $n_s = 3$, contrary to observational results [9]. This power law dependence was checked numerically as can be seen by Fig. 2. Hence, the model cannot describe the primordial era of our Universe.

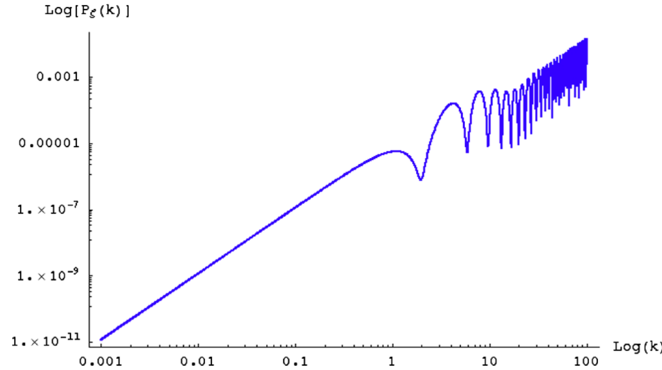

FIG. 2 (color online). The power spectrum $\mathcal{P}_\Phi$ calculated numerically. The numerical integration was carried out with $h = d = 3\times 10^2$ and $\alpha_0 = 1$. Since this is a log-log plot, one can immediately check that $\mathcal{P}_\Phi \propto k^2$ for small k.

V. CONCLUSION

In this paper we were able to obtain the simple equation for linear scalar perturbations of Ref. [1] for the case of a scalar field without potential. The simplification procedure was carried out without ever using any classical background equation. Instead, by a series of canonical transformations and redefinitions of the lapse function we are able to put the Hamiltonian in a form susceptible to quantization.

However, contrary to the perfect fluid case, the scalar field minisuperspace model has no natural way to define a time variable since its Hamiltonian constraint does not contain a linear term in the momenta. Nevertheless, if one assumes there is no backreaction, we have shown how to bypass this problem using the quantum background Bohmian trajectories. The quantum background dynamics in the Bohm-de Broglie interpretation naturally provides an evolutionary time to the perturbative sector, similarly to what is done at the semiclassical level through the classical background trajectories [20].

These perturbation equations were then used to calculate the spectrum index n_s of the background model of Ref. [8] yielding $n_s = 3$, incompatible with observations [9] ($n_s \approx 1$). This result is intimately related to the logarithmically dependence of the wave number to the potential crossing time, see Eq. (67). As a consequence, the model should be discarded. This is an example of an inflationary model without (almost) scale invariant scalar perturbations.

The no backreaction hypothesis we have used was justified through the assumption that the perturbations are in a quantum vacuum state initially [26]. One could verify the consistency of such hypothesis by checking whether the perturbations calculated under this assumption never departs the linear regime in the region where the background is influenced by quantum effects. This check was done in other frameworks (see Ref. [6]), where self-consistency was verified. This self-consistency check, however, was not implemented here because the model studied in Sec. IV does not present a scale invariant spectrum for long-wavelength perturbations, and the model should be discarded without the need of calculating the amplitude of perturbations.

We have also assumed that there is no quantum entanglement in such a way that the background disturbs the quantum evolution of the perturbations. This is a restriction on the possible wave functionals of the Universe, which should then satisfy condition (46). It should be interesting to investigate situations where entanglement is allowed when the background is in the quantum regime, which would imply modifications of Eq. (58) at the bounce. In this case, condition (46) reduces to condition (59) (no entanglement when the background becomes classical).

Some future investigations should be to apply the formalism to bouncing models obtained in the framework of quantum cosmology with scalar fields without potential described in Ref. [28] in order to evaluate their spectral index. We will also study the possibility to generalize the simplification of the perturbation equations obtained here to the case of scalar fields with an arbitrary potential term.

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[1] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[2] Patrick Peter, Emanuel J. C. Pinho, and Nelson Pinto-Neto, J. Cosmol. Astropart. Phys. 07 (2005) 014; arXiv:hep-th/0509232.

[3] E. J. C. Pinho and N. Pinto-Neto, Phys. Rev. D **76**, 023506 (2007).

[4] E. J. C. Pinho, PhD Thesis, CBPFIndex, Dezembro de, 2006.

[5] Patrick Peter, Emanuel J. C. Pinho, and Nelson Pinto-Neto, Phys. Rev. D **73**, 104017 (2006).

[6] Patrick Peter, Emanuel J. C. Pinho, and Nelson Pinto-Neto, Phys. Rev. D **75**, 023516 (2007).

[7] D. Langlois, Classical Quantum Gravity **11**, 389 (1994).

[8] Felipe T. Falciano, Nelson Pinto-Neto, and E. Sergio Santini, Phys. Rev. D **76**, 083521 (2007).

[9] E. Komatsu *et al.*, arXiv:astro-ph/0803.0547.

[10] J. A. de Barros and N. Pinto-Neto, Int. J. Mod. Phys. D **7**, 201 (1998).

[11] G. Veneziano, Phys. Lett. B **265**, 287 (1991).

[12] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).

[13] J. E. Lidsey, D. Wands, and E. J. Copeland, Phys. Rep. **337**, 343 (2000).

[14] G. Veneziano, in *The Primordial Universe*, edited by P. Binétruy *et al.*, Les Houches, session LXXI (EDP Science & Springer, Paris, 2000).

[15] David J. Mulryne, Reza Tavakol, James E. Lidsey, and George F. R. Ellis, Phys. Rev. D **71**, 123512 (2005).

[16] N. Pinto-Neto, E. Sergio Santini, and F. T. Falciano Phys. Lett. A **344**, 131 (2005).

[17] J. Acacio de Barros, N. Pinto-Neto, and M. A. Sagioro-Leal, Phys. Lett. A **241**, 229 (1998).

[18] F. G. Alvarenga, J. C. Fabris, N. A. Lemos, and G. A. Monerat, Gen. Relativ. Gravit. **34**, 651 (2002).

[19] N. Pinto-Neto and E. Sergio Santini, Phys. Lett. A **315**, 36 (2003).

[20] T. Banks, Nucl. Phys. **B249**, 332 (1985); T. P. Singh and T. Padmanabhan, Ann. Phys. (N.Y.) **196**, 296 (1989); D. Giulini and C. Kiefer Classical Quantum Gravity **12**, 403 (1995).

[21] B. Hatfield, *Quantum Field Theory of Point Particles and Strings*, Frontiers in Physics Vol. 75 (Addison-Wesley Publishing Company, Redwood City, CA, 1992).

[22] N. Pinto-Neto and E. Sergio Santini, Phys. Rev. D **59**, 123517 (1999).

[23] E. Sergio Santini, PhD Thesis, CBPF-Rio de Janeiro [arXiv:gr-qc/0005092].

[24] N. Pinto-Neto and E. Sergio Santini, Gen. Relativ. Gravit. **34**, 505 (2002).

[25] Shtanov, Phys. Rev. D **54**, 2564 (1996).

[26] J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).

[27] J. J. Halliwell and J. B. Hartle, Phys. Rev. D **41**, 1815 (1990); J. B. Hartle in *Proceedings of the 11th Nishinomiya-Yukawa Symposium*, edited by K. Kikkawa, H. Kunitomo, and H. Ohtsubo (World Scientific, Singapore, 1998).

[28] R. Colistete, J. C. Fabris, and N. Pinto-Neto, Phys. Rev. D **62**, 083507 (2000).