Scaling approach to quantum non-equilibrium dynamics of many-body systems

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Understanding non-equilibrium quantum dynamics of many-body systems is one of the most challenging problems in modern theoretical physics. While numerous approximate and exact solutions exist for systems in equilibrium, examples of non-equilibrium dynamics of many-body systems, which allow reliable theoretical analysis, are few and far between. In this paper we discuss a broad class of time-dependent interacting systems subject to external linear and parabolic potentials, for which the many-body Schrödinger equation can be solved using a scaling transformation. We demonstrate that scaling solutions exist for both local and nonlocal interactions and derive appropriate self-consistency equations. We apply this approach to several specific experimentally relevant examples of interacting bosons in one and two dimensions. As an intriguing result we find that weakly and strongly interacting Bose-gases expanding from a parabolic trap can exhibit very similar dynamics.

I. INTRODUCTION

Understanding time evolution of complex quantum systems, often in the presence of strong correlations between constituent particles, is crucial for solving many fundamental problems in physics, from expansion of the early universe, to heavy ion collisions, to pump and probe experiments in solids. New questions of dynamical evolution arise in recently realized artificial quantum many-body systems, such as ultracold atoms in optical potentials or photons in media with strong optical nonlinearities. Some of these systems are not coupled to external heat baths and have a limited life-time, thus many experiments require interpretation in terms of coherent quantum dynamics rather than properties of equilibrium states. On the positive side, these systems allow remarkable control of parameters and open exciting opportunities for doing controlled experiments exploring non-equilibrium many-body dynamics.

In the realm of many-body physics low-dimensional systems have a special place. They have dramatically enhanced quantum and thermal fluctuations and exhibit most surprising manifestations of strong correlations. Rigorous theorems provide strong constraints on long- range order and often such systems cannot be analyzed using mean-field approaches even at zero temperature. Nevertheless, equilibrium properties are well understood using methods specific to low dimensions, such as Coulomb-gas representation of vortices in two dimensions or effective low energy descriptions of one-dimensional systems including Luttinger liquid and sine-Gordon models (see e.g. ref. [1]). However, such analysis cannot be straightforwardly extended to non-equilibrium dynamics. Most equilibrium theories focus on the low-energy part of the spectrum while non-equilibrium dynamics can couple degrees of freedom at very different energy scales [2, 3, 4, 5, 6, 7, 8]. It would be highly valuable to have examples of many-body dynamics of low-dimensional strongly correlated systems amenable to an unbiased analytical treatment. These examples could be used not only for analyzing experimental systems, but also for testing theoretical calculations utilizing effective models or approximations and for checking validity of new numerical approaches. In this paper we propose such a class of non-equilibrium quantum problems with time-dependent Hamiltonians which allow for a scaling ansatz of many-body wave functions.

Scaling solutions in quantum dynamic were first discussed in the context of a single harmonic oscillator with a time-dependent frequency [9, 10, 11, 12, 13]. This problem can be reduced to a time-independent one by properly rescaling space and time. Scaling transformation of variables is possible due to the existence of a dynamical symmetry generated by dynamical invariants of the system [11, 12]. There are also extensions of this approach to single particle problems with potentials of the Coulomb and inverse square type [14, 15]. In many-body problems, scaling approach has been previously used in the context of mean-field solutions of bosonic systems, i.e. for a classical Gross-Pitaevskii equation [16, 17, 18, 19, 20] or in the analysis of hard-core bosons in one dimension, which can be mapped to non-interacting fermions [21]. Here we extend previous analysis to the case of full many-body solution of *interacting* quantum systems.

In this paper we consider systems with a finite number of particles in arbitrary dimension, and assume that the effective mass, the interaction constant and the external potential can be *time-dependent*. Particles can obey fermionic, bosonic or mixed statistics, they interact via pairwise interaction, and be subject to parabolic confining potential, to a linear potential, and to a complex chemical potential. While in general this represents a complicated dynamical problem, we will demonstrate that for a wide class of time-dependent variations of parameters one can map the non-equilibrium equations of motion to an equilibrium many-body Schrödinger equation. The mapping is based on scaling functions which relate correlation functions of time-dependent systems to correlation functions of systems in equilibrium. Hence several results known for equilibrium many-body systems can be directly translated to non-equilibrium situations. To our knowledge this is the first example of a scaling solution for interacting many-body systems with time-dependent Hamiltonians.

Our scaling solution does not impose any constraints on the wave function but requires the interaction and the external potentials to be dynamically tunable. In systems of cold atoms control of the effective interaction can be done using either Feshbach resonances or by changing the transverse confin-

ing potential, whereas the effective mass can be changed by application of the weak optical lattice [22]. In quantum optics, the time-dependent dispersion and Kerr nonlinearity can be achieved using electromagnetically induced transparency [23]. In this paper we propose applications of the scaling ansatz which are experimentally relevant in the context of both of these systems.

The paper is organized as follows. In section II we introduce a general formalism of scaling transformation for a many-body Schrödinger equation. In section III we compute momentum distributions for one- and two-dimensional bosonic gases with contact interactions released from a parabolic trap. Further details are given in the Appendices, where we also discuss relation of our work to classical integrability of time-dependent bosonic systems with contact interactions.

II. SCALING TRANSFORMATION – GENERAL APPROACH

Our starting point is the many-body Schrödinger equation for N interacting particles in D dimensions,

$$i\frac{\partial\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N;t)}{\partial t}=H(t)\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N;t), \quad (1)$$

$$H(t)=-\frac{1}{2m(t)}\sum_{i=1}^{N}\Delta_{x_i}^{(D)}-\mu(t)N+\mathbf{g}(t)\sum_{i=1}^{N}\mathbf{x}_i + \frac{m(t)\omega^2(t)}{2}\sum_{i=1}^{N}\mathbf{x}_i^2+\sum_{i\neq j}V(\mathbf{x}_i-\mathbf{x}_j;t),$$

where $\Delta_{x_i}^{(D)}$ is a D-dimensional Laplacian acting on the coordinate $\mathbf{x}_i=(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(D)})$ of the particle i ($\hbar=1$ here). The external parameters (chemical potential $\mu(t)$, linear potential $\mathbf{g}(t)$ and trapping frequency $\omega(t)$) and the many-body interaction potential $V(\mathbf{x};t)$ depend explicitly on time. The chemical potential $\mu(t)=\Re[\mu(t)]+i\Im[\mu(t)]$ can accommodate effects of dissipation via its imaginary part [39].

We address the following question: under which conditions Eq. (1) (the Ψ-system) can be transformed into the Schrödinger equation for a time-independent (Φ-system):

$$i\frac{\partial\Phi(\mathbf{y}_1,\ldots,\mathbf{y}_N;\tau)}{\partial\tau}=H_0\Phi(\mathbf{y}_1,\ldots,\mathbf{y}_N;\tau), \quad (2)$$

$$H_0=-\frac{1}{2m_0}\sum_{i=1}^{N}\Delta_{y_i}^{(D)}+\frac{m_0\omega_0^2}{2}\sum_i\mathbf{y}_i^2+\sum_{i\neq j}V_0(\mathbf{y}_i-\mathbf{y}_j).$$

We emphasize that so far in (2) ω_0 and m_0 are unspecified parameters; in particular the Φ-system can have vanishing confining potential even when the Ψ-system is confined. We assume that the time dependence of the pairwise interaction potential enters through a single time-dependent coupling $V(\mathbf{x};t)\equiv V(\mathbf{x})v(t)$ and $V_0(\mathbf{x})=V(\mathbf{x})v_0$. We further assume that the interactions have a scaling property and are characterized by the exponent α, which we take to be the same for both Ψ- and Φ-systems,

$$V(\lambda\mathbf{x})=\lambda^{\alpha}V(\mathbf{x}). \quad (3)$$

Most generic interaction potentials (or pseudo-potentials) satisfy a scaling law (3): s-wave interactions $V_s(\mathbf{x})\propto\delta(\mathbf{x})$ ($\alpha=-D$), any algebraic law, $V(\mathbf{x})\propto|\mathbf{x}|^{\alpha}$, including Coulomb ($\alpha=-1$), inverse square law ($\alpha=-2$) or dipole-dipole interactions ($\alpha=-3$). Other examples are ultracold fermions interacting via p-wave channel which gives rise to the δ' pseudo-potential ($\alpha=D-1$). Also logarithmic potentials can be treated; scaling of the logarithmic law produces a time-dependent shift to $\mu(t)$.

To express the solution of the time-dependent Schrödinger equation (1) in terms of the solution $\Phi(\mathbf{y}_1,\ldots,\mathbf{y}_N;\tau)$ of the static equation (2) we introduce the scaling ansatz

$$\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N;t) = e^{i[F(t)\sum_{i=1}^{N}\mathbf{x}_i^2+\mathbf{G}(t)\sum_{i=1}^{N}\mathbf{x}_i+M(t)N]} \times \frac{1}{R^N(t)}\Phi(\mathbf{y}_1,\ldots,\mathbf{y}_N;\tau), \quad (4)$$

with $\mathbf{y}_i=(\mathbf{x}_i/L(t))+\mathbf{S}(t)$ and $\tau\equiv\tau(t)$. Direct calculation shows (se appendix A), that this ansatz is valid if the scaling functions $R(t),L(t),F(t),\tau(t),\mathbf{G}(t),\mathbf{S}(t),M(t)$ satisfy a set of coupled differential equations,

$$\dot{R}(t) = \frac{1}{m(t)}DF(t)R(t)-\Im[\mu(t)]R(t), \quad (5)$$

$$\dot{L}(t) = \frac{2}{m(t)}F(t)L(t) \quad (6)$$

$$\dot{F}(t) = -\frac{2}{m(t)}F^2(t)-\frac{m(t)\omega^2(t)}{2}+\frac{m_0^2\omega_0^2}{2L^4(t)m(t)}, \quad (7)$$

$$\dot{\tau}(t) = \frac{m_0}{m(t)L^2(t)}, \quad (8)$$

$$\dot{M}(t) = -\frac{\mathbf{G}^2(t)}{2m(t)}-\Re[\mu(t)]+\frac{m_0^2\omega_0^2\mathbf{S}^2(t)}{2m(t)L^2(t)}, \quad (9)$$

$$\dot{\mathbf{S}}(t) = -\frac{\mathbf{G}(t)}{m(t)L(t)}, \quad (10)$$

$$\dot{\mathbf{G}}(t) = -\frac{2F(t)\mathbf{G}(t)}{m(t)}-\mathbf{g}(t)+\frac{m_0^2\omega_0^2\mathbf{S}(t)}{m(t)L^3(t)}, \quad (11)$$

$$L^{\alpha+2}(t) = \frac{m(t)}{m_0}\frac{v(t)}{v_0}. \quad (12)$$

It is not obvious a priori that equations (5-12) can be satisfied simultaneously for any reasonable time-dependencies of system parameters $m(t)$, $v(t)$, $\omega(t)$. Our next goal is to show that there is a number of non-trivial cases for which equations (6-12) are consistent with each other. First of all we note that equations (5) and (6) imply that $R(t)=[L(t)]^{D/2}\exp(-\int_0^t\Im[\mu(t)]dt)$. In the absence of dissipation ($\Im[\mu(t)]=0$) this condition is equivalent to the conservation of the norm of the wave function under the scaling transformation. Eq. (6) allows to express $F(t)$ via $L(t)$, $F(t)=\frac{m(t)}{2}\dot{L}/L$, which can the be substituted into the Eq. (7). This leads to the differential equation for $L(t)$,

$$\ddot{L}(t)+h(t)\dot{L}(t)+\omega^2(t)L(t)=\frac{m_0^2\omega_0^2}{m^2(t)L^3(t)}, \quad (13)$$

where $h(t) = m_0\dot{m}(t)/m(t)$. The term with the first derivative can be removed by the change of variables $L(t) = \exp[B(t)]y(t)$ with $\dot{B}(t) = -h/2$. For $y(t)$ we obtain

$$\ddot{y}(t) + \Omega^2(t)y(t) = \frac{\omega_0^2}{y^3(t)}\,, \tag{14}$$

where $\Omega^2(t) = \frac{1}{4}h^2 - \frac{1}{2}\dot{h} + \omega^2(t)$. Eq. (14) is the celebrated Ermakov equation [25] first discovered in 1880 [26]. This equation has been used primarily for tracking invariants of the time-dependent harmonic oscillator. In appendix B we show how one can use the non-linear superposition principle to reduce Eq. (14) to the linear equation. Once $L(t)$ is known, the remaining set of equations for $S(t), M(t), \mathbf{G}(t)$ can be solved directly.

In summary, to find time-dependent parameters which admit a scaling solution one can apply the following recipe: after specifying two time-dependent functions $\omega(t)$ and $m(t)$ one obtains a solution of the Ermakov equation (14) from which one determines time-dependent interaction strength $v(t)$ consistent with Eq. (12). Solutions for the functions $M(t), \mathbf{G}(t), \mathbf{S}(t)$ can then be obtained straightforwardly provided that functions $\mathbf{g}(t)$ and $\mu(t)$ are explicitly specified.

The initial conditions for systems (1) and (2) are related to each other through Eq. (4) applied at time $t = 0$:

$$\begin{aligned}\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N;0) &= e^{i[F(0)\sum_{i=1}^N \mathbf{x}_i^2 + \mathbf{G}(0)\sum_{i=1}^N \mathbf{x}_i + M(0)N]} \\ &\times \frac{1}{R^N(0)}\Phi\left(\frac{\mathbf{x}_1}{L(0)},\ldots,\frac{\mathbf{x}_N}{L(0)};\tau(0)\right).\end{aligned}$$

Generally at $t = 0$ the Hamiltonians controlling the dynamics of Ψ- and Φ-systems do not coincide. For example, they can have different confining potentials, or one system can be in a trap while the other one is in free space ($\omega_0 = 0$). In this paper we focus on a finite initial trapping potential, $\omega(0) = \omega_0 > 0$, for which we introduce the additional assumption that at $t = 0$ the two systems coincide. This means that we have $m(t = 0) = m_0, v(t = 0) = v_0$ $F(t = 0) = G(t = 0) = M(t = 0) = 0$. At $t > 0$ the parameters of the Ψ-system begin to change in time while the parameters of the Φ-system remain constant. Since the two systems coincide for $t < 0$, the initial state of the Ψ-systems at $t = 0$ should correspond to the equilibrium state of the Φ-system. Existence of the scaling solution in one dimension in the hard-core limit $v_0 \to \infty$ has been established previously [21]. Within our approach this can be understood as follows: the first equation of (12) is trivially satisfied, whereas other equations do not depend on the interaction strength and remain valid.

III. DYNAMICS OF BOSE-GAS WITH CONTACT INTERACTION RELEASED FROM THE TRAP

In this section we apply the scaling approach to an ultracold Bose gas with contact interaction which is prepared in a confined, weakly interacting initial state. The nontrivial dynamics comes from a sudden switching off of the confining potential from $\omega(t) = \omega_0$ at $t = 0$ to $\omega(t) = 0$ at $t > 0$. Solution of the scaling equation (13) for constant mass $m(t) = m_0$, is then given by $L(t) = \sqrt{(1 + \omega_0^2 t^2)}$, and consequently $F(t) = \frac{m_0\omega_0^2 t}{2}/L^2(t)$. In appendix B 3 we examine additional scenarios corresponding to varying mass which exhibit similar behavior of the scaling functions. We recall that the time-dependence of the interaction strength is determined by the the scaling function and exponent α, here $v(t) = v_0 L(t)^{\alpha+2}$.

To characterize the non-equilibrium dynamics it is convenient to deal with correlation functions which can be easily derived within the scaling approach (Appendix D). The dynamics of the momentum distribution, for example, can be related to the single-particle density matrix g_1 of the initial state,

$$\begin{aligned}n(\mathbf{p},t) &= [L(t)]^D \int_{-\infty}^{\infty} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{x}' g_1(\mathbf{x},\mathbf{x}';0) \\ &\times \mathrm{e}^{-i[F(t)L^2(t)(\mathbf{x}^2 - \mathbf{x}'^2) + L(t)\mathbf{p}(\mathbf{x}-\mathbf{x}')]}\,.\end{aligned} \tag{15}$$

From the asymptotic behavior of the scaling functions $L(t) \xrightarrow{\omega_0 t \gg 1} \omega_0 t$ and $F(t)L(t) = m(t)\dot{L}(t)/2 \xrightarrow{\omega_0 t \gg 1} m_0\omega_0/2$ we can extract the long-time limit of the momentum distribution using the stationary phase approximation (SPA),

$$n(\mathbf{p},t) \xrightarrow{\omega_0 t \gg 1} \left(\frac{2\pi}{m(t)\dot{L}(t)}\right)^D g_1\left(\frac{\mathbf{p}}{m(t)\dot{L}(t)}, \frac{\mathbf{p}}{m(t)\dot{L}(t)}; 0\right).$$

Hence the momentum distribution becomes fully determined by the *density* distribution [$\rho(\mathbf{x},t) = g_1(\mathbf{x},\mathbf{x},t)$] of the initial state.

For a quantitative description of dynamics we need to specify the initial correlation function, which we take from earlier analysis of effective theories for weakly interacting Bose-gases in harmonic traps [27, 28, 29]. An important characteristic for a condensed state with a sufficiently large number of particles is the Thomas-Fermi shape of the density profile, $\rho(\mathbf{x}) = \Theta(R_{TF} - |\mathbf{x}|)(\mu/v_0)\left(1 - (\mathbf{x}/R_{TF})^2\right)$, where $R_{TF} = \sqrt{2\mu/m_0}/\omega_0$ is the Thomas-Fermi radius.

First we analyze the one-dimensional case in the low-temperature regime when the coherence length is of the order of the Thomas-Fermi radius (Eq. (E1) of Appendix D). According to the scaling equation (12), for contact interactions, $V(\mathbf{x},t) = v(t)\delta(\mathbf{x})$ ($\alpha = -D$), the interaction must be tuned inversely proportional to the scaling function, $v(t) = v_0/L(t)$. In Fig. 1a results of numerical evaluation of the momentum distributions (15) for specific initial values are shown together with results from SPA. The behavior of the $\mathbf{p} = 0$ component is characterized by a steep decay on a time scale ω_0^{-1} followed by slowly dephasing oscillations, which are due to the finite extension of the density profile and the quadratic phase factor in (15). The corresponding period of oscillations P is determined by the Thomas-Fermi radius, $P \sim \frac{2\pi m_0}{\hbar R_{TF}^2}$. Oscillations as a function of $|\mathbf{p}|$ at constant t can be attributed to the finite Thomas-Fermi radius as well. Here the quadratic phase factor leads to the oscillation period growing with $|\mathbf{p}|$. In agreement with the SPA prediction, the momentum distribution relaxes to a semi-circle law. This is remarkable, since such a behavior has been previously associated with one-dimensional Bose-systems in the *strongly* interacting limit

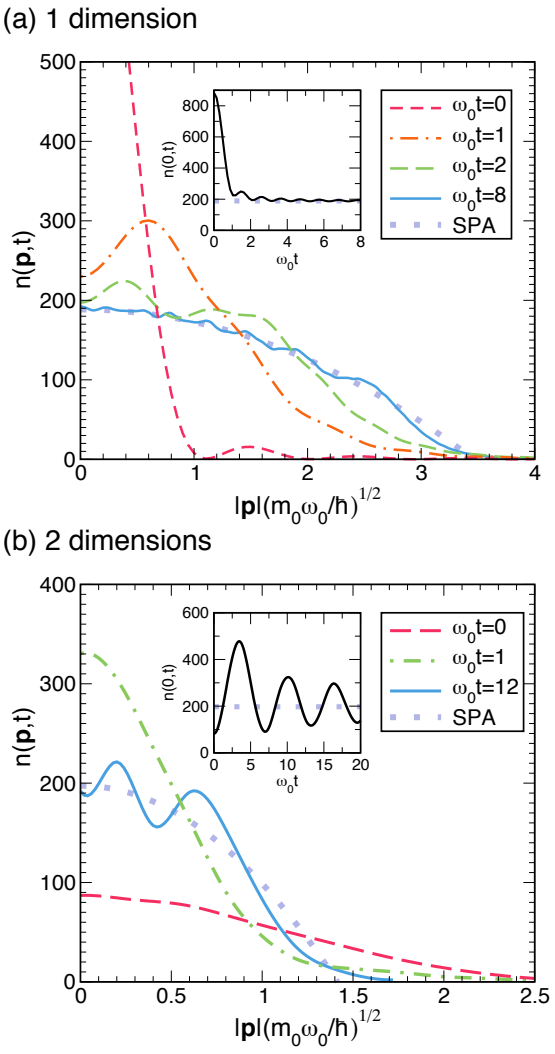

FIG. 1: Temporal evolution of momentum distribution functions following turning off the trap at $t = 0$. The insets show the time evolution of the $\mathbf{p} = 0$ component. The initial correlation functions are derived from effective theories (Refs. [27, 28, 29], see also appendix D). Dynamical evolution is obtained from numerical integration of eq. (15). The stationary phase approximation (SPA) represents the asymptotic $t \to \infty$ result. Numerical errors are of the order of the line thickness. In the one-dimensional case (a) the system parameters are $N = 140$, $k_B T = 0.1\hbar\omega_0$, $v_0 = 0.2\sqrt{\hbar^3\omega_0/m_0}$, $R_{TF} = 3.46\sqrt{\hbar/(m_0\omega_0)}$, $v(t) = v_0\sqrt{(1+\omega_0^2 t^2)}$. In the two-dimensional case (b) the interaction strength is constant, $v(t) = v_0$ and $N = 16$, $k_B T = 0.1\hbar\omega_0$, $v_0 = 0.2\hbar/m_0$, $R_{TF} = 1.41\sqrt{\hbar/(m_0\omega_0)}$.

($v_0 \to \infty$) [21] only. In our case the interaction strength is initially small and then even decreases in time. We note that this can not be understood as effect of dilution due to expansion of the system because the effective one-dimensional interaction parameter [30], $\gamma \propto v(t)/\rho(t) \propto v(t)L(t)$, remains constant.

In two dimensions $\alpha = -2$ and eq. (12) leads to interactions which are constant in time. When the initial state is weakly interacting (Appendix D), we choose an effective theory which incorporates effects of quantum and thermal fluctuations [28]. Results of numerical evaluation of Eq. (15) are shown in Fig. (1b). The momentum distribution evolves very much like in the one-dimensional case and is essentially determined by the initial density distribution and the associated Thomas-Fermi radius. Here the number of particles ($N = 16$) is set to be smaller than in the one-dimensional system. Therefore the asymptotic stationary phase solution is approached slowly and oscillations dominate in the analyzed time window $\omega_0 t \leq 20$. We checked that both in one and two dimensions the results are robust against variation of temperature and interactions as long as phase coherence is not destroyed.

The analysis of these examples leads to remarkable consequences. We note that the stationary phase regime is reached rather quickly with momentum distribution determined by the initial density distribution. Therefore specially designed initial density distributions (equilibrium or not) can be used to *create* specific momentum distributions, such as step-like fermionic ones, on demand. Previously such behavior of the momentum distribution function has been obtained for hard core bosons. It is remarkable that we find very similar dynamics for weakly interacting systems. This is opposite to what is realized in time-of-flight experiments of ultracold atoms released from a lattice [31], where the expansion at sufficiently large times can be regarded as free and momentum distributions get mapped to density profiles. By contrast in our case we find that the real space density profile in the trap determines momentum distribution after expansion (see eq. (15)). While we do not discuss the appropriate time evolution of $\omega(t)$, $m(t)$, and $v(t)$ here, we point out that the time-of-flight 'far-field' limit [31] may also be captured formally by our scaling approach when the asymptotics of $L(t)$ are linear and the contribution of the quadratic phase factor in Eq. (15), $m(t)\dot{L}(t)$, vanishes in the long-time limit.

IV. CONCLUSIONS AND OUTLOOK

We used scaling ansatz to show that certain quantum non-equilibrium problems with time-dependent parameters can be related to equilibrium problems with constant parameters provided that the time-dependent parameters satisfy a system of self-consistency equations. This approach is valid for rather general types of interactions and is not linked to the integrability of the model. However, an integrable structure, when it exists, is consistent with the scaling transformation. Solvability by the scaling ansatz is a consequence of the non-relativistic dynamical symmetry which received considerable attention recently in relation to the non-relativistic version of AdS/CFT correspondence [32]. The appearance of this symmetry in realistic many-body systems, which we discuss in this paper, can open intriguing connections to the concept of AdS/CFT correspondence.

We used scaling approach to analyze the problem of an abrupt switching off of a confining potential for bosoinc systems with contact interactions in $d = 1$ and 2. Such experiments can be performed using either ultracold atoms or photons in non-linear medium. We find that the asymptotic momentum distribution is essentially given by the initial density profile – a phenomenon which previously has been discussed

only in the (Tonks-Girardeau) limit of the infinitely strong repulsive one-dimensional Bose gas [21]. Possible future applications of the scaling ansatz include interaction quenches or transport phenomena (by considering finite linear potentials). Extensions of our method to systems with dissipation are also possible.

In our analysis we considered the situation when the scaling ansatz is obeyed exactly. We expect however that our results remain qualitatively valid even for systems with small deviations from the exactly scalable Hamiltonians. For example, weak lattice potentials should not have dramatic effects as long as the effective mass approximation is applicable. Therefore one could achieve a full description of time-of-flight experiments if the lattice potential and interactions are tuned accordingly. Moreover it is conceivable that on a phenomenological level the ansatz can be used even when the time- and space-dependencies of system parameters do not fully satisfy the consistency equations. The scaling solution could then be seen as a universality class of non-equilibrium systems, very much like a renormalization group fixed point at equilibrium. It would be interesting to address this conjecture in experiments.

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APPENDIX A: DERIVATION OF THE SCALING EQUATIONS

We consider the ansatz (4)

$$\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N;t)=\frac{1}{R(t)}\exp(i[F(t)\sum_{i=1}^{N}\mathbf{x}_i^2+\mathbf{G}(t)\sum_{i=1}^{N}\mathbf{x}_i+M(t)])\Phi(\frac{\mathbf{x}_i}{L(t)}+\mathbf{S}(t);\tau(t)) \tag{A1}$$

for the transformation between the many-body Schrödinger equation with time-dependent parameters (Eq. (1) and the equation (2) with time-independent coefficients. Calculating directly

$$\dot{\Psi} =(-\frac{\dot{R}}{R^2}+\frac{i\dot{F}}{R}\sum_{i=1}^{N}\mathbf{x}_i^2+\frac{i\dot{\mathbf{G}}}{R}\sum_{i=1}^{N}\mathbf{x}_i+i\frac{\dot{M}}{R})e^{i\phi(x_i,t)}\Phi(\mathbf{y}_i,\tau)) \tag{A2}$$

$$+ \frac{1}{R}e^{i\phi(x_i,t)}\sum_{i=1}^{N}\frac{\partial\Phi(\mathbf{y}_i;\tau)}{\partial\mathbf{y}_i}[\mathbf{x}_i(-\frac{\dot{L}}{L^2})+\dot{\mathbf{S}}(t)]+\frac{1}{R}e^{i\phi(x_i,t)}\frac{\partial\Phi(\mathbf{y}_i;\tau)}{\partial\tau}\dot{\tau}, \tag{A3}$$

where for the sake of brevity we introduced $\phi(\mathbf{x}_i,t)=F(t)\sum_{i=1}^{N}\mathbf{x}_i^2+\mathbf{G}(t)\sum_{i=1}^{N}\mathbf{x}_i+M(t)$ and where the dot denotes the derivative with respect to t, and

$$\frac{\partial\Psi(\mathbf{x}_i,t)}{\partial\mathbf{x}_i} = \frac{1}{R}\left(2iF\sum_i\mathbf{x}_i+\mathbf{G}\right)e^{i\phi(x_i,t)}\Phi(\mathbf{y}_i,t)+\frac{1}{R}e^{i\phi(x_i,t)}\frac{\partial\Phi(\mathbf{y}_i,\tau)}{\partial\mathbf{y}_i}, \tag{A4}$$

$$\Delta_{x_i}^{(D)}\Psi(\mathbf{x}_i,t) = [\left(\frac{2iFD}{R}+\frac{1}{R}(2iF\mathbf{x}_i+i\mathbf{G})(2iF\mathbf{x}_i+i\mathbf{G})\right)\Phi(\mathbf{y}_i,\tau)) \tag{A5}$$

$$+ \left(\frac{4iF\mathbf{x}_i+2i\mathbf{G}}{RL}\frac{\partial\Phi(\mathbf{y}_i;t)}{\partial\mathbf{y}_i}+\Delta_{y_i}^{(D)}\Phi(\mathbf{y}_i;t)\frac{1}{RL^2}\right)]e^{i\phi(x_i,t)}. \tag{A6}$$

Substituting this into the initial Schrödinger equation (1) with time-dependent coefficients and adding and subtracting the term $A(t)\sum_i\mathbf{x}_i^2$ with yet to be determined function $A(t)$ we regroup the different contributions in front of $\Phi(\mathbf{y}_i,\tau)$, $\partial\Phi(\mathbf{y}_i,\tau)/\partial\mathbf{y}_i$, and Δ_{y_i}. Each group has several contributions proportional to $\mathbf{x}_i^0,\mathbf{x}_i,\mathbf{x}_i^2$ which are linearly independent and must be treated separately. This is how conditions expressed by Eqs.(7) appear. The remaining equation has the form of a Schrödinger equation with time-dependent coefficients

$$i\frac{\partial\Phi(\mathbf{y}_i,\tau)}{\partial\tau}\dot{\tau}=-\frac{1}{2m(t)L^2(t)}\Delta_{y_i}\Phi(y_i,\tau)+\left[A(t)L^2(t)\sum_i\mathbf{y}_i^2+L^\alpha(t)v(t)V(\mathbf{y}_i-\mathbf{y}_j)\right]\Phi(\mathbf{y}_i,\tau). \tag{A7}$$

We note that to compensate the terms appearing after the change $\mathbf{x}_i\to\mathbf{y}_i$ in the quadratic potential we get terms proportional to ω_0^2 in the Eqs. (6-12). Now, requiring that the three unknown functions $\tau, L(t), A(t)$ satisfy

$$\dot{\tau}=\frac{m_0}{L^2(t)m(t)},\qquad v_0\dot{\tau}=v(t)L^\alpha(t),\qquad A(t)L^2(t)=\dot{\tau}\frac{m_0\omega_0^2}{2} \tag{A8}$$

we obtain the remaining conditions in the set of Eqs.(5-12). Under this conditions the Schrödinger equation for the function $\Phi(y,\tau)$ has no time-dependent coefficients. From the conditions (A8) above we determine the function

$$A(t)=\frac{m_0\omega_0^2\left[(v(t)m(t)\right]^{\frac{4}{\alpha+2}}}{2m(t)v_0^{\frac{4}{\alpha+2}}} \tag{A9}$$

Therefore we find that when pairwise potentials obey Eq. (3), and the systems of Eqs (5) is satisfied, Eq. (1) is indeed mapped to Eq. (2).

APPENDIX B: ANALYSIS OF THE SCALING EQUATIONS AND THEIR SOLUTIONS – THE ERMAKOV EQUATION AND DYNAMICAL SYMMETRY

1. General properties of the Ermakov and related equations

In this Appendix we briefly overview some general properties of the Ermakov (sometimes spelled as Yermakov) equation which plays such a fundamental role in our formalism. We also point out the relation of this equation with the Riccati equation and with the linear differential equation with variable coefficients. The Riccati equation directly appears in our approach in some limiting cases.

The Ermakov [26] equation is defined as follows

$$\ddot{y}(t) + f(t)y(t) = \frac{a}{y(t)^3}. \tag{B1}$$

Here a is some t-independent constant. If there is a nontrivial solution of the second order differential equation

$$\ddot{x}(t) + f(t)x(t) = 0 \tag{B2}$$

then the transformation

$$\xi = \int \frac{dt}{x^2(t)}, \qquad z = \frac{y}{x} \tag{B3}$$

puts the Ermakov equation into the form

$$\ddot{z} = az^{-3}. \tag{B4}$$

The solution for the initial equation then follows immediately

$$C_1 y^2 = ax^2 + x^2(C_2 + C_1 \int \frac{dt}{x^2})^2 \tag{B5}$$

where $C_{1,2}$ are arbitrary constants. If we take two solutions of the linear (Hill) equation to satisfy initial data $x_1(0) = x_1$, $\dot{x}_1(0) = \dot{x}_1$ while $x_2(0) = 0, \dot{x}_2 \neq 0$ then a general solution of the Ermakov equation is given by a nonlinear superposition principle,

$$y(t) = \sqrt{x_1^2(t) + \frac{1}{w^2}x_2^2(t)} \tag{B6}$$

where $w = x_1\dot{x}_2 - x_2\dot{x}_1$ is a constant Wronskian.

Now, provided the linear equation for $x(t)$ is satisfied, the function $u(t)$ defined as

$$x(t) = \exp(-\int_0^t u(t)dt) \tag{B7}$$

satisfies the Riccati equation,

$$\dot{u} - u^2 = f(t) \tag{B8}$$

This demonstrates that all three equations are closely related: Ermakov, linear second order differential equation with variable coefficients and the Riccati equation. Other remarkable equations are also connected to the Ermakov equation. For example (taking $a = 1$ for simplicity in (B4)) and defining $\xi(t) = z(t)^{-2}$ we obtain $\xi\ddot{x}i - (3/2)(\dot{\xi})^2 + 2\xi^4 = 0$. Now, defining $w(t)$ via $\xi(t) = \alpha\dot{w}/w$ with $\alpha^2 = -1/4$ we obtain a Kummer-Schwarz equation $\dot{w}\dddot{w} - (3/2)(\ddot{w})^2 = 0$.

In some limiting situations (e.g. $\omega_0 = 0$, see the next appendices) the Riccati equation appears naturally in our approach, so we sketch some of its properties here. The general Riccati equation with time-dependent coefficients

$$\dot{u}(t) = f(t)u^2(t) + g(t)u(t) + h(t) \tag{B9}$$

can be transformed into the second order differential equation

$$f(t)\ddot{y}(t) - [\dot{f}(t) + f(t)g(t)]\dot{y}(t) + f^2(t)h(t)y(t) = 0 \tag{B10}$$

by the following substitution $y(t) = \exp(-\int f(t)u(t)dt)$. In many cases a particular solution of (B10) is easier to find than the one for the (B9).

The Riccati equation has a remarkable property: if there is a known particular solution $u_0(t)$ of (B9), then the general solution of (B9) is given by

$$u(t) = u_0(t) + \Phi(t)\left[C - \int f(t)\Phi(t)dt\right]^{-1} \tag{B11}$$

$$\Phi(t) = \exp\left[\int (2f(t)u_0(t) + g(t))dt\right] \tag{B12}$$

where C is an arbitrary constant. The particular solution $u_0(x)$ corresponds to $C = \infty$.

The property (B11) allows the construction of many solutions of (B9) for given functions $f(t), g(t), h(t)$. If, for example, $f(t) = 1, g(t)$ is arbitrary and $h(t) = -(a^2 + ag(t))$ a particular solution is $u_0(t) = a$, and a general solution is then

$$u(t) = a + \Phi(t)[C - \int \Phi(t)]^{-1}, \quad \Phi(t) = \exp(2at + \int g(t)dx) \tag{B13}$$

for arbitrary C. For example for $f(x) = 1, g(x) = 0, h(x) = bx^n$ we obtain

$$u(t) = -\frac{\dot{w}(t)}{w(t)}, \quad w(t) = \sqrt{t}[C_1 J_{\frac{1}{2k}}(\frac{1}{k}\sqrt{b}t^k) + C_2 Y_{\frac{1}{2k}}(\frac{1}{k}\sqrt{b}t^k)], \tag{B14}$$

$$k = \frac{1}{2}(n+2), \qquad \text{for} \qquad n \neq 2 \tag{B15}$$

$$u(t) = \frac{\lambda}{t} - t^{2\lambda}(\frac{t}{2\lambda+1}t^{2\lambda} + C)^{-1}, \qquad \text{for} \qquad n = -2, \tag{B16}$$

where λ is a root of $\lambda^2 + \lambda + b = 0$.

2. Relation to dynamical symmetry

The Ermakov equation has the symmetry algebra isomorphic to $sl(2, R)$, which is isomorphic to the algebra $so(2, 1)$ of rotations on the surface of one-sheet hyperboloid. The property (B11) of the Riccati equation is related to the covariance of the Riccati equation with respect to the fractional-linear transformations which are generated by the action of $sl(2, R)$ algebra: the general solution can be expressed as a combination of particular solutions. The same algebra (more explicitly, one of its form, $su(1, 1)$) appears as a dynamical symmetry of the quantum harmonic oscillator, where Ermakov equation appears as well. This has been first found in [9]. There a single quantum harmonic oscillator with time-dependent frequency has been solved using the methods of (adiabatic) invariants. An adiabatic invariant in this case is a function of a solution of the Ermakov equation. This approach has led to appearance of the Ermakov-Pinney type equation [26] in quantum mechanics (see e.g. [13] for a recent review). In [10] the same equation appears as a certain consistency condition on the time-dependent rescaling of coordinate and time in the wave function of the oscillator. It became clear that these two approaches, one based on dynamical invariants and the other on the scaling of dynamical variables, are equivalent. Indeed the rescaling procedure can be regarded as a transformation, generated by a certain symmetry group, i.e. $sl(2, R)$. The generators of this symmetry are operators corresponding to dynamical invariants. Therefore the successiveness of applicability of scaling transformation implies the presence of *dynamical* symmetry generated by the dynamical invariants [11, 12]. For this symmetry to hold one has to have a special class of potential terms in the single-particle Hamiltonian [14]. Physically interesting potentials correspond to the contact interaction, harmonic, Coulomb and inverse square laws. That is why the scaling approach has been applied to a Calogero-Sutherland model [15] and *classical* Gross-Pitaevski type systems [16, 17, 18, 19, 20]. The appearance of the $su(1, 1)$ dynamical symmetry in our non-relativistic systems suggests a possible connection to non-relativistic version of the AdS/CFT correspondence [32]. In fact the Virasoro algebra of any conformal field theory contains $su(1, 1)$ as subalgebra.

3. Specific solutions for $\omega_0 > 0$

We compare examples for decreasing trapping potential and constant, increasing and decreasing masses.

(a) *Constant mass* – For the case of constant mass $m(t) = m_0$ we choose an exponential decrease of the potential $\omega(t) = \omega_0 e^{-t/\tau_\omega}$. The two independent solutions of the homogeneous equation (B2) read $x_1(t) = J_0(2\tau_\omega\sqrt{\omega(t)})$, $x_2(t) =$

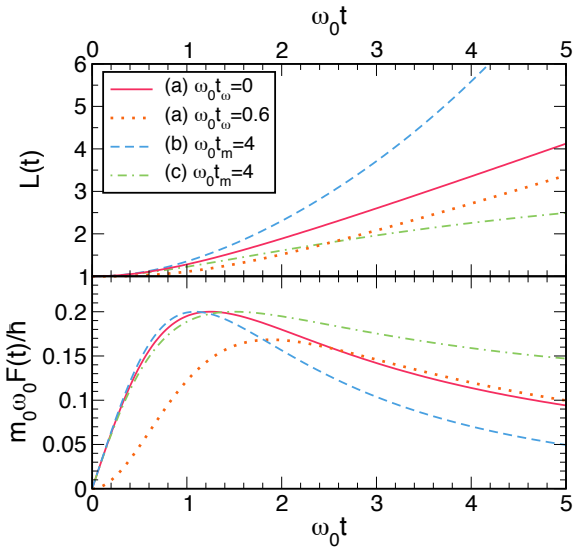

FIG. 2: Scaling functions for $\omega_0 > 0$ ($\hbar$ reinserted by dimensional analysis). Each curve corresponds to one of the cases (a)-(c) analyzed in the text.

$Y_0(2\tau_\omega\omega_0\sqrt{\omega(t)})$. In fig. 2 the resulting scaling functions obeying the initial conditions $L(0) = 1, F(0) = 0$ are plotted. For sufficiently small τ_ω the functions are well described by the limit $\tau_\omega \to 0$, for which the scaling solution reduces to

$$L(t) = \sqrt{(1 + \omega_0^2 t^2)}, \qquad F(t) = \frac{m_0\omega_0^2 t}{2}/L^2(t)\,. \tag{B17}$$

(b) *Increasing mass* – We choose $m(t) = m_0 e^{t/\tau_m}$ and, for sake of simplicity, $\omega(t > 0) = 0$. The solution then reads

$$L(t) = \sqrt{1 + (1 - e^{-t/\tau_m})\tau_m^2 m_0\omega_0^2}, \qquad F(t) = (1 - e^{-t/\tau_m})\tau_m^2 m_0\omega_0^2/L^2(t)\,, \tag{B18}$$

(plotted in fig. 2); this is similar to the scaling functions of the case (a), although the time is rescaled and in the limit $t \to \infty$ the functions converge to the values of the functions of case (a) at $t = \tau_m$.

(c) *Decreasing mass* – For $m(t) = m_0 e^{-t/\tau_m}$ the scaling functions take the form of case (b) when replacing τ_m by $-\tau_m$ (see fig. 2 for an illustration).

We emphasize that the solutions do not depend on the dimensionality of the system; only the interaction constants, which have to fulfill the consistency equation (12), will do so.

4. Specific solutions for $\omega_0 = 0$

Based on two examples we demonstrate within our formalism, that if we relate the non-equilibrium system in the trap to the system without trap (the case $\omega_0 = 0$ in the main text) we directly obtain a Riccati equation.

For $D = 1$ eq. 12 reads $L(t) = m_0(m(t)c(t))^{-1}$ (we define $c(t) = v(t)/v_0$) what we substitute in the equation for $L(t)$ to obtain $F(t) = -(m(t)/2)\frac{d}{dt}\log[c(t)m(t)/m_0]$. Consistency with the equation for $F(t)$ imposes the following relation between three time-dependent parameters

$$-\frac{\dot{m}(t)}{2}\frac{d}{dt}\log[c(t)m(t)] - \frac{m(t)}{2}\frac{d^2}{dt^2}\log[c(t)m(t)] = -\frac{m(t)}{2}(\frac{d}{dt}\log[c(t)m(t)])^2 - \frac{m(t)\omega(t)}{2}. \tag{B19}$$

By introducing $U(t) = \frac{d}{dt}\log(c(t)m(t))$ it reduces to the Riccati equation

$$\dot{U}(t) = \omega(t) - \frac{d}{dt}(\log[m(t)])U + U^2. \tag{B20}$$

The scaling ansatz (4) implies the relation between initial conditions of the two systems: $\Psi(t = 0) = \exp(iF(0)\sum_i x_i^2)\Phi(t = 0)$ provided that $L(t = 0) = 1$. The initial condition for the function $U(t)$ is not so important for us because of the special

property of the Riccati equation, related to the Bäcklund symmetry, which allows to interrelate solutions with different initial conditions via a rational function.

We note that the same equation describes the evolution of spin in a time-dependent magnetic field. A general way to solve it is to notice that under some change of variables it can be reduced to the second-order liner differential equation

$$\ddot{u} - P(t)\dot{u} + Q(t)u = 0, \qquad P(t) = -\frac{d}{dt}\log[m(t)], \qquad Q(t) = \omega(t). \tag{B21}$$

Numerous explicit solutions are possible if we specify some of the functions $\omega(t), m(t)$.

In the two-dimensional case we obtain from (5)-(12) that $R(t) \equiv L(t)$ and time-dependent parameters are connected by the constraint $c(t)m(t) = c_0$. Then $F(t) = (m(t)/2)\frac{d}{dt}\log[L(t)]$. Introducing $V(t) = \frac{d}{dt}\log L(t))$ and $h(t) = \frac{d}{dt}\log(m(t)/2)$ we obtain

$$-\frac{dV(t)}{dt} = \omega(t) + h(t)V(t) + V^2(t) \tag{B22}$$

which is a Riccati equation for the coordinate scaling function $L(t)$; its solution for given time-dependent parameters $m(t), \omega(t)$ then defines a solution for the time-rescaling function

$$\frac{d\tau(t)}{dt} = \frac{m_0}{m(t)L^2(t)} \tag{B23}$$

To be specific we list two examples of dynamical parameters:

(a) *Increasing mass* – From the form of the Riccati equation it is somewhat appealing to take $m(t) = m_0 e^{\alpha t}$, and constant $\omega(t) \equiv \Omega$. Then

$$c(t) = \phi(t)\exp[-\alpha t/2] \tag{B24}$$

where $\phi(t) = \sin(At + B)/C$ with A, B, C related to α and Ω. In particular, for $m(t) = e^{2t}$, where $\Omega = 1$,$A = B = C$ and $C \to 0$ we obtain $c(t) = (1 + t)e^{-t}$.

(b) *Constant mass* – For $m(t) \equiv m_0$ the equation can be transformed into the equation for the harmonic oscillator with time-dependent-frequency $\omega(t)$ for which many known solutions exist. Using these solutions we can extract the function $c(t)$. In particular, for constant $\omega(t) = \Omega$ the solution for some domain of parameters is

$$c(t) = \frac{1}{m_0\cos(\Omega t)}\,. \tag{B25}$$

In the simplest case of $m(t) = 1, \omega(t) = 0$ we obtain $c(t) = -1/(1 + t)$. This example is a many-body analogue of the solution of the Hamiltonian with potential $V(x) = c(t)\delta(x)$ found in ref. [14] for a single-particle Schrödinger equation. Direct application of this solution can be found in the ultracold Bose gas close to the confinement-induced resonance [33].

Other examples of solutions of (B9) can be found in the literature, see e.g. ref. [34].

APPENDIX C: CLASSICAL INTEGRABILITY OF THE NONLINEAR SCHRÖDINGER EQUATION WITH TIME-DEPENDENT PARAMETERS

It is instructive to check whether the exact scaling transformation we have studied in this paper is consistent with the property of integrability of the nonlinear Schrödinger equation (NSE). Here we address this question for the classical NSE.

In the zero curvature representation, the NSE

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + 2c|\Psi|^2\Psi \tag{C1}$$

is represented by the system of the first order differential equations

$$\frac{\partial F}{\partial x} = U(x,t,\lambda)F, \qquad \frac{\partial F}{\partial t} = V(x,t,\lambda)F, \qquad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{C2}$$

such that the matrices $U(x,t,\lambda)$ and $V(x,t,\lambda)$ which depend on the spectral parameter λ satisfy the condition

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0 \tag{C3}$$

which is equivalent to the compatibility condition of the system,

$$\frac{\partial^2 F}{\partial x \partial t} = \frac{\partial^2 F}{\partial t \partial x} \tag{C4}$$

and which is equivalent to the initial Schrödinger equation. In case of (C1) one can establish that

$$U = U_0 + \lambda U_1, \qquad V = V_0 + \lambda V_1 + \lambda^2 V_2 \tag{C5}$$

$$U_0 = \sqrt{c}(\bar{\Psi}\sigma_+ + \Psi\sigma_-), \qquad U_1 = \frac{1}{2} i\sigma_3 \tag{C6}$$

$$V_0 = ic|\Psi|^2\sigma_3 - i\sqrt{c}(\frac{\partial \bar{\Psi}}{\partial x}\sigma_+ - \frac{\partial \Psi}{\partial x}\sigma_-), \qquad V_1 = -U_0, \quad V_2 = -U_1 \tag{C7}$$

Conserved quantities are constructed from the matrices U, V in a known way. This method provides a direct way to various generalizations of NSE. In particular one can obtain some generalization where the interaction parameter c and the mass are explicitly time-dependent functions. Introducing generalization of (C5) as

$$\tilde{U} = \begin{pmatrix} -\frac{i}{2}\alpha(x,t) & \gamma(x,t)\bar{\Psi} \\ \gamma(x,t)\Psi & \frac{i}{2}\beta(x,t) \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} iA(|\Psi|^2, \lambda(x,t)) & B(\bar{\Psi}, \frac{\partial \bar{\Psi}}{\partial x}, \mu(x,t)) \\ B^*(\Psi, \frac{\partial \Psi}{\partial x}, \mu(x,t)) & -iD(|\Psi|^2, \lambda(x,t) \end{pmatrix} \tag{C8}$$

one can look for generalizations of integrable NSE by appropriately choosing the functions $\alpha(x,t), \beta(x,t), \gamma(x,t), \lambda(x,t), \mu(x,t), A, B, D$. Analysis of the zero-curvature condition (C3) in the case of inhomogeneous time-dependent functions leads to a set of equations between those functions and reveals a large class of solutions of the classical equations of motions for NSE with time-dependent coefficients. To get a consistency condition for a zero-curvature representation we conclude that the spectral parameter should be an inhomogeneous time-dependent function.

Some restricted form of this inhomogeneous time-dependent $\tilde{U} - \tilde{V}$ pair has been considered in ref. [35] where it was shown that a combination of space-time transformation together with a $U(1)$ gauge transformation of the linear equations for the $\tilde{U} - \tilde{V}$ pair and corresponding redefinition of the field variables brings the system into the form of a homogeneous time-independent NLS system, thus showing the integrability of a time-dependent system. We note that a similar analysis has been given in Ref. [36].

Although it is more difficult to show integrability on the quantum level directly, presumably the property of integrability is not violated in that case for specific choice of time-dependent parameters which correspond to our scaling equations. A related approach based on the inhomogeneity of spectral parameters for the quantum sine-Gordon model has been recently presented in ref. [37].

APPENDIX D: SCALING OF CORRELATION FUNCTIONS

With the scaling ansatz (4) the relation between the single-particle correlation functions in the time-dependent and time-independent systems is derived straightforwardly,

$$g_1^{(\Psi)}(\mathbf{x}, \mathbf{x}', t) = N \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d\mathbf{x}_2 \dots d\mathbf{x}_N \Psi^*(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N; t)\Psi(\mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_N; t) \tag{D1}$$

$$= \frac{1}{[L(t)]^D} g_1^{(\Phi)}\left(\frac{\mathbf{x}}{L(t)}, \frac{\mathbf{x}'}{L(t)}; 0\right) \exp\left(-iF(t)(\mathbf{x}^2 - \mathbf{x}'^2)\right). \tag{D2}$$

The labels in the g_1-function refer to the time-dependent (Ψ) and time-independent (Φ) systems. From this expression we can readily extract the density: $\rho^{(\Psi)}(\mathbf{x}, t) = g_1(\mathbf{x}, \mathbf{x}, t) = (1/L(t))\rho^{(\Phi)}(\mathbf{x}/L(t); 0)$. The momentum distribution of a time-dependent system, defined as

$$n^{(\Psi)}(\mathbf{p}, t) = \int_{-\infty}^{\infty} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{x}' e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})} g_1^{(\Psi)}(\mathbf{x}, \mathbf{x}', t), \tag{D3}$$

is then given by

$$n^{(\Psi)}(\mathbf{p}, t) = [L(t)]^D \int_{-\infty}^{\infty} d\mathbf{x} \int_{-\infty}^{\infty} d\mathbf{x}' g_1^{(\Phi)}(\mathbf{x}, \mathbf{y}; 0) \exp[-iF(t)L^2(t)(\mathbf{x}^2 - \mathbf{x}'^2) - iL(t)\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')]. \tag{D4}$$

Note that because of the quadratic term in the exponent the integrations are nontrivial.

For the two-particle density matrix we find analogously

$$\begin{aligned} g_2^{(\Psi)}(x_1,x_2,x_1',x_2';t) &= N(N-1)\int dx_3\dots dx_N \Psi^*(x_1,x_2,\dots,x_N;t)\Psi(x_1',x_2',\dots,x_N;t) \\ &= \frac{1}{L(t)^2} g_2^{(\Phi)}\left(\frac{x_1}{L(t)},\frac{x_2}{L(t)},\frac{x_1'}{L(t)},\frac{x_2'}{L(t)};0\right)\exp\left(-iF(t)(x_1^2+x_2^2-x_1'^2-x_2'^2)\right). \end{aligned} \tag{D5}$$

and the two-particle correlation function reads

$$\rho_2^{(\Psi)}(x,y;t) = g_2^{(\Psi)}(x,y,x,y;t) = \frac{1}{L^2(t)}\rho_2^{(\Phi)}\left(\frac{x}{L(t)},\frac{y}{L(t)};0\right). \tag{D6}$$

Other useful quantities such as non-equilibrium time-dependent correlation functions (e.g. $n^{(\Psi)}(p,t,t')$) or the multi-mode squeezing spectrum ($S(k,k';t,t') = \langle n^{(\Psi)}(p,t)n^{(\Psi)}(p',t')\rangle$) can also be easily computed using the scaling approach.

APPENDIX E: SOME TECHNICAL DETAILS RELATED TO THE DERIVATION OF 1D AND 2D MOMENTUM DISTRIBUTION AT EQUILIBRIUM

1. Trapped weakly interacting Bose gases

In order to describe a condensed Bose gas in a harmonic potential we adopt results of previous works [27, 28, 29] which consider phase fluctuations on top of the mean-field solution while density fluctuations are assumed to be negligible. This is a valid approximation for a sufficiently high number of weakly interacting particles at low temperatures. The temperature range where the density fluctuations are suppressed is $T_d \gg T \gg T_\phi$ where the temperature of quantum degeneracy is $T_d = N\hbar\omega_0$ and $T_\phi = T_d\hbar\omega_0/\mu$.

Generically, the single-particle correlation can be represented as

$$g_1(\mathbf{x},\mathbf{x}') = \sqrt{\rho(\mathbf{x})\rho(\mathbf{x}')}\exp\left(-\frac{1}{2}\langle(\phi(\mathbf{x})-\phi(\mathbf{x}'))^2\rangle\right), \tag{E1}$$

where $\langle\phi(\mathbf{x})\rangle$ denotes the average over phase fluctuations. We assume the validity of the Thomas-Fermi approximation for the density

$$\rho(\mathbf{x}) \to \rho_{TF}(\mathbf{x}) = \frac{\mu}{g}\left(1-(\frac{\mathbf{x}}{R_{TF}})^2\right)\theta(1-|\frac{\mathbf{x}}{R_{TF}}|), \tag{E2}$$

where $R_{TF} = \sqrt{2\mu/m_0}/\omega_0$ is the Thomas-Fermi radius.

In a 1D geometry, taking into account thermal fluctuations and neglecting contributions from quantum fluctuations, one obtains the phase average [27]

$$\langle(\phi(\mathbf{x}')-\phi(\mathbf{x}))^2\rangle = \frac{4T\mu}{\hbar^2\omega^2}\left|\ln\left[\frac{(1-\frac{\mathbf{x}'}{R_{TF}})(1+\frac{\mathbf{x}}{R_{TF}})}{(1+\frac{\mathbf{x}'}{R_{TF}})(1-\frac{\mathbf{x}}{R_{TF}})}\right]\right|. \tag{E3}$$

For the 2D case an expression similar to the 1D case can be derived. In this work we used the complete expression obtained by Xia et *al*. (Eq. (77) in Ref. [28]), which explicitly accounts for thermal *and* quantum fluctuations. As a result, at inter-particle distances much smaller than $2R_{TF}$ the correlations decay exponentially with a decay rate approximately given by $mk_BT/\left(2\pi\hbar^2\rho(0)\right)$. However, for the dynamics studied in this paper we did not find significant effects from quantum corrections.

2. One- and two-dimensional uniform Bose gases

For a one-dimensional Bose gas it was recently shown [38] that the effective field theory (Luttinger liquid) provides an extremely accurate description for a single-body correlation function at distances beyond the inter-particle separation. If we are

not interested in its large momentum behavior it is legitimate to use this effective theory. The single particle correlation function in time-independent theory is then well known (see e.g. [1]). For nonzero temperatures it is given by (we omit oscillating terms)

$$g_1^{(\Phi)}(x,x';0) = \langle \Phi^\dagger(x)\Phi(x')\rangle = \rho_0 B \left[\frac{\pi/\xi_T}{\rho_0 \sinh(\pi(x-x')/\xi_T)}\right]^{\frac{1}{2K}} \tag{E4}$$

where $\xi_T = \hbar v_s/T = \hbar^2\pi\rho/(m_0 KT)$, ρ_0 is the uniform equilibrium density, v_s is the sound velocity, K is a Luttinger parameter which is related to the interaction strength c and $B = (K/\pi)^{1/2K}$ is Popov's factor.

In the two-dimensional case, we consider a system below the Berezinskii-Kosterlitz-Thouless (BKT) transition. The correlation functions then decay algebraically with a temperature-dependent exponent, which tends to the universal value $1/4$ when approaching the BKT transition from below.