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Scale-independent approach to deformation and fracture of solid-state materials

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Abstract: As a scale-independent theory of deformation and fracture applicable to practical engineering problems, the field theoretical approach employed by physical mesomechanics is discussed in this paper. Being based on a fundamental physical principle known as gauge invariance, this approach does not rely on empirical concepts or phenomenology, and thereby is fundamentally scale independent. The derivation of physical-mesomechanical field equations is examined on a step-by-step basis, and the physical meaning of each step is clarified. Along the same line of argument, plastic deformation and transition to fracture is interpreted as an energy-dissipative process. Previously derived plastic deformation and fracture criteria are validated via detailed theoretical consideration and comparison with supporting experimental data.

Keywords: deformation dynamics, gauge invariance, fracture mechanics, electronic speckle-pattern interferometry

1 INTRODUCTION

The deformation and fracture behaviour of solid-state materials observed at the nano and microscopic scale are substantially different from behaviours observed at the macroscopic level of the same material. As an example, basic properties of material strength such as the yield stress and flow stress show strong size dependence at the nanoscale [**1**]. Conventional design concepts are usually developed for macroscopic objects. Consequently, nano- and micro-scale systems fabricated under a conventional design concept often do not meet the design requirements. The underlying problem here is the lack of full understanding of deformation and fracture on a solid physical basis. Conventional theories of plasticity rely on experimentally determined constitutive relations and mathematical techniques for calculating non-uniform distributions of stress and strain [**2**, **3**]. In most cases, the constitutive relation is observed at the macroscopic level, and thereby the theory is scale dependent. Continuum mechanics [**4**] derives differential equations describing materials' behaviour based on conservation laws of physics, but yet requires a constitutive relation to apply the theory to a particular case. It is crucially important to develop a scale-independent theory based on a solid physical basis.

In this respect, a field theoretical approach proposed by Panin *et al.* [**5**] as part of the general theory of deformation and fracture called physical mesomechanics [**6**] has great advantage. Based on a fundamental physical principle known as local (gauge) symmetry [**7**], this approach is capable of describing deformation dynamics without relying on empirical concepts or phenomenology; the theory yields an equation that expresses the external force acting on a unit volume of the material. Thus, by nature, the formalism is scale independent and universal. Egorushkin [**8**] applied this formalism to dislocation dynamics, whereas Panin *et al.* [**5**] applied it to plastic deformation dynamics at the macroscopic level. In both cases, interestingly, the formalism yields wave equations which represent synergetic interaction between translational and rotational modes of the dynamics. Wave characteristics have been experimentally observed in the strain field [**9**] and in the displacement field [**10**]. The universality of this approach enables the dynamics of all stages of deformation to be formulated, from the elastic to the fracturing stage, on a common theoretical basis. Here the plasticity is characterized as a stage where dynamics becomes energy dissipative hence the displacement is unrecoverable; fracture is the final stage of deformation where the dynamics becomes totally energy dissipative and the material is no longer able to react to the external load by deforming further. This leads to physical-mesomechanical criteria of plastic deformation and fracture of isotopic materials [**11**]. With these criteria, it is possible to visualize the plastic zone at the pre-fracture stage [**12**], and diagnose loading hysteresis of materials after the load is removed [**13**].

Essentially, the physical-mesomechanical formalism describes deformation by a linear transformation similar to that used by conventional continuum mechanics. To deal with the non-linearity in the plastic regime, it allows the transformation matrix to be coordinate dependent. The underlying idea is that even in the plastic regime, the material locally obeys the linear, elastic law, and the non-linear dynamics can be described by the interrelationship among these local linear transformations. From the gauge symmetrical viewpoint, this interrelationship makes the law of elasticity invariant under the coordinate-dependent transformation. This is a well-established formalism and can be found in other branches of physics such as general relativity and electrodynamics [**7**]. As the formalism commonly yields the so-called field equation, the corresponding theory is categorized as a field theory. The procedure to derive the physical-mesomechanical field equation is not necessarily simple. In addition, in the original paper by Panin *et al.* [**5**], some of the key equations are derived without step-by-step proofs. This situation seems to hinder this approach from being applied to engineering problems. Recent studies [**14**, **15**] have clarified the physical meaning of a number of intermediate steps necessary to derive the field equation. The aim of the current paper is to fill these previous gaps in the equational derivation through discussions of the physical interpretations on the intermediate steps and related concepts. Recent experimental results that support the theoretical development are also provided.

2 FORMULATION

2.1 Theoretical overview

The basic postulate of physical mesomechanics is that even in the plastic regime, deformation is

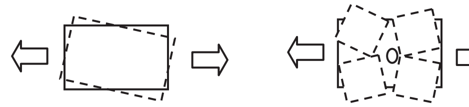

Fig. 1 Schematic view of material rotations of four segments around a defect

locally elastic. This makes complete sense because microscopically the force is always proportional to the displacement. What makes the macroscopic deformation non-linear is the existence of defects; as schematically illustrated in Fig. 1, if a defect exists segments around it rotate differently. Within each segment (called the deformation structural element), however, deformation is elastic and thereby can be expressed by a transformation matrix of conventional continuum mechanics: $\boldsymbol{\eta}' = \mathbf{U}(x^{\mu})\boldsymbol{\eta}$. Here, η and η' denote the line element vector before and after the deformation, and $\mathbf{U}$ is the transformation matrix that represents the deformation. Since the elastic deformation is different from one deformation structural element to another, the transformation matrix is a function of the space coordinates.

The above postulate raises a question: Are these local transformations completely independent of each other? The answer should be 'No', because as different parts of the same object experiencing non-linear deformation there must be some interrelationship among these local linear transformations, and this interrelationship should describe the non-linear part of the dynamics. This is where the concept of local symmetry and associated formalism needs to be considered and this will be elaborated in the following sections.

2.2 Deformation as a linear transformation

Consider a line element vector in an object under deformation, as shown in Fig. 2. Here, $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ denote the line element vector before and after the deformation, and $\boldsymbol{\xi}$ is the displacement vector. As $\boldsymbol{\xi}$ depends on the coordinates and the head and tail of $\boldsymbol{\eta}$ are located on different coordinate points, $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are not parallel to each other. With $\mathbf{U}$ being the transformation matrix to represent this transformation, $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}$ can be related by

$$\boldsymbol{\eta}' = \mathbf{U}\boldsymbol{\eta} \tag{1}$$

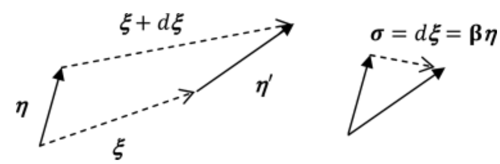

Fig. 2 Deformation as a linear transformation

The difference between $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$, $\boldsymbol{\delta\eta}$, is due to the difference in displacement between the coordinate points of the tail and head of $\boldsymbol{\eta}$. Hence, it can be expressed in terms of the distortion matrix $\boldsymbol{\beta}$

$$\delta\boldsymbol{\eta} = \boldsymbol{\eta}' - \boldsymbol{\eta} = \begin{pmatrix} \frac{\partial \xi_x}{\partial x} & \frac{\partial \xi_x}{\partial y} & \frac{\partial \xi_x}{\partial z} \\ \frac{\partial \xi_y}{\partial x} & \frac{\partial \xi_y}{\partial y} & \frac{\partial \xi_y}{\partial z} \\ \frac{\partial \xi_z}{\partial x} & \frac{\partial \xi_z}{\partial y} & \frac{\partial \xi_z}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \boldsymbol{\beta} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \boldsymbol{\beta\eta} \quad (2)$$

Equation (2) indicates $\boldsymbol{\eta}' = \boldsymbol{\eta} + \boldsymbol{\beta\eta} = (\mathbf{I} + \boldsymbol{\beta})\boldsymbol{\eta}$ where $\mathbf{I}$ is the unit matrix, and comparison of this matrix with equation (1) enables $\mathbf{U}$ and $\boldsymbol{\beta}$ to be related as

$$\mathbf{U} = \mathbf{I} + \boldsymbol{\beta} \quad (3)$$

Now consider if the force law is invariant under this transformation. In the theory of elasticity, force is proportional to the stretch or the differential displacement. Therefore, for the theory to be invariant, the differential must transform in the same fashion as the vector itself. Otherwise, after the transformation the elastic force law can not be written in the same form as before the transformation. This means that after the transformation, $\mathbf{U}(\mathrm{d}\boldsymbol{\xi})$ must represent the differential of the transformed displacement, i.e.

$$\mathbf{U}(\mathrm{d}\boldsymbol{\xi}) = \mathrm{d}(\mathbf{U}\boldsymbol{\xi}) = (\mathrm{d}\mathbf{U})\boldsymbol{\xi} + \mathbf{U}(\mathrm{d}\boldsymbol{\xi}) \quad (4)$$

Here the left-hand side is the transformation of the differential and the right-hand side is the differential of the transformed. Apparently, the condition that 'the differential transforms in the same fashion as the vector itself' holds only if $\mathrm{d}\mathbf{U} = 0$, or $\mathbf{U}$ is independent of the coordinates. From equation (3), this is equivalent to $\boldsymbol{\beta}$ being independent of the coordinates. Furthermore, since a component of $\boldsymbol{\beta}$ is the first derivatives of displacement, this indicates that the condition is equivalent to the first derivatives of displacement being coordinate independent, or the coordinate dependence of displacement being as high as the first order, i.e. the deformation is linear.

2.3 Coordinate-dependent transformation

When the deformation is non-linear, the first term on the right-hand side of equation (4) becomes non-zero, meaning that the differentials do not transform in the same form as the vector. In order to make the elastic force law invariant under the transformation, it becomes necessary to replace the usual differentiation, d, with a new differentiation operator, D, so that the right-hand side of equation (4) can be put in the form $\mathrm{D}'(\mathbf{U}\boldsymbol{\xi})$, i.e. 'differentiation of transformed'. Here a prime (′) is added to emphasize that it represents the differentiation 'after' the transformation. Equating this to the transformation of the new differential, the following equation is obtained

$$\mathbf{U}(\mathrm{D}\boldsymbol{\xi}) = \mathrm{D}'(\mathbf{U}\boldsymbol{\xi}) \quad (5)$$

The left-hand side of equation (5) represents the transformation of the differential and the right-hand side the differential of the transformed. From equation (5), it follows that $\mathbf{U}\mathrm{D} = \mathrm{D}'\mathbf{U}$, and therefore it is necessary that the newly defined differential transforms in the following fashion

$$\mathrm{D}' = \mathbf{U}\mathrm{D}\mathbf{U}^{-1} \quad (6)$$

as $\boldsymbol{\xi}$ transforms according to equation (1). Conversely, if D transforms according to equation (6), the theory becomes invariant under the coordinate-dependent transformation $\mathbf{U}(x^\mu)$. As a pair, transformations (1) and (6) are called the gauge transformation associated with this dynamics.

Comparison of equations (4) and (5) indicates that D must eliminate the extra term $\mathrm{d}\mathbf{U}$, leading to the following form of D_i

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} - \Gamma_i = \partial_i - \Gamma_i, \quad i = x, y, z \quad (7)$$

Here, $-\Gamma_i$ represents the removal of the extra term and is called the gauge term. The resultant derivatives, D_i, are called the covariant derivatives. Note that on the right-hand side of equation (7), the first term ∂_i represents the physically true infinitesimal change in the vector and the second term is the apparent change associated with the coordinate dependence of the transformation. Thus, in order to describe the underlying physics, this apparent term must be removed from the differentials. Figure 3 illustrates this schematically.

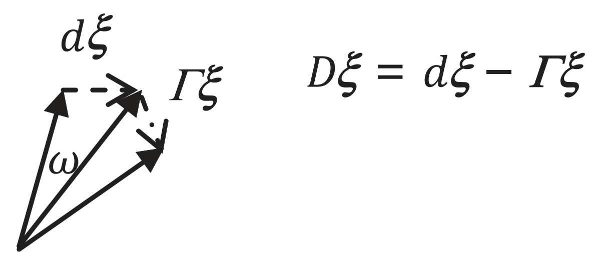

Fig. 3 Gauge term as part of differential not representing deformation

2.4 Field interaction and stress tensor

From the viewpoint that the gauge term represents the apparent change of the vector associated with coordinate dependence of the transformation matrix, the effect can be interpreted as an interaction of the vector with the field. Therefore, the covariant derivative can be expressed in terms of vector potential $\boldsymbol{A}$ as

$$D\xi_i = \left(\frac{\partial \xi_i}{\partial x} - \Gamma_x \xi_i\right) dx + \left(\frac{\partial \xi_i}{\partial y} - \Gamma_y \xi_i\right) dy + \left(\frac{\partial \xi_i}{\partial z} - \Gamma_z \xi_i\right) dz \equiv d\xi_i - A_i \quad (8)$$

In elastic deformation, the rotation matrix represents rigid body rotation of the material, which does not involve length change. In equation (8), the actual change in the length of differential displacement vector is all in dξ. Thus, $\boldsymbol{A}$ can be interpreted as representing the rotation of the local volume element. Splitting the distortion matrix into the symmetric part (strain matrix) and asymmetric part (rotation matrix), the differential displacement can be written in the following form

$$D\boldsymbol{\xi} = \boldsymbol{\beta}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} - \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = d\boldsymbol{\xi} - \begin{pmatrix} -\omega_z dy + \omega_y dz \\ \omega_z dx - \omega_x dz \\ -\omega_y dx + \omega_x dy \end{pmatrix} \quad (9)$$

Comparison of equations (8) and (9) indicates that the three components of $\boldsymbol{A}$ can be identified as

$$A_i = \omega_j \mathrm{d}x^k - \omega_k \mathrm{d}x^j,\ i, j, k = x, y, z \quad (10)$$

Now consider the interaction with the field via the vector potential. Figure 4 illustrates an infinitesimal differential displacement $\mathrm{D}\xi_s$ (difference between the displacement of two neighbouring points). Consider moving from point A to point B along two paths under the influence of the potential; the first path is to move along the x^ν axis and then the x^μ axis (clockwise), and the second path is counter-clockwise (Fig. 4). Because of the interaction with the field, the differentials between the two paths are potentially different. Dropping the second-order differentials, the clockwise case is

$$\mathrm{D}_\mu(\mathrm{D}_\nu \xi_s \mathrm{d}x^\nu)\mathrm{d}x^\mu = \partial_\mu \partial_\nu \xi_s \mathrm{d}x^\nu \mathrm{d}x^\mu - \partial_\mu(\Gamma_\nu \xi_s \mathrm{d}x^\nu)\mathrm{d}x^\mu + \Gamma_\mu \Gamma_\nu \xi_s \mathrm{d}x^\nu \mathrm{d}x^\mu$$

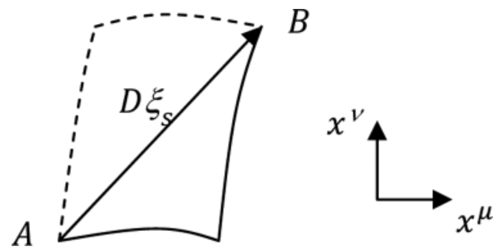

Fig. 4 Clockwise and counter-clockwise paths from the tail to the head of differential displacement

Here from the definition, $\Gamma_\nu \xi_s \mathrm{d}x^\nu$ can be interpreted as A_ν, and so

$$\mathrm{D}_\mu(\mathrm{D}_\nu \xi_s \mathrm{d}x^\nu)\mathrm{d}x^\mu = \partial_\mu \partial_\nu \xi_s \mathrm{d}x^\nu \mathrm{d}x^\mu - \partial_\mu A_\nu \mathrm{d}x^\mu + \frac{1}{\xi_s} A_\mu A_\nu$$

The counter-clockwise case can be expressed with the vector potential in the same fashion. Thus the difference between the clockwise and counter-clockwise case is

$$\mathrm{D}_\mu(\mathrm{D}_\nu \xi_s \mathrm{d}x^\nu)\mathrm{d}x^\mu - \mathrm{D}_\nu\left(D_\mu \xi_s \mathrm{d}x^\mu\right)\mathrm{d}x^\nu = \left(\partial_\nu A_\mu \mathrm{d}x^\nu - \partial_\mu A_\nu \mathrm{d}x^\mu\right) + \frac{1}{\xi_s}\left[A_\mu, A_\nu\right]$$

In the infinitesimal limit, $\mathrm{d}x^\nu = \mathrm{d}x^\mu = \mathrm{d}s$, and division of the above equation by ds leads to

$$\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right]_s \xi_s \mathrm{d}s = \left(\partial_\nu A_\mu - \partial_\mu A_\nu\right) + \frac{1}{\mathrm{d}s\xi_s}\left[A_\mu, A_\nu\right] \equiv F_{\mu\nu} \quad (11)$$

This quantity $F_{\mu\nu}$, known as the stress tensor, represents the strength of the interaction with the gauge field. Before considering its physical meaning, examine how the stress tensor transforms. With the use of equation (6)

$$F'_{\mu\nu} = \left[\mathrm{D}'_\mu, \mathrm{D}'_\nu\right] dx\xi_s = \left(\mathrm{D}'_\mu \mathrm{D}'_\nu - \mathrm{D}'_\nu \mathrm{D}'_\mu\right) dx\xi_s = \left(U\mathrm{D}_\mu U^{-1} U\mathrm{D}_\nu U^{-1} - U\mathrm{D}_\nu U^{-1} U\mathrm{D}_\mu U^{-1}\right) dx\xi_s = U\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right] U^{-1} dx\xi_s = UF_{\mu\nu}U^{-1}$$

Apparently, $F'_{\mu\nu} \neq F_{\mu\nu}$ and the stress tensor is not invariant under the gauge transformation. However, from the mathematical identity $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$, it is found that the trace of the inner product of the stress tensor is invariant

$$\frac{1}{4} tr\left(F'_{\mu\nu} F^{\mu\nu\prime}\right) = \frac{1}{4} tr\left(UF_{\mu\nu}U^{-1} UF^{\mu\nu}U^{-1}\right) = \frac{1}{4}\left(UF_{\mu\nu}F^{\mu\nu}U^{-1}\right) = \frac{1}{4} tr\left(U\left(F_{\mu\nu}F^{\mu\nu}U^{-1}\right)\right) = \frac{1}{4} tr\left(\left(F_{\mu\nu}F^{\mu\nu}U^{-1}\right)U\right) = \frac{1}{4} tr\left(F_{\mu\nu}F^{\mu\nu}\right) \quad (12)$$

Now consider the meaning of the stress tensor. From equation (10), apparently A_μ and A_ν commute with each other. Thus the stress tensor for $\mu=x$, $\nu=y$, as an example, becomes

$$\begin{aligned}F_{xy}&=\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}=\left(\frac{\partial\omega_y}{\partial y}dz-\frac{\partial\omega_z}{\partial y}dy\right)-\left(\frac{\partial\omega_z}{\partial x}dx-\frac{\partial\omega_x}{\partial x}dz\right)\\&=\frac{1}{2}\left\{\frac{\partial}{\partial y}\left(\frac{\partial\xi_x}{\partial z}-\frac{\partial\xi_z}{\partial x}\right)dz-\frac{\partial}{\partial y}\left(\frac{\partial\xi_y}{\partial x}-\frac{\partial\xi_x}{\partial y}\right)dy\right\}\\&\quad-\frac{1}{2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial\xi_y}{\partial x}-\frac{\partial\xi_x}{\partial y}\right)dx-\frac{\partial}{\partial x}\left(\frac{\partial\xi_z}{\partial y}-\frac{\partial\xi_y}{\partial z}\right)dz\right\}\\&=\frac{1}{2}\left(\frac{\partial\xi_x}{\partial y}-\frac{\partial\xi_y}{\partial x}\right)=-\omega_z\end{aligned}$$

Here, note that A_i actually represents displacement ξ_i. In general, F_{ij} represents rotation as

$$F_{ij}=-\omega_k,\quad i,j,k=x,y,z \tag{13}$$

At this point, it is necessary to introduce the time terms. Using the relativistic four-vector notations [**7**], the stress tensor with the time components included can be written as

$$F_{0i}=\frac{\partial A_0}{\partial x^i}-\frac{\partial A_i}{\partial x^0}=\frac{\partial A_0}{\partial x^i}-\frac{1}{c}\frac{\partial A_i}{\partial t}=-\frac{1}{c}\frac{\partial\xi_i}{\partial t} \tag{14}$$

Here $(x^0, x^1, x^2, x^3)=(ct, x, y, z)$ and c is the phase velocity of the wave characteristics. In the present case, c can be interpreted as the transverse wave velocity, which is given as the square root of the reciprocal of the product of the density and the shear modulus. See the appendix for more details.

$$c=\sqrt{1/(\varepsilon\mu)} \tag{15}$$

2.5 Lagrangian formalism and field equations

Now that the rule of transformation on the gauge in conjunction with the transformation of the line element vector has been found, the part of the true dynamics associated with the coordinate dependence of the transformation matrix can be expressed through the analysis of the vector potential. For this purpose, the Lagrangian formalism works well. As discussed above, $F_{\mu\nu}F^{\mu\nu}$ is invariant under the present gauge transformation. From the definition, the inner products of covariant vectors are apparently invariant. Thus the Lagrangian of the following form is found invariant

$$L=\frac{1}{4\mu}F_{\mu\nu}F^{\mu\nu}-g^{ij}\mathrm{D}_\mu\eta_i^\alpha\mathrm{D}_\nu\eta_j^\beta C_{\alpha\beta}^{\mu\nu} \tag{16}$$

Here $1/\mu$ is the shear modulus and $C_{\alpha\beta}^{\mu\nu}$ is the dimensionless elastic constant of the material [**5**]. (NB The shear modulus $1/\mu$ is proportional to the spatial derivative of rotation ω, which is already a spatial derivative of displacement (e.g. $\omega_z=\partial\xi_y/\partial x-\partial\xi_x/\partial y$), this force is proportional to the second-order spatial derivative of displacement unlike the usual shear force which is proportional to the shear strain, a first-order derivative of displacement.) Substitution of the explicit form of $F_{\mu\nu}$ shown in equations (13) and (14) into the first term of equation (16) indicates that this term represents the energy of the gauge field. The second term is basically the work done by the elastic field of the material. Application of the least action principle leads to the following Euler–Lagrange equation with respect to the vector potential A_μ

$$\partial_\nu\frac{\partial L}{\partial(\partial_\nu A_\mu)}-\frac{\partial L}{\partial A_\mu}=0 \tag{17}$$

Equation (17) yields the following field equations

$$\nabla\cdot\boldsymbol{v}=j^0 \tag{18}$$

$$\nabla\times\boldsymbol{\omega}=-\frac{1}{c^2}\frac{\partial\boldsymbol{v}}{\partial t}-\boldsymbol{j} \tag{19}$$

where $\boldsymbol{\omega}$ is the angle of rotation, $j^0\equiv g^{ij}\eta_i^\alpha\eta_j^\alpha$, and $j\equiv g^{ij}\eta_i^\alpha\mathrm{D}_V\eta_j^\beta C_{\alpha\beta}^{\mu\nu}$. From equations (13) and (14), $\boldsymbol{\omega}$ is found to be related to the velocity $\boldsymbol{v}$ as follows

$$\nabla\times\boldsymbol{v}=\frac{\partial\boldsymbol{\omega}}{\partial t} \tag{20}$$

Taking the divergence of equation (19) with the substitution of equation (15) to c in equation (19) and the use of equation (18) along with the mathematical identity $\nabla\cdot(\nabla\times\boldsymbol{\omega})=0$ leads to the following relationship between j^0 and $\boldsymbol{j}$

$$\varepsilon\frac{\partial j^0}{\partial t}=-\frac{1}{\mu}\nabla\cdot\boldsymbol{j} \tag{21}$$

Assuming that the density ε is constant over time, the left-hand side of equation (21) can be interpreted as the change of momentum over time. On the other hand, the right-hand side of equation (21) represents the differential of $\boldsymbol{j}$, which is basically the longitudinal force [**14**]. Thus, equation (21) can be interpreted as 'the change in momentum in a unit volume is caused by the difference in the longitudinal force acting on the leading and tailing edge of the unit volume'; i.e. Newton's second law. With these notations, equation (19) can be rewritten as

$$\varepsilon\frac{\partial\boldsymbol{v}}{\partial t}=-\frac{1}{\mu}(\nabla\times\boldsymbol{\omega})-\frac{1}{\mu}\boldsymbol{j} \tag{22}$$

Equation (22) is the equation of motion governing the dynamics of a unit volume [**14**]. The left-hand side is the product of the mass and acceleration. The right-hand side is the external force acting on the unit volume where the first term represents the restoring force proportional to the shear modulus $1/\mu$ and the second represents the longitudinal force. When the deformation is in the elastic regime, the longitudinal force is due to the elastic response of the surrounding materials represented by the elastic constant $C_{\alpha\beta}^{\mu\nu}$ (equation (16)). When the deformation enters the plastic regime, this longitudinal force becomes velocity damping force through the mechanism explained in the next section.

3 PLASTIC DEFORMATION AS AN ENERGY-DISSIPATING PROCESS

3.1 Longitudinal force

The dynamics in the plastic regime is well understood by viewing equation (21) as an equation of continuity, which reads 'the change in the quantity $\varepsilon j^0 = (\varepsilon\nabla \cdot \boldsymbol{\nu})$ over time in a closed volume is equal to the net flow of εj^0 into the volume by means of $\boldsymbol{j}/\mu$. Here the quantity $\nabla \cdot \boldsymbol{\nu}$ is basically the longitudinal change in displacement per unit volume, i.e. the stretch corresponding to displacement in unit time. Thus its flow represents displacement of the stretched pattern, or in a one-dimensional picture, 'a stretched spring' flows as a whole. In other words, when additional force is applied to an already stretched spring, it moves as a whole rather than being further stretched. In order for this to happen, it is necessary that the spring is detached at the point where it is supported. If this happens, the displacement is naturally unrecoverable. Figure 5 illustrates the situation schematically. Figure 5(a) shows that in the elastic regime, the momentum of the closed volume changes over time due to the differential longitudinal force $\nabla \cdot \boldsymbol{j}/\mu$ (in the one-dimensional picture, $\nabla \cdot \boldsymbol{j} = \partial j_x/\partial x$). Figure 5(b) shows that in the plastic regime, the entire stretch moves out of the closed volume; this causes net momentum loss for the closed volume because the material near the leading edge of the stretch has higher velocity than the tailing edge [**14**].

The interpretation of equation (21) as an equation of continuity enables it to be rewritten as

$$\frac{1}{\mu}\boldsymbol{j} = \boldsymbol{W}_{\mathbf{d}}\varepsilon j^0 \tag{23}$$

where $\boldsymbol{W}_{\mathbf{d}}$ is interpreted as the drift velocity of the quantity $\varepsilon j^0 = \nabla \cdot (\varepsilon \boldsymbol{v})$ which can be viewed as the

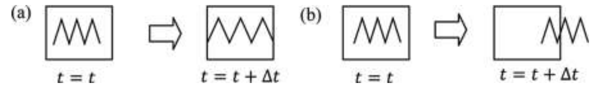

Fig. 5 Schematic illustration of one-dimensional $\nabla \cdot \boldsymbol{j}$; (a) elastic case and (b) plastic case

divergence of momentum density. As Fig. 5(b) indicates, when the deformation is plastic, this quantity εj^0 flows in the same direction as the velocity $\boldsymbol{v}$ causing the momentum loss. Thus being proportional to the velocity, the corresponding force $\boldsymbol{j}/\mu$ can be interpreted as velocity damping force, i.e. the work done by the external force is dissipated, as opposed to stored as the spring energy in the elastic case (Fig. 5(a)). It seems that equation (21) represents the momentum conservation that visco-plastic constitutive models [**16**] are based on. Connection between the present formalism and visco-plasticity is the subject of future study.

It is interesting to consider $\boldsymbol{W_d}\varepsilon j^0$ that appears on the right-hand side of equation (23) in the elastic case. Setting the direction of elastic elongation along the x-axis, this quantity can be expressed as

$$\frac{1}{\mu} j_x = W_d \frac{\varepsilon \partial v_x}{\partial x} \tag{24}$$

where v_x is the x-component of the velocity; $v_x = \dot{\xi}_x$. Since it is elastic deformation, the displacement can be expressed as a longitudinal elastic wave with angular frequency ω_0, wave number k_0 and amplitude ξ_0 in the form of $\xi_0 e^{i(\omega_0 t - k_0 x)}$. Therefore

$$\frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x}\dot{\xi}_x = i\omega_0 \frac{\partial \xi_x}{\partial x} = \frac{i\omega_0}{-ik_0}\frac{\partial^2 \xi_x}{\partial x^2} = -v_{\mathrm{ph}}\frac{\partial^2 \xi_x}{\partial x^2}$$

where v_{ph} is the phase velocity, which can be expressed in terms of the Young's modulus and density as $v_{\mathrm{ph}} = \sqrt{(E/\varepsilon)}$. By viewing $W_{\mathrm{d}} = v_{\mathrm{ph}}$, equation (24) becomes

$$\frac{1}{\mu} j_x = -v_{\mathrm{ph}}{}^2 \varepsilon \frac{\partial^2 \xi_x}{\partial x^2} = -E\frac{\partial^2 \xi_x}{\partial x^2} \tag{25}$$

Equation (25) represents the differential force at the leading and tailing boundary of a unit volume due to the elastic force. Thus, it is found that in the linear elastic limit where ω represents rigid body rotation and hence $\nabla \times \boldsymbol{\omega} = 0$, equation (22) reduces to the equation of longitudinal elastic wave.

3.2 Plastic deformation and fracture criteria

The argument made in the preceding sections indicates that the onset of plastic deformation of a linear material can be characterized by coordinate-dependent rotation, i.e. the left-hand side of equation (19) is non-zero. In addition, the transverse restoring force and longitudinal damping force represented by the two terms on the right-hand side of equation (22) are active. Thus, the plastic deformation criterion [**11**] of an initially linear material can be given by

$$\frac{1}{\mu}(\nabla\times\boldsymbol{\omega}) \neq 0 \text{ and } \frac{1}{\mu}\boldsymbol{j} \neq 0 \tag{26}$$

It is expected that as plastic deformation develops, the transverse restoring force weakens and, instead, the longitudinal damping force tends to dominate. At the point where the restoring force vanishes completely, the material totally loses its restoring mechanism. This can be interpreted as the pre-fracturing stage. Eventually, both the restoring force and damping force vanish, and this is when the material fractures. Thus pre-fracturing criterion and fracture criterion [**11**] can be given, respectively, by

$$\frac{1}{\mu}(\nabla\times\boldsymbol{\omega}) = 0 \text{ and } \frac{1}{\mu}\boldsymbol{j} \neq 0 \tag{27}$$

and

$$\frac{1}{\mu}(\nabla\times\boldsymbol{\omega}) = 0 \text{ and } \frac{1}{\mu}\boldsymbol{j} = 0 \tag{28}$$

3.3 Supporting experimental results

In order to test the plastic deformation and fracture criteria discussed above, a number of tensile experiments have been conducted with the use of an optical interferometric technique known as electronic speckle pattern interferometry. Figure 6 is the result of an experiment conducted under research collaboration with O. Umezawa and K. Sunaga of Yokohama National University. The experiment was conducted with a 5 mm wide, 30 mm long, and 0.3 mm thick aluminum alloy specimen; the loading characteristic is shown along with typical interferometric fringe patterns observed: (a) prior to the yield point, (b) in the middle of the plastic regime, and (c) immediately before the fracture. In this experiment, the interferometer is sensitive to the horizontal component of in-plane displacement. Thus, dark fringes represent the locations on the specimen where the horizontal component of the displacement is integral multiples of a unit value determined by the interferometer's setting (0.45 μm). The tensile load was applied vertically. It can be seen that in stage (a) the fringes are slightly slanted but nearly horizontal and, relatively speaking, evenly distributed; in stage (b) the fringes are curved and diagonal, and in stage (c) they are horizontal and their density increases toward the upper end of the specimen where the fringes are so dense that a bright X-shaped pattern is apparent. In accordance with criteria (26) and (27), these observations can be explained as follows. As the interferometer is sensitive to the horizontal component of displacement (ξ_x), the horizontal (x) and vertical (y) component of the transverse restoring force represented by the fringe pattern can be expressed as

$$F_x \propto (\nabla\times\omega)_x = \frac{\partial\omega_z}{\partial y} - \frac{\partial\omega_y}{\partial z} \to \frac{\partial\omega_z}{\partial y} \to -\frac{\partial}{\partial y}\left(\frac{\partial\xi_x}{\partial y}\right)$$

$$F_y \propto (\nabla\times\omega)_y = \frac{\partial\omega_x}{\partial z} - \frac{\partial\omega_z}{\partial x} \to -\frac{\partial\omega_z}{\partial x} \to \left(\frac{\partial^2\xi_x}{\partial x\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\xi_x}{\partial x}\right)$$

Thus, generally speaking, if the fringes are horizontal, $\partial\xi_x/\partial x = 0$ and $F_y = 0$. If the fringes are slanted (i.e. $\partial\xi_x/\partial x \neq 0$, $\partial\xi_x/\partial y \neq 0$) and unevenly distributed in the vertical direction ($\partial/\partial y \neq 0$), $F_x \neq 0$ and $F_y \neq 0$. From this viewpoint, the above observations are explained as follows: in stage (a) the specimen exerts some level of F_x and F_y. However, judging from the fact that the fringes are more or less evenly distributed along the y axis, the forces are small. In stage (b), both F_x and F_y are substantial. As the deformation develops toward fracture, F_y practically vanishes and there is some level of F_x over a large part of the specimen. The bright pattern due to the dense fringes represents the damping force proportional to $\partial\xi_x/\partial x$. Previous studies [**17**, **18**] have indicated that this bright pattern of dense fringes has a strong correlation with the shear band due to the Portevin–Le Chatelie effect. It is therefore considered that this pattern results from the associated

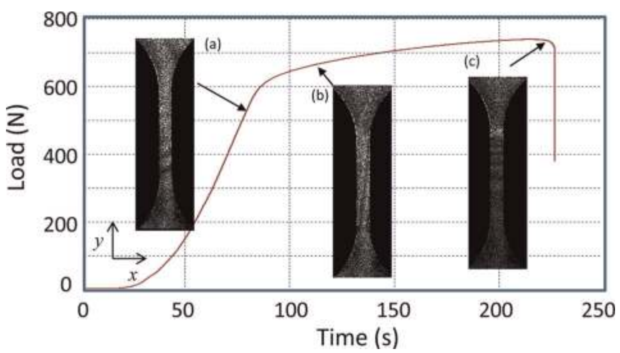

Fig. 6 Tensile experiment with Al alloy (2 per cent Mg) specimen. Pulling rate: 20 μm/s. The loading characteristic is shown along with typical interferometric fringe patterns observed: (a) prior to the yield point, (b) in the middle of the plastic regime, and (c) immediately before the fracture

dislocation dynamics. These are consistent with the intuitive trend of the deformation.

4 CONCLUSIONS

In summary, the derivation of physical-meso-mechanical field equations has been analysed on a step-by-step basis, and the physical meaning of the quantities involved in the equational derivation has been discussed. The vector potential associated with the gauge term has been identified as representing the rotation matrix of local transformation. This identification has confirmed the validity of the previous interpretation of transverse force associated with the rotational nature of material rotation, and longitudinal force associated with the momentum change of unit volume over unit time. The transition from the elastic to the plastic regime has been characterized as the longitudinal force of the material changing from being proportional to the displacement to being proportional to the local velocity. The plastic deformation criterion and fracture criterion derived from these ideas have been verified through comparison with a tensile experiment conducted with the use of an optical interferometric technique.

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APPENDIX

Four-vector notation

In order to describe deformation dynamics with the above-mentioned formalism, it is convenient to use the four-vector notation used in the relativistic theories. Consider an infinitesimal volume element in an elastic material. According to Newton's second law, the acceleration of the volume is equal to the net external force exerted by surrounding volumes. Figure 7 illustrates the dynamics as a one-dimensional model. If the normal force is dominant (Fig. 7(a)), the external force acting on the

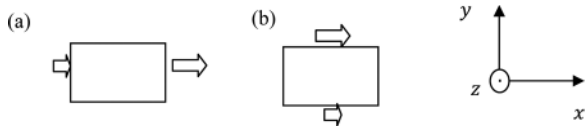

Fig. 7 Phase velocity in four-vector notation: (a) normal force dominant; (b) shear force dominant

volume at the leading edge at $s + \mathrm{d}s$ is $f(s + \mathrm{d}s) = k_\mathrm{n}\mathrm{d}\xi(s + \mathrm{d}s)$ and that at the tailing edge is $f(s) = k_\mathrm{n}\mathrm{d}(s)$. Here k_n denotes the normal elastic constant (stiffness) and ξ the displacement; hence $\mathrm{d}\xi = (\partial\xi/\partial s)\mathrm{d}s$ is the differential displacement or the stretch. Consequently, the net force can be expressed in terms of the displacement as $f(s + \mathrm{d}s) - f(s) = (\partial f/\mathrm{d}s)\mathrm{d}s = k_\mathrm{n}(\partial^2\xi/\partial s^2)\mathrm{d}s = EA(\partial^2\xi/\partial s^2)\mathrm{d}s$. Here A is the cross-sectional area of the volume and E is the Young's modulus. Since this is equal to the acceleration term

$$m(\mathrm{d}u/\mathrm{d}t) = m\{(\partial u/\partial t) + (\partial u/\partial s)(\mathrm{d}s/\mathrm{d}t)\} \cong m\{(\partial u/\partial t) = m(\partial^2\xi/\partial t^2) = \varepsilon A ds(\partial^2\xi/\partial t^2)$$

(NB The second-order terms of derivatives $(\partial u/\partial s)(\mathrm{d}s/\mathrm{d}t)$ are neglected above) the following equation is obtained

$$\frac{\partial^2\xi}{\partial t^2} = \frac{E}{\varepsilon}\frac{\partial^2\xi}{\partial s^2} \tag{29}$$

Here ε is the density. Equation (29) is the well-known equation of longitudinal elastic wave. By expressing the differential ds in terms of a Cartesian coordinate, equation (29) can be written as

$$\frac{\partial^2\xi}{\partial t^2} - \frac{E}{\varepsilon}(\nabla^2\xi) = 0 \tag{30}$$

With the use of covariant and contravariant gradient $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$, $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$, display equation equation (30) can be written as

$$(\partial_\mu)(\partial^\mu\xi) = \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} - \nabla^2\xi = 0 \tag{31}$$

Comparison of equations (30) and (31) indicates that c is the phase velocity of the longitudinal elastic wave yielded by equation (29)

$$c = \sqrt{E/\varepsilon} \tag{32}$$

When shear force is dominant (Fig. 7(b)), the same argument can be repeated to yield an equation of transverse, elastic wave with the phase velocity being associated with the shear modulus in place of the Young's modulus E in equation (32).