Scale relativity and gauge invariance*

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Abstract

The theory of scale relativity extends Einstein's principle of relativity to scale transformations of resolutions. It is based on the giving up of the axiom of differentiability of the space-time continuum. As a consequence, space-time becomes fractal, i.e., explicitly resolution-dependent. The requirement that this geometry satisfies the principle of scale relativity leads to introduce scale laws having a Galilean form (constant fractal dimension), then a log-Lorentzian form. In this framework, the Planck length-time scale becomes a minimal impassable scale, invariant under dilations. Then we attempt to construct a generalized scale relativity which includes scale-motion coupling. In this last framework, one can reinterpret gauge invariance as scale invariance on the internal resolutions. This approach allows one to set new constraints in the standard model, concerning in particular the Higgs boson mass, which we find to be $\sqrt{2}m_W = 113.73 \pm 0.06$ GeV in a large class of models.

1 Introduction

The theory of scale relativity [10] studies the consequences of giving up the hypothesis of space-time differentiability. One can show [10] [12] [3] that a continuous but nondifferentiable space-time is necessarily fractal. Here the word fractal [7] is taken in a general meaning, as defining a set, object or space that shows structures at all scales, or on a wide range of scales. More precisely, one can demonstrate [13] that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathcal{L}$ is strictly increasing and tends to infinity when the resolution interval tends to zero, i.e. $\mathcal{L} = \mathcal{L}(\varepsilon)_{\varepsilon\to 0} \to \infty$. This theorem naturally leads to the proposal that the concept of fractal space-time [15] [16] [10] [5] is the geometric tool adapted to the research of such a new description.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. We thus need to complete the standard laws of physics (which are essentially laws of motion in classical physics) by laws of scale, intended to describe the new

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resolution dependence. We have suggested [9] that the principle of relativity can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's formulation of the principle of relativity, by requiring that the laws of nature be valid in any reference system, whatever its state. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity and acceleration). In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of scale-covariance, requiring that the equations of physics keep their form under resolution transformations.

Among the consequences of this approach, one is able to recover the main axioms of quantum mechanics [10] [12] [6] [17] [18] from a geometric description using the concepts of general relativity (metric element, geodesics), in accordance with Conrad's [4] account of the profound unity that underlies the general relativistic and quantum mechanical behavior. In the present paper, we shall briefly review various levels of development of the theory, then consider some of its consequences in the domains of gauge field theories.

2 Galilean scale relativity

As we shall first see, simple fractal scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws. Indeed, let us consider a non-differentiable coordinate $\mathcal{L}$. Our basic theorem that links non-differentiability to fractality implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. As a first step, one can assume that $\mathcal{L}(\varepsilon)$ satisfies the simplest possible scale differential equation one may write, namely, a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, $d\mathcal{L}/d\ln\varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown but, still taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d\ln\varepsilon} = a + b\mathcal{L} + \dots \tag{1}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0\,[1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta}]. \tag{2}$$

Here δ is the scale dimension, i.e., $\delta = D - D_T$, the fractal dimension minus the topological dimension.

The Galilean structure of the group of scale transformation that corresponds to this law can be verified in a straightforward manner from the fact that $\delta\mathcal{L} = \mathcal{L} - \mathcal{L}_0$ transforms in a scale transformation $\varepsilon \to \varepsilon'$ as

$$\ln\frac{\delta\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln\frac{\delta\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon)\ln\frac{\varepsilon}{\varepsilon'} \quad ; \quad \delta(\varepsilon') = \delta(\varepsilon). \tag{3}$$

This transformation has exactly the structure of the Galileo group, as confirmed by the law of composition of dilations $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes $\ln \rho'' = \ln \rho + \ln \rho'$, with $\rho = \varepsilon'/\varepsilon$, $\rho' = \varepsilon''/\varepsilon'$ and $\rho'' = \varepsilon''/\varepsilon$.

Note however that such a structure, which remains Galilean from the viewpoint of scales, goes beyond general relativity from the viewpoint of the metric. Indeed, the above explicit resolution-dependent behavior of the length measured along a fractal coordinate can be written in terms of an explicit scale dependence of the metric itself [8]. Consider indeed, in order to simplify the argument, the one dimensional case ($dx = dy = dz = 0$). In general relativity, the proper time is no longer identical to the time coordinate, but is related to it through the metric element g_{00}, given in the central body case by:

$$ds^2 = g_{00}\, dt^2 = \left(1 - \frac{r_s}{r}\right) dt^2, \tag{4}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius. The correction in g_{00} with respect to the Minkowski value 1 is the manifestation of curvature and is understood as the gravitational potential at the Newtonian limit.

We can make a similar interpretation here for the fractal behavior and consider the resolution-dependent correction $\mathcal{L}_0(\lambda/\varepsilon)^\delta$ to the classical coordinate $\mathcal{L}_0$ as defining a metric element. Recall that, after derivation, we have in previous works expressed the above behavior in terms of a elementary displacement, itself made of two terms, a mean and a fluctuation, $dX^\mu = dx^\mu + d\xi^\mu$, such that $dx^\mu = u^\mu ds$, $< d\xi >= 0$ and $< d\xi^2 >= \lambda_c ds$. The new view here is that $d\xi^2$ can be interpreted as being nothing but a metric correction. In the case where not only space but also time is fractal (that gives rise to relativistic quantum mechanics, i.e. to the Klein-Gordon equation, see [12] Sec. 5.1), one obtains:

$$dS^2 = \left(1 - \frac{\tau}{\delta t}\right) dt^2 = \left(1 - \frac{\lambda}{r}\right) dt^2, \tag{5}$$

where $r = c\delta t$, $\tau = \hbar/E$ and $\lambda = c\tau$, i.e., in rest frame λ becomes the Compton length λ_c. In other words, we have now demonstrated the form of the metric that we had conjectured in [8] (basing ourselves at that time on electron-positron pair creation at the Compton scale). This form is the same as the Schwarzschild one, but with the black hole radius replaced by the Compton length and a different interpretation of the radial variable, then a different interpretation of the nature of the horizon: here, when δt becomes smaller than τ, the signature of the metric changes from $(- + ++)$ to $(+ + ++)$, simply indicating a change toward a regime where the fluctuations dominate (and they indeed combine in an Euclidean way, behaving as errors).

This result therefore provides full support to the suggestion by Sidharth [20] that particles can be understood as quantum mechanical Kerr-Newman black holes whose black hole radius is their Compton length, and also to the equivalent proposal by Conrad [4] that the superpositional collapse in quantum mechanics and the gravitational collapse in general relativity are actually associated phenomena.

3 Special scale-relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group. The same is true for scale laws. Indeed, if one looks for the general linear laws of scale that come under the principle of scale relativity, one finds that, once they are expressed in logarithm form, they have the structure of the Lorentz group [9]. Therefore, in special scale relativity, we have suggested to substitute to the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ the more general log-Lorentzian law:

$$\ln\frac{\varepsilon'}{\lambda} = \frac{\ln\rho + \ln(\varepsilon/\lambda)}{1 + \ln\rho\ln(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}\,. \tag{6}$$

The scale dimension (or "djinn", δ) becomes itself a variable. More precisely, the couple it makes with the fractal coordinate $[\delta, \ln(\mathcal{L}/\lambda)]$ becomes a scale vector. In the simplified case when $\delta(\lambda) = 1$ (i.e., fractal dimension 2 at transition scale) and $\mathcal{L}(\lambda) = \lambda$, it writes:

$$\delta(\varepsilon) = \frac{1}{\sqrt{1 - \ln^2(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}}, \tag{7}$$

where λ is the fractal-nonfractal transition scale (e.g., the Compton length of a particle). In such a law, there appears a minimal scale of space-time resolution which is invariant under dilations and contractions, and plays the same role for scales as that played by the velocity of light for motion.

Toward the small scales, this invariant length-scale is naturally identified with the Planck scale, $\lambda_P = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point. Some consequences of this new interpretation of the Planck length-time-scale have been considered elsewhere [9][12], concerning in particular the unification of fundamental fields.

4 Generalized scale-relativity and gauge invariance

The theory of scale relativity also allows one to get new insights about the physical meaning of gauge invariance [11] [12]. In the scale laws recalled hereabove, only scale transformations at a given point were considered. But we may also wonder about what happens to the structures in scale-space of a scale-dependent object such as an electron or another charged particle, when it is displaced. Consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$, where λ is the Compton length of the particle) for a given position of the particle. In a displacement, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. In other words, $\varepsilon = \varepsilon(x, t)$ is now a function of the space-time coordinates, and we expect the occurrence of *dilations of resolutions induced by translations*, so that we are led to introduce a covariant derivative:

$$e\,\frac{D\varepsilon}{\varepsilon} = e\,\frac{d\varepsilon}{\varepsilon} - A_\mu dx^\mu, \tag{8}$$

where a four-vector A_μ must be introduced since dx^μ is itself a four-vector and $d\ln\varepsilon$ a scalar (in the case of a global dilation), and where e is a charge.

However, if one wants such a "field" A_μ to be physical, it should be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$, we get the same expression as in Eq.(8), but with A_μ replaced by A'_μ. Therefore, we obtain the relation:

$$A'_\mu = A_\mu + e\,\partial_\mu \ln\rho, \tag{9}$$

which depends on the relative "state of scale", $\bar{V} = \ln\rho = \ln(\varepsilon/\varepsilon')$, that is now a function of the coordinates.

One may therefore identify A_μ with a 4-potential, and Eq.(9) with the property of gauge invariance. Now we know that applying a gauge transformation to the field implies to change also the wave function of the associated charged particle, that becomes:

$$\psi' = \psi\, e^{i4\pi\alpha \ln\rho} \tag{10}$$

where α is a coupling constant ($e^2 = 4\pi\alpha\hbar c$).

Now, while in Galilean scale relativity, the scale ratio ρ is unlimited, in the more general framework of special scale relativity it is limited by the Planck-scale/Compton-scale ratio. This limitation implies the quantization of charge, following a general mass-charge relation [12]: $\alpha \ln(m_P/m) = k/2$, where k is integer. In order to compare such a relation with experimental data, one should account for the electroweak theory, according to which the electromagnetic coupling is only 3/8 of its high energy value (plus radiative corrections). We get:

$$\frac{8}{3}\,\alpha_{em}\,\ln\left(\frac{m_P}{m_e}\right) = 1 \tag{11}$$

where $\alpha_{em} = 1/137.036$ is the low energy fine structure constant and m_e is the electron mass. This relation is indeed implemented by the experimental values with a relative precision of 2×10^{-3}, becoming 10^{-4} when accounting for threshhold effects [12].

The above result correspond to a unique global dilation and therefore build a U(1) symmetry. However, this approach can be generalized, since we are led in general scale relativity to define four different and independent dilations along the four space-time resolutions. The above U(1) field is then expected to be embedded into a larger field, in agreement with the electroweak and grand unification theories, and the charge e to be one element of a more complicated, "vectorial" charge. Some hints about such a generalization are given in the following section.

More generally, we shall be led to look for the general non-linear scale laws that satisfy the principle of scale relativity. Such a generalized framework implies working in a five-dimensional space-time-djinn in which scla and motion are treated on the same footing. The development of such a "general scale-relativity" lies outside the scope of the present paper and will be considered in forthcoming works.

5 Generalization to electroweak theory

The framework of generalized scale-relativity provides one with possibilities so suggest a new version of the electroweak theory and, as a consequence, to make a theoretical prediction of the value of the Higgs boson mass [14]. The (summarized) argument is as follows.

In today's electroweak scheme, the Higgs boson is considered to be separated from the electroweak field. Moreover, a more complete unification is mainly seeked in terms of attempts of "grand" unifications with the strong field. However, one may wonder whether, maybe in terms of an effective theory at intermediate energy, one could achieve a more tightly unified purely electroweak theory. Recall indeed that in the present standard model, the weak and electromagnetic fields are mixed, but there remains four free parameters, which can e.g. be taken to be the Higgs boson mass, the vacuum expectation value of the Higgs field and the Z and W masses. In the attempt sketched out hereafter, the Higgs field is assumed to be a part of the total field, so that only two free parameters would be left. As a consequence, the Higgs boson mass and the W/Z mass ratio could be derived in such a model.

Recall that the structure of the present electroweak boson content is as follows. There is a SU(2) gauge field, involving three fields of null mass (i.e. $2 \times 3 = 6$ degrees of freedom), a U(1) null mass field (2 d.f.) and a Higgs boson complex doublet (4 d.f.), which makes 12 degrees of freedom in all. Through the Glashow-Salam-Weinberg mechanism, 3 of the 4 components of the Higgs doublet become longitudinal components of the weak field which therefore acquires mass ($3 \times 3 = 9$ d.f.), while the photon remains massless (2 d.f.), so that there remains a Higgs scalar which is nowadays experimentally searched (1 d.f.).

Let us now consider four independant scale transformations on the four space-time resolutions, i.e., $(\ln \varepsilon_x, \ln \varepsilon_y, \ln \varepsilon_z, \ln \varepsilon_t)$, considered as intrinsic to the description and being now variable with space-time coordinates. This means that the scale space (i.e., here the gauge space) is at least four-dimensional (but note that this is not the final word on the subject, since this does not yet include the fifth "djinn" dimension δ). Moreover, the mixing relation between the B [U(1)] and W_3 [SU(2)] fields may also be interpreted as a rotation in the full gauge space. Therefore we expect the appearance of a 6 component antisymmetric tensor field (linked to the rotations in this space), corresponding in the simplest case to a SO(4) group. Such a zero mass field would yield 12 degrees of freedom by itself alone, so that it is able to include the electromagnetic and weak fields, but also the residual Higgs field.

We shall tentatively explore the possibility that the Higgs boson appears as a separated scalar only as a low energy approximation, while in the new framework it would be one of the components of the unified field (in analogy with energy appearing as scalar at low velocity, while it is ultimately a component of the energy-momentum four-vector).

Such an attempt is supported by the form of the electroweak Lagrangian (we adopt Aitchison's [1] notations). Its Higgs scalar boson part writes (in terms of the residual massive scalar σ):

$$L_H = \frac{1}{2}\,\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\,m_H^2 \sigma^2 - \frac{1}{8}\lambda^2 \sigma^4. \tag{12}$$

The vacuum expectation value v of the Higgs field is computed from the square (mass term) and quartic term, so that the Higgs mass is related to v and λ as:

$$m_H = \sqrt{2}\, v\, \lambda. \tag{13}$$

A prediction of the constant λ would therefore lead to a prediction of the Higgs mass. Now, a non-Abelian field writes in terms of its potential :

$$F^{\alpha\mu\nu} = \partial^\mu W^{\alpha\nu} - \partial^\nu W^{\alpha\mu} - g\, c^\alpha_{\beta\gamma} W^{\beta\mu} W^{\gamma\nu}, \tag{14}$$

where g is the (now unique) charge and $c^{\alpha}_{\beta\gamma}$ the structure coefficients of the Lie algebra associated to the gauge group. Its Lagrangian writes:

$$L_W = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{15}$$

Therefore, it includes W^4 terms coming from the W^2 terms in the field. Now our ansatz consists of identifying some of these W^4 terms, of coefficient $-\frac{1}{4}\, g^2\, (c^{\alpha}_{\beta\gamma})^2$, with the Higgs boson ϕ^4 term of coefficient $-\frac{1}{2}\,\lambda^2$ (where ϕ is the initial scalar doublet).

Namely, let us first separate the six components of the total field in two subsystems, $[W_1, W_2, W_3]$ and $[B_1, B_2, B_3]$. The three W's can be identified with the standard SU(2) field and, say, B_1 with the $\mathrm{U(1)_Y}$ field. Two vectorial fields remain, B_2^{μ} and B_3^{μ}. They will contribute in a non-vanishing way to the quartic term in the Lagrangian by their cross product. At the approximation (considered here) where their space components are negligible, we find:

$$-B_{2\mu}\, B_{3\nu}\, B_3^{\mu}\, B_2^{\nu} = -[B_2^0\, B_3^0]^2. \tag{16}$$

Finally, we make the identification of these time components with the residual Higgs boson, $B_2^0 = B_3^0 = \sigma$. This allows a determination of the constant λ according to the relation:

$$\lambda^2 = \frac{g^2\, c^2}{2}, \tag{17}$$

where the squared Lie coefficient c^2 is equal to 1 in the case of an SO(4) group. Provided the global charge is identical to the SU(2) charge, and since the W mass is given by $m_W = gv/\sqrt{2}$, one finally obtains a Higgs boson mass :

$$m_H = \sqrt{2c^2}\, m_W. \tag{18}$$

More generally, one must make the sum of all the terms that contribute to the final Higgs boson, and since the c's take the values 0, ± 1 for a large class of groups, we expect $m_H = \sqrt{2k}\, m_W$ with k integer. In particular, using the recently precisely determined W boson mass, $m_W = 80.42 \pm 0.04$ GeV [19],the simplest case $k = 1$ yields a theoretical prediction [14]:

$$m_H = \sqrt{2}\, m_W = 113.73 \pm 0.06 \text{ GeV}, \tag{19}$$

which is in agreement with current constraints and with a possible recent detection at CERN [2]. Although this calculation is still incomplete and although the self-consistency of this model remains to be established, we hope that at least some of its ingredients could reveal to be useful in more complete attempts [Lehner and Nottale, in preparation].

6 Conclusion

After having summarized the main lines of development of the theory, we have, in the present contribution, shown how they can be applied to the question of the nature of gauge invariance and of gauge quantum fields. This proposal allows one to bring new constraints to the standard model, in terms of a general relation that links coupling constants and mass scales and of a relation between the Higgs mass and the weak gauge boson mass.

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References

[1] Aitchison I., 1982, An informal introduction to gauge field theories (Cambridge University Press)

[2] ALEPH collaboration, 2000, Phys. Lett. B, in press

[3] Ben Adda F., Cresson J., 2000, C.R.Acad.Sci. Paris, Serie I, 330, 261

[4] Conrad M., 2000, in "Sciences of the Interface", International Symposium in honor of O. Rössler, ZKM Karlsruhe, 18-21 May 2000, Ed. H. Diebner

[5] El Naschie M.S., 1992, Chaos, Solitons & Fractals, 2, 211

[6] El Naschie M.S., 1995, in Quantum mechanics, Diffusion and Chaotic Fractals, eds. M.S. El Naschie, O.E. Rossler and I. Prigogine, (Pergamon), pp. 93, 185

[7] Mandelbrot B., 1983, The fractal geometry of nature (Freeman)

[8] Nottale L., 1989, Int. J. Mod. Phys. A4, 5047

[9] Nottale L., 1992, Int. J. Mod. Phys. A7, 4899

[10] Nottale L., 1993, Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity (World Scientific)

[11] Nottale L., 1994, in "Relativity in General", (Spanish Relativity Meeting 93), Eds. J. Diaz Alonso & M. Lorente Paramo (Editions Frontières), p.121.

[12] Nottale L., 1996, Chaos, Solitons & Fractals 7, 877

[13] Nottale L., 1997, A&A 327, 867

[14] Nottale L., 2000, in "Sciences of the Interface", International Symposium in honor of O. Rössler, ZKM Karlsruhe, 18-21 May 2000, Ed. H. Diebner

[15] Nottale, L., & Schneider, J., 1984, J. Math. Phys., 25, 1296

[16] Ord G.N., 1983, J. Phys. A: Math. Gen., 16, 1869

[17] Ord G.N., 1996, Chaos, Solitons & Fractals 7, 821

[18] Ord G.N., Deakin A.S., 1996, Phys. Rev. A 54, 3772

[19] Particle Data Group, 2000, The European Physical Journal, C15, 1 (http://www-pdg.lbl.gov/)

[20] Sidharth B.G., 2000, in "Frontiers of fundamental physics", Proceeding of Birla Science Center Fourth International Symposium, Hyderabad, 9-13 December 2000