Scale free properties of random k-trees

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December 27, 2009

Abstract

Scale free graphs have attracted attention as their non-uniform structure that can be used as a model for many social networks including the WWW and the Internet. In this paper, we propose a simple random model for generating scale free k-trees. For any fixed integer k, a k-tree consists of a generalized tree parameterized by k, and is one of the basic notions in the area of graph minors. Our model is quite simple and natural; it first picks a maximal clique of size $k + 1$ uniformly at random, it then picks k vertices in the clique uniformly at random, and adds a new vertex incident to the k vertices. That is, the model only makes uniform random choices twice per vertex. Then (asymptotically) the distribution of vertex degree in the resultant k-tree follows a power law with exponent $2 + 1/k$, the k-tree has a large clustering coefficient, and the diameter is small. Moreover, our experimental results indicate that the resultant k-trees have extremely small diameter, proportional to $o(\log n)$, where n is the number of vertices in the k-tree, and the $o(1)$ term is a function of k.

Key words: scale free graph, small world network, clustering coefficient, k-tree, Apollonian network.

1 Introduction

Small world networks are the focus of recent interest because of their potential as models for interaction networks of complex systems in real world since early

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works by Watts & Strogatz [12] and Barabási & Albert [1]. Connected graphs with a power law degree sequence (SF), high clustering coefficient (CC) and low diameter (SW) are said to be small world networks or scale free networks (see, e.g., [10]). In more detail, these properties are as follows:

(SF) The degree distribution of G follows a power law distribution. That is, the number of vertices of degree i is proportional to i^α for some fixed α. It is known that α is between 2 and 3 in real social networks.

(CC) Two neighbors of any node of G are also likely to joined by an edge. More precisely, the clustering coefficient $CC(v)$ at v is defined as follows:

$$CC(v) = \frac{|\{u \sim w : u, w \in N(v)\}|}{\binom{d(v)}{2}},$$

where $u \sim w$ means that they are joined by an edge. The clustering coefficient $CC(G)$ of the graph G is the average clustering coefficient $CC(v)$ for all vertices v in G.

(SW) Any two nodes of G are connected by a relatively short path.

The possibility of generating small world networks using discrete random graph processes has been studied by many authors and in many contexts. The study of such processes dates back at least, to Yule [14] in 1924. Many models of such process exist. For details see, for example, the surveys [5, 9] and the monograph [2]. Interest in such models in computer science follows from the work of Barabási and Albert [1] who observed a power law degree sequence for a subgraph of the World Wide Web, and of Faloutsos, Faloutsos and Faloutsos [7] who observed power law behavior for the internet graph. Small world networks have many of the properties required by peer-to-peer (P2P) networks. P2P networks are by nature decentralized, and the possibility to structure such networks randomly is attractive.

The results of this paper are as follows. In Theorem 1 we obtain the precise expressions for the expected number of vertices of degree i for any feasible i as a function of k; and give the associated power law. We also obtain a precise estimate of the expected clustering coefficient, as a function of k. This allows a value of k to be chosen to give the required clustering coefficient. The precise dependence of the diameter on the value of k is unknown. We give an experimental study of this dependence.

We assume that the reader is familiar with the notion of probability and graph theory. In this paper we prove Theorem 1, briefly discuss Theorem 2 and give experimental results supporting Theorem 1 and our hypothesis regarding diameter.

We believe the random k-tree model analyzed in this paper is interesting for several reasons. The model satisfies all three small-world properties (SF), (CC), and (SW). Many random processes have properties (SF) and (SW), and some have (CC) and (SF), for example see the discussion below for [11]. Few models, except the one we propose, enjoy all three properties. In real life, a network based on k-trees (for large k) is particularly attractive as it is locally highly connected, and thus robust under edge and vertex failure. Moreover, simulations suggest that the diameter is $O(\log n/\log k)$ where n is the network size. The model is easy to generate, and our simulations (see Section 5) suggest that the properties of the model are achieved by quite small instances. The model can easily be extended to give a formal analysis of a random instance of a packing process known as an Apollonian network. Previously only a mean field analysis of random Apollonian networks has been made [8, 15].

Recently, Shigezumi et al. [11] also proposed a model of scale free graphs which satisfies the two properties (SF) and (CC) with high probability. Their model, based on time sequential data, the *scale free interval graph*, employs interval graphs as basic graphs. A graph is an interval graph if and only if there is a one-to-one mapping between vertices and intervals such that two vertices are joined by an edge if and only if the corresponding intervals share a common point. In their model, each vertex in the graph corresponds to a time period, and its lifespan is determined by a simple rule: longer life tends to survive in the next generation.

2 Random k-trees, model and results

In the area of graph algorithms, k-trees form a well known graph class that generalizes trees and plays an important role in graph minor area (see [3, 4] for further details). There are several equivalent definition of k-trees, and we employ one of them as follows; for any fixed positive integer k,
(0) a complete graph K_k of k vertices is a k-tree,
(1) for a k-tree G of n vertices, a new k-tree G' of $n+1$ vertices is obtained by adding a new vertex v incident to a clique of size k in G.
We note that a complete graph K_{k+1} of $k+1$ vertices is a k-tree, which is obtained by adding a vertex to K_k.

For each time $t = 1, 2, \ldots$, our model is an algorithm that generates a sequence of k-trees of $k + t$ vertices as follows.

Algorithm 1: Generation of k-trees

Input : Positive integer k.
Output: A series of k-trees $G_k(1), G_k(2), \ldots$.
begin
    $t = 1$; let $G_k(t)$ be K_{k+1}; output $G_k(t)$;
    for $t = 2, 3, \ldots$ **do**
        pick $D_t = K_{k+1}$ from $G_k(t-1)$ uniformly at random with probability $\frac{1}{t-1}$;
        pick $f_t = K_k$ from D_t uniformly at random with probability $\frac{1}{k+1}$;
        let $G_k(t)$ be the graph obtained from $G_k(t-1)$ by adding a new vertex v_t incident to every vertex in f_t;
        output $G_k(t)$;
    end
end

By the definition, it is clear that $G_k(t)$ is a k-tree of $k + t$ vertices. We remark that the algorithm only makes two uniform random choices at each step t.

Let $\boldsymbol{X}(t)$ be a random variable, and $X(t) = \mathbf{E}\boldsymbol{X}(t)$ be its expectation, then $\lim_{t\to\infty} X(t)/t$ is the *limiting expected proportion* of $\boldsymbol{X}(t)$. The *limiting expected clustering coefficient* $c(k)$ is defined by

$$c(k) = \lim_{t\to\infty} \mathbf{E}CC(G_k(t)).$$

Our first theorem states that the simple combination of two uniform random choices makes a scale free k-tree with properties (SF) and (CC).

Theorem 1 *Let $k \geq 2$. For a graph $G_k(t)$, we denote by n_i the number of vertices of degree i. Then the graph $G_k(t)$ has the following properties.*

1. *The limiting expected proportion n_i of vertices of degree $i = k + \ell - 1$ is given by*

$$n_{k+\ell-1} = \frac{(\ell-1)!k^{\ell-1}(k+1)}{\prod_{j=1\ldots\ell}((j+1)k+1)}. \qquad (1)$$

   This expression has power law asymptotic

$$n_i \propto i^{-(2+\frac{1}{k})}. \qquad (2)$$

2. *The limiting expected clustering coefficient* $c(k)$ *is given by*

$$c(k) = \sum_{\ell \geq 1} \frac{\binom{k}{2} + (k-1)(\ell-1)}{\binom{(k-1)+\ell}{2}} \frac{(\ell-1)!k^{\ell-1}(k+1)}{\prod_{j=1\ldots\ell}((j+1)k+1)}. \tag{3}$$

We have $c(k) \geq 1/2$ *for* $k \geq 2$*, and* $c(k) \to 1$ *if* $k \to \infty$*.*

We give a short combinatorial proof of this theorem, and also the following theorem for the finite process $G_k(t)$, which states that our model also has the small world property (SW). This is an advantage compared to the scale free interval graphs of [11]; their model generates scale free interval graphs of n vertices with diameter $\Theta(n)$.

We say that a sequence of events $\mathcal{E}_t$ occurs *with high probability* (**whp**) if $\lim_{t\to\infty} \Pr(\mathcal{E}_t) = 1$.

Theorem 2 *The following properties hold* **whp**

1. *Let* $\boldsymbol{N}(i,t)$ *denote the number of vertices of degree* i *in* $G_k(t)$*. Then* $\boldsymbol{N}(i,t) = tn_i(1+o(1))$ *for* $i \leq t^a$*, where* a *is some positive constant.*

2. $CC(G_k(t)) = c(k)(1+o(1))$.

3. *The diameter of* $G_k(t)$ *is* $O(\log t)$.

We also study k-trees of finite size experimentally. We show that the resultant k-tree has a very strong small world property (SW). The experimental results indicate that the diameter of the resultant k-tree of n vertices is proportional to $o(\log n)$, as k increases, and we conjecture a diameter of $O(\log n/\log k)$.

Finally we mention a connection with Apollonian networks. Processes related to the one we consider have been studied in the context of Apollonian packing by e.g. [8, 15]. In the Apollonian model, a random k-clique which *has never previously been selected* is extended to a $k+1$-clique by adding an extra vertex.

In [15], the evolution of degree of a given vertex is approximated using mean field theory. The authors show that, for $k \geq 3$ the degree sequence follows a power law with coefficient $2 + 1/(k-2)$.

We note that the power law coefficient $2+1/(k-2)$ for the random Apollonian model differs from the value of $2+1/k$ obtained for our model. For example, a power law of 3 is obtained in our model when $k = 1$, by choosing a random endpoint of a random edge. Indeed, this is the preferential attachment model of [1]. In the Apollonian model a power law of 3 occurs when $k = 3$, i.e., when the network grows by subdividing a triangular face.

3 Proof of Theorem 1

To prove the theorem, we first show the following lemma:

Lemma 1 *Let v_t be the vertex added to $G_k(t)$ at time t. For any $t' \geq t$, let ℓ be the number of K_{k+1} that contain v_t. Then the clustering coefficient at v_t in $G_k(t')$ is*

$$CC(v_t) = \frac{\binom{k}{2} + (k-1)(\ell-1)}{\binom{(k-1)+\ell}{2}}.$$

Proof. Suppose that at time $t > 1$, we add a vertex v_t and join it to each vertex $u_1, \ldots, u_k$ in the clique f_t of size k chosen in step 5 in the clique D_t of size $k+1$ chosen in step 4. Then $G_k(t)$ contains $k+t$ vertices. We call each induced clique K_k in $G_k(t')$ a *face* of $G_k(t')$, and define the *degree* of a face f by the number of K_{k+1} containing f, that is denoted by $\deg_{t'}(f)$. At time t, we add a new clique $Q = K_{k+1}$ by joining v_t to an existing face f_t. Thus $\deg_t(f_t) = \deg_{t-1}(f_t) + 1$ since f_t is in Q.

We define *face degree* $\text{Deg}_{t'}(v)$ of a vertex v by the total face degree of all faces incident with v. That is, $\text{Deg}_{t'}(v) = \sum_{v \in f} \deg_{t'}(f)$. Initially, when v_t is added at time t, $\text{Deg}_t(v_t) = k$ as there are k faces containing v_t, i.e., $Q = K_{k+1}$ contains k K_k subgraphs with distinguished vertex v (delete any of the k edges incident with v). Extending a face f to K_{k+1} adds one to $\deg(f)$ (since it is now in an extra K_{k+1}) and $k-1$ extra faces at v_t of face degree 1. Thus $\text{Deg}_{t'}(v_t) = k\ell$, where ℓ is the number of K_{k+1} that contain v_t.

At time t', we denote the set of neighbors of v by $N_{t'}(v)$, and define $\text{d}_{t'}(v) = |N_{t'}(v)|$ (that is, $\text{d}_{t'}$ is the ordinary degree of v in $G_k(t)$). When v_t is added to $G_k(t)$, we have $\text{d}_t(v_t) = \text{Deg}_t(v_t) = k$. Each time a face containing v_t is extended the face degree of v_t increases by k, but the vertex degree of v_t only increases by 1. Hence $\text{d}_{t'}(v_t) = (k-1) + \frac{\text{Deg}_{t'}(v_t)}{k}$.

Now we define *triangle degree* Δ_v of v by the number of K_3 in the subgraph induced by $\{v\} \cup N(v)$. That is, $CC(v)$ is given by $\frac{\Delta_v}{\binom{d(v)}{2}}$. Initially, when v_t is added to $G_k(t)$ it is contained in a unique $K_{k+1}(= Q)$, and the k edges incident at v induce $\binom{k}{2}$ triangles. Suppose face $f_t = K_k$, incident with v_t, is extended to a K_{k+1} at step t'. Face f already has $k-1$ edges vu_i with $i = 1, \ldots, k-1$, each of which will form a new triangle $(v_{t'}v_t, v_{t'}u_i, v_tu_i)$ with the new vertex $v_{t'}$. Thus $\Delta_{v_t} = \binom{k}{2} + (k-1)(d_{t'}(v) - k)$. Therefore if $\text{Deg}_{t'}(v_t) = k\ell$ then $d_{t'}(v_t) = (k-1) + \ell$ and $\Delta_{v_t} = \binom{k}{2} + (k-1)(\ell-1)$. Since $CC(v_t) = \frac{\Delta_{v_t}}{\binom{d(v_t)}{2}}$, the lemma follows. $\square$

We now turn to the clustering coefficient of a graph $G_k(t)$, which is defined by $CC(G_k(t)) = \sum_v \frac{CC(v)}{k+t}$. Let $f_{\ell k}$ be the limiting proportion of vertices of face degree ℓk and $n_{k-1+\ell}$ the limiting proportion of vertices of degree $k-1+\ell$. Then we have $f_{\ell k} = n_{k-1+\ell}$ and hence

$$\lim_{t\to\infty} CC(G(t)) = \sum_{\ell\geq 1} f_{\ell k} \frac{\binom{k}{2} + (k-1)(\ell-1)}{\binom{(k-1)+\ell}{2}}. \tag{4}$$

Next we analyze $f_{\ell k}$:

Lemma 2

$$f_{\ell k} = \frac{(\ell-1)! k^{\ell-1}(k+1)}{\prod_{j=1\ldots\ell}((j+1)k+1)}. \tag{5}$$

Proof. For $t \geq t'$, the relationship between vertex degree and face degree of a vertex $v_{t'}$ is given by $d_t(v_{t'}) = (k-1) + \frac{\mathrm{Deg}_t(v_{t'})}{k}$. Thus it suffices to study face degree $\mathrm{Deg}_t(v_{t'})$ of the vertices $v_{t'}$ of $G_k(t)$.

Let $\boldsymbol{F}_i(G_k(t))$ be the number of vertices of face degree i in $G_k(t)$ at the end of time t, and let $F_i(t)$ be its expected value.

Recall that we make $G_k(t+1)$ from $G_k(t)$ by picking a $D_{t+1} = K_{k+1}$ uniformly at random from $G_k(t)$ with probability $\frac{1}{t}$, and then picking a face $f_{t+1} = K_k$ uniformly at random from D_{t+1} with probability $\frac{1}{k+1}$. This process in fact picks faces proportional to their degree. This can be seen as follows. Suppose face f has degree i and thus occurs in i distinct K_{k+1}. Then

$$\Pr(f \text{ is chosen}) = \frac{i}{(k+1)t}.$$

Similarly, $\Pr(\text{face incident with } v \text{ chosen}) = \frac{\mathrm{Deg}_t(v)}{(k+1)t}$.

On adding vertex v_{t+1}, the number of vertices of face degree i is updated as follows:

$$\begin{aligned}\boldsymbol{F}_i(G_k(t+1)) &= \boldsymbol{F}_i(G_k(t)) + 1(i=k) + \textstyle\sum_{\mathrm{Deg}_t(v)=i-k} 1(v \text{ is in chosen face}) \\ &\qquad - \textstyle\sum_{\mathrm{Deg}_t(v)=i} 1(v \text{ is in chosen face}),\end{aligned}$$

where $1(H)$ is the indicator for the event H. On taking expectations over the random choices made by the process on the given graph $G_k(t)$, we obtain

$$F_i(G_k(t+1)) = \boldsymbol{F}_i(G_k(t)) + \frac{(i-k)\boldsymbol{F}_{i-k}(G_k(t))}{(k+1)t} - \frac{i\,\boldsymbol{F}_i(G_k(t))}{(k+1)t} + 1(i=k).$$

On taking expectations over all processes $G_k(t)$, we obtain the following recurrences, which are valid for $i = \ell k$, $\ell \geq 1$.

$$\begin{aligned} F_k(t+1) &= F_k(t) + 1 - \frac{kF_k(t)}{(k+1)t} \\ F_i(t+1) &= F_i(t) + \frac{(i-k)F_{i-k}(t)}{(k+1)t} - \frac{iF_i(t)}{(k+1)t} \qquad (i > k). \end{aligned}$$

Now we use the following lemma on real sequences [6, Lemma 3.1]:

Lemma 3 ([6, Lemma 3.1]) *If* (α_t), (β_t) *and* (γ_t) *are real sequences satisfying the relation*

$$\alpha_{t+1} = \left(1 - \frac{\beta_t}{t}\right)\alpha_t + \gamma_t,$$

where $\lim_{t\to\infty} \beta_t = \beta > 0$ *and* $\lim_{t\to\infty} \gamma_t = \gamma$, *then* $\lim_{t\to\infty} \frac{\alpha_t}{t}$ *exists and equals* $\frac{\gamma}{1+\beta}$.

Using Lemma 3, we have

$$\lim \frac{F_{\ell k}(t)}{t} = \frac{(\ell-1)!k^{\ell-1}(k+1)}{\prod_{j=1\ldots\ell}((j+1)k+1)} = f_{\ell k}.$$

□

Theorem 1(i) now follows Lemma 2, and taking the limit of equation (5) gives the claimed power law. Theorem 1(ii) follows from inserting equation (5) into relationship equation (4). It can be seen directly that for $k \geq 2$, $c(k) \geq \frac{1}{2}$. For the value of $c(k)$, when $k \to \infty$, we see that

$$c(k) \to \sum_{\ell\geq 1} \frac{1}{\ell(\ell+1)} = \sum_{\ell \geq 1}\left(\frac{1}{\ell} - \frac{1}{\ell+1}\right) = \lim_{\ell\to\infty}\left(1 - \frac{1}{\ell+1}\right) = 1.$$

4 Proof of Theorem 2

We give a brief outline of the proof. Recall that $\boldsymbol{F}_i(G_k(t))$ be the number of vertices of face degree i in $G_k(t)$ at the end of time t. The **whp** convergence of $\boldsymbol{F}_i(G_k(t))$ to $f_i t(1+o(1))$ can be established by standard methods e.g. [13]. This holds for $i \leq t^a$, where a is some positive constant. This establishes that the proportion of vertices of degree i in the finite process $G_k(t)$ is close to its limiting value. The value of the clustering coefficient follows directly from this.

As regards the diameter, a crude calculation suffices to establish a **whp** upper bound of $O(\log t)$. Consider a shortest (edge) path $v_t, u_1, ..., u_i, v_0$ back from v_t to a root vertex v_0 in $G_1(t)$. As half of the K_{k+1} in $G_k(t)$ were added by time $t/2$,

$$\Pr(v_t \text{ chooses a face } f \text{ in } G_k(t/2)) = \frac{\deg_t(G_k(t/2))}{(k+1)t} \geq \frac{1}{2}.$$

Thus the expected distance to the root must be (at least) halved by the edge $v_t u_1$. Whatever the label s of $u_1 = v_s$, this halving occurs independently at the next step. This must terminate **whp** after $c \log t$ steps, for some suitably large constant c, as we now prove.

Let Z_i be an indicator variable for the event that the distance to the root halves at step i, (conditional on not being at the root), or $Z_i = 1$ identically, if we have arrived at the root. Then $\Pr(Z_i = 1) \geq 1/2$, and $S_j = Z_1 + \cdots + Z_j$ stochastically dominates the binomial random variable $B \sim \text{Bin}(j, 1/2)$. As $\Pr(B < j/4) = O(e^{-j/16})$, then after $j = c \log_2 t$ steps, where $c > 4$ we conclude **whp** that we have arrived. Thus **whp** $\textsc{diam}(G_k(t)) = O(\log t)$.

5 Experimental Results

Algorithm 1 can be implemented easily. In this section, we give experimental results for the three properties (SF, CC, SW) of our model. The properties (SF) and (CC) were checked on a standard PC. To check the diameter property (SW) for large n we used a supercomputer (SGI Altix 4700: 96 Processors with 2305GB Memory).

Property (SW) This property implies that any two nodes on the network is connected by a relatively short path. The experimental results are shown in Figure 1. The figure implies that any pair of two nodes in a scale free k-tree of n vertices in our model seems to be joined by a very short path, possibly even of length $O(\log n / \log k)$. To observe this, we also plot the number of vertices and the value of (diameters$\times$ $\log k$) in Figure 2. From these experimental results, we conjecture that the diameter of a random k-tree is proportional to $\Theta(\log n / \log k)$.

Property (SF) As shown in Theorem 1(1), the distribution of degrees follows power law on the resultant k-tree in asymptotically. The experimental results imply that convergence to the asymptotic degree distribution occurs rapidly. In

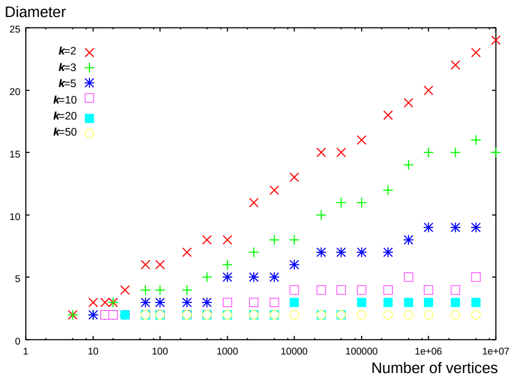

Figure 1: Diameter of scale free k-trees for $k = 2, 3, 5, 10, 20$, and 50.

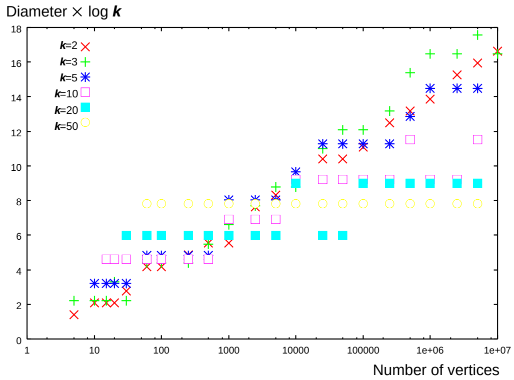

Figure 2: (Diameter$\times \log k$) for scale free k-trees for $k = 2, 3, 5, 10, 20$, and 50.

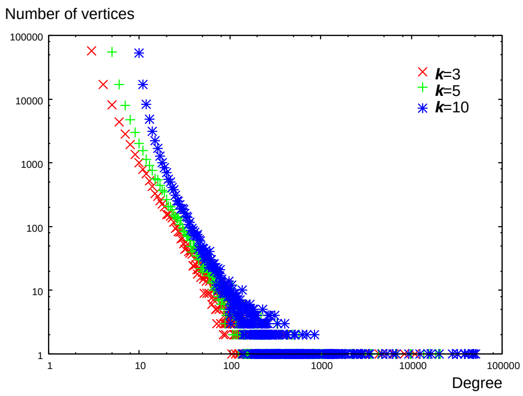

Figure 3: Degree distribution for scale free k-trees for $k = 3, 5$, and 10.

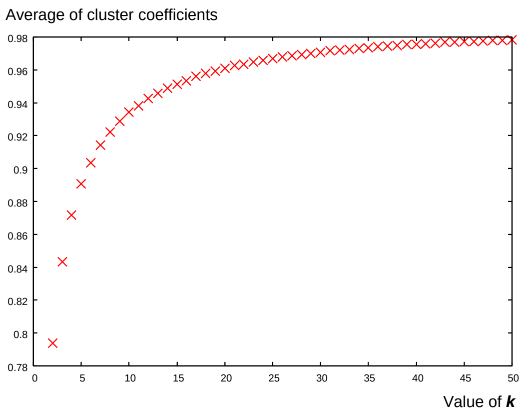

Figure 4: Average cluster coefficient.

Figure 3, we randomly generate a k-tree of $n = 100000$ vertices for $k = 3, 5$ and 10.

Property (CC) As shown in Theorem 1(2), the limiting expected clustering coefficient $c(k)$ converges to 1 for sufficiently large k. In Figure 4, we generate k-tree of $n = 10000$ vertices and note the convergence to the asymptotic result.

6 Conclusions

The model of random k-trees we propose exhibits small world properties of scale-free degree sequence, large clustering coefficient and small diameter. The model is closely related to Apollonian networks, and is easily extended to provide a formal analysis of random Apollonian networks. We also suggest the study of a more general model which allows the construction of partial k-trees. This would be achieved by modifying step 6 of Algorithm 1.

It would be interesting to establish the precise diameter D as a function of k. At present we have $D = O(\log n)$, with high probability, for an n vertex network. It seems reasonable to suppose that $D = O(\log n/f(k))$ where $f(k) \to \infty$ with k. On the basis of simulations it seems the diameter is at most of order $\log n/\log k$. The exact functional form of $f(k)$ is, however, unknown to us.

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