Scaling limit results for the sum of many inverse Lévy subordinators

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Abstract

The first passage time process of a Lévy subordinator with heavy-tailed Lévy measure has long-range dependent paths. The random fluctuations that appear in a natural scheme of summation and time scaling of such stochastic processes are shown to converge weakly. The limit process, which is neither Gaussian nor stable and which does not have the self-similarity property, is possibly of independent interest as a random process that arises under the influence of coexisting Gaussian and stable domains of attraction.

Keywords: Long-range dependence, weak convergence, fractional Brownian motion.

1 Introduction and statement of results

A Lévy subordinator $\{X_t, t \geq 0\}$ is a real-valued random process with independent and stationary increments and increasing pure-jump trajectories. The inverse process $\{T_x, x \geq 0\}$ defined by the first passage times $T_x = \inf\{t \geq 0 : X_t > x\}$ has nondecreasing, trajectories, where the lengths of the flat pieces of $\{T_x\}$ correspond to the jump sizes of $\{X_t\}$. The dependence structure in the paths of the inverse process is entirely different from that of the Lévy subordinator, since big jumps in the Lévy process may cause strong dependencies that last over a considerable period of evolution of the path of its inverse. In this paper we take a scaling approach to study the nature of the random fluctuations that build up as a result of such long-memory effects. By superposing a large number of paths of the inverse Lévy process and simultaneously scale the time parameter of the process, we obtain scaling limit results for the centered and normalized superposition process.

In somewhat more detail, our starting point is a Lévy subordinator with Lévy measure $\nu(dx)$ of regularly varying tail with index $1+\beta$, $0<\beta<1$. In particular, $\mu := \int x\nu(dx) < \infty$. The initial distribution of the subordinator process is chosen such that the resulting inverse process has stationary increments and expected value $E(T_x) = x/\mu$. Letting $\{T_x^i\}_{i\geq 1}$ be a collection of independent copies of $\{T_x\}$, our main result is the derivation of a limit process for the summation scheme

$$\frac{1}{a}\sum_{i=1}^{m}(T_{ax}^i - \frac{1}{\mu}ax), \quad x \geq 0,$$

as both m and $a = a_m$ tend to infinity in such a way that m is of the same order of magnitude as a^β, modulo slowly varying functions. The reason for this choice of scaling is to attempt to trace the superposition process on a time scale that captures the size of the fluctuations around its mean. In the asymptotic limit appears a non-Gaussian, non-stable process with long-range dependence, which has also been obtained earlier in a different but related context in Gaigalas and Kaj [7].

To give a heuristic context for the topics of interest in this work, let us recall the following limit result for Lévy processes. Writing $\alpha = 1+\beta$, the centered and scaled process $(X_t - \mu t)/t^{1/\alpha}$ converges in distribution as $t \to \infty$ to a random variable Z_α, having a stable distribution with stable index α. If we write Γ_x for the overshoot at x, so that $X_{T_x} = x + \Gamma_x$, then

$$\frac{T_x - x/\mu}{x^{1/\alpha}} = -\frac{X_{T_x} - \mu T_x}{T_x^{1/\alpha}}\Big(\frac{T_x}{x}\Big)^{1/\alpha}\frac{1}{\mu} + \frac{\Gamma_x}{\mu x^{1/\alpha}}.$$

In this relation, $T_x/x \to 1/\mu$ as $x \to \infty$ by the law of large numbers. It can be shown moreover that the second term on the right hand side is a remainder term with $\Gamma_x/x^{1/\alpha} \to 0$ as $x \to \infty$. Therefore $(T_x - x/\mu)/x^{1/\alpha}$ converges in distribution to $-Z_\alpha/\mu^{1+1/\alpha}$ as $x \to \infty$. Proceeding heuristically, with $m \sim a^\beta$ we may now rewrite the superposition process either as

$$\frac{1}{a}\sum_{i=1}^{m}(T_{ax}^i - \frac{1}{\mu}ax) \sim \frac{1}{a^{1-\beta/2}}\int_0^{ax}\frac{1}{m^{1/2}}\sum_{i=1}^{m}(dT_u^i - \frac{1}{\mu}du)$$

or

$$\frac{1}{a}\sum_{i=1}^{m}(T_{ax}^{i}-\frac{1}{\mu}ax)\sim\frac{1}{m^{1/(1+\beta)}}\sum_{i=1}^{m}\frac{T_{ax}^{i}-ax/\mu}{a^{1/(1+\beta)}}.$$

The first representation emphasizes a sequence of random variables in the domain of attraction of a Gaussian law ($m\to\infty$ with a fixed). The second representation highlights a sequence in the domain of attraction of a stable law with index $1+\beta$ ($a\to\infty$ with m fixed), which is the type of convergence just discussed above. In our case of interest the two domains of attraction coexist and both of them influence the resulting limit process.

The main result (Theorem 2 below) is studied in parallel to and compared with a scaling regime of Gaussian predominance, leading to fractional Brownian motion in the limit (Theorem 1). Similar scaling limit results where fractional Brownian fluctuations appear have been established earlier for a variety of models, typically with motivation of modeling random variation in aggregated data traffic streams. For an introduction and overview of these topics and discussion of the modeling context, as well as detailed statements and derivations of such results, see Taqqu [13] and Willinger *et al.* [14].

After introducing the model in detail and stating our results in Section 1 of the paper, the proofs are given in Section 2. The main technique we use for the study of the one-dimensional distributions of the scaled processes and their limit behavior is that of double transforms in the sense of taking Laplace transforms in the time variable of the logarithmic moment generating function of the random variables. The finite-dimensional distributions are then obtained from recursive sets of integral equations for the finite-dimensional cumulant functions.

1.1 A Lévy subordinator and its inverse

Let $\{\widetilde{X}_t, t\geq 0\}$, $\widetilde{X}_0=0$, denote a Lévy subordinator with right-continuous paths, having drift zero and Lévy measure $\nu(a,b)=\int_a^b\nu(dx)$ with no atom at zero, such that

$$\int_0^\infty(1\wedge x)\nu(dx)<\infty\quad\text{and}\quad\mu=\int_0^\infty x\nu(dx)<\infty,\tag{1}$$

which implies that the first moment is finite, $E(\widetilde{X}_t)=\mu t<\infty$. The Laplace transform is given by $-\ln E(e^{-u\widetilde{X}_t})=t\Phi(u)$, $u\geq 0$, with Laplace exponent

$$\Phi(u)=\int_0^\infty(1-e^{-ux})\,\nu(dx).$$

Let $X_t=X_0+\widetilde{X}_t$ denote the corresponding delayed subordinator process with general initial distribution X_0 assumed to be independent of $\{\widetilde{X}_t\}$. We will study the case when $X_0>0$ has distribution function

$$P(X_0\leq x)=\frac{1}{\mu}\int_0^x\int_y^\infty\nu(ds)\,dy,\tag{2}$$

for which $E(e^{-uX_0})=\frac{1}{\mu u}\Phi(u)$ and so

$$E(e^{-uX_t})=\frac{1}{\mu u}\Phi(u)\,\exp\{-t\Phi(u)\}\quad u\geq 0.\tag{3}$$

Next we introduce the first passage process of the subordinator. Useful references are Bertoin [2], [3]. Van Harn and Steutel, [9], investigate stationarity properties of delayed subordinators and derive closely related results to those in Lemma 1 and Lemma 3 below. The entrance time of the Lévy process $\{X_t\}$ into a set B is defined by $T_B = \inf\{t \geq 0 : X_t \in B\}$. For any open set B, T_B is a stopping time. The first passage time $T_x = T_{(x,\infty)}$ strictly above a level x is the entrance time into (x, ∞), that is

$$T_x = \inf\{t \geq 0 : X_t > x\}, \quad x \geq 0,$$

which is a right-continuous function with left limits. Since $X_t \uparrow \infty$ as $t \uparrow \infty$, we have $T_x < \infty$ for all x and $P(T_x \leq t) = P(X_t > x)$, $x, t \geq 0$. Also,

$$E(T_x) = \int_0^\infty P(T_x > t)\, dt = E \int_0^\infty 1_{\{X_t \leq x\}}\, dt.$$

Hence

$$\int_0^\infty u e^{-ux} E(T_x)\, dx = \int_0^\infty E(e^{-uX_t})\, dt = \frac{1}{\mu u}$$

in view of (3), and therefore

$$E(T_x) = \frac{1}{\mu} x.$$

We call $\{T_x\}$ the inverse Lévy subordinator and the process

$$\widetilde{T}_x = \inf\{t \geq 0 : \widetilde{X}_t > x\}, \quad x \geq 0,$$

the pure inverse Lévy subordinator. The path-regularity of $\{T_x\}$ is determined by the distribution of the small jumps of $\{X_t\}$, manifest in the asymptotic behavior of $\nu(dx)$ for x close to 0. The nature of the paths varies considerably with compound Poisson processes as one extreme case. These are the subordinators for which the Lévy measure is finite on the positive half line and the passage time process has piece-wise constant trajectories of lengths drawn from the probability distribution $\nu(0, x]/\nu(0, \infty)$, $x \geq 0$, and with exponentially distributed jumps. On the other hand, if the number $\sigma = \sup\{\alpha > 0 : \lim_{\lambda \to \infty} \lambda^{-\alpha} \Phi(\lambda) = \infty\}$, known as the lower index of the subordinator, is positive (and ≤ 1 because of assumption (1)) then the inverse process $\{T_x\}$ is a.s. γ-Hölder continuous on any compact interval, for each index $\gamma < \sigma$. This property is shown in Bertoin [2], Ch. 3 ($X_0 = 0$).

The scaling problem studied in this work involves weak convergence in the sense of convergence of finite-dimensional distributions plus a tightness property. Only the asymptotic behavior of $\nu(x, \infty)$ as $x \to \infty$ is relevant for the distributional convergence. For tightness we make the additional assumption that the Lévy measure lower index σ is positive, and establish tightness in the space $C = C[0, \infty)$ of continuous random processes. The path space is such that each $C[0, T]$, $T > 0$, is equipped with the topology of convergence in supremum norm.

We will prove below the following

Lemma 1 *The inverse subordinator process $\{T_x, x \geq 0\}$ has stationary increments.*

1.2 Scaling limit theorem

Our basic assumption is that the Lévy measure ν is regularly varying at infinity with index $1+\beta$, $0<\beta<1$, i.e.

$$\int_x^\infty \nu(dy) \sim \frac{1}{x^{1+\beta}} L(x), \quad x \to \infty, \tag{4}$$

where L is a slowly varying function and we write $f(x) \sim g(x)$ if f and g are positive functions and $f(x)/g(x) \to 1$ as $x \to \infty$. The summation schemes to be applied involve speeding up the time parameter using a rescaling sequence $a_m \to \infty$, either such that

$$\frac{mL(a_m)}{a_m^\beta} \to \infty, \quad m \to \infty, \tag{5}$$

or such that

$$\frac{mL(a_m)}{a_m^\beta} \to c^\beta \mu, \quad m \to \infty, \tag{6}$$

where c, $0<c<\infty$, is an additional parameter that signifies the relative change of scales of size and time. In addition, we assume that the lower index of ν is strictly positive,

$$\sigma = \sup\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\} > 0. \tag{7}$$

Theorem 1 *Under assumption (4), let a_m be a sequence such that $a_m \to \infty$ as $m \to \infty$ and (5) holds, and define b_m by*

$$b_m^2 = m a_m^{2-\beta} L(a_m)/\mu. \tag{8}$$

Then, in the sense of weak convergence of random processes in C,

$$\Big\{\frac{1}{b_m}\sum_{i=1}^m (T^i_{a_m x} - \frac{1}{\mu} a_m x),\ x \ge 0\Big\} \Rightarrow \{\mu^{-1}\sigma_\beta B_H(x),\ x\ge 0\}, \tag{9}$$

where

$$\sigma_\beta^2 = \frac{2}{\beta(1-\beta)(2-\beta)}, \quad H = 1-\beta/2,$$

and B_H is standard fractional Brownian motion with Hurst index H, i.e.

$$\log E \exp\Big\{\sum_{i=1}^n \theta_i B_H(x_i)\Big\} = \frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n \theta_i\theta_j (x_i^{2-\beta} + x_j^{2-\beta} - (x_i - x_j)^{2-\beta}),$$

where $0 = x_0 \le x_1 \le \ldots \le x_n$, $n \ge 1$.

Theorem 2 *Under the assumption (4), if a_m is a sequence such that $a_m \to \infty$ and (6) holds for some constant $c>0$ as $m\to\infty$, then*

$$\Big\{\frac{1}{a_m}\sum_{i=1}^m (T^i_{a_m x} - \frac{1}{\mu} a_m x),\ x \ge 0\Big\} \Rightarrow \{-\mu^{-1} c Y_\beta(x/c),\ x\ge 0\}, \tag{10}$$

in the sense of weak convergence in C*. Here* $\{Y_\beta(x), x \geq 0\}$ *is a zero mean stochastic process with continuous paths and finite-dimensional distributions characterized by the cumulant generating function*

$$\log E \exp\Big\{\sum_{i=1}^n \theta_i (Y_\beta(x_i) - Y_\beta(x_{i-1}))\Big\} = \frac{1}{\beta}\sum_{i=1}^n \theta_i^2 \int_0^{\Delta x_i}\int_0^v e^{\theta_i u} u^{-\beta}\, du dv$$
$$+\frac{1}{\beta}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \theta_i\theta_j \exp\Big\{\sum_{k=i+1}^{j-1}\theta_k \Delta x_k\Big\}$$
$$\times \int_0^{\Delta x_i}\int_0^{\Delta x_j} e^{\theta_j u} e^{\theta_i v}(x_{j-1} - x_i + u + v)^{-\beta}\, du dv, \tag{11}$$

where $0 = x_0 \leq x_1 \leq \ldots \leq x_n$*, and* $\Delta x_i = x_i - x_{i-1}$*,* $i = 1, \ldots, n$*.*

Remarks **a)** Assumption (7) is used for proving tightness. Convergence of the finite-dimensional distributions holds in general.

b) The process $\{Y_\beta(x)\}$ has been derived in Gaigalas and Kaj [7] as a limit process in the setting of a superposition of independent renewal processes with stationary increments and heavy-tailed inter-renewal distribution, and in Kaj and Taqqu [10] for an infinite source Poisson process with heavy-tailed activity periods. In both cases the motivation is partly from modeling the total traffic load generated by many independent sources at an arrival point in a data traffic network. In these references condition (5) is called *fast connection rate* and (6) *intermediate connection rate*. They are compared to an alternative third scaling regime of *slow* connection rate, for which the limit process turns out to be a stable Lévy process with stable index $\alpha = 1 + \beta$, see also Mikosch *et al.* [11] or Willinger *et al.* [14].

c) Proofs of the following properties among others can be found in Gaigalas and Kaj [7]. The process $\{Y_\beta\}$ has stationary increments and continuous trajectories. The process is not self-similar. The higher moments are of the order $E(Y_\beta^k(x)) \sim \text{const}\, x^{k-\beta}$, $k \geq 2$, for large x. Specifically, the second-order properties (mean, variance, covariance) are the same (modulo constants) as those for the Gaussian fractional Brownian motion, whereas higher moments are different. For example, $\{Y_\beta\}$ is positively skewed. The paths are γ-Hölder continuous for all $\gamma < 1 - \beta/2$ (not $\gamma < 1$ as claimed in [7]). In addition, representations for $\{Y_\beta\}$ as integrals with respect to compensated Poisson measures are derived i Gaigalas [6] and in Kaj and Taqqu [10].

d) The results of Theorem 1 and Theorem 2 can be stated in greater generality by considering a Lévy subordinator process with drift. Starting again from the Lévy subordinator $\{\widetilde{X}_t\}$ with expected value μt, let $\eta > -\mu$ be a drift parameter and define a new initial value $X_0^{(\eta)}$ by letting $P(X_0^{(\eta)} > x) = \eta P(X_0 > x)/(\mu + \eta)$. Then define $X_t^{(\eta)} = X_0^{(\eta)} + \widetilde{X}_t + \eta t$ and form the inverse process $T_x^{(\eta)} = \inf\{t \geq 0 : X_t^{(\eta)} > x\}$. It is straightforward to check as above that $\{T_x^{(\eta)}\}$ has stationary increments and expected value $x/(\mu+\eta)$. Furthermore, the limit results in Theorems 1 and 2 (the case $\eta = 0$) remain true in this more general situation if μ is everywhere replaced by $\mu + \eta$. bf e) A referee of this work made the following interesting observations: The renewal processes studied in [7] can be viewed as discrete local time processes of discrete regenerative sets (ranges of compound Poisson subordinators). In this light, the present situation is the natural analoque for continuous local time processes of perfect regenerative sets (ranges of subordinators that are not compound Poisson). One can expect the scaling limits to transfer since they are large-time asymptotics which should not depend on the local structure. Some relevant references for the connections of regenerative sets and subordinators are Fristedt citefristedt, and Gnedin and Pitman [8].

2 Demonstration of the results

We focus on the proof of Theorem 2. The proof of Theorem 1 can be carried out in parallel. Each step in the latter case turns out to be simpler and therefore we give only a summary of the arguments. As a preliminary for the proof of Theorem 2 we observe the following properties of the functions introduced in (11), which are straightforward to verify.

Lemma 2 *Relation (11) defines a consistent family of finite-dimensional distributions, such that for any $c > 0$*

$$\begin{aligned}\log E \exp\Big\{\sum_{i=1}^{n} \theta_i (cY_\beta(x_i/c) - cY_\beta(x_{i-1}/c))\Big\} \\ = c^\beta \log E \exp\Big\{\sum_{i=1}^{n} \theta_i (Y_\beta(x_i) - Y_\beta(x_{i-1}))\Big\}.\end{aligned}$$

The main part of the proofs of Theorem 1 and Theorem 2 consists in establishing convergence of the scaled n-point cumulant functions

$$\begin{aligned}\log E \exp\Big\{\sum_{i=1}^{n} \theta_i \frac{1}{b_m} \sum_{k=1}^{m} (T^{(k)}_{a_m x_i} - \frac{1}{\mu} a_m x_i)\Big\} \\ = mE\Big[\exp\Big\{\sum_{i=1}^{n} \frac{\theta_i}{b_m} (T_{a_m x_i} - \frac{1}{\mu} a_m x_i)\Big\} - 1\Big] + \mathcal{O}(1/m) \qquad (12)\end{aligned}$$

toward the corresponding functionals of the limit processes. As a preparation we study the joint distribution (T_x, Γ_x), where $\{\Gamma_x\}$ is the overshoot process, and other properties of the one-dimensional marginal distributions of T_x.

2.1 Marginal distributions and the overshoot process

The overshoot process $\{\Gamma_x, x \geq 0\}$ associated with the first passage time $\{T_x\}$ is defined for $x \geq 0$ by

$$\Gamma_x = X_{T_x} - x,$$

and represents at time x the remaining time until the next point of increase of the inverse subordinator. The following Lemma is a special case of a result valid for general Lévy processes adaptated to the case of a general initial distribution X_0. For a proof see Theorem 49.2 in Sato [12].

Lemma 3 *For $u > 0$, $\theta < \Phi(u)$, and $v > 0$ with $v \neq u$,*

$$\int_0^\infty ue^{-ux} E(e^{\theta T_x - v\Gamma_x})\,dx = \frac{u}{u-v}\left(\frac{\Phi(v)}{\mu v} - \frac{\Phi(v)-\theta}{\Phi(u)-\theta}\,\frac{\Phi(u)}{\mu u}\right).$$

Proof of Lemma 1 In Lemma 3, take $u > 0$ and $v > 0$, $u \neq v$, and let $\theta = 0$. We obtain

$$\int_0^\infty ue^{-ux} E(e^{-v\Gamma_x})\,dx = \frac{u}{u-v}\left(\frac{\Phi(v)}{\mu v} - \frac{\Phi(v)}{\Phi(u)}\,\frac{\Phi(u)}{\mu u}\right) = \frac{\Phi(v)}{\mu v}.$$

Hence, for any $x \geq 0$, $\Gamma_x \overset{d}{=} X_0$. Consequently, for each x the increment process $T_{x+y} - T_x$, $y \geq 0$, begins with a flat period for a duration of time having the distribution X_0, which is just the same behavior as the original process T_x, $x \geq 0$. To formalize the argument, note

$$P(T_{x+y} - T_x > t) = P(\Gamma_x < y, T_{x+y} - T_x > t) = P(\Gamma_x < y, X_{T_x+t} - X_{T_x} < y - \Gamma_x).$$

Since $\Gamma_x = X_{T_x} - x$ is independent of $X_{T_x+t} - X_{T_x}$ and $X_{T_x+t} - X_{T_x} \overset{d}{=} \widetilde{X}_t$ it follows that

$$P(T_{x+y} - T_x > t) = P(X_0 < y, X_{T_x+t} - X_{T_x} < y - X_0) = P(X_t < y) = P(T_y > t).$$

□

Lemma 4 *For $u > 0$ and $\theta < \Phi(u)$,*

$$\int_0^\infty ue^{-ux} E(e^{\theta T_x})\,dx = 1 + \frac{\theta}{\Phi(u)-\theta}\,\frac{\Phi(u)}{\mu u}. \tag{13}$$

Also, for $u > 0$ and $\theta > -\mu u$,

$$\int_0^\infty ue^{-ux}\, E(e^{\theta(T_x - x/\mu)} - 1)\,dx = \frac{\theta^2}{(\mu u + \theta)^2}\left[\frac{\mu u}{\Phi(u+\theta/\mu) - \theta} - 1\right] \tag{14}$$

and

$$\int_0^\infty ue^{-ux}\, E(e^{\theta(\widetilde{T}_x - x/\mu)} - 1)\,dx = \frac{\theta}{\mu u + \theta}\left[\frac{\mu u}{\Phi(u+\theta/\mu) - \theta} - 1\right]. \tag{15}$$

Proof. Relation (13) follows by letting $v \to 0$ in Lemma 3 and using that $\Phi(v)/v \to \mu$ in this limit.

The remaining calculations, involving the random variables $T_x - x/\mu$ and $\widetilde{T}_x - x/\mu$, follow from (13) and the analogous expression

$$\int_0^\infty ue^{-ux}\, E(e^{\theta \widetilde{T}_x})\,dx = \frac{\Phi(u)}{\Phi(u)-\theta},$$

where we note $\phi(u + \theta/\mu) < \mu u + \theta$ for all θ such that $u + \theta/\mu > 0$. □

Lemma 5 *The function*

$$E(e^{\theta(T_x - x/\mu)} - 1), \quad x \geq 0,$$

is nonnegative for any real parameter θ and differentiable and nondecreasing with respect to the variable x. The derivative with respect to x is given by

$$\frac{d}{dx}E(e^{\theta(T_x - x/\mu)} - 1) = \theta e^{-\theta x/\mu} E(e^{\theta \widetilde{T}_x} - e^{\theta T_x})/\mu \geq 0.$$

Proof. The nonnegativity is obvious from Jensen's inequality. It follows from (14), (15) and the uniqueness property of Laplace transforms that $E(e^{\theta(T_x - x/\mu)} - 1)$ is obtained as the convolution of $E(e^{\theta(\widetilde{T}_x - x/\mu)} - 1)$ with the exponential $e^{\theta x/\mu}$. Hence

$$E(e^{\theta(T_x - x/\mu)} - 1) = \frac{\theta}{\mu}\int_0^x e^{-\theta(x-y)/\mu} E(e^{\theta(\widetilde{T}_y - y/\mu)} - 1)\, dy.$$

The left hand side is differentiable in x with derivative

$$\begin{aligned}\frac{d}{dx}E(e^{\theta(T_x - x/\mu)} - 1) &= -\frac{\theta}{\mu}E(e^{\theta(T_x - x/\mu)} - 1) + \frac{\theta}{\mu}E(e^{\theta(\widetilde{T}_x - x/\mu)} - 1)\\ &= \frac{\theta}{\mu}e^{-\theta x/\mu}E(e^{\theta\widetilde{T}_x} - e^{\theta T_x}).\end{aligned}$$

Now we observe that the processes T_x and $\widetilde{T}_x$ can be constructed on the same probability space simply by a shift of size X_0 so that T is a copy of $\widetilde{T}$ with the first point of increase in X_0 rather than in 0. In particular $P(\widetilde{T}_x \geq T_x) = 1$. Hence $\theta E(e^{\theta\widetilde{T}_x} - e^{\theta T_x}) \geq 0$ for any θ.

□

Lemma 6 *For $x > 0$,*

i) $E(\widetilde{T}_x) \leq \dfrac{e^2(e-1)^{-1}}{\nu(x,\infty)}$,

ii) $0 \leq E(\widetilde{T}_x) - x/\mu \leq \dfrac{e}{\Phi(1/x)}$,

iii) $\dfrac{d}{dx}\mathrm{Var}(T_x) = \dfrac{2}{\mu}E(\widetilde{T}_x - x/\mu) \geq 0$,

iv) $\mathrm{Var}(T_x) \leq \dfrac{2e}{\mu}\displaystyle\int_0^x \Phi(1/y)^{-1}\, dy.$

Proof. Inequality i) follows from

$$E(\widetilde{T}_x) = \int_0^\infty P(\widetilde{X}_t \leq x)\, dt \leq \int_0^\infty eE(e^{-\widetilde{X}_t/x})\, dt = e\int_0^\infty e^{-t\Phi(1/x)}\, dt = e/\Phi(1/x)$$

and

$$\Phi(1/x) = \frac{1}{x}\int_0^\infty e^{-u/x}\nu(u,\infty)\, du \geq \frac{1}{x}\int_0^x e^{-u/x}\nu(u,\infty)\, du \geq \nu(x,\infty)(1-e^{-1}).$$

For ii), it was noticed in the proof of Lemma 5 that the processes T_x and $\widetilde{T}_x$ could be constructed such that $\widetilde{T}_x \geq T_x$ almost surely. Hence $E(\widetilde{T}_x) \geq E(T_x) = x/\mu$. Moreover,

$$E(\widetilde{T}_x) - x/\mu \leq \int_{x/\mu}^{\infty} P(\widetilde{X}_t \leq x)\,dt \leq e \int_{x/\mu}^{\infty} e^{-t\Phi(1/x)}\,dt \leq e/\Phi(1/x).$$

To prove iii) and iv), differentiate twice with respect to θ in (14) to obtain

$$\int_0^{\infty} ue^{-ux}\mathrm{Var}(T_x)\,dx = \frac{2}{(\mu u)^2}\Big(\frac{\mu u}{\Phi(u)} - 1\Big). \tag{16}$$

Similarly, using (15),

$$\int_0^{\infty} ue^{-ux}E(\widetilde{T}_x - x/\mu)\,dx = \frac{1}{\mu u}\Big(\frac{\mu u}{\Phi(u)} - 1\Big),$$

hence by partial integration

$$\int_0^{\infty} ue^{-ux}\int_0^x E(\widetilde{T}_y - y/\mu)\,dy\,dx = \frac{1}{\mu u^2}\Big(\frac{\mu u}{\Phi(u)} - 1\Big).$$

By identification of the Laplace transforms,

$$\mathrm{Var}(T_x) = \frac{2}{\mu}\int_0^x E(\widetilde{T}_y - y/\mu)\,dy.$$

The two inequalities in (ii) now imply iii) and iv).

□

2.2 The marginal distribution under scaling

The first scaling properties that we will need are a weak law of large numbers and an elementary renewal type theorem for $\widetilde{T}_x$.

Lemma 7 *As $a \to \infty$, we have*

i) $\dfrac{1}{a}\widetilde{T}_{ax} \to \dfrac{x}{\mu}$ *in distribution*

ii) $\dfrac{1}{a}E(\widetilde{T}_{ax}) \to \dfrac{x}{\mu}$.

Proof. By the law of large numbers for Lévy processes,

$$P(\widetilde{X}_{at}/a \leq x) \to 1_{\{\mu t \leq x\}}, \quad a \to \infty.$$

Moreover,

$$P(\widetilde{X}_{at}/a \leq x) \leq e^{\gamma x}E(e^{-\gamma \widetilde{X}_{at}/a}) = e^{\gamma x}e^{-at\Phi(\gamma/a)}, \quad \gamma > 0.$$

Fix θ. Take $\theta_0 > \theta \vee 0$ and γ_0 such that $\gamma_0\mu > \theta_0$. Since $a\Phi(\gamma/a) \to \mu\gamma$ we can find a_0 for which

$$P(\widetilde{X}_{at}/a \le x) \le e^{\gamma_0 x} e^{-\theta_0 t}, \quad a \ge a_0.$$

Thus, by dominated convergence as $a \to \infty$,

$$E(e^{\theta \widetilde{T}_{ax}/a}) = 1 + \int_0^\infty \theta e^{\theta t} P(\widetilde{X}_{at}/a \le x)\, dt \to 1 + \int_0^{x/\mu} \theta e^{\theta t}\, dt = e^{\theta x/\mu}.$$

The same proof yields

$$E(\widetilde{T}_{ax})/a = \int_0^\infty P(\widetilde{X}_{at}/a \le x)\, dt \to x/\mu,$$

which is ii). □

We are now prepared to prove a limit property of the centered variable $T_x - x/\mu$ under scaling, which is crucial for the distributional convergence in Theorem 2.

Lemma 8 *If the sequence $a = a_m$ is such that (6) holds for some $c > 0$, then as $m \to \infty$,*

$$m\, E(e^{\theta(\widetilde{T}_{ax} - ax/\mu)/a} - e^{\theta(T_{ax} - ax/\mu)/a}) \to \frac{c^\beta}{\mu\beta} \int_0^x \theta e^{-\theta t/\mu} t^{-\beta}\, dt \tag{17}$$

and

$$m\, \frac{d}{dx} E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) \to \frac{c^\beta}{\mu^2\beta} \int_0^x \theta^2 e^{-\theta t/\mu} t^{-\beta}\, dt. \tag{18}$$

Proof. It is enough to prove (17) since (18) then follows directly from Lemma 5.

Recall from (2) the relation $P(X_0 \le x) = \frac{1}{\mu}\int_0^x \nu(u, \infty)\, dy$ where we use the notation $\nu(y, \infty) = \int_y^\infty \nu(dv)$. For fixed x condition on X_0 to get

$$P(T_x < t < \widetilde{T}_x) = P(X_0 > x) P(t < \widetilde{T}_x) + \frac{1}{\mu} \int_0^x P(\widetilde{T}_{x-y} < t < \widetilde{T}_x) \nu(y, \infty)\, dy.$$

Multiply this identity by $\theta e^{\theta t}$ and integrate over $t \ge 0$ to obtain

$$E(e^{\theta \widetilde{T}_x} - e^{\theta T_x}) = P(X_0 > x) E(e^{\theta \widetilde{T}_x} - 1) + \frac{1}{\mu} \int_0^x E(e^{\theta \widetilde{T}_x} - e^{\theta \widetilde{T}_{x-y}})\, \nu(y, \infty)\, dy.$$

Hence

$$\begin{aligned} m\, E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta T_{ax}/a}) &= m P(X_0 > ax) E(e^{\theta \widetilde{T}_{ax}/a} - 1) \\ &+ \frac{1}{\mu} \int_0^x E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta \widetilde{T}_{a(x-y)}/a})\, am\nu(ay, \infty)\, dy. \end{aligned} \tag{19}$$

By (4),

$$\frac{1}{\mu} am\, \nu(ay, \infty) \to c^\beta y^{-1-\beta}.$$

By (4) and (6), and using the direct half of Karamata's theorem,

$$m P(X_0 > ax) \to \beta^{-1} c^\beta x^{-\beta},$$

cf. Bingham *et al.* (1987) Thm. 1.5.11 ii) (using in their notation $f(x) = \nu(x, \infty)$, $\rho = -(1+\beta)$, $\sigma = 0$). If we assume for the moment that the order can be interchanged in which we integrate over y and take the limit $m, a \to \infty$, then applying the above asymptotic results as well as Lemma 7 i),

$$\begin{aligned} &m\, E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta T_{ax}/a}) \\ &\quad \to \beta^{-1} c^\beta y^{-\beta}(e^{\theta x/\mu} - 1) + \int_0^x (e^{\theta x/\mu} - e^{\theta(x-y)/\mu}) c^\beta y^{-1-\beta}\, dy \\ &\quad = e^{\theta x/\mu} \frac{c^\beta}{\mu\beta} \int_0^x \theta e^{-\theta t/\mu} t^{-\beta}\, dt, \end{aligned}$$

which is the desired relation (17). In the remaining part of the proof we verify the validity of this limit operation by deriving an upper bound for the integrand $E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta \widetilde{T}_{a(x-y)}/a})\, am\nu(ay, \infty)$ in (19), which is dy-integrable over $(0, x]$.

Using

$$|E(e^{\theta \widetilde{T}_x} - e^{\theta \widetilde{T}_{x-y}})| \le |\theta| E[(e^{\theta \widetilde{T}_x} \vee 1)\, |\widetilde{T}_x - \widetilde{T}_{x-y}|]$$

and Hölder's inequality we have, for each integer $k \ge 2$,

$$|E(e^{\theta \widetilde{T}_x} - e^{\theta \widetilde{T}_{x-y}})| \le |\theta| E[(e^{\theta \widetilde{T}_x} \vee 1)^{k/(k-1)}]^{(k-1)/k} E[|\widetilde{T}_x - \widetilde{T}_{x-y}|^k]^{1/k}. \qquad (20)$$

Now,

$$\begin{aligned} E[|\widetilde{T}_x - \widetilde{T}_{x-y}|^k] &= E\int_0^\infty \dots \int_0^\infty 1_{\{\widetilde{T}_{x-y} < t_1, \dots, t_k < \widetilde{T}_x\}}\, dt_1 \dots dt_k \\ &= k! \int \dots \int_{t_1 < \dots < t_k} P(\widetilde{T}_{x-y} < t_1, \dots, t_k < \widetilde{T}_x)\, dt_1 \dots dt_k \\ &= k! \int \dots \int_{t_1 < \dots < t_k} P(x - y < \widetilde{X}_{t_1} < \dots < \widetilde{X}_{t_k} < x)\, dt_1 \dots dt_k. \end{aligned}$$

For the event $x - y < \widetilde{X}_{t_1} < \dots < \widetilde{X}_{t_k} < x$ to occur it is necessary, in addition to $X_{t_1} \le x$, that all increments $\widetilde{X}_{t_j} - \widetilde{X}_{t_{j-1}}$, $2 \le j \le k$ are less than y in size. Hence the right hand side is at most

$$k! \int \dots \int_{t_1 < \dots < t_k} P(\widetilde{X}_{t_1} < x, \widetilde{X}_{t_j} - \widetilde{X}_{t_{j-1}} < y, 2 \le j \le k)\, dt_1 \dots dt_k,$$

which equals

$$\begin{aligned} &k! \int_0^\infty dt_1 P(\widetilde{X}_{t_1} < x) \int_{t_1}^\infty dt_2 P(\widetilde{X}_{t_2 - t_1} < y) \dots \int_{t_{k-1}}^\infty dt_k P(\widetilde{X}_{t_k - t_{k-1}} < y) \\ &\quad = k! E(\widetilde{T}_x) E(\widetilde{T}_y)^{k-1}, \end{aligned}$$

since the increments of $X(t)$ are independent and stationary. Combined with (20) this yields,

$$|E(e^{\theta \widetilde{T}_x} - e^{\theta \widetilde{T}_{x-y}})| \le 2|\theta| E[(e^{\theta \widetilde{T}_x} \vee 1)^{k/(k-1)}]^{(k-1)/k} (k!)^{1/k} E(\widetilde{T}_x)^{1/k} E(\widetilde{T}_y)^{1-1/k}.$$

By Lemma 7, for $a \geq a_0$ and sufficiently large a_0, we may assume

$$E[(e^{\theta \widetilde{T}_{ax}/a} \vee 1)^{k/(k-1)}]^{(k-1)/k} E(\widetilde{T}_{ax}/a)^{1/k} \leq 2(e^{\theta x/\mu} \vee 1)(x/\mu)^{1/k}$$

thus,

$$|E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta \widetilde{T}_{a(x-y)}/a})| \leq 4|\theta|(k!)^{1/k}\,(e^{\theta x/\mu} \vee 1)\,(x/\mu)^{1/k}\,E(\widetilde{T}_{ay}/a)^{1-1/k}.$$

Writing $C_{\theta,k}(x) = 4|\theta|(k!)^{1/k}\,(e^{\theta x/\mu} \vee 1)\,(x/\mu)^{1/k}$, we have obtained

$$\begin{aligned} &|E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta \widetilde{T}_{a(x-y)}/a})|\, am\nu(ay,\infty) \\ &\quad \leq C_{\theta,k}(x)\, E(\widetilde{T}_{ay}/a)^{1-1/k}\, am\nu(ay,\infty), \quad 0 < y \leq x,\ a \geq a_0. \end{aligned} \tag{21}$$

We split the further task of estimating the right hand side in the above expression in the two cases $ay > a_0$ and $ay \leq a_0$.

For $ay > a_0$ and suitably modified a_0 we are allowed to use once again Lemma 7 i) to conclude $E(\widetilde{T}_{ay})/a \leq 2y/\mu$. Furthermore, since the function $\nu(x,\infty)$ is regularly varying at infinity with index $-(1+\beta)$, we have for any $\epsilon > 0$ the Potter type bound

$$am\nu(ay,\infty) \leq 2y^{-1-\beta}\max(y^{\epsilon}, y^{-\epsilon})$$

(Bingham *et al.* (1987), Ch. 1.5). Thus,

$$E(\widetilde{T}_{ay}/a)^{1-1/k}\, am\nu(ay,\infty) \leq 4y^{-1/k-\beta}\max(y^{\epsilon}, y^{-\epsilon}).$$

Choose

$$\epsilon < 1-\beta, \quad k > (1-\beta-\epsilon)^{-1} \tag{22}$$

to obtain a dominating function for (21) which is integrable in y over $[0,x]$.

For the remaining case $ay \leq a_0$, Lemma 6 i) implies

$$E(\widetilde{T}_{ay}/a)^{1-1/k}\, am\nu(ay,\infty) \leq (e^2/(e-1))^{1-1/k}\, m\, a^{1/k}\nu(ay,\infty)^{1/k}.$$

Using a property of slowly varying functions (Bingham *et al.* (1987), Prop 1.3.6), for any $\epsilon > 0$, $L(a)a^{\epsilon} \to \infty$ as $a \to \infty$. Hence we may assume $a^{-\epsilon} \leq L(a)$. Also note

$$\nu(ay,\infty) \leq \frac{1}{ay}\int_{ay}^{\infty} u\,\nu(du) \leq \frac{\mu}{ay}.$$

Thus, using (6),

$$m\, a^{1/k}\nu(ay,\infty)^{1/k} \leq \frac{mL(a)}{a^{\beta}} a^{\epsilon+\beta+1/k}\,(\mu/ay)^{1/k} \leq 2c^{\beta}\mu^{1+1/k}a^{\epsilon+\beta}y^{-1/k}.$$

Now apply $a \leq a_0/y$ to obtain from (21),

$$|E(e^{\theta \widetilde{T}_{ax}/a} - e^{\theta \widetilde{T}_{a(x-y)}/a})|\, am\nu(ay,\infty) \leq C_{\theta,k}(x) 10 c^{\beta}\mu^{1+1/k}a_0^{\epsilon+\beta}\,\frac{1}{y^{\epsilon+\beta+1/k}},$$

which is integrable, chosing again ϵ and k according to (22). □

Lemma 9 *For a_m such that (6) holds for some $c > 0$,*

$$mE(e^{\theta(T_{a_m x} - a_m x/\mu)/a_m} - 1) \to \frac{c^\beta}{\beta\mu^2}\int_0^x \int_0^y \theta^2 e^{-\theta s/\mu} s^{-\beta}\, ds dy, \quad m \to \infty.$$

Proof. By Lemma 5, $mE(e^{\theta(T_{ax} - ax/\mu)/a} - 1)$ is nonnegative and increasing in x. The limit function on the right hand side is also nonnegative and increasing. Hence the lemma follows from weak convergence of measures if we can prove

$$\int_0^\infty e^{-ux}\frac{d}{dx} mE(e^{\theta(T_{ax} - ax/\mu)/a} - 1)\, dx \to \int_0^\infty e^{-ux}\left(\frac{c^\beta}{\mu^2\beta}\int_0^x \theta^2 e^{\theta s} s^{-\beta}\, ds\right) dx. \tag{23}$$

To find the Laplace transform on the right hand side note that

$$\frac{\theta^2\Gamma(1-\beta)}{\beta(u-\theta)^{1-\beta}} = \frac{\theta^2}{\beta}\int_0^\infty e^{-ux} e^{\theta x} x^{-\beta}\, dx, \quad \theta < u.$$

Multiplication of the transform by $1/u$ corresponds to integration of $e^{\theta x}x^{-\beta}$. Hence

$$\int_0^\infty e^{-ux}\left(\frac{1}{\beta}\int_0^x \theta^2 e^{\theta s} s^{-\beta}\, ds\right) dx = \frac{\Gamma(1-\beta)\theta^2}{\beta u(u-\theta)^{1-\beta}}, \quad \theta < u,$$

and hence (23) is equivalent to

$$\int_0^\infty u e^{-ux} mE(e^{\theta(T_{ax} - ax/\mu)/a} - 1)\, dx \to \frac{\Gamma(1-\beta)c^\beta\theta^2}{\beta u(u+\theta/\mu)^{1-\beta}\mu^2}, \quad \theta > -\mu u. \tag{24}$$

To help analyze the Laplace transform in (24) we introduce the additional notation

$$I(u) = \mu u - \Phi(u) = \int_0^\infty (e^{-ux} - 1 + ux)\, \nu(dx) \geq 0.$$

Writing $I(u) = u^2\int_0^\infty e^{-ux}U(x)\, dx$ with $U(x) = \int_x^\infty \nu(y,\infty)\, dy$, it follows from Karamata's Tauberian Theorem (Thm. 1.7.6 in Bingham *et al.* (1987)) that

$$aI(u/a) \sim \frac{\Gamma(1-\beta)L(a/u)u^{1+\beta}}{\beta a^\beta}, \quad a \to \infty \tag{25}$$

Relation (14) of Lemma 4 now shows

$$\begin{aligned}
&\int_0^\infty u e^{-ux} mE(e^{\theta(T_{ax} - ax/\mu)/a} - 1)\, dx \\
&= \frac{m\theta^2}{(\mu u+\theta)^2}\,\frac{aI((u+\theta/\mu)/a)}{\mu u - aI((u+\theta/\mu)/a)} \\
&\sim \frac{mL(a)}{a^\beta}\,(u+\theta/\mu)^{-(1-\beta)}\,\frac{\Gamma(1-\beta)\theta^2}{\beta\mu^3 u}\left(1 + \mathcal{O}\Big(\frac{L(a)}{a^\beta}\Big)\right) \\
&\sim \frac{\Gamma(1-\beta)c^\beta\theta^2}{\beta u(u+\theta/\mu)^{1-\beta}\mu^2}, \quad \theta > -\mu u,
\end{aligned}$$

which proves (24) and hence the lemma. □

We are now able to conclude convergence of the marginal distributions.

Lemma 10 *Under the assumptions of Theorem 2, for any $x \geq 0$*

$$\frac{1}{a_m}\sum_{i=1}^{m}(T^i_{a_m x} - \frac{1}{\mu}a_m x) \xrightarrow{d} -\frac{1}{\mu}c\,Y_\beta(x/c)$$

Proof. Writing

$$\Lambda^{(m)}(\theta; x) = mE(e^{\theta(T_{ax}-ax/\mu)/a} - 1),$$

Lemma 9 shows that

$$\log E\exp\Big\{\theta\frac{1}{b_m}\sum_{k=1}^{m}(T^{(k)}_{a_m x} - \frac{1}{\mu}a_m x)\Big\}\,dx$$
$$= \log\Big(1 + \frac{1}{m}\Lambda^{(m)}(\theta; x)\Big)^m \to \frac{c^\beta}{\beta\mu^2}\int_0^x\int_0^y \theta^2 e^{-\theta s/\mu}s^{-\beta}\,dsdy.$$

This proves the lemma since the limit process Y_β has the property

$$\log E(e^{\theta Y_\beta(x)}) = \frac{1}{\beta}\int_0^x\int_0^y \theta^2 e^{\theta s}s^{-\beta}\,dsdy$$

and so, as noticed in Lemma 2,

$$\log E(e^{-\theta c Y_\beta(x/c)/\mu}) = \frac{c^\beta}{\beta\mu^2}\int_0^x\int_0^y \theta^2 e^{-\theta s/\mu}s^{-\beta}\,dsdy.$$

□

2.3 Multivariate distribution

The proofs of convergence of the finite-dimensional distributions are based on the following recursive equations for moment generating functions.

Lemma 11 *Fix $n \geq 2$ and a sequence of time points $0 \leq x_1 \leq \ldots \leq x_n$. The moment generating function of the finite-dimensional distributions of the stationary inverse Lévy subordinator process $\{T_x\}$ satisfies the recurrence relation*

$$E\exp\Big\{\sum_{i=1}^{n}\theta_i T_{x_i}\Big\} = E\exp\Big\{\sum_{i=2}^{n}\theta_i T_{x_i}\Big\}$$
$$+\frac{\theta_1}{\sum_{i=1}^{n}\theta_i}\int_0^{x_1} E\Big[\exp\Big\{\sum_{i=2}^{n}\theta_i \widetilde{T}_{x_i - x}\Big\}\Big]\,d_x E\Big[\exp\Big\{T_x\sum_{i=1}^{n}\theta_i\Big\}\Big], \tag{26}$$

where $\widetilde{T}_x$ is the corresponding pure inverse Lévy process. Moreover,

$$E\exp\Big\{\sum_{i=1}^{n}\theta_i \widetilde{T}_{x_i}\Big\} = E\exp\Big\{\sum_{i=2}^{n}\theta_i \widetilde{T}_{x_i}\Big\}$$
$$+\frac{\theta_1}{\sum_{i=1}^{n}\theta_i}\int_0^{x_1} E\Big[\exp\Big\{\sum_{i=2}^{n}\theta_i \widetilde{T}_{x_i - x}\Big\}\Big]\,d_x E\Big[\exp\Big\{\widetilde{T}_x\sum_{i=1}^{n}\theta_i\Big\}\Big], \tag{27}$$

Proof. We have

$$E\exp\Big\{\sum_{i=1}^n \theta_i T_{x_i}\Big\} - E\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\} = E\Big[\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\}\Big(e^{\theta_1 T_{x_1}} - 1\Big)\Big].$$

Since

$$e^{\theta_1 T_{x_1}} - 1 = \int_0^\infty 1_{\{u\le T_{x_1}\}}\theta_1 e^{\theta_1 u}\,du = \int_0^\infty 1_{\{X_u \le x_1\}}\theta_1 e^{\theta_1 u}\,du,$$

it follows that

$$\begin{aligned} &E\Big[\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\}\Big(e^{\theta_1 T_{x_1}} - 1\Big)\Big] \\ &\quad= E\Big[\int_0^\infty 1_{\{X_u\le x_1\}}\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\}\theta_1 e^{\theta_1 u}\,du\Big] \\ &\quad= E\Big[\int_0^\infty 1_{\{X_u\le x_1\}}\exp\Big\{\sum_{i=2}^n \theta_i (T_{x_i} - T_{X_u})\Big\}\,\theta_1\exp\Big\{(\sum_{i=1}^n \theta_i) T_{X_u}\Big\}\,du\Big]. \end{aligned}$$

Here, $T_{X_u} = u$. For any $u > 0$ and $i \ge 2$, on the set $\{X_u \le x_1\}$ we have

$$\{T_{x_i} - T_{X_u} \le t\} = \{T_{x_i} \le u + t\} = \{X_{u+t} > x_i\}.$$

Since $\{X_t\}$ has independent increments the rightmost event has the same probability as

$$\{X_u + \widetilde{X}_t > x_i\} = \{\widetilde{T}_{x_i - X_u} \le t\},$$

where $X_u \le x_1$ is assumed independent of $\widetilde{X}_t$. Thus, on $\{X_u \le x_1\}$ the increment $T_{x_i} - T_{X_u}$ has the same distribution as $\widetilde{T}_{x_i - X_u}$. It follows that

$$\begin{aligned} &E\exp\Big\{\sum_{i=1}^n \theta_i T_{x_i}\Big\} - E\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\} \\ &\quad= \theta_1 E\Big[\int_0^\infty 1_{\{X_u\le x_1\}} E\Big[\exp\Big\{\sum_{i=2}^n \theta_i \widetilde{T}_{x_i - X_u}\Big\}|X_u\Big]\exp\Big\{(\sum_{i=1}^n \theta_i)u\Big\}\,du\Big] \\ &\quad= \theta_1 E\Big[\int_0^{x_1} E\Big[\exp\Big\{\sum_{i=2}^n \theta_i \widetilde{T}_{x_i - x}\Big\}\Big]\exp\Big\{(\sum_{i=1}^n \theta_i) T_x\Big\}\,dT_x\Big], \end{aligned}$$

where the integration after variable substitution $x = X_u$ is with respect to the increasing function of bounded variation $\{T_x, x \ge 0\}$. (Intuitively, the time-change X_u picks out the rightmost point of each flat piece of T_x.) Moreover, if we change to the measure

$$d_x\Big(\exp\Big\{T_x\sum_{i=1}^n \theta_i\Big\}\Big) = \Big(\sum_{i=1}^n \theta_i\Big)\exp\Big\{T_x \sum_{i=1}^n \theta_i\Big\}\,dT_x$$

we obtain

$$E\exp\Big\{\sum_{i=1}^n \theta_i T_{x_i}\Big\} - E\exp\Big\{\sum_{i=2}^n \theta_i T_{x_i}\Big\}$$

$$= \frac{\theta_1}{\sum_{i=1}^n \theta_i} E\Big[\int_0^{x_1} E\Big[\exp\Big\{\sum_{i=2}^n \theta_i \widetilde{T}_{x_i - x}\Big\}\Big]\, d_x\Big(\exp\Big\{T_x \sum_{i=1}^n \theta_i\Big\}\Big)\Big]$$

$$= \frac{\theta_1}{\sum_{i=1}^n \theta_i} \int_0^{x_1} E\Big[\exp\Big\{\sum_{i=2}^n \theta_i \widetilde{T}_{x_i - x}\Big\}\Big]\, d_x E\Big[\exp\Big\{T_x \sum_{i=1}^n \theta_i\Big\}\Big],$$

which is (26). Start with $\widetilde{T}$ rather than T to get (27). □

For $n \geq 1$ and $1 \leq k \leq n$, put $\bar{\theta}_{k,n} = (\theta_k, \ldots, \theta_n)$ and $\bar{x}_{k,n} = (x_k, \ldots, x_n)$, where $0 = x_0 \leq x_1 \leq \ldots \leq x_n$ and let

$$\Phi_{n-k+1}(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) = E \exp\Big\{\sum_{i=k}^n \theta_i (T_{x_i} - x_i/\mu)\Big\} \tag{28}$$

denote the multivariate moment generating functions for the centered process $\{T_x - x/\mu\}_{x\geq 0}$. Similarly, let $\widetilde{\Phi}_{n-k+1}(\bar{\theta}_{k,n};\, \bar{x}_{k,n})$, $1 \leq k \leq n$, denote the corresponding functions for the pure process $\{\widetilde{T}_x - x/\mu\}_{x\geq 0}$. The subtraction $\bar{x}_{k,n} - u = (x_k - u, \ldots, x_n - u)$ is interpreted component-wise in the next statement and in the sequel.

Lemma 12 *The moment generating functions defined in (28) satisfy the integral equation*

$$\Phi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) = \Phi_{n-1}(\bar{\theta}_{2,n};\, \bar{x}_{2,n}) e^{-\theta_1 x_1/\mu}$$
$$+ \frac{\theta_1}{\sum_{i=1}^n \theta_i} \int_0^{x_1} e^{-\theta_1(x_1 - x)/\mu}\, \widetilde{\Phi}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x)\, \Phi_1\Big(\sum_{i=1}^n \theta_i;\, dx\Big)$$
$$+ \frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1(x_1 - x)/\mu}\, \widetilde{\Phi}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x)\, \Phi_1\Big(\sum_{i=1}^n \theta_i;\, x\Big)\, dx.$$

Proof. By Lemma 11,

$$\Phi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) = \Phi_{n-1}(\bar{\theta}_{2,n};\, \bar{x}_{2,n}) e^{-\theta_1 x_1/\mu} + \frac{\theta_1}{\sum_{i=1}^n \theta_i}$$
$$\times \int_0^{x_1} e^{-\theta_1(x_1 - x)/\mu} \widetilde{\Phi}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \exp\Big\{-\frac{x}{\mu}\sum_{i=1}^n \theta_i\Big\}\, d_x E\Big[\exp\Big\{T_x \sum_{i=1}^n \theta_i\Big\}\Big],$$

which, by observing

$$\exp\Big\{-\frac{x}{\mu}\sum_{i=1}^n \theta_i\Big\}\, d_x E\Big[\exp\Big\{T_x \sum_{i=1}^n \theta_i\Big\}\Big]$$
$$= d_x E\Big[\exp\Big\{(T_x - x/\mu)\sum_{i=1}^n \theta_i\Big\}\Big] + \frac{1}{\mu}\sum_{i=1}^n \theta_i\, E\Big[\exp\Big\{(T_x - x/\mu)\sum_{i=1}^n \theta_i\Big\}\Big]\, dx$$
$$= \Phi_1\Big(\sum_{i=1}^n \theta_i;\, dx\Big) + \frac{1}{\mu}\sum_{i=1}^n \theta_i\, \Phi_1\Big(\sum_{i=1}^n \theta_i;\, x\Big)\, dx,$$

may be rewritten in the form stated in the lemma. □

According to (12) we must find the limits of the scaled function

$$m(\Phi_n(\bar\theta_{1,n}/a;\, a\bar x_{1,n})-1)=mE\Big[\exp\Big\{\sum_{i=1}^n\theta_i(T_{ax_i}-ax_i/\mu)/a\Big\}-1\Big]$$

as m, a and b tend to infinity, when a and b satisfy either (5) together with (8) or condition (6). The first case is *FBM scaling* leading to fractional Brownian motion in the limit, as in Theorem 1, and the second case is the *intermediate scaling* studied in Theorem 2.

For $n\geq 1$, $m\geq 1$ and $a,b>0$ we introduce

$$\Lambda_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n}) \;=\; m(\Phi_n(\bar\theta_{1,n}/b;\,a\bar x_{1,n})-1),$$

as well as

$$\widetilde\Lambda_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n}) \;=\; \frac{am}{b}(\widetilde\Phi_n(\bar\theta_{1,n}/b;\,a\bar x_{1,n})-1)$$

and

$$\Xi_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n}) \;=\; \widetilde\Lambda_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n})-\Lambda_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n}). \tag{29}$$

Our strategy for finding the corresponding limit functions is to derive for fixed m sequences of integral equations, which are recursive in n. As already pointed out we give the detailed proof only for Theorem 2.

2.4 Multivariate distribution under the intermediate scaling

We study the asymptotic limits of $\Lambda_n^{(m)}$ and $\widetilde\Lambda_n^{(m)}$ as $m\to\infty$ under assumption (6). For simplicity the constant in (6) is set to $c=1$. The general case $c\neq 1$ then follows from Lemma 2. We begin with a system of equations for the functions $\Xi_n^{(m)}$ defined in (29), which will be used to determine corresponding limit functions as $m\to\infty$.

Lemma 13 *We have*

$$\begin{aligned}\Xi_n^{(m)}(\bar\theta_{1,n};\,\bar x_{1,n}) \;=\;& \Xi_1^{(m)}\Big(\sum_{i=1}^n\theta_i;\,x_1\Big)+e^{-\theta_1x_1/\mu}\,\Xi_{n-1}^{(m)}(\bar\theta_{2,n};\bar x_{2,n})\\ &+\Big(\frac{\theta_1}{\sum_{i=1}^n\theta_i}-1\Big)\int_0^{x_1}e^{-\theta_1(x_1-x)/\mu}\,\Xi_1^{(m)}\Big(\sum_{i=1}^n\theta_i;\,dx\Big)+\frac{1}{m}R^{(m)},\end{aligned}$$

where

$$\begin{aligned}R^{(m)} \;=\;& \frac{\theta_1}{\sum_{i=1}^n\theta_i}\int_0^{x_1}e^{-\theta_1(x_1-x)/\mu}\,\widetilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\bar x_{2,n}-x)\,\Xi_1^{(m)}\Big(\sum_{i=1}^n\theta_i;\,dx\Big)\\ &+\frac{\theta_1}{\mu}\int_0^{x_1}e^{-\theta_1(x_1-x)/\mu}\,\widetilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\bar x_{2,n}-x)\,\Xi_1^{(m)}\Big(\sum_{i=1}^n\theta_i;\,x\Big)\,dx\end{aligned}$$

Proof. After inserting the scaling parameters m and a into the equation obtained in Lemma 12 and sorting the terms appropriately, it is seen that the scaled functions $\Lambda_n^{(m)}$ and $\widetilde{\Lambda}_n^{(m)}$ satisfy

$$\Lambda_n^{(m)}(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) = \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\, \bar{x}_{2,n})e^{-\theta_1 x_1/\mu} + I_1^{(m)} + I_2^{(m)} + I_3^{(m)} + \frac{1}{m}R_1^{(m)},$$

where

$$\begin{aligned}
I_1^{(m)} &= \frac{\theta_1}{\sum_{i=1}^n \theta_i}\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big)\\
I_2^{(m)} &= \frac{\theta_1}{\mu}\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\,\widetilde{\Lambda}_{n-1}^{(m)}(\bar{\theta}_{2,n};\bar{x}_{2,n}-x)\,dx\\
I_3^{(m)} &= \frac{\theta_1}{\mu}\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x\Big)\,dx\\
R_1^{(m)} &= \frac{\theta_1}{\sum_{i=1}^n \theta_i}\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\,\widetilde{\Lambda}_{n-1}^{(m)}(\bar{\theta}_{2,n};\bar{x}_{2,n}-x)\,\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big)\\
&\quad + \frac{\theta_1}{\mu}\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\,\widetilde{\Lambda}_{n-1}^{(m)}(\bar{\theta}_{2,n};\bar{x}_{2,n}-x)\,\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x\Big)\,dx
\end{aligned}$$

By partial integration,

$$I_3^{(m)} = \Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x_1\Big) - \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big).$$

Hence

$$\begin{aligned}
\Lambda_n^{(m)}(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) &= \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\, \bar{x}_{2,n})e^{-\theta_1 x_1/\mu} + \Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x_1\Big)\\
&\quad + I_2^{(m)} + I_4^{(m)} + \frac{1}{m}R_1^{(m)}, \qquad (30)
\end{aligned}$$

where now

$$I_4^{(m)} = \Big(\frac{\theta_1}{\sum_{i=1}^n \theta_i} - 1\Big)\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big)$$

Similarly,

$$\begin{aligned}
\widetilde{\Lambda}_n^{(m)}(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) &= \widetilde{\Lambda}_{n-1}^{(m)}(\bar{\theta}_{2,n};\bar{x}_{2,n})e^{-\theta_1 x_1/\mu} + \widetilde{\Lambda}_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x_1\Big)\\
&\quad + I_2^{(m)} + \widetilde{I}_4^{(m)} + \frac{1}{m}R_2^{(m)}, \qquad (31)
\end{aligned}$$

with

$$\widetilde{I}_4^{(m)} = \Big(\frac{\theta_1}{\sum_{i=1}^n \theta_i} - 1\Big)\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\widetilde{\Lambda}_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big)$$

and with $R_2^{(m)}$ being the modification of $R_1^{(m)}$ obtained by replacing $\Lambda_1^{(m)}$ by $\widetilde{\Lambda}_1^{(m)}$. By subtracting (30) from (31) and using $R^{(m)} = R_2^{(m)} - R_1^{(m)}$, we obtain the desired equation for $\Xi_n^{(m)}$, hence concluding the proof of the lemma. □

Lemma 14 *For each $n \geq 1$, the limit functions*

$$\Xi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) = \lim_{m\to\infty} \Xi_n^{(m)}(\bar{\theta}_{1,n};\, \bar{x}_{1,n})$$

exist and are given by

$$\begin{aligned}\Xi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) \;&=\; \sum_{j=1}^{n} \exp\Big\{\sum_{i=1}^{j-1}(\theta_i + \ldots + \theta_n)(x_i - x_{i-1})/\mu\Big\} \\ &\qquad \times \frac{1}{\mu}\int_{x_{j-1}}^{x_j} (\theta_j + \ldots + \theta_n) e^{-(\theta_j+\ldots+\theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta}\, du.\end{aligned}$$

For $n \geq 2$ they solve the recursive system

$$\begin{aligned}&\Xi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) - \Xi_1\Big(\sum_{i=1}^{n}\theta_i;\, x_1\Big) \\ &\qquad = e^{-\theta_1 x_1/\mu}\Big(\Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \Xi_1\Big(\sum_{i=2}^{n}\theta_i\,;\, x_1\Big)\Big). \qquad (32)\end{aligned}$$

Proof. By Lemma 5 and 17,

$$\Xi_1^{(m)}(\theta;\, x) = \frac{\mu}{\theta}\frac{d}{dx} mE(e^{\theta(T_{ax}-ax/\mu)/a} - 1) \to \frac{1}{\beta\mu}\int_0^x \theta e^{-\theta u/\mu} u^{-\beta}\, du = \Xi_1(\theta;\, x). \qquad (33)$$

This observation and a similar argument as in (23) shows

$$\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, \Xi_1^{(m)}\Big(\sum_{i=1}^{n}\theta_i;\, dx\Big) \to \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, \Xi_1\Big(\sum_{i=1}^{n}\theta_i;\, dx\Big).$$

It then follows from Lemma 13, by induction on n, that all limit functions Ξ_n exist and satisfy

$$\begin{aligned}\Xi_n(\bar{\theta}_{1,n};\, \bar{x}_{1,n}) - \Xi_1\Big(\sum_{i=1}^{n}\theta_i;\, x_1\Big) \;&=\; e^{-\theta_1 x_1/\mu}\, \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) \\ +\Big(\frac{\theta_1}{\sum_{i=1}^{n}\theta_i} - 1\Big)&\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, \Xi_1\Big(\sum_{i=1}^{n}\theta_i;\, dx\Big) \\ =\; e^{-\theta_1 x_1/\mu}\Big(\Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) &- \frac{1}{\mu}\Big(\sum_{i=2}^{n}\theta_i\Big)\int_0^{x_1} \exp\Big\{-x\Big(\sum_{i=2}^{n}\theta_i\Big)/\mu\Big\}\beta^{-1}x^{-\beta}\, dx\Big),\end{aligned}$$

which is the relation (32). It is now straightforward to verify the explicit form of the solution given in the lemma. □

The remaining proofs of the convergence of multivariate distributions in Theorem 2 are organized in four consecutive lemmas, leading up to the identification of the cumulant generating function (11) in Theorem 2.

Lemma 15 *The limit functions* $\Lambda_n = \lim_{m\to\infty} \Lambda_n^{(m)}$*,* $n \geq 1$*, exist and we have*

$$\Lambda_1(\theta; x) = \frac{\theta^2}{\beta\mu^2} \int_0^x \int_0^u e^{-\theta v/\mu} v^{-\beta}\, dv du \tag{34}$$

and for $n \geq 2$*, recursively*

$$\begin{aligned}\Lambda_n(\bar\theta_{1,n};\, \bar x_{1,n}) \;=\;& \Lambda_{n-1}(\bar\theta_{2,n}; \bar x_{2,n} - x_1) + \Lambda_1\Big(\sum_{i=1}^n \theta_i;\, x_1\Big)\\ &+\Big(\frac{\theta_1}{\sum_{i=1}^n \theta_i} - 1\Big) \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1\Big(\sum_{i=1}^n \theta_i;\, dx\Big)\\ &- \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, d\Lambda_{n-1}(\bar\theta_{2,n}; \bar x_{2,n} - x)\\ &+\frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, \Xi_{n-1}(\bar\theta_{2,n}; \bar x_{2,n} - x)\, dx.\end{aligned}$$

Proof. This follows for $n = 1$ from Lemma 10 and for $n \geq 2$ from (30) and a further partial integration of the term $I_2^{(m)}$, which gives

$$\begin{aligned}\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar x_{1,n}) \;=\;& \Lambda_{n-1}^{(m)}(\bar\theta_{2,n}; \bar x_{2,n} - x_1) + \Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, x_1\Big)\\ &+\Big(\frac{\theta_1}{\sum_{i=1}^n \theta_i} - 1\Big) \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1^{(m)}\Big(\sum_{i=1}^n \theta_i;\, dx\Big)\\ &- \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, d\Lambda_{n-1}^{(m)}(\bar\theta_{2,n}; \bar x_{2,n} - x)\\ &+\frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu}\, \Xi_{n-1}^{(m)}(\bar\theta_{2,n}; \bar x_{2,n} - x)\, dx + \frac{1}{m} R^{(m)}.\end{aligned}$$

□

Lemma 16 *For any* $0 \leq s \leq x_1$ *and* $n \geq 1$*,*

$$-\frac{d}{ds}\Lambda_n(\bar\theta_{1,n};\, \bar x_{1,n} - s) = \frac{1}{\mu}\Big(\sum_{i=1}^n \theta_i\Big)\, \Xi_n(\bar\theta_{1,n};\, \bar x_{1,n} - s).$$

Proof. The representations (33) and (34) show that the lemma is true for $n = 1$. For $n \geq 2$, by Lemma 15,

$$\begin{aligned}&\frac{d}{ds}\Lambda_n(\bar\theta_{1,n};\, \bar x_{1,n} - s) = \frac{\theta_1}{\sum_{i=1}^n \theta_i} \frac{d}{ds}\Lambda_1\Big(\sum_{i=1}^n \theta_i;\, x_1 - s\Big)\\ &\quad+\Big(\frac{\theta_1}{\sum_{i=1}^n \theta_i} - 1\Big)\frac{\theta_1}{\mu} \int_0^{x_1-s} e^{-\theta_1(x_1-s-x)/\mu} \Lambda_1\Big(\sum_{i=1}^n \theta_i;\, dx\Big)\\ &\quad- e^{-\theta_1(x_1-s)/\mu} \frac{d}{ds}\Lambda_{n-1}(\bar\theta_{2,n}; \bar x_{2,n} - s) + \frac{\theta_1}{\mu} e^{-\theta_1(x_1-s)/\mu}\, \Xi_{n-1}(\bar\theta_{2,n}; \bar x_{2,n} - s)\end{aligned}$$

and hence

$$-\frac{d}{ds}\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n}-s)=\frac{\theta_1}{\mu}\Xi_1\Big(\sum_{i=1}^n\theta_i;\,x_1-s\Big)$$
$$+e^{-\theta_1(x_1-s)/\mu}\Big(\frac{1}{\mu}\sum_{i=1}^n\theta_i\Big)\Big(\frac{\theta_1}{\sum_{i=1}^n\theta_i}-1\Big)\frac{\theta_1}{\mu}\int_0^{x_1-s}e^{\theta_1 x/\mu}\,\Xi_1\Big(\sum_{i=1}^n\theta_i;\,x\Big)\,dx$$
$$+e^{-\theta_1(x_1-s)/\mu}\left(\frac{\theta_1}{\mu}\Xi_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-s)-\frac{d}{ds}\Lambda_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-s)\right).$$

Here,

$$-\frac{\theta_1}{\mu}\int_0^{x_1-s}e^{\theta_1 x/\mu}\,\Xi_1\Big(\sum_{i=1}^n\theta_i;\,x\Big)\,dx$$
$$=e^{\theta_1(x_1-s)}\Xi_1\Big(\sum_{i=1}^n\theta_i\,;\,x_1-s\Big)-\sum_{i=1}^n\theta_i\int_0^{x_1-s}\exp\Big\{-v\sum_{i=2}^n\theta_1\Big\}v^{-\beta}\,dv,$$

and therefore

$$-\frac{d}{ds}\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n}-s)=\frac{1}{\mu}\sum_{i=1}^n\theta_i\Xi_1\Big(\sum_{i=1}^n\theta_i;\,x_1-s\Big)$$
$$+e^{-\theta_1(x_1-s)/\mu}\left(\frac{\theta_1}{\mu}\Xi_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-s)-\frac{d}{ds}\Lambda_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-s)\right.$$
$$\left.-\frac{1}{\mu}\sum_{i=1}^n\theta_i\,\Xi_1\Big(\sum_{i=2}^n\theta_i\,;\,x_1-s\Big)\right).$$

If we now assume as induction hypothesis that the lemma is true for index $n-1$, in the sense

$$-\frac{d}{ds}\Lambda_{n-1}(\bar\theta_{2,n};\,\bar x_{2,n}-s)=\frac{1}{\mu}\Big(\sum_{i=2}^n\theta_i\Big)\,\Xi_{n-1}(\bar\theta_{2,n};\,\bar x_{2,n}-s),$$

then it follows from (32) that the statement of the lemma is true for index n. □

Lemma 17 *The finite-dimensional distributions of the sequence of random processes studied in Theorem 2 (with $c=1$) converge to those of a limit process Y_β, such that the collection of logarithmic moment generating functions*

$$\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n})=\log E\exp\Big\{\sum_{i=1}^n\theta_iY_\beta(x_i)\Big\},\quad n\ge1$$

is the unique solution to the closed system of linear integral equations

$$\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n})=\Lambda_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-x_1)+\Lambda_1\Big(\sum_{i=1}^n\theta_i;\,x_1\Big)$$
$$+\Big(\frac{\theta_1}{\sum_{i=1}^n\theta_i}-1\Big)\int_0^{x_1}e^{-\theta_1(x_1-x)/\mu}\Lambda_1\Big(\sum_{i=1}^n\theta_i;\,dx\Big)$$
$$-\Big(\frac{\theta_1}{\sum_{i=2}^n\theta_i}+1\Big)\int_0^{x_1}e^{-\theta_1(x_1-x)/\mu}\,d\Lambda_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-x),\quad n\ge2,$$

with Λ_1 as in (34).

Proof. This follows by combining Lemma 15 and Lemma 16. □

Lemma 18 *The cumulant function for the increments of Y_β,*

$$\Gamma_n(\bar\theta_{1,n};\,\bar x_{1,n}) = \log E\exp\Big\{\sum_{i=1}^n \theta_i(Y_\beta(x_i)-Y_\beta(x_{i-1}))\Big\}$$

has the explicit form given in (11).

Proof. We have

$$\Gamma_n(\bar\theta_{1,n};\,\bar x_{1,n}) = \Lambda_n((\theta_1-\theta_2,\dots,\theta_{n-1}-\theta_n,\theta_n),\bar x_{1,n})$$

so by Lemma 17

$$\begin{aligned}\Gamma_n(\bar\theta_{1,n};\,\bar x_{1,n}) &= \Gamma_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-x_1) + \Lambda_1\Big(\sum_{i=1}^n\theta_i;\,x_1\Big)\\ &+\frac{\theta_2}{\theta_1}\int_0^{x_1} e^{-(\theta_1-\theta_2)(x_1-x)/\mu}\Lambda_1\Big(\sum_{i=1}^n\theta_i;\,dx\Big)\\ &-\frac{\theta_1}{\theta_2}\int_0^{x_1} e^{-(\theta_1-\theta_2)(x_1-x)/\mu}\,d\Gamma_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-x),\quad n\ge 2.\end{aligned}$$

It may now be checked that the functions in (13) solves the above system of equations. For details, see Gaigalas, Kaj [7], Section 6.3. □

2.5 Multivariate distribution under FBM scaling

In this section we discuss briefly the convergence of the finite-dimensional distributions in Theorem 1. Recall that for standard fractional Brownian motion B_H,

$$\log E\exp\Big\{\sum_{i=1}^n\theta_i\sigma_\beta B_H(x_i)\Big\} = \frac12\sigma_\beta^2\sum_{i=1}^n\sum_{j=1}^n\theta_i\theta_j\,\mathrm{Cov}(B_H(x_i),B_H(x_j))$$

with

$$\mathrm{Cov}(B_H(x),B_H(y)) = \frac12(x^{2-\beta}+y^{2-\beta}-(x-y)^{2-\beta}).$$

In the scaling regime defined by (5) and (8) we have

$$\frac{a}{b}=\sqrt{\frac{a^\beta\mu}{mL(a)}}\to 0,\qquad \frac{am}{b}=\sqrt{\frac{a^\beta m\mu}{L(a)}}\to\infty.$$

By analyzing in this case the recursive equations for $\widetilde\Lambda_n^{(m)}(\bar\theta_{1,n};\bar x_{1,r})$ it follows that

$$\widetilde\Lambda_n(\bar\theta_{1,n};\bar x_{1,n}) = \lim_{m\to\infty}\widetilde\Lambda_n^{(m)}(\bar\theta_{1,n};\bar x_{1,r}) = \frac{\delta_\beta^2}{\mu}\sum_{i=1}^n\theta_i x_i^{1-\beta},\quad \delta_\beta^2=\frac{1}{\beta(1-\beta)}. \tag{35}$$

Moreover, the limit functions $\Lambda_n(\bar\theta_{1,n};\bar x_{1,r}) = \lim_{m\to\infty}\Lambda_n^{(m)}(\bar\theta_{1,n};\bar x_{1,r})$ satisfy, in analogy to the result of Lemma 17,

$$\Lambda_1(\theta;\,x) = \lim_{m\to\infty}\Lambda_1^{(m)}(\theta;\,x) = \frac{1}{2}\sigma_\beta^2\mu^{-2}\theta^2x^{2-\beta}$$

and for $n\geq 2$,

$$\begin{aligned}\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n}) &= \Lambda_{n-1}(\bar\theta_{2,n};\,\bar x_{2,n}) + \frac{\theta_1}{\sum_{i=1}^n\theta_i}\Lambda_1\Big(\sum_{i=1}^n\theta_i;\,x_1\Big)\\ &\quad+\frac{\theta_1}{\mu}\int_0^{x_1}\widetilde\Lambda_{n-1}(\bar\theta_{2,n};\bar x_{2,n}-x)\,dx.\end{aligned}$$

Thus, using (35),

$$\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n}) = \Lambda_{n-1}(\bar\theta_{2,n};\,\bar x_{2,n}) + \frac{\sigma_\beta^2}{2\mu^2}\sum_{j=1}^n\theta_1\theta_j\frac{1}{2}\Big[x_1^{2-\beta}+x_j^{2-\beta}-(x_j-x_1)^{2-\beta}\Big].$$

Hence

$$\Lambda_n(\bar\theta_{1,n};\,\bar x_{1,n}) = \frac{\sigma_\beta^2}{2\mu^2}\sum_{i=1}^n\sum_{j=1}^n\theta_i\theta_j\frac{1}{2}(x_i^{2-\beta}+x_j^{2-\beta}-(x_i-x_j)^{2-\beta}).$$

2.6 Proof of tightness in C

To complete the proofs of our results we establish tightness of the sequences

$$Y^{(m)}(x) = \frac{1}{b_m}\sum_{i=1}^m(T^{(i)}_{a_mx}-\frac{a_mx}{\mu})$$

studied in Theorems 1 and 2, by applying a standard moment criterion. Since $Y^{(m)}(x)$ has stationary increments, to prove that that $\{Y^{(m)}\}$ is tight in C it is enough to find $\gamma>1$, an integer m_0 and a constant K such that for fixed T,

$$\mathrm{Var}(Y^{(m)}(x)) = \frac{m}{b_m^2}\mathrm{Var}(T_{a_mx})\leq Kx^\gamma \tag{36}$$

for $0<x<T$ and $m\geq m_0$ (Billingsley (1968), Thm. 12.3).

By Lemma 6 iii), the variance of T_x is a non-decreasing function in x. Hence we may apply Karamata's Tauberian theorem (Bingham *et al.* [4] Theorem 1.7.1) to show that $\mathrm{Var}(T_x)$ is regularly varying in infinity with index $2-\beta$. Indeed, recalling the previously used notation $I(u)=\mu u-\Phi(u)$, the asymptotic property (25) implies

$$\int_0^\infty ue^{-ux}\mathrm{Var}(T_x)\,dx = \frac{2}{(\mu u)^2}\sum_{n=1}^\infty\Big(\frac{I(u)}{\mu u}\Big)^n\sim\frac{2\Gamma(1-\beta)L(1/u)}{\beta\mu^2u^{2-\beta}},\quad u\to 0,$$

hence

$$\mathrm{Var}(T_x)\sim\frac{2\Gamma(1-\beta)x^{2-\beta}L(x)}{\Gamma(3-\beta)\beta\mu^3} = \frac{\sigma_\beta^2}{\mu^3}L(x)x^{2-\beta},\quad x\to\infty. \tag{37}$$

The next step is to apply the Potter bounds for regularly varying functions (Bingham *et al.*, Ch 1.5) to obtain for any $\epsilon > 0$ an a_0, such that

$$\frac{\mathrm{Var}(T_{ax})}{\mathrm{Var}(T_a)} \le (1+\epsilon)\max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon}), \quad a \ge a_0,\ ax \ge a_0.$$

Hence for $m \ge m_0$ so large that $a \ge a_0$, $ax \ge a_0$,

$$m\mathrm{Var}(T_{ax})/b^2 \le (1+\epsilon) m\mathrm{Var}(T_a)/b^2 \max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon})$$

But in either case of the FBM scaling (5), (8) or the intermediate scaling (6), the asymptotic relation (37) yields

$$m\mathrm{Var}(T_a)/b^2 \to \sigma_\beta^2/\mu^2, \quad m \to \infty,$$

and so, eventually choosing a larger $m_1 \ge m_0$,

$$m\mathrm{Var}(T_{ax})/b^2 \le (1+\epsilon)(\sigma_\beta^2+\epsilon)\max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon}), \quad m \ge m_1.$$

With $\epsilon < 1-\beta$ this yields (36) for $ax \ge a_0$.

It remains to prove (36) for $a \ge a_0$ and $ax < a_0$. By Lemma 6 iv),

$$\begin{aligned} m\mathrm{Var}(T_{ax})/b^2 &\le \frac{2e}{\mu}\frac{m}{b^2}\int_0^{ax} \Phi(1/y)^{-1}\,dy \\ &\le \frac{2e}{\mu}\frac{ma^{2-\beta}}{b^2}\frac{(1/ax)^{1-\beta}}{\Phi(1/ax)}\,x^{2-\beta}. \end{aligned}$$

Take ϵ such that $(1-\beta-\sigma)\vee 0 < \epsilon < 1-\beta$. As in the proof of Lemma 8, $a^{-\epsilon} \le L(a)$. Thus,

$$m\mathrm{Var}(T_{ax})/b^2 \le 2e\frac{ma^{2-\beta}L(a)}{\mu b^2}\frac{(1/ax)^{1-\beta-\epsilon}}{\Phi(1/ax)}\,x^{2-\beta-\epsilon}.$$

In Theorem 1, $ma^{2-\beta}L(a)/\mu b^2 = 1$. In Theorem 2, $ma^{2-\beta}L(a)/\mu b^2 \to c^\beta$. Since σ is the lower index associated with Φ and $1-\beta-\epsilon < \sigma$, the ratio $\Phi(u)/u^{1-\beta-\epsilon} \to \infty$ as $u \to \infty$. Thus, for $ax \le a_0$, we can find a constant K such that (36) holds with $\gamma = 2-\beta-\epsilon > 1$. □

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