

# Scale-free Networks on Lattices

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We suggest a method for embedding scale-free networks, with degree distribution  $P(k) \sim k^{-\lambda}$ , in regular Euclidean lattices. The embedding is driven by a natural constraint of minimization of the total length of the links in the system. We find that all networks with  $\lambda > 2$  can be successfully embedded up to an (Euclidean) distance  $\xi$  which can be made as large as desired upon the changing of an external parameter. Clusters of successive chemical shells are found to be compact (the fractal dimension is  $d_f = d$ ), while the dimension of the shortest path between any two sites is smaller than one:  $d_{min} = \frac{\lambda-2}{\lambda-1-1/d}$ , contrary to all other known examples of fractals and disordered lattices.

Many social, biological, and communication systems can be properly described by complex networks whose nodes represent individuals or organizations and links mimic the interactions among them [1]. An important class of complex networks are the *scale-free* networks, which exhibit a power-law connectivity distribution. Examples of scale-free networks include the Internet [2,3], WWW [4,5], metabolic [6] and cellular networks [7]. Most of the work done on scale free networks concerns off-lattice systems (graphs) where the Euclidean distance between nodes is irrelevant. However, real-life networks are often embedded in Euclidean space (e.g., the Internet is embedded in the two-dimensional network of routers, neuronal networks are embedded in a three-dimensional brain, etc.). Indeed, in the case of the Internet, indications for the relevance of embedding space is given in [8].

In this Letter we develop a method for generating scale-free networks on Euclidean lattices and study some of its properties. As a guiding principle we impose the natural restriction that the total length of links in the system be minimal.

Our model is defined as follows. To each site of a  $d$ -dimensional lattice, of size  $R$ , and with periodic boundary conditions, we assign a random connectivity  $k$  taken from the scale-free distribution

$$P(k) = Ck^{-\lambda}, \quad m < k < K, \quad (1)$$

where the normalization constant  $C \approx (\lambda - 1)m^{\lambda-1}$  (for  $K$  large) [9]. We then select a site at random and connect it to its closest neighbors until its (previously assigned) connectivity  $k$  is realized, or until all sites up to a distance

$$r(k) = Ak^{1/d} \quad (2)$$

have been explored. (Links to some of the neighboring sites might prove impossible, in case that the connectivity quota of the target site is already filled.) This process is repeated for all sites of the lattice. We show that following this method networks with  $\lambda > 2$  can be successfully embedded up to an (Euclidean) distance  $\xi$  which can be

made as large as desired upon the changing of the external parameter  $A$ .

Suppose that one attempts to embed a scale-free network, by the above recipe, in an *infinite* lattice,  $R \rightarrow \infty$ . Sites with a connectivity larger than a certain cutoff  $k_c(A)$  cannot be realized, because of saturation of the surrounding sites. Consider the number of links  $n(r)$  entering a generic site from a surrounding neighborhood of radius  $r$ . Sites at distance  $r'$  are linked to the origin with probability  $P(k' > (r'/A)^d)$ :

$$P\left(k' > \left(\frac{r'}{A}\right)^d\right) = C \int_{\left(\frac{r'}{A}\right)^d}^{\infty} k^{-\lambda} dk \quad (3)$$

$$\sim \begin{cases} 1 & r' < A. \\ \left(\frac{r'}{A}\right)^{d(1-\lambda)} & r' > A. \end{cases}$$

Hence

$$n(r) \sim \int_0^r dr' r'^{d-1} P\left(k' > \left(\frac{r'}{A}\right)^d\right) \quad (4)$$

$$\sim \frac{\lambda - 1}{d(\lambda - 2)} A^d - \frac{A^{d(\lambda-1)}}{d(\lambda - 2)} r^{d(2-\lambda)}.$$

The cutoff connectivity is then

$$k_c = \lim_{r \rightarrow \infty} n(r) \sim \frac{1}{\lambda - 2} A^d. \quad (5a)$$

The cutoff connectivity implies a cutoff length

$$\xi = r(k_c) \sim (\lambda - 2)^{-1/d} A^{2/d}. \quad (5b)$$

The embedded network is *scale-free* up to distances  $r < \xi$ , and repeats itself (statistically) for  $r > \xi$ , similar to the infinite percolation cluster above criticality: The infinite cluster in percolation is *fractal* up to the coherence length  $\xi$  and repeats thereafter [10–12].

When the lattice is finite,  $R < \infty$ , the number of sites is finite,  $N \sim R^d$ , which imposes a maximum connectivity [13,14]

$$K \sim mN^{1/(\lambda-1)} \sim R^{d/(\lambda-1)}. \quad (6a)$$

This implies a finite-size cutoff length

$$r_{max} = r(K) \sim AR^{1/(\lambda-1)}. \quad (6b)$$

The interplay between the three length scales,  $R$ ,  $\xi$ ,  $r_{max}$ , determines the nature of the network. If the lattice is finite, then the maximal connectivity is  $k_{max} = K$  only if  $r_{max} < \xi$ . Otherwise ( $r_{max} > \xi$ ) the lattice repeats itself at length scales larger than  $\xi$ . As long as  $\min(r_{max}, \xi) \ll R$ , the finite size of the lattice imposes no serious restrictions. Otherwise ( $\min(r_{max}, \xi) \gtrsim R$ ) finite-size effects become important. We emphasize that in all cases the degree distribution (up to the cutoff) is scale-free.

In Fig. 1(a) we show typical networks that result from our embedding method, for  $\lambda = 2.5$  and  $5$  in two-dimensional lattices (in this Letter, we limit our numerical results to  $d = 2$ ). The larger  $\lambda$  is the more closely the network resembles the embedding lattice, because longer links are rare [15]. In Fig. 1(b) we show the same networks as in part (a) where successive chemical shells are depicted in different shades. Chemical shell  $l$  consists of all sites at minimal distance (minimal number of connecting links)  $l$  from a given site. For our choice of parameters,  $\lambda = 5$  happens to fall in the region of  $\xi > r_{max}$ , while for  $\lambda = 2.5$ ,  $\xi < r_{max}$ . In the latter case we clearly see (Fig.1(b),  $\lambda = 2.5$ ) the (statistical) repetition of the network beyond the length scale  $\xi$ .

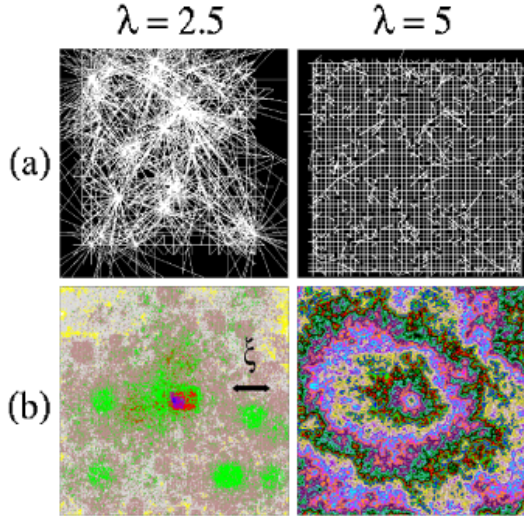


FIG. 1. Spatial structure of connectivity network. (a) shown is the typical map of links for a system of  $50 \times 50$  sites generated from connectivity distributions with  $\lambda = 2.5$  and  $\lambda = 5$ . (b) shown are shells of equidistant sites to the central one in a lattice of  $300 \times 300$  sites. Note that for  $\lambda = 5$ , shells are concentric and continuous fractals; but for  $\lambda = 2.5$ , shells are broken.

The degree distribution resulting from our embedding method is illustrated in Fig. 2. In Fig. 2(a),  $\xi < r_{max}$  and

the distribution terminates at the cutoff  $k_c$ . The scale-free distribution is altered slightly, for  $k < k_c$ , due to saturation effects, but the overall trend is highly consistent with the original power-law. The scaling in the inset confirms that  $k_c \sim A^d$ . In Fig. 2(b),  $\xi > r_{max}$  and the cutoff  $K$  in the distribution results from the finite number of sites in the system. The scaling in the inset in Fig. 2(b) confirms the known relation  $K \sim mR^{d/(\lambda-1)}$  [13,14].

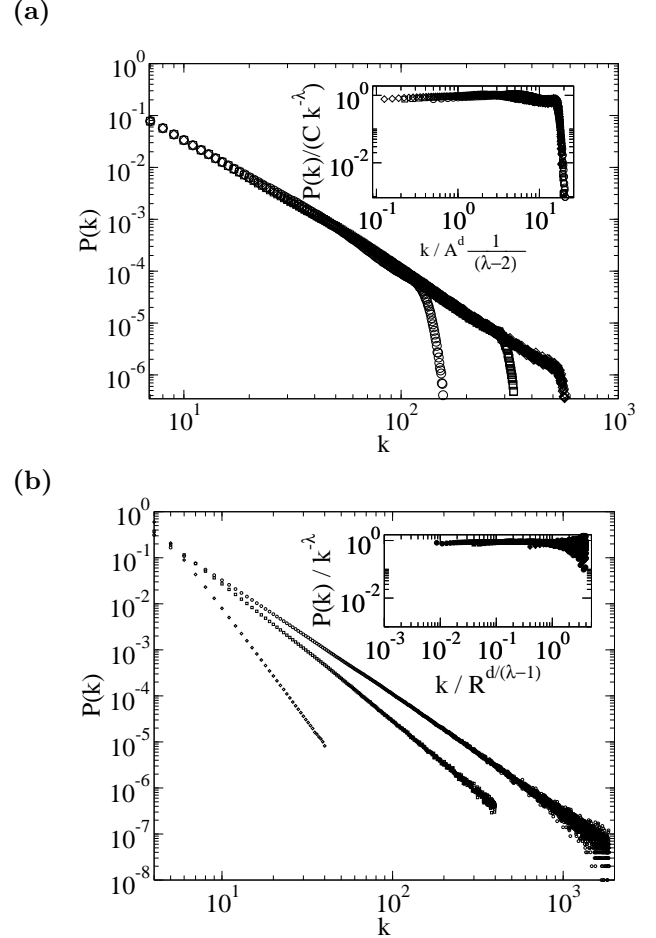


FIG. 2. (a) The resulting connectivity distribution obtained from simulations performed on two dimensional systems of size  $R = 400$ ,  $\lambda = 2.5$  and for several values of  $A$ : (circles)  $A=2$ , (squares)  $A=3$  and (diamonds)  $A=4$ ; they all end at a cutoff  $k_c(A)$ . For this case  $r_{max} > \xi$ . In the inset we show scaling collapse using same data. The threshold takes place at  $k_c \sim A^d \frac{1}{\lambda-2}$  and confirms the validity of our theoretical estimations. (b) Power law distribution of site connectivity in the network is shown for  $R = 100$ ,  $A = 10$  and for different values of  $\lambda$ :  $\lambda = 2.5$  (circles),  $3.0$  (squares), and  $5.0$  (diamonds). Note that in all cases the distribution achieves its (natural) cutoff  $K$ . In the inset we show the corresponding collapse supporting  $K \sim R^{\frac{d}{\lambda-1}}$ . For this case,  $r_{max} < \xi$ .

The different regimes are summarized in Fig. 3.

$$\sim \begin{cases} r^d & r < A, \\ c(\lambda)Ar^{d-1} & r > A, \end{cases}$$

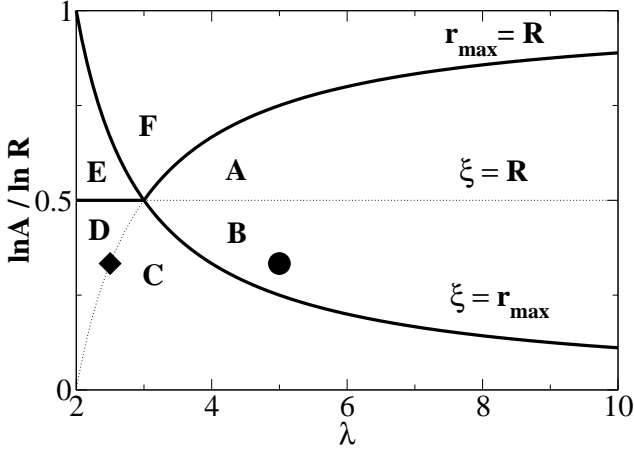


FIG. 3. This diagram shows the six regions where different behavior of the network is found: for region A:  $r_{max} < R < \xi$ , B:  $r_{max} < \xi < R$ , C:  $\xi < r_{max} < R$ , D:  $\xi < R < r_{max}$ , E:  $R < \xi < r_{max}$ , F:  $R < r_{max} < \xi$ . The diagram can be mapped into only four regions where the cutoff  $k_c$  and where size effect  $K$  are expected. A and B: no cutoff and no size effect; C and D: cutoff and no size effect; E: cutoff and size effect; F: no cutoff but size effect. The two symbols indicate the parameters corresponding to Fig. 1b, (full diamond)  $\lambda = 2.5$  and (full circle)  $\lambda = 5$ .

We now address the geometrical properties of the networks, arising from their embedding in Euclidean space. To this aim, it is useful to consider the spatial arrangement of the networks as measured both in an Euclidean metric and in *chemical space*. The chemical distance  $l$  between any two sites is the length of the minimal path between them (*minimal* number of links). Thus if the distance between the two sites is  $r$ , then  $l \sim r^{d_{min}}$  defines the minimal length exponent  $d_{min}$ . We will see that  $d_{min} < 1$  (for  $d > 1$ ), contrary to all naturally occurring fractals and disordered media. Sites at chemical distance  $l$  from a given site constitute its  $l$ -th chemical shell. The number of (connected) sites within radius  $r$  scales as  $m(r) \sim r^{d_f}$ , defining the fractal dimension  $d_f$ . Likewise, the number of (connected) sites within chemical radius  $l$  scales as  $m(l) \sim l^{d_l}$ , which defines the fractal dimension  $d_l$  in chemical space. The two fractal dimension are related:  $d_{min} = d_l/d_f$  [10–12].

To study  $d_f$ , we compute the perimeter  $S(r)$ , the number of sites that connect the interior cluster of a region of radius  $r$  to sites outside. The fractal dimension then follows from the scaling relation  $S(r) \sim r^{d_f-1}$ . We focus on the regime  $\xi > r_{max}$ . Consider a shell  $dr'$ , of radius  $r'$ . A site of connectivity  $k'$  within the shell is connected to the outside (to a distance larger than  $r - r'$ ) with probability  $P(k' > (\frac{r-r'}{A})^d)$ , eq. (3). Thus,

$$S(r) = \int_0^r dr' r'^{d-1} P(k' > (\frac{r-r'}{A})^d) \quad (7)$$

where  $c(\lambda) \sim 1 + 1/[d(\lambda - 1) + 1]$ . In other words, the network is compact,  $d_f = d$  at large distances  $r > A$ , and super-compact,  $d_f = d + 1$ , at  $r < A$ . Results for  $d_f$  are presented in Fig. 4 and are in good agreement with Eq. (7). The slight slope observed for  $r > A$  is due to analytical corrections, of order  $r^{-1}$ , to the scaling  $S(r) \sim r^{d-1}$ , and can be obtained from a more careful analysis of Eq. (7).

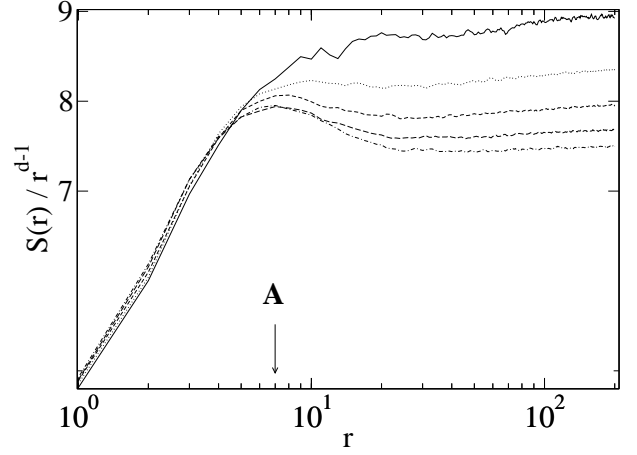


FIG. 4. Plot of scaled perimeter as a function of the Euclidean distance from the central site, for several values of  $\lambda$ :  $\lambda = 3.0$  (top), 3.25, 3.5, 3.75 and 4.0 (bottom). The simulations were performed with  $A = 7$ . Note that the position where the curves split,  $r \simeq A$ , is consistent with our analytical results (Eq. (7)). Also, the asymptotic values shown for large  $r$  are consistent with  $c(\lambda)A$ .

In order to compute  $d_{min}$  (or  $d_l$ ), we regard the chemical shells as being roughly smooth, at least in the regime  $\xi > r_{max}$ , as suggested by Fig. 1b ( $\lambda = 5$ ). Let the width of shell  $l$  be  $\Delta r(l)$ , then

$$l = \int dl = \int \frac{dr}{\Delta r(l)} \sim r^{d_{min}}, \quad (8)$$

since  $\Delta l = 1$ . The number of sites in shell  $l$ ,  $N(l)$ , is, on the one hand,  $N(l) \sim r(l)^{d-1} \Delta r(l)$ . On the other hand, since the maximal connectivity in shell  $l$  is  $K(l) \sim N(l)^{1/(\lambda-1)}$ , the thickness of shell  $(l + 1)$  is  $\Delta r(l + 1)$  which is determined by the length of the largest link to the next shell i.e.,  $r[K(l)]$ , and thus,  $\Delta r(l + 1) \sim r[K(l)] \sim AK(l)^{1/d}$ . Assuming (for large  $l$ ) that  $\Delta r(l + 1) \sim \Delta r(l)$ , we obtain

$$\Delta r(l) \sim r^{\frac{d-1}{d(\lambda-1)-1}}. \quad (9)$$

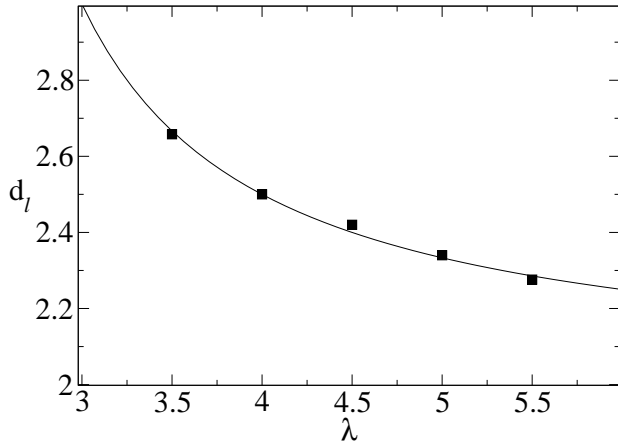
Using this expression in (8), yields

$$d_{min} = \frac{\lambda - 2}{\lambda - 1 - 1/d}. \quad (10)$$

Thus, above  $d = 1$ , the dimensions  $d_{min}$  and  $d_l = d_{min}d_f$  are anomalous for all values of  $\lambda$ .

In Fig. 5a we plot  $d_l$  as measured from simulations, and compared with the analytical result Eq. (10). The scaling suggested in Fig. 5b,  $N(l) \sim l^{d_l-1}\Phi(l^{d_l}/R^d)$ , is valid only for  $\xi > r_{max}$ . For  $R \rightarrow \infty$ , we expect that the network is scale-free up to length scale  $\xi$  and the analogous scaling will be  $N(l) \sim l^{d_l-1}\Psi(l^{d_l}/\xi^d)$ , where  $\Psi(x \gg 1) \sim x^{(d-d_l)/d_l}$ .

(a)



(b)

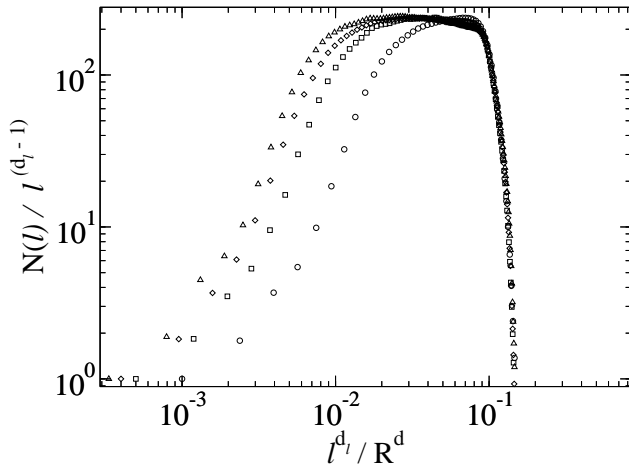


FIG. 5. (a) The chemical dimension  $d_l$  as a function of  $\lambda$ . Note the good agreement between theoretical estimations (continuous line) and simulations results (full squares). (b) The shape of the  $\Phi(l^{d_l}/R^d)$  scaling function is shown for  $\lambda = 4$  and several lattice sizes:  $R=1000$  (circle),  $2000$  (square),  $2500$  (diamond) and  $3000$  (triangle).

In summary, we propose a method for embedding scale-free networks in Euclidean lattices. The method is based on a natural principle of minimizing the total length of links in the system. This principle enables us to embed the scale-free in Euclidean space without additional external exponents such as assumed by Manna and Sen [16] and Xulvi-Brunet and Sokolov [17]. We have shown that while the fractal dimension  $d_f$  of the network is the same as the Euclidean dimension, the chemical dimension  $d_l > d_f$  for all values of  $\lambda$ , yielding  $d_{min} < 1$  for all  $\lambda$  and  $d > 1$ .

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