

SCALING INVARIANT HARDY TYPE INEQUALITIES WITH NON-STANDARD REMAINDER TERMS

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Abstract. We consider the Hardy inequality on \mathbb{R}^N , the critical Hardy inequality on a ball, and the Rellich inequality on \mathbb{R}^N . These three Hardy type inequalities can be refined by adding remainder terms. Our remainder terms are expressed by a distance from the families of “virtual” extremals. A key ingredient is the critical Hardy inequality on \mathbb{R}^N which was proved by Machihara, Ozawa and Wadade [21].

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