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Scaling in electron scattering from a relativistic Fermi gas

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Within the context of the relativistic Fermi gas model, the concept of "y scaling" for inclusive electron scattering from nuclei is investigated. Specific kinematic shifts of the single-nucleon response in the nuclear medium can be incorporated with this model. Suggested generalizations beyond the strict Fermi gas model, including treatments of separated longitudinal and transverse responses, are also explored.

I. INTRODUCTION

In the past few years considerable attention has been paid to the scaling behavior of the nuclear response for high-momentum-transfer electron scattering at excitation energies falling below the quasielastic peak. The concept of y scaling in nuclear physics was introduced by West[1] as a nonrelativistic analog of x scaling in high-energy physics. There it was shown that a collection of pointlike, nonrelativistic, charged "nucleons" with negligible final-state interactions leads to an inclusive quasielastic electron scattering cross section that can be written in terms of a factor containing a single-nucleon cross section times a specific function. In the limit of large momentum transfers, this function scales; that is, it becomes a function of only one variable, usually represented by y. Indeed, inclusive quasielastic electron scattering cross sections do appear to scale[2–5] even though the original assumption of negligible final-state interactions[6] appears to be questionable under the kinematical conditions which were used in obtaining some of these cross sections.

In the analysis of these experiments, it was clear that, at the large momentum transfers where y scaling should occur, the kinematical aspects of special relativity and the effects of the finite size of nucleons cannot be ignored. As a result, extensions of y scaling to allow for relativistic kinematics have been introduced, specifically involving some relativistic analog of the y variable and the use of a factorized cross section that incorporates relativity into the treatment of the elastic single-nucleon cross section which multiplies the scaling function.[2,5] We note that there are problems associated with introducing relativity into descriptions of inclusive electron scattering. The simplest of these is that of the explicit energy dependence which occurs in the electron-nucleus cross section as a result of the Lorentz transformation of the single-nucleon current from the rest frame to frames moving with velocities corresponding to the spectrum in the nuclear momentum distribution. This energy dependence complicates any rigorous attempt to factorize the electron-nucleus cross section by extraction of a unique single-nucleon current. A more serious practical and conceptual problem is the question of the off-mass-shell extrapolation of the single-nucleon current which is a dynamical consequence of attempting a relativistic description of a system of interacting nucleons. Since the full functional complexity of the off-shell current cannot be determined experimentally, and no fundamental calculation of the subnucleonic degrees of freedom which give rise to this complexity exists to give theoretical guidance, there is no unique solution to this problem at present. Some attempts have been made to provide scaling variables which contain prescriptive corrections for off-shell effects and for possible effects arising from final-state interactions.[7]

Our aim in the present work is to investigate the straightforward kinematical effects of relativity without becoming embroiled in dynamical effects such as those due to the off-shell behavior of the single-nucleon response. By extension, we suggest that similar results may be obtained when a fully relativistic theory of nuclear dynamics is available. The natural starting point for such an examination is the relativistic Fermi gas model[8,9] in which the nucleus is described as a noninteracting gas of nucleons. In the simplest form of this model, where the momentum distribution is a step function, the quasielastic electron scattering cross section can be written analytically as in the nonrelativistic case. A simple prescriptive extension of the model to allow for more realistic momentum distributions, which we will refer to as the generalized Fermi gas model, is easily calculable and can be used to provide pseudodata to be analyzed in the context of the scaling variable and factorizations which we obtain from an examination of the simplest form of the model.

For clarity, it is useful to provide an overview of what must by its nature be a somewhat technical argument. The starting point of our study of the relativistic aspects of scaling is an examination of the form of the inclusive cross section for the simple relativistic Fermi gas model. From this a dimensionless scaling variable is chosen so that a parabolic scaling function can be defined which is identical to the nonrelativistic Fermi gas model. The relationship of our scaling variable to other simple scaling variables is examined. Extraction of our scaling function from the electron-nucleon cross section does not result in a residual factor which is simply proportional to the single-nucleon cross section, but in one which contains some dependence on the nuclear Fermi momentum (which, however, reduces to the single-nucleon cross section in the limit of vanishing Fermi momentum). The cross section resulting from the generalized Fermi gas model is used to generate pseudodata; these are divided by the residual factor obtained from the simple Fermi gas model in order to study the approach to scaling of the resulting function. The implications of such a treatment for the study of scaling of separated longitudinal and transverse response functions rather than for complete cross sections are also examined.

II. FORMALISM AND APPLICATION TO $^{16}O(e,e')$

A. Basic formalism

We begin by summarizing some of the basic formalism for inclusive electron scattering from nuclei in which an electron with four-momentum $K^\mu=(\epsilon,\mathbf{k})$ is scattered through an angle θ to four-momentum $K'^\mu=(\epsilon',\mathbf{k}')$. The four-momentum transferred in the process is then $Q^\mu=(K-K')^\mu=(\omega,\mathbf{q})$, where $\omega=\epsilon-\epsilon'$, $q=|\mathbf{q}|=|\mathbf{k}-\mathbf{k}'|$, and $Q^2=\omega^2-q^2\le 0$ (we employ the conventions and metric of Ref. 10), together with the notation where four vectors are indicated by using capital letters). Of course we shall consider only the extreme relativistic limit where $|\mathbf{k}|\cong\epsilon\gg m_e$ and $|\mathbf{k}'|\cong\epsilon'\gg m_e$, with m_e the electron mass. In the one-photon-exchange approximation, the double-differential cross section in the laboratory system can be written in the form

$$\frac{d^2\sigma}{d\Omega\, d\epsilon'}=\sigma_M\left[\left[\frac{Q^2}{q^2}\right]^2 R_L(q,\omega)+\left[\frac{1}{2}\left|\frac{Q^2}{q^2}\right|+\tan^2\frac{\theta}{2}\right]R_T(q,\omega)\right], \tag{1}$$

where $L\,(T)$ refer to responses with longitudinal (transverse) projections (i.e., with respect to the momentum transfer direction) of the nuclear currents, and where the Mott cross section is given by

$$\sigma_M=\left[\frac{\alpha\cos(\theta/2)}{2\epsilon\sin^2(\theta/2)}\right]^2, \tag{2}$$

with α the fine-structure constant. This cross section is obtained by contracting leptonic and hadronic current-current interaction electromagnetic tensors and so is proportional to $\eta_{\mu\nu}W^{\mu\nu}$. The leptonic tensor may be calculated in the standard way involving traces of Dirac γ matrices, yielding

$$\eta_{\mu\nu}=K_\mu K'_\nu+K'_\mu K_\nu-g_{\mu\nu}K\cdot K' . \tag{3}$$

Contracting this with a general hadronic tensor $W^{\mu\nu}$ and rewriting the cross section in Eq. (1), we have the following for the two response functions (summation convention on repeated indices):

$$R_L(q,w)=W^{00}$$

and

$$R_T(q,w)=-\left[g_{ij}+\frac{q_iq_j}{q^2}\right]W^{ij} . \tag{4}$$

The next step is to provide a model for the hadronic response tensor. To begin with we use the relativistic Fermi gas model in which the scattering is assumed to involve a struck nucleon of momentum $P=[E(\mathbf{p}),\mathbf{p}]$ with corresponding (on-shell) energy $E(\mathbf{p})\equiv(\mathbf{p}^2+m_N^2)^{1/2}$ lying below the Fermi momentum p_F and, having supplied energy and momentum ω and q, respectively, to the nucleon, resulting in a four-momentum $(P+Q)^\mu$ lying above the Fermi surface. Conserving energy and momentum and integrating over the momenta in the Fermi sea, this leads to the following expression for the hadronic tensor:

$$W^{\mu\nu}=\frac{3\mathcal{N}m_N^2}{4\pi p_F^3}\int\frac{d^3p}{E(\mathbf{p})E(\mathbf{p}+\mathbf{q})}\times\theta(p_F-|\mathbf{p}|)\theta(|\mathbf{p}+\mathbf{q}|-p_F)\times\delta\{\omega-[E(\mathbf{p}+\mathbf{q})-E(\mathbf{p})]\}\times f^{\mu\nu}(P+Q,P) , \tag{5}$$

where $f^{\mu\nu}(P+Q,P)$ is the single-nucleon response tensor obtained by Lorentz transforming the measured response involving the system where the struck nucleon is at rest to the system where the struck nucleon has four-momentum P (see, for example, Refs. 11):

$$f^{\mu\nu}(P+Q,P)=-W_1(\tau)\left[g^{\mu\nu}-\frac{Q^\mu Q^\nu}{Q^2}\right]+W_2(\tau)\frac{1}{m_N^2}\left[P^\mu-\frac{P\cdot Q}{Q^2}Q^\mu\right]\times\left[P^\nu-\frac{P\cdot Q}{Q^2}Q^\nu\right] . \tag{6}$$

Here we have used a standard dimensionless four-momentum transfer $\tau\equiv|Q^2|/4m_N^2\ge 0$, where m_N is the nucleon mass, and may write the nucleon responses in terms of the Sachs form factors G_E and G_M (Refs. 12 and 13)

$$W_1(\tau)=\tau G_M^2(\tau) , \tag{7a}$$

$$W_2(\tau)=\frac{1}{1+\tau}[G_E^2(\tau)+\tau G_M^2(\tau)] \ . \tag{7b}$$

In Eq. (5) we have written the particle number in a generic form as $\mathcal{N}$, although it should be understood that for use in describing the response of a given nucleus we should add one copy of Eq. (5) having G_E^p, G_M^p with $\mathcal{N}\rightarrow Z$ to another having G_E^n, G_M^n with $\mathcal{N}\rightarrow N$.

The response functions may then be obtained by substituting Eq. (6) into Eq. (5), performing the integrations and then making use of Eqs. (4). For convenience we wish to cast the results in terms of dimensionless variables,

$$\left.\begin{array}{l}\kappa=q/2m_N\\ \lambda\equiv\omega/2m_N\end{array}\right\}\Longrightarrow\tau=\kappa^2-\lambda^2 \ ,$$
$$\eta\equiv|\mathbf{p}|/m_N, \quad \varepsilon\equiv E(\mathbf{p})/m_N=\sqrt{1+\eta^2} \ , \tag{8}$$
$$\eta_F\equiv p_F/m_N, \quad \varepsilon_F=\sqrt{1+\eta_F^2} \ .$$

The responses then take on the relatively simple forms

$$R_{L,T}=\frac{3\mathcal{N}}{4m_N\kappa\eta_F^3}(\varepsilon_F-\Gamma)\theta(\varepsilon_F-\Gamma)\begin{cases}\dfrac{\kappa^2}{\tau}\{[(1+\tau)W_2(\tau)-W_1(\tau)]+W_2(\tau)\Delta\} & \text{for } L \ ,\\ 2W_1(\tau)+W_2(\tau)\Delta & \text{for } T \ ,\end{cases} \tag{9}$$

where we have defined a function

$$\Delta\equiv\frac{\tau}{\kappa^2}[\tfrac{1}{3}(\varepsilon_F^2+\varepsilon_F\Gamma+\Gamma^2)+\lambda(\varepsilon_F+\Gamma)+\lambda^2]-(1+\tau) \tag{10a}$$

with

$$\Gamma\equiv\max[(\varepsilon_F-2\lambda),\gamma_-\equiv\kappa\sqrt{1+1/\tau}-\lambda] \ . \tag{10b}$$

In Fig. 1 we show the behavior of γ_- as a function of λ for fixed κ. The minimum ($\gamma_-=1$) at

$$\lambda=\lambda_0\equiv\tfrac{1}{2}[(1+4\kappa^2)^{1/2}-1] \tag{11}$$

corresponds to the quasielastic peak, for then $\lambda=\tau$ or, equivalently, $\omega=|Q^2|/2m_N$. The Pauli-blocked regime, Fig. 1(a), occurs at low momentum transfers $[(1+\kappa^2)^{1/2}\le\varepsilon_F\Longrightarrow\kappa\le\eta_F\Longrightarrow q\le 2p_F]$, where $\gamma_-=\eta_F$ has only one solution for $\lambda\ge 0$ and where $\Gamma=\varepsilon_F-2\lambda$ for some values of λ. The non-Pauli-blocked regime, Fig. 1(b), occurs when $\kappa>\eta_F$ ($q>2p_F$) with two values of λ where $\gamma_-=\varepsilon_F$ and with $\Gamma=\gamma_-$ in the entire range $\{\lambda\ge 0\}$. We shall consider only this second regime in the following as our focus here is on scaling at high momentum transfers.

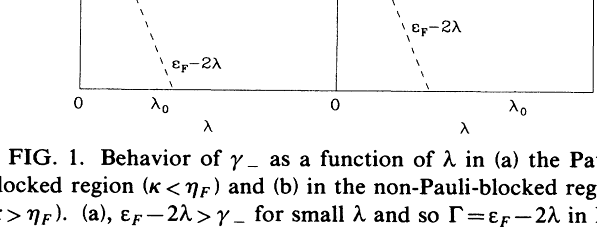

FIG. 1. Behavior of γ_- as a function of λ in (a) the Pauli-blocked region ($\kappa<\eta_F$) and (b) in the non-Pauli-blocked region ($\kappa>\eta_F$). (a), $\varepsilon_F-2\lambda>\gamma_-$ for small λ and so $\Gamma=\varepsilon_F-2\lambda$ in Eq. (9); (b), $\Gamma=\gamma_-$ for all λ. We focus on the non-Pauli-blocked region in this paper since we are dealing with scaling in the large-q limit.

Nonrelativistically (indicated by nr), the factor $(\varepsilon_F-\gamma_-)$ which then enters in Eqs. (9) is a parabola in λ for constant κ; specifically, the responses nonrelativistically are proportional to $(1-\psi_{\rm nr}^2)$, where $\psi_{\rm nr}=[(\lambda/\kappa)-\kappa]/\eta_F\equiv y_{\rm nr}/p_F$ is a dimensionless form of the y-scaling variable, using p_F to remove the dimensions in $y_{\rm nr}$. Our strategy here is to map the more general results in Eqs. (9) onto a parabola as well. To this end we define a generalized dimensionless scaling variable

$$\psi\equiv\left[\frac{1}{\xi_F}(\gamma_--1)\right]^{1/2}\begin{cases}+1, & \lambda\ge\lambda_0 \ ,\\ -1, & \lambda\le\lambda_0 \ ,\end{cases} \tag{12}$$

where $\xi_F\equiv\varepsilon_F-1$ and where, by definition, $\psi=0$ on the quasielastic peak (see above). We are then led to define a generalized scaling function

$$S(\psi;\eta_F)\equiv(1-\psi^2)\theta(1-\psi^2)\frac{3\xi_F}{\eta_F^3} \ , \tag{13}$$

involving only the single kinematic variable ψ in addition to the explicit dependence on the Fermi momentum. The factor $3\xi_F/\eta_F^3$ has been chosen so that the scaling function has a sensible limit when $\eta_F\rightarrow 0$ [see Eq. (19)] and yet satisfies a particular sum rule [see Eqs. (21)]. If one were to attempt to extract a sort of *superscaling* relation in which various different nuclear targets (consequently with different values of η_F) were to be studied in a unified way, then a different definition for the scaling function, where $3\xi_F/\eta_F^3$ is removed from Eq. (13) and instead incorporated in Eqs. (14), would be appropriate. The double-differential cross section in Eq. (1) may then be written in the forms

$$\frac{d^2\sigma}{d\Omega\,d\epsilon'}=C(\theta,\tau,\psi;\eta_F)S(\psi;\eta_F) \tag{14a}$$

$$=\frac{\mathcal{N}}{4m_N\kappa}\sigma_M X(\theta,\tau,\psi;\eta_F)S(\psi;\eta_F) \ , \tag{14b}$$

which, after some rearrangements, leads us to an expression for X:

$$X(\theta,\tau,\psi;\eta_F)=\left[W_2(\tau)+2W_1(\tau)\tan^2\frac{\theta}{2}\right]+W_2(\tau)\left\{\left[\frac{\tau(1+\tau)}{\kappa^2}-1\right]+\left[\frac{3}{2}\left[\frac{\tau}{\kappa^2}\right]+\tan^2\frac{\theta}{2}\right]\Delta\right\}, \quad (15)$$

where Δ can be reexpressed in the more compact form

$$\Delta=\frac{\tau}{\kappa^2}(1-\psi^2)\xi_F\left[\kappa\left[1+\frac{1}{\tau}\right]^{1/2}+\tfrac{1}{3}(1-\psi^2)\xi_F\right]. \quad (16)$$

We can immediately see that the formalism here has the correct limiting behavior under circumstances where $\mathcal{N}=1$ and $\eta_F\to 0$, which should reduce to the one-body problem, that of the single-nucleon (sn) itself. Since η_F is relatively small $[\approx(\frac{1}{4})]$, it provides the one quantity in the problem to use as an expansion parameter, giving

$$\Delta=\tfrac{1}{2}(1-\psi^2)\eta_F^2+O(\eta_F^4), \quad (17a)$$

$$1-\frac{\tau(1+\tau)}{\kappa^2}=2\left[\frac{\tau}{1+\tau}\right]^{1/2}\psi\eta_F+O(\eta_F^2) \quad (17b)$$

and yielding the following approximation for X:

$$X(\theta,\tau,\psi;\eta_F)=\left[W_2(\tau)+2W_1(\tau)\tan^2\frac{\theta}{2}\right]-2\left[\frac{\tau}{1+\tau}\right]^{1/2}W_2(\tau)\psi\eta_F+O(\eta_F^2). \quad (18)$$

In obtaining numerical results for the nuclear problem we may, of course, use the complete expressions given in Eqs. (14)–(16); the small-η_F limits (here and in the developments to follow) are included to reduce the complexity of the results and to help indicate the relative orders-of-magnitude in the problem. We are then able to conclude that

$$S(\psi;\eta_F)\to 4m_N\kappa f_{\rm rec}^{-1}\delta(\epsilon'-\epsilon f_{\rm rec}^{-1}), \quad (19)$$

where the recoil factor is given by

$$f_{\rm rec}\equiv 1+2\epsilon\sin^2(\theta/2)/m_N$$

[equivalently, for use in Eq. (19), this may be written ϵ/ϵ']. With

$$\tau=[\epsilon\sin(\theta/2)/m_N]^2 f_{\rm rec}^{-1}$$

we then obtain for this limit

$$\left[\frac{d\sigma}{d\Omega}\right]_{sn}=\int d\epsilon'\lim_{\eta_F\to 0}\left[\frac{d^2\sigma}{d\Omega\,d\epsilon'}\right]_{\mathcal{N}=1}=\sigma_M f_{\rm rec}^{-1}\left[W_2(\tau)+2W_1(\tau)\tan^2\frac{\theta}{2}\right], \quad (20)$$

which is the expected single-nucleon cross section.[13]

Other limiting properties of these factorized results can be extracted from the general expressions presented above. Firstly, we may consider the following integral over the scaling function:

$$\Sigma\equiv\int_0^\infty d\lambda\, S(\psi;\eta_F) \quad (21a)$$

$$=\frac{2\kappa}{[1+(2\kappa)^2]^{1/2}}\left[1-\frac{3}{10}\left[1+\frac{1}{[1+2\kappa)^2]^2}\right]\eta_F^2+O(\eta_F^4)\right] \quad (21b)$$

$$\underset{\kappa\to 0}{\longrightarrow}2\kappa[1+O(\eta_F^2)] \quad (21c)$$

$$\underset{\kappa\to\infty}{\longrightarrow}1+O(\eta_F^2). \quad (21d)$$

This constitutes a sort of sum rule, although it can only be applied with any hope of being realized in fact for the longitudinal part of the cross section, since for the transverse part there are important contributions other than those which stem from the assumed one-body knockout mechanism (such as Δ production or knockout involving two-body meson-exchange currents). Secondly, we may use the following expression for λ in terms of κ and the scaling variable ψ,

$$\lambda=\tfrac{1}{2}[(1+\{2\kappa+\psi[\xi_F(2+\xi_F\psi^2)^{1/2}]\}^2)^{1/2}-(1+\xi_F\psi^2)], \quad (22)$$

to obtain limits for the full width of the parabola (i.e., for the span of energy transfers over which the Fermi gas response is nonzero):

$$\Delta\lambda\equiv\lambda(\psi=+1)-\lambda(\psi=-1) \quad (23a)$$

$$\underset{\kappa\to 0}{\longrightarrow}2\kappa\eta_F/\varepsilon_F=2\kappa\eta_F[1+O(\eta_F^2)] \quad (23b)$$

$$\underset{\kappa\to\infty}{\longrightarrow}\eta_F. \quad (23c)$$

Thus we see for both the sum rule in Eqs. (21) and the width of the quasielastic response in Eqs. (23) that the leading κ dependence is quite different at very low and very high values of κ. For example, at low κ the width of the quasielastic peak grows linearly with κ; however, at relativistic momentum transfers the peak width becomes constant. Correspondingly, the sum rule as defined here at first grows with κ and then stabilizes to unity for asymptotic values of κ. If we wished to maintain a sum rule at low values of κ (still, however, with $\kappa>\eta_F$), say, for the longitudinal response alone to yield the Coulomb sum rule, then S would have to be divided by κ and so would no longer scale.

B. Relationships to other scaling variables

In addition to West's nonrelativistic scaling variable there is another set of scaling variables which we call $\psi_r^{(i)}$, $i=1,2,3$ based on a different set of assumptions. They correspond to the minimum momentum a nucleon can have and still satisfy energy conservation in the impulse approximation.[5] The three discussed here are related by different approximations. The most sophisticated of these we call $\psi_r^{(3)} = \sqrt{2\xi_F}\, y^{(3)}/m_N$ again made dimensionless by dividing out the factor

$$\{2m_N[E(\mathbf{p}_F) - m_N]\}^{1/2} .$$

It is a solution of the equation[5]

$$M_A + \omega = [m_N^2 + (y^{(3)} + q)^2]^{1/2} + (M_{A-1}^2 + y^{(3)2})^{1/2} . \tag{24}$$

In fact scaling is generally viewed[7] as a concept that is valid only in the limit of large q where terms depending on the separation energy or on final-state interaction effects become small. Note, however, that the spirit of our approach is somewhat different: We are using the Fermi gas model to motivate choices for the variables which will also work at rather modest values of q (of course, with $q > 2p_F$), where "modest" depends on just how far below the quasielastic peak we attempt to go. Therefore, it is natural to consider an approximate form for Eq. (24). If we ignore separation-energy effects taking $M_A = Am_N$, $M_{A-1} = (A-1)m_N$ for simplicity, and define $\tilde{\lambda} \equiv \lambda - \tau/A$, then

$$\psi_r^{(3)} = \left[\frac{2}{\xi_F}\right]^{1/2} \frac{1}{1+4\tilde{\lambda}/A}[(1+2\lambda/A)\sqrt{\tilde{\lambda}(\tilde{\lambda}+1-1/A)} - \kappa(1-1/A+2\tilde{\lambda}/A)] \tag{25a}$$

$$\underset{\kappa\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}}\psi[(\xi_F\psi^2+2)^{1/2} - \sqrt{\xi_F}\psi] + \left[\frac{\xi_F}{2}\right]^{1/2} \psi^2[\sqrt{\xi_F}\psi(\xi_F\psi^2+2)^{1/2} - \xi_F\psi^2 - 1]\frac{1}{A} + O(1/A^2) \tag{25b}$$

$$= \psi[1 - \psi\eta_F(1+1/A)/2 + O(1/A^2, \eta_F^2)] . \tag{25c}$$

If we also ignore the recoil contributions in Eq. (24), i.e., assume $A \to \infty$ so

$$M_A - (M_{A-1}^2 + y^2)^{1/2} \approx m_N ,$$

we get the defining equation for $\psi_r^{(2)} = \sqrt{2\xi_F}\, y^{(2)}/m_N$ (see Ref. 7)

$$m_N + \omega = [m_N^2 + (y^{(2)} + q)^2]^{1/2} , \tag{26}$$

giving

$$\psi_r^{(2)} = \left[\frac{2}{\xi_F}\right]^{1/2} [\sqrt{\lambda(\lambda+1)} - \kappa] \tag{27a}$$

$$\underset{\kappa\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}}\psi[(\xi_F\psi^2+2)^{1/2} - \sqrt{\xi_F}\psi] \tag{27b}$$

$$= \psi[1 - \psi\eta_F/2 + O(\eta_F^2)] . \tag{27c}$$

Finally, we may rewrite $\psi_r^{(2)}$, multiplying and dividing by $\sqrt{\lambda(\lambda+1)} + \kappa$, to obtain

$$\psi_r^{(2)} = \left[\frac{2}{\xi_F}\right]^{1/2} \frac{\lambda(\lambda+1) - \kappa^2}{\sqrt{\lambda(\lambda+1)} + \kappa} . \tag{28}$$

For kinematics near the quasielastic peak, $\sqrt{\lambda(\lambda+1)} \approx \kappa$, we have

$$\psi_r^{(2)} \approx \frac{1}{\sqrt{2\xi_F}}\left[\frac{\lambda(\lambda+1)}{\kappa} - \kappa\right]$$

$$\equiv \psi_r^{(1)} \tag{29a}$$

$$\underset{\kappa\to 0}{\longrightarrow} \frac{\eta_F}{\sqrt{2\xi_F}}\psi_{\rm nr}$$

$$= \psi_{\rm nr}[1 + O(\eta_F^2)] \tag{29b}$$

$$\underset{\kappa\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}}\psi[(\xi_F\psi^2+2)^{1/2} - \sqrt{\xi_F}\psi]$$

$$= \psi[1 - \psi\eta_F/2 + O(\eta_F^2)] . \tag{29c}$$

This quantity $\psi_r^{(1)}$ is frequently used as an approximation in discussing the general behavior of relativistic scaling variables, although in most state-of-the-art treatments of scaling a quantity such as that defined in Eq. (25a) is usually employed in representing the data.[2,5,7]

Figure 2 shows a comparison of $\psi_{\rm nr}$ and $\psi_r^{(i)}$ with ψ by expressing each scaling variable as a function of κ and ψ and plotting them at constant κ. To obtain some feeling for the orders of magnitude with dimensional variables, one can multiply the ψ's (approximately) by p_F to obtain

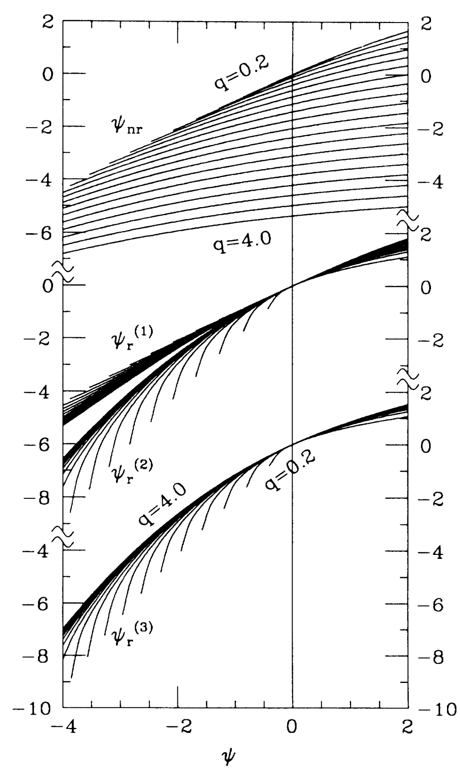

FIG. 2. A comparison of various scaling variables $\psi_{\rm nr}$ and $\psi_r^{(i)}$, $i = 1,2,3$ as defined in the text expressed as functions of κ and ψ. Each line corresponds to a fixed value of q (or equivalently κ) ranging from $q = 0.2$ GeV/c to $q = 4.0$ GeV/c in steps of 0.2 GeV/c. In all cases η_F has been taken to be 0.25 and for the $i = 3$ case we have chosen $A = 16$.

equivalent y's. Thus, for example, if the Fermi momentum is $m_N/4 \approx 235$ MeV/c, then $\psi = -3$ corresponds to $y \approx -700$ MeV/c. As is clear from the figure, the nonrelativistic scaling variable is not a good approximation at high momentum transfers, as expected. The three relativistic scaling variables considered here all approach limiting relationships to ours when $\kappa \to \infty$, although for finite κ they can differ significantly, either with our variable or with each other. $\psi_r^{(1)}$ and $\psi_r^{(2)}$ eventually coalesce when $\kappa \to \infty$, but it is clear that even at $q = 4.0$ GeV/c ($\kappa = 2.13$) they differ at large negative ψ. $\psi_r^{(3)}$ which includes effects of recoil (treated approximately here) does not go to exactly the same limit [cf. Eqs. (25b) and (27b)]; however, for the $A = 16$ case chosen here, the high-q values of $\psi_r^{(3)}$ and $\psi_r^{(2)}$ are not markedly different. For say $A = 2$ or 3 these recoil effects are much more significant. Clearly any of these differently motivated scaling variables are rather similar to ours for modest negative ψ, but then deviate from a linear relationship when the product $\psi\eta_F$ becomes appreciable.

Finally, let us remark that there is another commonly used relativistic scaling variable[14,4]

$$\psi_r' = \psi \left[1 + \tfrac{1}{2}\xi_F \psi^2 \right]^{1/2} , \tag{30}$$

which stems from using $(\gamma_-^2 - 1)^{1/2}$ rather than $\sqrt{\gamma_- - 1}$ as we have in this work [see Eq. (12)]. Since no other variable except ψ occurs in this redefinition, if the response scales with one choice, it will scale with the other. Our choice maps the relativistic Fermi gas response to a universal parabola which goes to zero when $\psi = \pm 1$, whereas the choice in Eq. (30) does not yield properties which are quite as simple.

C. Application to models for ^{16}O(e,e') responses

Having discussed the concept of scaling within the context of the relativistic Fermi gas, we now employ part of our formalism for a wider class of problems where interactions are present. Specifically, we make the assumption that the factor C in Eqs. (14) can be used to divide into the double-differential cross section in obtaining a generalization of the scaling function S. Since the nuclear-structure-related effects in the dividing factor C (as represented by the η_F dependence) are not too large, we may hope that this is a good way to estimate the corrections which occur in going beyond the lowest order where such nuclear effects are generally ignored. Furthermore, as we have nontrivial dependences on the kinematic variables (κ, λ, and τ) with specific combinations of the nucleon form factors [see Eqs. (15)–(19)], we can study how various model cross sections approach the asymptotic scaling behavior. Throughout this section we consider the electron scattering angle to be fixed at 10°, since our main focus is on large q, small ω where the cross sections only become practically measurable at reasonably forward angles.

We first consider a generalization of the simple relativistic Fermi gas model[9] to allow for momentum distributions other than the θ-function one used above [see Eq. (5)]. In particular, we employ the momentum density distributions of Ref. 15. These were obtained for ^{16}O using a Brueckner-Hartree-Fock (BHF) calculation of two-nucleon ground-state correlations in a harmonic oscillator basis, with either Reid soft-core (RSC) or super-soft-core (SSC) potentials. The resulting momentum densities were used to generate more realistic cross sections (pseudodata) with which to test the pure Fermi gas scaling prescription of Eqs. (14). We considered two dividing functions: $C(\theta,\tau,\psi;\eta_F)$, the full dividing function with $X(\theta,\tau,\psi;\eta_F)$ given by Eq. (15), and the dividing function of zeroth order in η_F, with $X(\theta,\tau,\psi;\eta_F)$ evaluated at $\eta_F = 0$. It is clear from Eq. (18) and (14) that the zeroth-order dividing function is simply proportional to the single-nucleon Rosenbluth cross section, that is, the naive factor that we might be tempted to use in the absence of the arguments presented in the last section.

On the one hand, one can show analytically that as $\kappa \to \infty$, for single-nucleon cross sections that have the same τ dependence at large τ and for general momentum densities, the model cross sections will scale using either of these two prescriptions, i.e., the scaling functions obtained using Eq. (15) will be the same function of ψ in the limit as $\kappa \to \infty$. Furthermore, it is easily seen that the asymptotic scaling function is simply related to the momentum density by

$$S(\psi) \underset{\kappa \to \infty}{\longrightarrow} \int_{\gamma_- = \xi_F \psi^2 + 1}^{\infty} d\varepsilon\, n[(\varepsilon^2 - 1)^{1/2}] , \tag{31}$$

where, for simplicity, the neutron and proton densities are assumed to be equal. On the other hand, the approach to scaling is markedly different depending on which prescription is used. Indeed, for rapid approach to scaling using the zeroth-order dividing function, the cross sections must be calculated at large electron angles, i.e., 90° or greater, where the cross sections become mainly transverse. This dependence is demonstrated in Fig. 3 which shows the scaling functions obtained for pseudodata calculated with the RSC momentum density using various values of κ and the two prescriptions. For negative ψ and for both prescriptions as κ increases from zero the scaling functions *increase* until for some κ they reach a maximum. As κ increases further, the scaling functions *decrease* until the asymptotic result is reached. Thus the scaling functions exhibit "false scaling;" there is a range of κ where the scaling function seems to have leveled off before slowly descending towards the true asymptotic answer. The region of false scaling and the magnitude of the scaling violation varies as a function of θ and the dividing function used. For the zeroth-order dividing function (proportional to the single-nucleon cross section) the false-scaling regime occurs for $q \approx 8$ GeV/c at $\theta = 10°$ and $q \approx 1.7$ GeV/c for $\theta = 90°$. For the exact dividing function the false-scaling regime occurs for $q \approx 1.6$ GeV/c for $\theta = 10°$. As far as the violation in scaling is concerned, it is clear from the figure that even at $q = 4.0$ GeV/c, the zeroth-order results lie significantly above the asymptotic answer; the scaling function using the exact result is much closer. For example, at $\psi = -2$ and $q = 2.0$ GeV/c, the result using the former is twice as large as the asymptotic answer while the result using the latter is only 30% too large. We get errors of the same order of magnitude when differentiating Eq. (31) to compute the momentum density from the scaling functions at $q=2.0$ GeV/c [cf. Eq. (46) below]. The zeroth-order dividing function gives results that approach the asymptotic answer so slowly that even when $q=100$ GeV/c the scaling function at $\psi=-2$ is still 50% too high. In contrast, already when $q=10$ GeV/c, the scaling function using the exact dividing function is only 5% high at $\psi=-2$. As one increases the angle, the functions scale steadily faster, until at $\theta=90°$ the zeroth-order dividing function gives results that are comparable to using the exact dividing function on $\theta=10°$ degree data. At 180° scattering, where the cross section is completely transverse, the approach to scaling is very rapid. Clearly, the longitudinal part of the cross section causes the scaling violations, and in the next section we will show some of the problems that this introduces. Finally, we note that we chose a momentum density with a reasonably high momentum tail. Indeed, the results are similar if we use the SSC density which is softer. The scaling functions are different specific functions, with the SSC results falling off with increasing $|\psi|$ more rapidly than the RSC results shown here; however, the slow approach to each asymptotic scaling function using the zeroth-order dividing function is qualitatively the same.

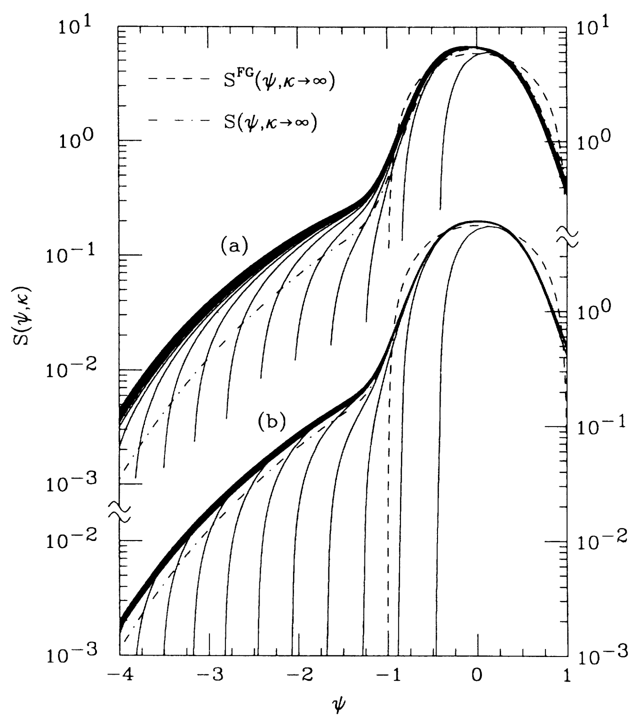

FIG. 3. The scaling function derived from the total $(L+T)$ cross section for ^{16}O at $\theta=10°$ for three-momentum transfers ranging from 0.2 to 4.0 GeV/c by steps of 0.2 GeV/c and progressing from right to left in the plots. The cross sections were calculated using an extension of the Fermi gas model to allow for arbitrary momentum densities; here the momentum density of Ref. 15 which includes two-nucleon ground-state correlations generated by the Reid soft-core potential was employed. The prescription of Eq. (14b) with two different dividing functions $X(\theta,\tau,\psi;\eta_F)$ was taken in generating the two sets of plots. (a) The dividing function X proportional to the single-nucleon cross section [$\eta_F=0$ in Eq. (15)] was used. (b) The exact form of X as given in Eq. (15) was used. The dot-dashed line is the asymptotic answer in this model; the dashed line is the same quantity but for the Fermi gas model. In all cases η_F has been taken to be 0.25.

Other authors have pointed out the possible slow approach to scaling. Weinstein and Negele,[16] for example, showed nonrelativistically using time-ordered perturbation theory that the dynamic structure function of quantum systems interacting via hard-core potentials of radius a exhibit y scaling only for momentum transfers on the order of $qa\approx 25$ (corresponding to $q\approx 6$ GeV/c for electron scattering from nuclei). Furthermore, the asymptotic scaling function they derive is 40–60 % lower than the one obtained naively from integrating the momentum density.

In a parallel effort we have investigated a model in which specific non-nucleonic degrees of freedom have been included.[17] Our analysis retains sets of diagrams involving single one-pion-exchange contributions both to meson-exchange currents (MEC) and to final-state-interactions (FSI) effects. The response functions are obtained using the Fermi gas model, but now for one-nucleon ejection via the pion-exchange-based two-body operators together with the familiar one-body contributions discussed above. With this particular $1p$-$1h$ model the responses are limited to $-1\le\psi\le+1$, just as in the case of the one-body Fermi gas. Consequently, no statement can be made about the large-, negative-ψ region, although in the region $-1\le\psi\le 0$ the resulting scaling functions are rather similar to the BHF curves shown in Fig. 3. Interestingly, even with the MEC and FSI included, the approach to scaling is about the same as when ground-state correlations alone are included.

D. Longitudinal and transverse scaling functions

Finally, we may try to apply these ideas to the individual longitudinal and transverse parts of the cross section. It is convenient to rewrite Eq. (1) in the following form:

$$\frac{d^2\sigma}{d\Omega\, d\epsilon'}=\sigma_0(\tilde{R}_T+\mathcal{E}\tilde{R}_L)\ , \tag{32}$$

where

$$\mathcal{E}\equiv[1+2\kappa^2\tan^2(\theta/2)/\tau]^{-1} \tag{33a}$$

is the longitudinal polarization of the virtual photon, with $0\le\mathcal{E}\le 1$, and where

$$\sigma_0=\frac{\alpha^2}{|Q^2|}\left[\frac{\epsilon'}{\epsilon}\right]\frac{2}{1-\mathcal{E}} \tag{33b}$$

is proportional to the virtual photon flux. The responses are simply related to those in Eq. (1): $\tilde{R}_L=(\tau/\kappa^2)R_L$ and $\tilde{R}_T=(\frac{1}{2})R_T$. They can again be written in a factorized form where scaling functions $S_{L,T}$ are isolated from kinematic factors, where the latter (the C's) are only weakly dependent on the properties of the nucleus:

$$\tilde{R}_{L,T}=C_{L,T}(\tau,\psi;\eta_F)S_{L,T}(\psi;\eta_F)\ . \tag{34}$$

Following the approach taken above where the relativistic Fermi gas is used to motivate the choices [i.e., employing Eq. (9)], we are led to divide the responses $\tilde{R}_{L,T}$ by the kinematic factors,

$$C_{L,T}^{(1)}(\tau,\psi;\eta_F)=\frac{\mathcal{N}}{4m_N\kappa}\begin{cases}G_E^2(\tau)+W_2(\tau)\Delta \quad \text{for } L & (35a)\\ \tau G_M^2(\tau)+\frac{1}{2}W_2(\tau)\Delta \quad \text{for } T\ , & (35b)\end{cases}$$

and so we obtain the scaling functions $S_{L,T}^{(1)}$. To the extent that the reaction mechanism (one-body nucleonic versus two-body, non-nucleonic, etc.) is the same in the L and T responses aside from the single-nucleon responses which we have removed (at least within the context of the present model), then these two scaling functions should be the same. From what we know about the nature of the reaction mechanisms and how they differ in the L and T channels in the region of excitation energies above the quasielastic peak, we could only hope to find such a universal scaling relation when $-1\le\psi\le 0$.

Of course, by construction, the relativistic Fermi gas responses exactly scale to the same universal function. For the responses calculated using the BHF density the scaling is also excellent over most of the region $-1\le\psi\le 0$, with differences between the longitudinal and transverse scaling functions as defined above occurring only near $\psi=-1$. Thus, for modest values of the scaling variable (namely, for modest excursions from the quasielastic peak), the scaling functions defined in Eqs. (34) and (35) appear to be good choices when individual L/T separations are to be considered.

As we continue to larger excursions from the quasielastic peak, however, the region beyond $\psi=-1$ presents special problems for the longitudinal scaling function defined above. To see this clearly, let us go to the asymptotic limit where $\tau\gg 1$ with ψ not too large in magnitude and fixed. Then we have

$$W_2(\tau)\underset{\tau\to\infty}{\longrightarrow} G_M^2(\tau)\ , \tag{36a}$$

$$\Delta\underset{\tau\to\infty}{\longrightarrow}\rho(1-\psi^2)\xi_F \tag{36b}$$

[see Eqs. (7b) and (16)], where in this limit we have used the fact that

$$\tau/\kappa\underset{\tau\to\infty}{\longrightarrow}\rho\equiv\tfrac{1}{2}[(2+\xi_F\psi^2)^{1/2}-\sqrt{\xi_F}\psi]^2\ . \tag{37}$$

For the transverse factor in Eq. (35b), we find the limiting behavior

$$C_T^{(1)}(\tau,\psi;\eta_F)\underset{\tau\to\infty}{\longrightarrow}\frac{\mathcal{N}}{4m_N}\rho G_M^2(\tau)\ , \tag{38}$$

which is well behaved. On the other hand, for the longitudinal factor in Eq. (35a) the limiting behavior is

$$C_L^{(1)}(\tau,\psi;\eta_F)\underset{\tau\to\infty}{\longrightarrow}\frac{\mathcal{N}}{4m_N\kappa}[G_E^2(\tau)+\rho(1-\psi^2)\xi_F G_M^2(\tau)]\ , \tag{39}$$

and for some value of the scaling variable in the region $\psi<-1$ this may be *zero*. For instance, taking $\eta_F=0.25$ and using a universal dipole parametrization[18] of the nucleon form factors, $C_L^{(1)}$ becomes zero at $\psi=-1.75$. Obviously this choice for the longitudinal factor is not a good one for use in the region $\psi<-1$. It does highlight an important aspect of *any* discussion of individual L and T scaling functions. While the transverse one may be well behaved, the longitudinal one has important G_M^2 effects introduced by the kinematic shifts of the single-nucleon response which must be considered together with the G_E^2 contributions. In the transverse case, where a similar problem might have arisen with effects involving G_E^2 occurring together with the expected G_M^2 contributions, the former decrease relative to the latter [proportionally to τ^{-1}, see Eq. (35b)] and so go away asymptotically. Moreover, in the total $(L+T)$ scaling discussed above the scaling factor C defined in Eqs. (14) is also well behaved.

Since we have identified an important difference in attempting to discuss the longitudinal and transverse scaling functions separately, let us pursue this a little further to suggest an alternative way to approach the problem. Let us extend the Fermi gas prescription for the responses by replacing the θ-function momentum distribution by a general function $n(\eta)$. In fact, the BHF responses employed above were calculated this way. Now as a second choice of scaling factors let us use

$$C_{L,T}^{(2)}(\tau,\psi;\eta_F)=\frac{\mathcal{N}}{4m_N\kappa}\begin{cases}G_E^2(\tau) \quad \text{for } L\ , & (40a)\\ \tau G_M^2(\tau) \quad \text{for } T\ , & (40b)\end{cases}$$

that is, the same as in Eqs. (35) but without the terms involving Δ. We note from Eq. (15) that this is not the same as extracting only the single-nucleon cross section [see Eq. (20)], since additional effects involving η_F are still included via the term $\tau(\tau+1)/\kappa^2-1$. Proceeding this way we obtain for the functions which may scale asymptotically:

$$S_L^{(2)}(\tau,\psi;\eta_F)=I(\gamma_-)+\frac{W_2(\tau)}{G_E^2(\tau)}\tilde{I}(\tau,\gamma_-)\ , \tag{41a}$$

$$S_T^{(2)}(\tau,\psi;\eta_F)=I(\gamma_-)+\frac{W_2(\tau)}{2\tau G_M^2(\tau)}\tilde{I}(\tau,\gamma_-)\ , \tag{41b}$$

where, of course, ψ and γ_- are related by Eq. (12). The integrals I and $\tilde{I}$ are given by [cf. Eq. (31)]

$$I(\gamma_-)\equiv\int_{\gamma_-}^{\infty}d\varepsilon\, n[(\varepsilon^2-1)^{1/2}]\ , \tag{42a}$$

$$\tilde{I}(\tau,\gamma_-)\equiv\frac{2\sqrt{\tau(\tau+1)}}{\kappa}\int_{\gamma_-}^{\infty}d\varepsilon\, n[(\varepsilon^2-1)^{1/2}](\varepsilon-\gamma_-)\times\left[1+\frac{\varepsilon-\gamma_-}{2\kappa\sqrt{1+1/\tau}}\right]\ , \tag{42b}$$

and, in the limit where $\tau\gg 1$ with ψ not too large in magnitude and fixed, the latter has the limiting behavior

$$\tilde{I}(\tau,\gamma_-)\underset{\tau\to\infty}{\longrightarrow}\tilde{I}^{\infty}(\gamma_-)\equiv 2\rho\int_{\gamma_-}^{\infty}d\varepsilon\, n(\sqrt{\varepsilon^2-1})(\varepsilon-\gamma_-)\ , \tag{43}$$

where ρ is given in Eq. (37). Thus, in this limit

$$S_T^{(2)} \underset{\tau\to\infty}{\longrightarrow} I(\gamma_-) \tag{44}$$

involving only the first integral, whereas

$$S_L^{(2)} \underset{\tau\to\infty}{\longrightarrow} I(\gamma_-)+\left[\frac{G_M^2}{G_E^2}\right]\tilde{I}^{\infty}(\gamma_-) , \tag{45}$$

where both integrals occur. The transverse scaling function (and hence the entire $L+T$ scaling function asymptotically where the transverse contributions dominate) has the attractive property of being simply related to the momentum distribution:

$$-\frac{dS_T^{(2)}}{d\gamma_-} \underset{\tau\to\infty}{\longrightarrow} n[(\gamma_-^2-1)^{1/2}] . \tag{46}$$

The behavior of the longitudinal scaling function, however, is different:

$$S_L^{(2)} \underset{\tau\to\infty}{\longrightarrow} S_T^{(2)}+\left[\frac{2G_M^2}{G_E^2}\right]\rho\int_{\gamma_-}^{\infty} d\gamma'_- S_T^{(2)}[\tau\to\infty,\psi'(\gamma'_-)] \tag{47a}$$

$$=S_T^{(2)}(\tau\to\infty,\psi) + \left[\frac{2G_M^2}{G_E^2}\right]2\xi_F\rho\int_{-\infty}^{\psi} d\psi' \, |\psi'| \, S_T^{(2)}(\tau\to\infty,\psi') , \tag{47b}$$

where the latter relation is meant to be used for negative ψ. Thus, if we neglect the τ dependence in the *ratio* G_M^2/G_E^2, we see that both $S_L^{(2)}$ and $S_T^{(2)}$ scale, namely, become functions only of ψ; however, they approach *different* scaling functions. Only the asymptotic transverse scaling function is simply related to the momentum distribution through Eq. (46).

Applying these ideas to the ^{16}O responses calculated using the RSC momentum density we get the following numerical results. The peak of the longitudinal asymptotic scaling function is 1.3 times *larger* than the asymptotic transverse peak. This ratio is constant for $-1<\psi\le 0$. However, it grows steadily as ψ becomes still more negative, e.g., the longitudinal asymptotic scaling function is four times larger than the transverse at $\psi=-2$. The SSC momentum density gives the same qualitative results: the S_L,S_T ratio is 1.25 at the peak and 3.45 at $\psi=-2$.

Finally, we make contact with the violations in scaling found in the previous section for the total cross section. In the generalized Fermi gas model the total scaling function using the dividing function of zeroth order in η_F is

$$S(\tau,\psi;\eta_F)=I(\gamma_-)+\frac{W_2(\tau)}{W_2(\tau)+2\tan^2(\theta/2)W_1(\tau)} \times\left[\left[\frac{\tau(\tau+1)}{\kappa^2}-1\right]I(\gamma_-) + \left[\frac{3}{2}\frac{\tau}{\kappa^2}+\tan^2\frac{\theta}{2}\right]\tilde{I}(\tau,\gamma_-)\right] . \tag{48}$$

The second term represents the scaling violation in this model. For asymptotically large τ from Eqs. (17b) and (43) the term in the square brackets becomes constant, while from Eqs. (7) if one assumes the different nucleon form factors have the same asymptotic τ dependence the multiplying factor vanishes:

$$\frac{1}{1+2\dfrac{W_1(\tau)}{W_2(\tau)}\tan^2\dfrac{\theta}{2}} \sim O(1/\tau) . \tag{49}$$

Therefore the scaling violations in the generalized Fermi gas model disappear for τ's characterized by

$$2\frac{W_1(\tau)}{W_2(\tau)}\tan^2\frac{\theta}{2}\approx 1 \Longrightarrow \frac{1}{\tau}\approx 2\tan^2\frac{\theta}{2} . \tag{50}$$

For $\theta=10^\circ$ this occurs for four-momentum transfers on the order of 225 $(\mathrm{GeV}/c)^2$.

III. CONCLUSIONS

In this work we have investigated the idea of y scaling from the perspective of the relativistic Fermi gas. This relatively simple model, involving only on-shell nucleons, has the merit of maintaining a certain level of consistency which is very hard to achieve in other treatments of the problem. For instance, clearly we would like to treat the off-shell electromagnetic vertex and the initial-state and final-state interaction effects in a consistent manner; presently, lacking a true dynamical description of the interacting relativistic many-body problem at a microscopic level, this is not a fully realized goal. Despite the simplicity of the relativistic Fermi gas, it does, however, allow us to proceed from its initial formulation all the way to obtaining expressions for the electromagnetic response functions without making any further approximations. Only for this model are the kinematic consequences of relativity strictly enforced. In particular, we have explored the nontrivial kinematic variation of the electromagnetic interaction with a nucleon in the momentum distribution provided by the relativistic gas. Such effects occur even though the nucleons are entirely on shell and presumably also would be present in more sophisticated models of the nucleon and nuclear structure were we able to handle the relativistic aspects of the problem.

Given this recognition of an important kinematic variation for how a moving nucleon responds electromagnetically, we have organized the relativistic Fermi gas responses into a product of two factors. One is a function only of a single variable ψ which becomes (in this context) a natural choice for a scaling variable. The other involves the single-nucleon electromagnetic response with its Q^2 dependences, together with relatively weak dependence on ψ. The cross section is then to be divided by the latter to obtain the former, and this in the Fermi gas model is the scaling function. The fact that the dividing function contains nontrivial dependence on the Fermi gas through explicit p_F dependence is the kinematic variation referred to above. In fact, no strong demands are placed on knowing the nuclear structure contained in the model; rather, we need only the average properties it supplies, in particular, that it is characterized by a typical nuclear momentum scale (p_F). For these natural choices, the relativistic Fermi gas responses exactly scale to a universal function.

With this basis we have proceeded to study how well our choices work for generalizations of the simple Fermi gas. We have used momentum distributions provided by BHF treatments of the ground state of ^{16}O to obtain response functions to treat as pseudodata in discussing scaling. Only in this way can we truly span the entire kinematical behavior all the way from the nonrelativistic limit to the ultrarelativistic limit, where $q \rightarrow \infty$. Upon dividing these calculated cross sections by the Fermi-gas-motivated single-nucleon factor (i.e., with the nontrivial kinematic variation included), we obtain functions which now may or may not scale. First, we found that for moderate excursions from the quasielastic peak the scaling is excellent. Furthermore, when pionic effects are included in the form of meson-exchange currents and a specific class of final-state interaction contributions, the scaling and approach to scaling were found to be very similar to the BHF results. For more extreme values of ψ below the peak we found, as expected, that asymptotically we reach a scaling regime again. The approach to scaling, however, is not especially fast, as has also been noted by other authors, and in our context, depends importantly on whether the single-nucleon kinematic variation is not included (where it is very slow) or included as we do here (where it is considerably faster). In the former kinematically uncorrected case asymptopia is only reached when $\tan^2(\theta/2) \gg 2m_N/q$. Moreover, in that case there is a peculiar regime of false scaling at finite but still large momentum transfer before the final, slow relaxation to the asymptotic answer. Our approach greatly diminishes the difference between false scaling and asymptopia.

Finally we investigated the possibility of having separated longitudinal and transverse responses scale to a universal function. We found that the transverse contributions are well behaved, whereas the longitudinal ones are not. In some prescriptions for scaling[4] the kinematic dependences on G_E^2 and G_M^2 in the individual L and T responses are quite different and would inevitably lead to different asymptotic scaling functions even given a common nuclear momentum distribution. In fact, this is just the reason for the slow approach to scaling and for the regime of false scaling found in the total $(L+T)$ cross section.

In conclusion, even within the context of a simple, but relativistically consistent, model we have seen important effects stemming from the requirement that the electromagnetic response of a nucleon imbedded in a nuclear ground-state momentum distribution be Lorentz transformed from the measured response of a nucleon in its rest frame. It seems to us rather likely that similar effects should be present even in more sophisticated nuclear models and so should be investigated once these models become capable of handling special relativity.

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