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Scaling limits of anisotropic Hastings–Levitov clusters[1]

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Abstract. We consider a variation of the standard Hastings–Levitov model HL(0), in which growth is anisotropic. Two natural scaling limits are established and we give precise descriptions of the effects of the anisotropy. We show that the limit shapes can be realised as Loewner hulls and that the evolution of harmonic measure on the cluster boundary can be described by the solution to a deterministic ordinary differential equation related to the Loewner equation. We also characterise the stochastic fluctuations around the deterministic limit flow.

Résumé. Dans cet article, on presente une étude d'une version du modèle de Hastings–Levitov HL(0) où la croissance est anisotrope. Deux limites d'échelle naturelles sont établies, et nous décrivons précisément les effets de l'anisotropie. Nous montrons que les formes limites du modèle peuvent être réalisées comme remplissages associés à l'équation de Loewner et que l'évolution de la mesure harmonique sur la frontière des agrégats tend vers un certain flot deterministe. Nous caractérisons enfin les fluctuations stochastiques autour de ce flot.

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1. Introduction

1.1. *Generalized* HL(0) *clusters*

In this paper we consider growing sequences of compact sets in the complex plane $\mathbb{C}$ obtained by composing random conformal mappings. Let D_0 denote the exterior unit disk

$$D_0 = \{z \in \mathbb{C}_\infty \colon |z| > 1\},$$

and let $K_0 = \mathbb{C} \setminus D_0$ be the closed unit disk. We write $\mathbb{T}$ for its boundary, the unit circle; we frequently identify $\mathbb{T}$ with the interval $[0, 1)$. Let $D_1 \subset D_0$ be simply connected, that is, a set whose complement in D_0 is connected. We assume that $P = D_1^c \setminus K_0$ has diameter $d \in (0, 1]$ and $1 \in \overline{P}$. The set P models an incoming particle, which is attached to the unit disk at 1. There exists a unique conformal mapping

$$f_P : D_0 \to D_1 \tag{1}$$

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with expansion at infinity of the form $f_P(z) = C(P)z + c_0 + c_{-1}/z + \cdots$ for some $C(P) > 0$. The value $C(P) = \mathrm{cap}(K_0 \cup P)$ is called the logarithmic capacity.

Suppose $P_1, P_2, \ldots$ is a sequence of particles (or, equivalently, let $f_{P_1}, f_{P_2}, \ldots$ be the sequence of associated conformal mappings) with $\mathrm{diam}(P_j) = d_j$. Let $\theta_1, \theta_2, \ldots$ be a sequence of angles. Define rotated copies of the maps $\{f_{P_j}\}$ by setting

$$f_{P_j}^{\theta_j}(z) = \mathrm{e}^{\mathrm{i}\theta_j} f_{P_j}\bigl(\mathrm{e}^{-\mathrm{i}\theta_j} z\bigr), \quad j = 1, 2, \ldots.$$

Take $\Phi_0(z) = z$, and recursively define

$$\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}^{\theta_n}(z), \quad n = 1, 2, \ldots. \tag{2}$$

This generates a sequence of conformal maps $\Phi_n : D_0 \to D_n = \mathbb{C} \setminus K_n$, where $K_{n-1} \subset K_n$ are growing compact sets, which we usually call clusters. Loosely speaking we add, at the nth step, a particle of diameter $d_n|\Phi_{n-1}'(\mathrm{e}^{\mathrm{i}\theta_n})|$ to the previous cluster K_{n-1} at the point $\Phi_{n-1}(\mathrm{e}^{\mathrm{i}\theta_n})$.

By constructing the sequences $\{\theta_j\}$ and $\{d_j\}$ in different ways, it is possible to describe a wide class of growth models. The most well-known are the Hastings–Levitov family of models $\mathrm{HL}(\alpha)$, indexed by a parameter $\alpha \in [0, 2]$. Here the θ_j are chosen to be independent random variables distributed uniformly on the unit circle which, by conformal invariance, corresponds to the attachment point being distributed according to harmonic measure at infinity. The particle diameters are taken as $d_j = d/|\Phi_{j-1}'(\mathrm{e}^{\mathrm{i}\theta_j})|^{\alpha/2}$.

In this paper, we study a variant of the HL(0) model in which $\theta_1, \theta_2, \ldots$ are independent identically distributed random variables on the unit circle $\mathbb{T}$ with common law ν and $d_j = d$. We shall refer to this growth model as anisotropic Hastings–Levitov, $\mathrm{AHL}(\nu)$. Our limit results are not sensitive to the shapes of particles P_j and, in fact, we are even able to relax the restraint $d_j = d$, to allow for $P_1, P_2, \ldots$ to be chosen so that $d_1, d_2, \ldots$ are independent identically distributed random variables (independent of $\{\theta_j\}$) with law σ, satisfying certain conditions to be stated later.[2]

1.2. *Background and motivation*

The motivation behind studying these clusters comes from growth processes that arise in physics, such as diffusion-limited aggregation (DLA) [27], anisotropic diffusion-limited aggregation [14] and the Eden model [8]. In 1998, Hastings and Levitov [12] formulated a conformal mapping approach to modelling Laplacian growth of which DLA and the Eden model are special cases. They defined the family of growth models, $\mathrm{HL}(\alpha)$, whose construction is described in the previous section. The $\alpha = 2$ version is a candidate for off-lattice DLA. In this case, the diameters of the mapped particles are (more or less) the same.

The Hastings–Levitov model has been widely discussed in the physics literature. In the original paper [12], Hastings and Levitov studied the model numerically and found evidence for a phase transition in the growth behaviour at $\alpha = 1$. Further numerical investigations can be seen in, for example, the papers [7] and [19].

Unfortunately, the Hastings–Levitov model has proved difficult to analyse rigourously, particularly in the $\alpha > 0$ case. We give a brief review of the known results. In 2005, Rohde and Zinsmeister [24] established the existence of limit clusters for $\alpha = 0$ when the aggregate is scaled by capacity, and showed that the Hausdorff dimension of the limit clusters is 1, almost surely. They also considered a regularised version of $\mathrm{HL}(\alpha)$ for $\alpha > 0$ and estimated the growth rate of the capacity and length of the clusters. In 2009, Johansson Viklund and Sola [13] studied Loewner chains driven by compound Poisson processes. Certain cases of these were found to correspond to HL(0) clusters with random particle sizes, and the existence of (one-dimensional) limit clusters was established. The 2009 paper of Norris and Turner explored the evolution of harmonic measure on the boundary of HL(0) clusters and showed that this converges to the coalescing Brownian flow. We would finally like to mention the 2001 and 2002 papers of Carleson and Makarov ([5,6]), where the Loewner–Kufarev equation is used to describe deterministic versions of Laplacian growth.

[2] One way of constructing such a sequence is by fixing a deterministic measurable mapping $d \mapsto f_{P(d)}$, such that $\mathrm{diam}(P(d)) = d$, and then choosing an independent identically distributed sequence $d_1, d_2, \ldots$ with law σ.

In this paper we have modified the setup of the Hastings–Levitov model in the $\alpha = 0$ case. The use of more general distributions for the angles is a way of introducing anisotropy or localization in the growth. This is similar in spirit to the work of Popescu, Hentschel and Family [22], who study numerically a variant of HL(2), where the angles are distributed according to a certain density with m-fold symmetry. They suggest that such anisotropic Hastings–Levitov models may provide a description for the growth of bacterial colonies where the concentration of nutrients is directional. We discuss their work further in the next section.

Allowing for non-uniform angular distributions results in scaling limits in which the anisotropy is reflected. We consider two different natural scaling limits where we scale the particle sizes. We prove a shape theorem that describes the global macroscopic behaviour of the cluster: in the case of uniformly distributed angles, the shape is a disk (as was previously known [23]); but in the anisotropic case the limit shapes can be realised as non-trivial Loewner hulls. For the anisotropic case we also show that the evolution of harmonic measure on the cluster boundary is deterministic with small random fluctuations, unlike in the uniform case where the behaviour is purely stochastic.

1.3. *Outline of the paper*

Our paper is organised as follows. In Section 2, we review some background material concerning the Loewner equation and the coalescing Brownian flow and describe the general framework of our paper. We also discuss some examples of angular distributions that lead to interesting anisotropic behaviour in the growth. In Section 3, we establish continuity properties of the Loewner–Kufarev equation with respect to measures, and use this to prove a shape theorem for the limit clusters. In Section 4, we consider the evolution of harmonic measure on the cluster boundary. For general measures, we first prove that the flow on the boundary is described by a deterministic ordinary differential equation, and then obtain a description of the stochastic fluctuations around this deterministic flow. Finally, we show that uniformly chosen angles lead to purely stochastic behaviour, even if the particle sizes are chosen randomly.

2. Preliminaries

In this section we review some background material that is needed for our proofs.

2.1. *Loewner chains driven by measures*

A decreasing Loewner chain is a family of conformal mappings

$$f_t : D_0 \to \mathbb{C} \setminus K_t, \quad \infty \mapsto \infty, \qquad f_t'(\infty) > 0,$$

onto the complements of a growing family of compact sets, called hulls, with

$$K_{t_1} \subset K_{t_2} \quad \text{for } t_1 < t_2.$$

We always take K_0 to be the closed unit disk. The capacity of each K_t is given by

$$\operatorname{cap}(K_t) = \lim_{z\to\infty} \frac{f_t(z)}{z}.$$

Let $\mathcal{P} = \mathcal{P}(\mathbb{T})$ denote the class of probability measures on $\mathbb{T}$. Under some natural assumptions on the function $t \mapsto \operatorname{cap}(K_t)$, such a chain can be parametrized in terms of families $\{\mu_t\}_{t\geq 0}$, $\mu_t \in \mathcal{P}(\mathbb{T})$. More precisely, the conformal mappings f_t satisfy the Loewner–Kufarev equation

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} \, \mathrm{d}\mu_t(\zeta) \tag{3}$$

with initial condition $f_0(z) = z$. Conversely, if $t \mapsto \|\mu_t\|$ is locally integrable (which is immediate for probability measures) then the solution to (3) exists and is a Loewner chain. See [5] for a general discussion.

The classical example is the case of pure point masses

$$\mu_t = \delta_{\xi(t)},$$

where $\xi = \xi(t)$ is a unimodular function, that is, a complex-valued function with modulus equal to 1. The Loewner–Kufarev equation then reduces to the equation

$$\partial_t f_t(z) = z f_t'(z) \frac{z + \xi(t)}{z - \xi(t)}, \tag{4}$$

originally introduced by Loewner in 1923. The function $\xi(t)$ is usually called the driving function. The particular choice $\xi(t) = 1$ produces as solutions the basic slit mappings $f_{d(t)} : D_0 \to D_0 \setminus (1, 1 + d(t)]$, with slit lengths $d(t)$ given by the explicit formula

$$d(t) = 2\mathrm{e}^t \bigl(1 - \sqrt{1 - \mathrm{e}^{-t}}\bigr) - 2. \tag{5}$$

We can recover (the slit version of) the HL(0) mappings Φ_n by driving the Loewner equation with a non-constant point mass at

$$\xi(t) = \exp\left(\mathrm{i} \sum_{j=1}^{n} \theta_j \chi_{[T_{j-1}, T_j]}(t)\right), \tag{6}$$

where the times T_j relate to the slit lengths d via the formula (5).

Choosing absolutely continuous driving measures

$$\mathrm{d}\mu_t = h_t(\zeta) |\mathrm{d}\zeta|$$

results in the growth of the clusters no longer being concentrated at a single point. In the simplest case $\mathrm{d}\mu_t(\zeta) = |\mathrm{d}\zeta|/2\pi$, the Loewner–Kufarev equation reduces to

$$\partial_t f_t(z) = z f_t'(z),$$

and we see that $f_t(z) = \mathrm{e}^t z$, so that $K_t = \mathrm{e}^t K_0$. We shall see that absolutely continuous driving measures arise naturally in connection with the anisotropic HL(0) clusters.

We can realize more general particles than slits using a driving function in the following way. Consider a particle P such that $\partial P \cap D_0$ can be described by a crosscut β of D_0 (see [21] for the definition of crosscut). We parametrize the crosscut $\beta(t)$ according to capacity, that is, $\mathrm{cap}(K_0 \cup \beta[0, t)) = \mathrm{e}^t$, $t \in [0, \mathrm{lcap}(P))$, where

$$\mathrm{lcap}(P) := \log\bigl(\mathrm{cap}(K_0 \cup P)\bigr).$$

Under rather mild conditions on the crosscut β, we can then find a driving function for the Loewner equation that produces a family $f_t : D_0 \to D_0 \setminus \beta[0, t)$. A sufficient but not necessary condition for this to hold is that the function $\beta = \beta(t)$ is smooth. As $t \to \mathrm{lcap}(P)$, the conformal maps f_t converge uniformly on compact subsets of D_0 to the mapping $f_P : D_0 \to D_0 \setminus P$. If $\xi : [0, \mathrm{lcap}(P)) \to \mathbb{T}$ denotes the driving function for a single particle, we obtain a driving function for the cluster similarly to (6).

There is a useful relation between the diameter of the particle, its capacity, and the driving function. Set

$$R(P) := \sqrt{\mathrm{lcap}(P)} + \sup_{0 \le t \le \mathrm{lcap}(P)} \bigl|\xi(t)\bigr|.$$

Then, as is proved in [18], Lemma 2.1, we have

$$d(P) = \mathrm{diam}(P) \asymp R(P). \tag{7}$$

Moreover, one can prove that there exists a constant $c < \infty$ such that

$$c^{-1}h^2 \le \operatorname{lcap}(P) \le chd$$

for small d, h, where $h = \sup\{|z|: z \in P\} - 1$. Indeed, the first inequality follows by comparing with the slit map solution to the Loewner equation. The second follows from a harmonic measure estimate and the identity $\operatorname{lcap}(P) = \mathbb{E}[\log|B_\tau|]$, where B_t is a planar Brownian motion started from ∞ and τ is the hitting time of $K_0 \cup P$. In particular we see that there are natural sequences of particles such that $\operatorname{lcap}(P) \asymp d^2$ as $d \to 0$. We shall make this assumption in certain sections of this paper.

We will sometimes need to consider the radial Loewner equation lifted to the real line. Let f_t be a solution to (3) and suppose that the boundary of the hull K_t contains an arc $\{e^{2\pi i x}: x \in [a, b]\}$ of the unit circle. We set $g_t(e^{2\pi i x}) = f_t^{-1}(e^{2\pi i x})$ and define $\gamma_t(x) = -i \log g_t(e^{2\pi i x})/2\pi$ for $x \in [a, b]$. The function γ_t then satisfies the differential equation

$$\partial_t \gamma_t(x) = \frac{1}{2\pi} \int_0^1 \cot\bigl(\pi\bigl(\gamma_t(x) - y\bigr)\bigr)\, d\mu_t\bigl(e^{2\pi i y}\bigr) \tag{8}$$

with $\gamma_0(x) = x$ (see [17], Chapter 4). This is well-defined as long as $\gamma_t(x)$ is outside the support of μ_t. However, we may interpret the integral in the sense of principal values, that is, as a multiple of the Hilbert transform of the measure μ_t (see [10], Chapter 3),

$$H[\mu_t](x) = \text{p.v.} \frac{1}{2\pi} \int_0^1 \cot\bigl(\pi(x - y)\bigr)\, d\mu_t\bigl(e^{2\pi i y}\bigr).$$

In this way, for nice enough measures, we obtain a differential equation defining a flow on all of $\mathbb{T}$.

2.2. *Coalescing Brownian flow and harmonic measure on the cluster boundary*

The coalescing Brownian flow (also known as the Arratia flow and the Brownian web) can loosely be defined as a family of coalescing Brownian motions, starting at all possible points in continuous space–time. Arratia [1] first considered this object in 1979 as a limit for discrete coalescing random walks. Since then it has been studied by, amongst others, Tóth and Werner [26], Fontes, Isopi, Newman and Ravishankar [9] and recently Norris and Turner [20]. One of the difficulties in studying the coalescing Brownian flow is constructing a suitable topological space on which a unique measure with the necessary properties exists. In this section we outline the construction of Norris and Turner [20] and show how the coalescing Brownian flow relates to the evolution of harmonic measure on the boundary of the AHL(ν) clusters.

The coalescing Brownian flow is constructed as a measure on the space of flow maps described briefly below; full details and proof are provided in [20]. Let $\mathcal{R}$ be the set of non-decreasing, right-continuous functions $f^+ : \mathbb{R} \to \mathbb{R}$ that satisfy the property

$$f^+(x + n) = f^+(x) + n, \quad x \in \mathbb{R}, n \in \mathbb{Z}.$$

Write $\mathcal{L}$ for the analogous set of left-continuous functions and let $\mathcal{D}$ be the set of all pairs $f = \{f^-, f^+\}$, where f^- is the left-continuous modification of f^+. Let $\mathcal{D}_0$ denote the set of all circle maps that have a lifting in $\mathcal{D}$. In what follows, we usually think of the maps f^+ and f^- as the result of extending the inverse conformal maps g_t to the unit circle as a map from the circle to itself, and then lifting to the real line. The reason for working with right and left continuous modifications is that our conformal maps sometimes send two distinct points on the unit circle to the same point on the circle, so that the inverse cannot directly be defined as a continuous map of the circle into itself. Using the space $\mathcal{D}$ allows us to handle this difficulty by considering instead pairs of mappings – right or left continuous depending on which point in the pre-image we choose.

Write $I = I_1 \oplus I_2$ if I_1, I_2 are disjoint intervals with $\sup I_1 = \inf I_2$ and $I = I_1 \cup I_2$. The set of cadlag weak flows D° consists of flows $\phi = (\phi_I: I \subseteq [0, \infty))$, where $\phi_I \in \mathcal{D}_0$ and I ranges over all non-empty finite intervals that satisfy

$$\phi_{I_2}^- \circ \phi_{I_1}^- \le \phi_I^- \le \phi_I^+ \le \phi_{I_2}^+ \circ \phi_{I_1}^+, \quad I = I_1 \oplus I_2,$$

and, for all $t \in (0, \infty)$,

$$\phi_{(s,t)} \to \text{id} \quad \text{as } s \uparrow t, \qquad \phi_{(t,u)} \to \text{id} \quad \text{as } u \downarrow t.$$

Note that at continuity points, the above conditions translate into the standard definition of a flow map, i.e. setting $\phi_{ts} = \phi^+_{(s,t]}$, for $s \le t \le u$, $\phi_{us}(x) = \phi_{ut}(\phi_{ts}(x))$ and $\phi_{ss}(x) = x$. In [20] a metric d_D is defined on D° under which D° is complete and separable.

The coalescing Brownian flow is constructed as a measure on the space D°, having the properties described below. For $e = (s, x) \in [0, \infty) \times \mathbb{R}$, let $D_e = D_x([s, \infty), \mathbb{R})$ denote the Skorokhod space of cadlag paths starting from x at time s. Write μ_e for the distribution on D_e of a standard Brownian motion starting from e.

For any countable sequence $E = (e_k\colon k \in \mathbb{N})$ in $[0, \infty) \times \mathbb{R}$, where $e_k = (s_k, x_k)$ say, let $D_E = \prod_{k=1}^\infty D_{e_k}$ be the complete separable product metric space with metric

$$d_E(\xi, \xi') = \sum_{k=1}^{\infty} 2^{-k}\bigl\{d\bigl(\xi^{e_k}, \xi'^{e_k}\bigr) \wedge 1\bigr\},$$

where d denotes appropriate instances of the Skorokhod metric.

There exists a unique probability measure μ_E on D_E under which the coordinate processes on D_E are coalescing Brownian motions. Define a measurable map $Z^{e,+}\colon D^\circ \to D_e$ by setting

$$Z^{e,+}(\phi) = \bigl(\phi^+_{(s,t]}(x)\colon t \ge s\bigr),$$

and a measurable map $Z^{E,+}\colon D^\circ \to D_E$ by

$$Z^{E,+}(\phi)^{e_k} = Z^{e_k,+}(\phi).$$

There exists a unique Borel probability measure μ_A on D° such that, for any finite set $F \subset [0, \infty) \times \mathbb{R}$, we have

$$\mu_A \circ \bigl(Z^{F,+}\bigr)^{-1} = \mu_F.$$

We call any D°-valued random variable with law μ_A a coalescing Brownian flow on the circle (see Fig. 1).

We now show how the evolution of harmonic measure on the boundary of AHL(ν) clusters can be described using the space D°. Recall the construction of the Hastings–Levitov clusters from the Introduction. Let P be a closed, connected, simply connected subset of D_0 of diameter $d \in (0, 1]$ such that $P \cap K_0 = \{1\}$. Write g_P for the inverse

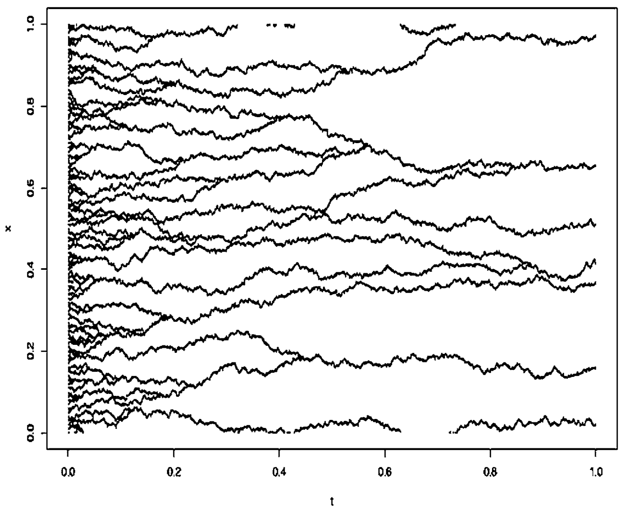


Fig. 1. A realisation of the coalescing Brownian flow, with only paths starting at time 0 shown.

mapping from $D_1 \to D_0$. This map extends locally in a continuous way to the boundary of ∂D_1. There exists a unique pair of mappings $\gamma_P = \{\gamma_P^-, \gamma_P^+\} \in \mathcal{D}$ such that γ_P restricts to a continuous map from the open interval $(0, 1)$ to itself, and such that

$$g_P\big(e^{2\pi i x}\big) = e^{2\pi i \gamma_P(x)}, \quad x \in (0,1),$$

where we suppress the superscripts $\pm$ to lighten the notation. Set $\Gamma_n = g_{P_n} \circ \cdots \circ g_{P_1}$, where $g_{P_n} = (f_{P_n}^{\theta_n})^{-1}$, so that $\Gamma_n : D_n \to D_0$.

The reason for using the flow space D° in the context of the Hastings–Levitov model will shortly become apparent. It provides us with a technical framework for handling both the discontinuities that arise if we wish to embed our process of compositions of conformal maps as a jump process in continuous time, as well as the spatial discontinuities that arise from the fact that the conformal maps g_P and Γ_n cannot be extended continuously to the whole circle. In the latter case, the "attachment points" of the particles correspond to discontinuities.

The extension of Γ_n to the boundary $\partial K_n = \partial D_n$ now gives a natural parametrization of the boundary of the nth cluster by the unit circle. It has the property that, for $\xi, \eta \in \partial K_n$, the normalized harmonic measure ω (from ∞) of the positively oriented boundary segment from ξ to η is given by $\Gamma_n(\eta)/\Gamma_n(\xi) = e^{2\pi i \omega}$. For $m, n \in \mathbb{N}$ with $m < n$, set

$$\Gamma_{nm} = g_{P_n} \circ \cdots \circ g_{P_{m+1}}|_{\partial K_0}.$$

Set $\Gamma_{nn} = \mathrm{id}$. The circle maps Γ_{nm} have the flow property

$$\Gamma_{nm} \circ \Gamma_{mk} = \Gamma_{nk}, \quad k \le m \le n.$$

The map Γ_{nm} expresses how the harmonic measure on ∂K_m is transformed by the arrival of new particles up to time n. Suppose $0 < T_1 < T_2 < \cdots$ are times of a Poisson process, independent of $\{\theta_j\}$ and $\{d_j\}$, with rate to be specified later. Embed Γ in continuous time by defining, for an interval $I \subseteq [0, \infty)$, $\Gamma_I = \Gamma_{nm}$ where m and n are the smallest and largest integers i, respectively, for which $T_i \in I$. Then $(\Gamma_I\colon\ I \subseteq [0,\infty))$ is a random variable in D°. We denote its law by μ_A^P.

In the paper [20], Norris and Turner showed that for HL(0) clusters (i.e. clusters where the particles P_j have constant diameters $d_j = d$, and θ_j is uniformly distributed on the circle), in the case of symmetric particles, $\mu_A^P \to \mu_A$ weakly on D° as $d \to 0$, where the Poisson process $\{T_i\}$ has rate $\rho(P) \asymp d^{-3}$, defined by

$$\rho(P) \int_0^1 \big(\gamma_P(x) - x\big)^2 \,\mathrm{d}x = 1.$$

If P is not symmetric, the same result holds once the definition of Γ_I is modified to

$$\Gamma_I\big(e^{2\pi i x}\big) = e^{-2\pi i \beta t}\, \Gamma_{nm}\big(e^{2\pi i (x + \beta s)}\big),$$

where $s = \inf I$ and $t = \sup I$ and $\beta = \beta(P)$ is defined by

$$\beta(P) = \rho(P) \int_0^1 \big(\gamma_P(x) - x\big)\,\mathrm{d}x = \mathcal{O}\big(d^{-1}\big).$$

In other words the following result about the evolution of harmonic measure on the cluster boundary holds.

Let $x_1, \ldots, x_n$ be a positively oriented set of points in $\mathbb{R}/\mathbb{Z}$ and set $x_0 = x_n$. Set $K_t = K_{\lfloor \rho(P) t \rfloor}$. For $k = 1, \ldots, n$, write ω_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k. Let $(B_t^1, \ldots, B_t^n)_{t \ge 0}$ be a family of coalescing Brownian motions in $\mathbb{R}/\mathbb{Z}$ starting from $(x_1, \ldots, x_n)$. Then, in the limit $d \to 0$, $(\omega_t^1, \ldots, \omega_t^n)_{t \ge 0}$ converges weakly in $D([0,\infty), [0,1]^n)$ to $(B_t^1 - B_t^0, \ldots, B_t^n - B_t^{n-1})_{t \ge 0}$.

In this paper, we extend the study of Norris and Turner to cover random arrival points θ_j with non-uniform law ν. In this case, the evolution of harmonic measure on the boundary is dominated by a non-trivial deterministic drift of order d^2, and the stochastic behaviour is seen only as fluctuations about this of order d^3.

2.3. *Some examples*

We give two examples of anisotropic growth to illustrate our results. We consider the case of slit mappings with deterministic length d for convenience.

2.3.1. *Angles chosen in an interval*

For $\eta \in (0,1]$, let θ_j be chosen uniformly in $[0,\eta]$. We build clusters K_n as before, at each step setting $d_j = d$ for $j = 1,\dots,n$. For fixed $t \in (0,\infty)$, if $n = \lfloor \mathrm{lcap}(P)^{-1} t \rfloor$, the hull K_n produced by the discrete iteration model then converges (in a sense to be made precise) as $d \to 0$, to the hulls obtained by solving the Loewner equation at time t driven by the measure

$$\mathrm{d}\nu\big(\mathrm{e}^{2\pi\mathrm{i}x}\big) = \frac{\chi_{[0,\eta]}(x)\,\mathrm{d}x}{\eta}.$$

A computation involving the power series expansion of the Schwarz–Herglotz kernel,

$$\frac{z+\mathrm{e}^{2\pi\mathrm{i}x}}{z-\mathrm{e}^{2\pi\mathrm{i}x}} = 1 + 2\sum_{j=1}^{\infty} \mathrm{e}^{2\pi\mathrm{i}jx} z^{-j},$$

shows that

$$\int_0^1 \frac{z+\mathrm{e}^{2\pi\mathrm{i}x}}{z-\mathrm{e}^{2\pi\mathrm{i}x}}\,\mathrm{d}\nu = 1 + 2\sum_{j=1}^{\infty} \frac{\sin(\pi\eta j)}{\pi\eta j}\big(\mathrm{e}^{-\pi\mathrm{i}\eta} z\big)^{-j}.$$

A closed expression for this series is given in [11], Formula 1.448; we now find that the explicit form of the Loewner equation for this choice of measure is

$$\partial_t f_t(z) = z f_t'(z)\left(1 + \frac{2}{\eta}\arctan\left[\frac{\mathrm{e}^{\mathrm{i}\pi\eta}\sin(\pi\eta)}{z - \mathrm{e}^{\mathrm{i}\pi\eta}\cos(\pi\eta)}\right]\right). \tag{9}$$

Construct the flow $\Gamma \in D^\circ$ that describes the evolution of harmonic measure on the cluster boundary, with rate $\mathrm{lcap}(P)^{-1} \asymp d^{-2}$. Then, as $d \to 0$, $\Gamma \to \phi$ in (D°, d_D), where $\phi_{(s,t]}(x)$ is the solution to the ordinary differential equation

$$\dot{\phi}_{(s,t]}(x) = \frac{1}{2\pi^2\eta}\log\left|\frac{\sin(\pi\phi_{(s,t)}(x))}{\sin(\pi(\phi_{(s,t]}(x)-\eta))}\right|$$

with $\phi_{(s,s]}(x) = x$; a derivation will be provided in Section 4. In the special case $\eta = 1/2$, we obtain the equation

$$\dot{\phi}_{(s,t]} = \frac{1}{\pi^2}\log\big|\tan\big(\pi\phi_{(s,t]}(x)\big)\big|.$$

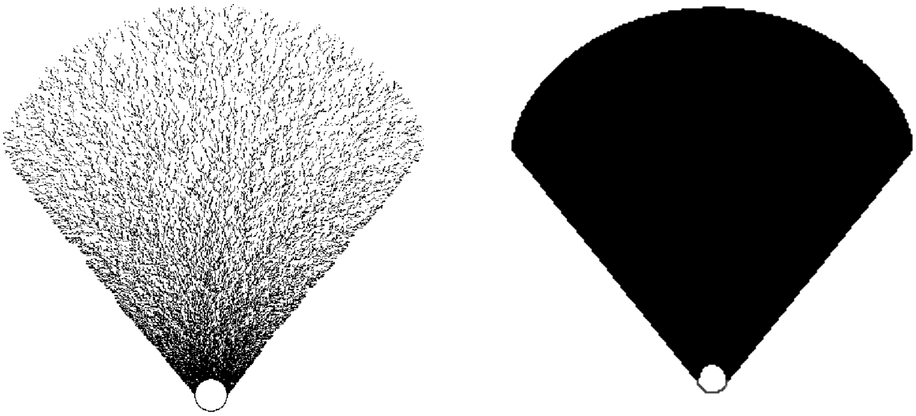
These results are illustrated in Fig. 2.

The spaces between the flow lines represent the proportion of harmonic measure carried by the fingers of the clusters attached between the corresponding points on the circle. Note the absence of random fluctuations in the region $(1/2, 1)$ in the simulation in the figure; this phenomenon will be discussed in Section 4.

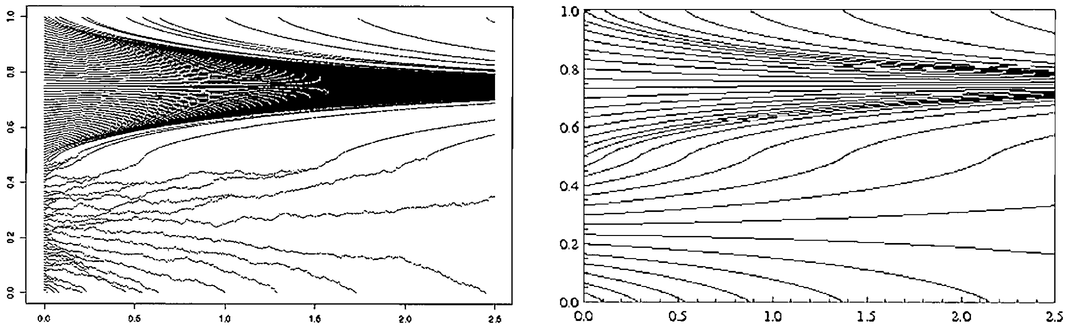
2.3.2. *Angles chosen from a density with m-fold symmetry*

For fixed $m \in \mathbb{N}$, choose θ_j distributed according to the density

$$\mathrm{d}\nu\big(\mathrm{e}^{2\pi\mathrm{i}x}\big) = 2\sin^2(m\pi x)\,\mathrm{d}x.$$



(a) AHL(ν) cluster (left) and the corresponding Loewner hull (right).



(b) Evolution of harmonic measure on the boundary of AHL(ν) (left) and the solution to the corresponding deterministic ODE (right).

Fig. 2. Simulations of AHL(ν) and associated limits, for $d = 0.02$ after 25,000 repetitions, corresponding to $\mathrm{d}\nu(x) = 2\chi_{[0,1/2]}\,\mathrm{d}x$.

This type of density with m-fold symmetry is considered in [22] as an example of a choice of angular distribution that introduces certain preferred directions in the cluster growth. The clusters converge, under the same scaling limits as above, to the hulls of the Loewner chain described by the equation

$$\partial_t f_t(z) = z f_t'(z)\left(1 - \frac{1}{z^m}\right).$$

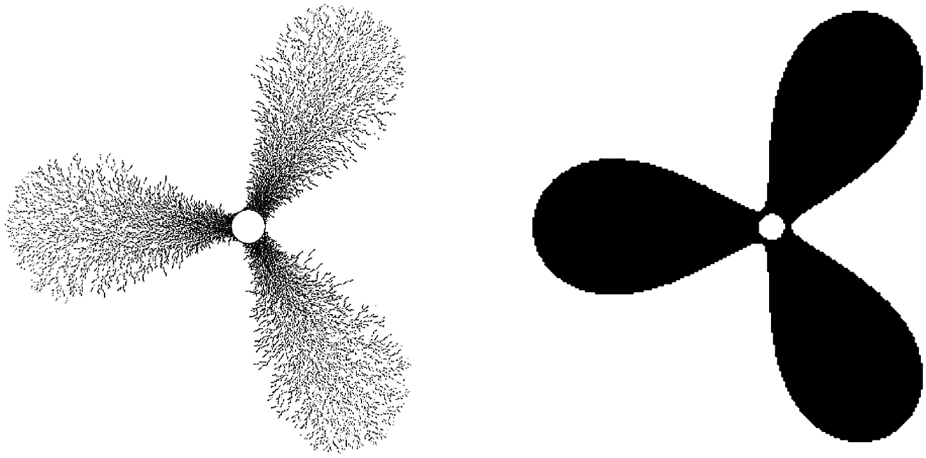
In the limit in this case, the evolution of harmonic measure on the cluster boundary is determined by the solutions to the ODE

$$\dot{\phi}_{(s,t]}(x) = -\frac{1}{2\pi}\sin\big(2\pi m \phi_{(s,t]}(x)\big), \quad \phi_{(s,s]}(x) = x.$$

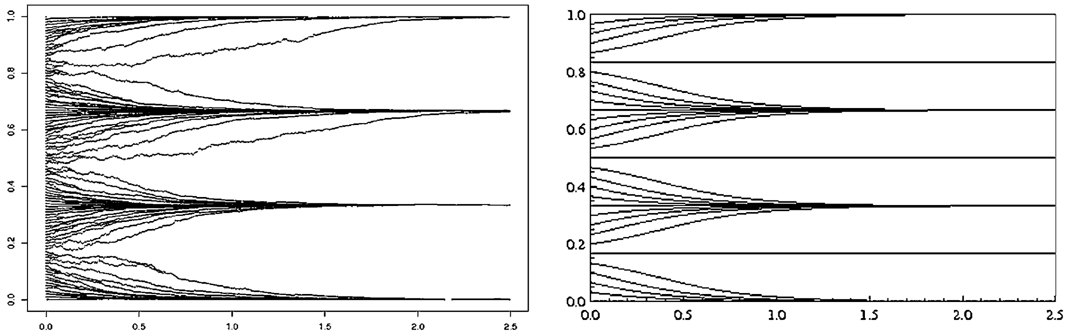
These results are illustrated in Fig. 3.

3. A shape theorem

In this section, we consider a scaling limit where the particle sizes converge to zero. The goal is to describe the macroscopic shape of the limiting cluster, that is, to prove a shape theorem. This generalizes a result we first learned about from Rohde [23], see also [19]: simulations of standard HL(0) clusters show that if the basic slit length d is chosen to be small, and the number of compositions is large, then the clusters K_n look rounded. In fact, if we let $d \to 0$ and $n \asymp d^{-2}$, then the laws of resulting HL(0) clusters do indeed converge to that of a closed disk cK_0.



(a) AHL(ν) cluster (left) and the corresponding Loewner hull (right).



(b) Evolution of harmonic measure on the boundary of AHL(ν) (left) and the solution to the corresponding deterministic ODE (right).

Fig. 3. Simulations of AHL(ν) and associated limits, for $d = 0.02$ after 25,000 repetitions, corresponding to $\mathrm{d}\nu(x) = 2\sin^2(3\pi x)\,\mathrm{d}x$.

Similarly, comparing the AHL(ν) clusters with the hulls generated by the Loewner equation driven by the time-independent measure ν, we see that, as the particle diameters d tend to zero and the number of compositions increases at a rate proportional to d^{-2}, the shapes converge (even for random particle sizes). Indeed, in Theorem 2 we prove that the discrete clusters converge to the Loewner hulls. We begin with a technical result about solutions to the Loewner–Kufarev equation.

3.1. *Continuity properties of the Loewner equation*

In this section, we show that solutions to the Loewner–Kufarev equation (3) are "close" at time T if the driving measures are "close" in some suitable sense. For conformal mappings, the notion of closeness is to be understood in the sense of uniform convergence on compact subsets of D_0.

Let Σ denote the space of conformal mappings $f : D_0 \to \mathbb{C}$ with expansions at infinity of the form

$$f(z) = c_1 z + c_0 + c_{-1}/z + \cdots, \quad c_1 > 0,$$

equipped with the topology induced by uniform convergence on compact subsets of D_0. Denote by $\Pi(\Sigma)$ the space of probability measures on Σ.

In [2], it is shown that if the Loewner equation is driven by continuous functions that are close in the uniform metric, then the corresponding solutions are close as conformal mappings. This was extended to cover Skorokhod space functions in [13].

The following proposition deals with the case of general driving measures.

Proposition 1. *Let $0 < T < \infty$. Let $\mu^n = \{\mu_t^n\}_{t\geq 0}, n = 1, 2, \dots,$ and $\mu = \{\mu_t\}_{t\geq 0}$ be families of measures in $\mathcal{P}$. Let m denote Lebesgue measure on $[0, \infty)$, and suppose that the measures $\mu_t^n \times m$ converge weakly on $S = \mathbb{T} \times [0, T]$ to the measure $\mu_t \times m$ as $n \to \infty$.*

Then the solutions $\{f_T^n\}$ to (3) corresponding to the sequence $\{\mu^n\}$ converge to f_T, the solution corresponding to μ, uniformly on compact subsets of D_0.

Proof. The proof is similar to the continuity lemmas of [2] and [13].

Fix a compact set $K \subset D_0$, and let $\varepsilon > 0$ be given. For $t \in [0, T]$, consider the backward Loewner flow given by

$$\dot{h}_t(z) = -h_t(z) \int_{\mathbb{T}} \frac{\zeta + h_t(z)}{\zeta - h_t(z)} \,\mathrm{d}\mu_{T-t}(\zeta), \quad h_0(z) = z. \tag{10}$$

It is a well-known result in the Loewner theory that this backward flow, when evaluated at time $t = T$, coincides with the mapping f_T arising from (3); that is, $h_T(z) = f_T(z)$, $z \in K$ (see [17], Chapter 4.2). An analogous statement holds for the solutions corresponding to the measures μ^n.

We shall use the convenient shorthand notations

$$v(s, \nu, z) = -z \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \,\mathrm{d}\nu_{T-s}(\zeta) \tag{11}$$

and

$$w(x, z) = z \frac{z + x}{z - x}. \tag{12}$$

For $z \in K$ fixed, set $u(t) = h_t(z)$ and $u^n(t) = h_t^n(z)$. Integrating (10) with respect to t and using $u(0) = u^n(0)$, we obtain, for $t \in [0, T]$,

$$\begin{aligned} \left|u(t) - u^n(t)\right| &\leq \left|u(t) - u(0) - \int_0^t v\bigl(s, \mu^n, u(s)\bigr) \,\mathrm{d}s\right| \\ &\quad + \left|\int_0^t v\bigl(s, \mu^n, u(s)\bigr) \,\mathrm{d}s - u^n(t) + u^n(0)\right| \\ &= \left|\int_0^t v\bigl(s, \mu, u(s)\bigr) \,\mathrm{d}s - \int_0^t v\bigl(s, \mu^n, u(s)\bigr) \,\mathrm{d}s\right| \\ &\quad + \left|\int_0^t v\bigl(s, \mu^n, u(s)\bigr) \,\mathrm{d}s - \int_0^t v\bigl(s, \mu^n, u^n(s)\bigr) \,\mathrm{d}s\right|. \end{aligned}$$

The first term may be written out as

$$\begin{aligned} &\left|\int_0^t v\bigl(s, \mu, u(s)\bigr) \,\mathrm{d}s - \int_0^t v\bigl(s, \mu^n, u(s)\bigr) \,\mathrm{d}s\right| \\ &\quad = \left|\int_0^t \int_{\mathbb{T}} u(s) \frac{u(s) + x}{u(s) - x} \,\mathrm{d}\mu_{T-s}(x) \,\mathrm{d}s - \int_0^t \int_{\mathbb{T}} u(s) \frac{u(s) + x}{u(s) - x} \,\mathrm{d}\mu_{T-s}^n(x) \,\mathrm{d}s\right|, \end{aligned}$$

and since the integrand is a continuous function on $\mathbb{T} \times [0, T]$, our assumption of weak convergence implies, in particular, that the right-hand side is smaller than ε for all $n \geq N$ for some N (which depends on the point z).

We estimate the second term by

$$\left|\int_0^t v\big(s, \mu^n, u(s)\big)\,\mathrm{d}s - \int_0^t v\big(s, \mu^n, u^n(s)\big)\,\mathrm{d}s\right|$$
$$\le \int_0^t \int_{\mathbb{T}} \big|w\big(x, u(s)\big) - w\big(x, u^n(s)\big)\big|\,\mathrm{d}\mu^n_{T-s}(x)\,\mathrm{d}s.$$

We now use the inequality

$$\big|w(x,z) - w\big(x,z'\big)\big| \le \big(\sup\big|\partial_z w(x,z)\big|\big)\big|z - z'\big|$$

together with standard growth estimates on conformal mappings of D_0 to obtain that

$$\big|w\big(x, u(s)\big) - w\big(x, u_n(s)\big)\big| \le C(T,K)\big|u(s) - u^n(s)\big|$$

for some constant $C(T,K)$ that does not depend on n (the lack of normalization of the mappings accounts for the dependence on T). This in turn leads to the estimate

$$\int_0^t \int_{\mathbb{T}} \big|w\big(x, u(s)\big) - w\big(x, u^n(s)\big)\big|\,\mathrm{d}\mu^n_{T-s}(x)\,\mathrm{d}s$$
$$\le C(T,K)\int_0^t \big\|\mu^n_{T-s}\big\|\big|u(s) - u^n(s)\big|\,\mathrm{d}s$$
$$= C(T,K)\int_0^t \big|u(s) - u^n(s)\big|\,\mathrm{d}s.$$

Putting everything together, we find that

$$\big|u(t) - u^n(t)\big| \le \varepsilon + C(T,K)\int_0^t \big|u(s) - u^n(s)\big|\,\mathrm{d}s.$$

We are now in a position to apply Grönwall's Lemma, and we obtain

$$\big|u(t) - u^n(t)\big| \le C'(T,K)\varepsilon, \quad t \in [0,T].$$

Thus $|u(T) - u^n(T)| < C'(T,K)\varepsilon$ for $n \ge N$, and this means that $f^n_T(z)$ converges to $f_T(z)$.

We have thus established the pointwise convergence of $\{f_n\}$ to f on the compact set K. We now observe that the mappings f_n are in Σ, with uniformly bounded coefficients c_1. Since the sequence $\{f_n\}$ is then locally bounded by growth-type estimates (see [13], Section 3), it follows from Vitali's theorem that the convergence is in fact uniform on each K, and the proof is complete. □

Now set $S = \mathbb{T} \times [0,\infty)$ and let $\mathcal{M} = \mathcal{M}(S)$ be the set of locally bounded Borel measures on S. A sequence $\mu, \mu^n \in \mathcal{M}$ is said to converge vaguely if

$$\int \varphi\,\mathrm{d}\mu^n \to \int \varphi\,\mathrm{d}\mu, \quad \forall \varphi \in \mathcal{C}_c(S),$$

where $\mathcal{C}_c(S)$ is the set of continuous functions in S with compact support. Weak convergence is defined the same way with compactly supported continuous φ replaced by bounded continuous φ. A random measure on S is a measurable mapping from some probability space into $\mathcal{M}$. In the next section we shall need the following lemma contained in [15], Theorem 15.7.6.

Lemma 1. *Let μ, μ^n, $n = 1, 2, \ldots$, be bounded measures on S. The sequence $\mu^n \to \mu$ with respect to the weak topology as $n \to \infty$ if and only if $\mu^n \to \mu$ in the vague topology and $\mu^n(S) \to \mu(S)$.*

3.2. *Statement and proof of the shape theorem*

In this section, we prove the convergence of the random measures generating AHL(ν) to the desired deterministic measure when the particle diameter d tends to zero and the number of compositions tends to infinity at a rate proportional to d^{-2}. In view of the continuity result of the previous section, the weak convergence of the AHL(ν) mappings then follows.

Let $P_1, P_2, \dots$ be chosen to be identical with $\operatorname{diam}(P) = d$. Assume additionally that the particle shape is chosen with capacity $\operatorname{lcap}(P)$ of order d^2 (see (7)). Note that our results can be shown to hold when the P_j are random, under additional conditions that are given in the remark at the end of this subsection. Let $\theta_1, \theta_2, \dots$ be $\mathbb{T}$-valued independent random variables with law ν.

Theorem 2. *Let Φ denote the solution to the Loewner–Kufarev equation driven by the measures $\{\nu_t\}_{t\geq 0} = \{\nu\}_{t\geq 0}$ and evaluated at time T, for some fixed $T \in (0,\infty)$.*

Set $n = \lfloor \operatorname{lcap}(P)^{-1} T \rfloor$, and define the conformal map

$$\Phi_n = f_{P_1}^{\theta_1} \circ \cdots \circ f_{P_n}^{\theta_n}.$$

Then Φ_n converges to Φ uniformly on compacts almost surely as $d \to 0$.

Proof. Let $\varepsilon > 0$ be given. For $k = 1, \dots, n$, set

$$T_k = k \operatorname{lcap}(P)$$

and

$$\Xi_n(t) = \sum_{k=1}^{n} \chi_{[T_{k-1},T_k)}(t) \xi_k(t),$$

where $\xi_k(t), t \in [T_{k-1}, T_k)$, is the (rotated) driving function for the particle P_k. We set $\xi^n(t) = \exp(\mathrm{i}\Xi_n(t))$. Then $\delta_{\xi^n(t)}$ is the measure that drives the evolution of the AHL clusters. That is, the mapping Φ_n is the solution to the Loewner–Kufarev equation

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} \, \mathrm{d}\delta_{\xi^n(t)}(\zeta) \tag{13}$$

with $f_0(z) = z$, evaluated at time $t = T_n$. Integrating with respect to Lebesgue measure in time, m, we see that we need to show that the random measures

$$\mu^P = \delta_{\xi^n(t)} \times m_{[0,T_n]} \in \mathcal{M}(S)$$

converge almost surely to $\mu = \nu \times m_{[0,T]}$ as $d \to 0$ with respect to the weak topology. Note that $\mu^P(S) = T_n \to T = \mu(S)$. By Lemma 1 it remains to prove convergence of the random variables $\langle \mu^P, \varphi\rangle$ to $\langle \mu, \varphi\rangle$, as $d \to 0$, for $\varphi \in \mathcal{C}_c(S)$.

As before, we identify the circle with the interval $[0,1)$, that is, $S = [0,1) \times [0,\infty)$. For $\varphi \in \mathcal{C}_c(S)$,

$$\begin{aligned} \left|\langle \mu, \varphi\rangle - \langle \mu^P, \varphi\rangle\right| &= \left| \int_0^T \int_{\mathbb{T}} \varphi(\theta,t)\,\mathrm{d}\mu - \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \varphi\big(\xi_k(t),t\big)\,\mathrm{d}m(t) \right| \\ &\leq \left| \int_0^T \int_{\mathbb{T}} \varphi(\theta,t)\,\mathrm{d}\mu - \operatorname{lcap}(P) \sum_{k=1}^{n} \varphi(\theta_k, T_k) \right| \\ &\quad + \left| \operatorname{lcap}(P) \sum_{k=1}^{n} \varphi(\theta_k, T_k) - \sum_{k=1}^{n} \int_{T_{k-1}}^{T_k} \varphi\big(\xi_k(t),t\big)\,\mathrm{d}m(t) \right|. \end{aligned} \tag{14}$$

The second term on the right-hand side can be bounded by

$$\sum_{k=1}^{n}\int_{T_{k-1}}^{T_k}\bigl|\varphi(\theta_k,T_k)-\varphi\bigl(\xi_k(t),t\bigr)\bigr|\,\mathrm{d}m(t).$$

Note that $T_k - T_{k-1} = \mathrm{lcap}(P)$ and by (7),

$$\sup_{T_{k-1}\le t<T_k}\bigl|\xi_k(t)-\mathrm{e}^{2\pi\mathrm{i}\theta_k}\bigr|\le C\,\mathrm{diam}(P).$$

Hence

$$\max_{1\le k\le n}\sup_{T_{k-1}\le t\le T_k}\bigl|\xi_k(t)-\mathrm{e}^{2\pi\mathrm{i}\theta_k}\bigr|\to 0$$

almost surely as $d\to 0$. Since φ is compactly supported, and hence uniformly continuous on S, we have

$$\max_{1\le k\le n}\sup_{T_{k-1}\le t\le T_k}\bigl|\varphi(\theta_k,T_k)-\varphi\bigl(\xi_k(t),t\bigr)\bigr|<\varepsilon$$

for d sufficiently small. It follows that, almost surely,

$$\sum_{k=1}^{n}\int_{T_{k-1}}^{T_k}\bigl|\varphi\bigl(\xi_k(t),t\bigr)-\varphi(\theta_k,T_k)\bigr|\,\mathrm{d}m(t)<\varepsilon\sum_{k=1}^{n}\int_{T_{k-1}}^{T_k}\mathrm{d}m(t)\le c\varepsilon$$

as soon as d is sufficiently small.

We turn to the first term on the right-hand side in (14). We apply the strong law of large numbers for independent random variables (see for instance [16], Corollary 4.22) to $X_k=\varphi(\theta_k,T_k)-\int_0^1\varphi(\theta,T_k)\,\mathrm{d}\nu$ to obtain

$$\mathrm{lcap}(P)\sum_{k=1}^{n}X_k\to 0$$

almost surely. As $\mathrm{lcap}(P)=T_k-T_{k-1}$, it follows that

$$\mathrm{lcap}(P)\sum_{k=1}^{n}\biggl(\int_0^1\varphi(\theta,T_k)\,\mathrm{d}\nu\biggr)\to\int_0^T\int_0^1\varphi(\theta,t)\,\mathrm{d}\mu$$

almost surely, by continuity of φ.

Hence the sequence of random variables $\langle\mu^P,\varphi\rangle$ converges almost surely to $\langle\mu,\varphi\rangle$ for each fixed $\varphi\in\mathcal{C}_c(S)$. Note that $\mathcal{C}_c(S)$ is separable when equipped with the supremum norm. Consequently, there exists a countable dense subset $K\subset\mathcal{C}_c(S)$ and for each test function $\varphi_j\in K$, an event of probability one on which $\langle\mu^P,\varphi_j\rangle$ converges to $\langle\mu,\varphi_j\rangle$. Let V be the countable intersection of these events and the event that $\mu^P(S)\to T$. Then $\mathbb{P}(V)=1$ and the random variables $\langle\mu^P,\varphi_j\rangle$, $\varphi_j\in K$, all converge on V. Moreover, by the density of K in the uniform norm we have that for each $\varepsilon>0$ and $\varphi\in\mathcal{C}_c(S)$, there exists $\tilde\varphi\in K$ such that

$$\begin{aligned}\bigl|\langle\mu^P,\varphi\rangle-\langle\mu,\varphi\rangle\bigr|&\le\bigl|\langle\mu^P,\varphi\rangle-\langle\mu^P,\tilde\varphi\rangle\bigr|\\&\qquad+\bigl|\langle\mu^P,\tilde\varphi\rangle-\langle\mu,\tilde\varphi\rangle\bigr|+\bigl|\langle\mu,\tilde\varphi\rangle-\langle\mu,\varphi\rangle\bigr|\\&\le\bigl|\langle\mu^P,\tilde\varphi\rangle-\langle\mu,\tilde\varphi\rangle\bigr|+\bigl(\mu^P(S)+\mu(S)\bigr)\|\varphi-\tilde\varphi\|_{\infty,S}\\&\le\bigl|\langle\mu^P,\tilde\varphi\rangle-\langle\mu,\tilde\varphi\rangle\bigr|+\bigl(\mu^P(S)+\mu(S)\bigr)\varepsilon,\end{aligned}$$

and on the event V this last expression converges to $2T\varepsilon$, as $d\to 0$. Hence, we have almost sure convergence of μ^P to μ with respect to the vague topology. Consequently, by Lemma 1 the random measures μ^P converge almost surely to μ with respect to the weak topology.

In view of Proposition 1, the corresponding conformal mappings converge uniformly on compact sets, and the proof is complete. □

Remark 1. *The setup in the theorem can easily be adapted to allow for random particle sizes tending to zero in probability. For example, we could take* $\operatorname{lcap}(P_k^n) = \lambda_k/n$ *for bounded i.i.d. random variables* λ_k *and obtain almost sure convergence of the corresponding conformal mappings. The proof is essentially the same, except that we apply an ergodic theorem* [4], *Theorem* 1, *instead of the law of large numbers. We can also relax the condition on the sequence* λ_k *to square-integrability, and then obtain convergence in law of the conformal mappings, by adapting the proof in* [4] *appropriately.*

Remark 2. *Instead of choosing* θ_j *as i.i.d. random variables, one could also take* $\{\theta_j\}_j$ *to be a Markov chain satisfying some natural conditions. By examining our proof, and applying a stronger version of* [4], *Theorem* 1, *we obtain a result similar to Theorem* 2, *with the limiting* ν *uniform on* $\mathbb{T}$. *This is a consequence of the fact that the invariant measure on* $\mathbb{T}$ *under rotation is Lebesgue measure.*

4. The evolution of harmonic measure on the cluster boundary

In this section we establish a scaling limit for the evolution of harmonic measure on the cluster boundary. We show that it can be approximated by the solution to a deterministic ordinary differential equation related to the Loewner equation and we also characterise the stochastic fluctuations around the deterministic limit flow.

For notational simplicity, we assume that the diameters $\{d_j\}$ of the particles are constant and equal to some $d > 0$ which tends to zero to obtain limit results. All the proofs can be directly adapted for $\{d_j\}$ with laws σ with finite third moment $\sigma_3 \to 0$. We also assume that ν has density h_ν on $\mathbb{R}$, periodic with period 1, which is twice differentiable. This restriction is purely for technical reasons and, through smoothing, any non-atomic Borel measure can be sufficiently well approximated by a measure with a twice differentiable density.

Recall the construction of the map γ_P and the flow $(\Gamma_I\colon I \subseteq [0,\infty))$ from Section 2.2. Define the function β_ν and the constant $\rho(P)$ by

$$\beta_\nu(x) = \int_0^1 \tilde{\gamma}_P(x-z) h_\nu(z)\,\mathrm{d}z,$$

$$1 = \rho(P) \int_0^1 \tilde{\gamma}_P(z)^2\,\mathrm{d}z,$$

where $\tilde{\gamma}_P^{\pm}(x) = \gamma_P^{\pm}(x) - x$. It is shown in [20] that $\rho(P) \asymp d^{-3}$.

Suppose that the Poisson process $\{T_i\}$, used in the construction of Γ_I, has rate $\operatorname{lcap}(P)^{-1}$ and let $X \in D^\circ$ be a lifting of Γ onto the real line. Then for fixed $e = (s,x) \in [0,\infty) \times \mathbb{R}$, $X_t^{e,\pm} = X_{(s,t]}^{\pm}(x)$ satisfies the integral equation

$$\begin{aligned} X_t^{e,\pm} &= x + \int_{(s,t]\times[0,1)} \tilde{\gamma}_P^{\pm}\big(X_r^{e,\pm} - z\big)\mu(\mathrm{d}r,\mathrm{d}z) \\ &= x + M_{ts}^{\pm} + \operatorname{lcap}(P)^{-1} \int_{(s,t]} \beta_\nu\big(X_r^{e,\pm}\big)\,\mathrm{d}r, \qquad t \ge s, \end{aligned}$$

where μ is a Poisson random measure on $[0,\infty) \times [0,1)$, equipped with the Borel σ-algebra, with intensity $\operatorname{lcap}(P)^{-1} h_\nu(z)\,\mathrm{d}z\,\mathrm{d}r$, and where $M_{ts}^{\pm}$ is a martingale (see, for example, [25], Proposition 19.5) satisfying

$$M_{ts}^{\pm} = \int_{(s,t]\times[0,1)} \tilde{\gamma}_P^{\pm}\big(X_r^{e,\pm} - z\big)\big(\mu(\mathrm{d}r,\mathrm{d}z) - \operatorname{lcap}(P)^{-1} h_\nu(z)\,\mathrm{d}r\,\mathrm{d}z\big).$$

In what follows, we suppress the superscripts $e, \pm$.

Recall (see Section 2.1) that there are natural sequences of particles P for which $\operatorname{lcap}(P) \asymp d^2$. We assume that this holds in what follows. It is also shown in [20] that there exists some universal constant $0 < C_3 < \infty$ such that

$$\left|\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z\right| \le C_3 d^2,$$

and so, by restricting to a subsequence if necessary, we assume that

$$\operatorname{lcap}(P)^{-1}\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z \to c_0$$

for some $c_0 \in \mathbb{R}$. Note that for symmetric particles, $\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z = 0$ in which case $c_0 = 0$.

Proposition 2. *As* $d \to 0$, $|\operatorname{lcap}(P)^{-1}\beta_\nu(x) - b(x)| \to 0$, *uniformly in* x, *where*

$$b(x) = c_0 h_\nu(x) + \frac{1}{2\pi}\int_0^1 \cot(\pi z)\big(h_\nu(x-z) - h_\nu(x)\big)\,\mathrm{d}z.$$

Furthermore, if P *is chosen so that*

$$d^{-1/2}\left|\operatorname{lcap}(P)^{-1}\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z - c_0\right| \to 0$$

as $d \to 0$, *then* $d^{-1/2}|\operatorname{lcap}(P)^{-1}\beta_\nu(x) - b(x)| \to 0$, *uniformly in* x, *as* $d \to 0$.

Proof. It is shown in [17], Section 3.5, that there exists some universal constant $c < \infty$ such that if $cd \le z \le 1 - cd$, then

$$\left|\tilde{\gamma}_P(z) - \frac{\operatorname{lcap}(P)}{2\pi}\cot(\pi z)\right| \le \frac{cd\operatorname{lcap}(P)}{2\pi\sin^2(\pi z)}. \tag{15}$$

If $z \in (-cd, cd)$, then $\gamma_P(-cd) \le \gamma_P(z) \le \gamma_P(cd)$ and hence

$$\left|\tilde{\gamma}_P(z)\right| \le \left|\tilde{\gamma}_P(cd)\right| \vee \left|\tilde{\gamma}_P(-cd)\right| + 2cd.$$

From this it can be deduced that there exists some $c' > 0$, such that $\|\tilde{\gamma}_P\|_\infty < c'd$. Now,

$$\beta_\nu(x) = h_\nu(x)\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z + \int_0^1 \tilde{\gamma}_P(z)\big(h_\nu(x-z) - h_\nu(x)\big)\,\mathrm{d}z,$$

and so, if d is sufficiently small that $cd < 1/2$ then it follows that

$$\begin{aligned}
&\left|\frac{\beta_\nu(x)}{\operatorname{lcap}(P)} - b(x)\right| \\
&\quad\le \left|\operatorname{lcap}(P)^{-1}\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z - c_0\right|\left|h_\nu(x)\right| \\
&\qquad + \operatorname{lcap}(P)^{-1}\int_{-cd}^{cd}\left|\tilde{\gamma}_P(z)\right|\left|h_\nu(x-z) - h_\nu(x)\right|\mathrm{d}z \\
&\qquad + \operatorname{lcap}(P)^{-1}\int_{cd}^{1/2}\left|\tilde{\gamma}_P(z) - \frac{\operatorname{lcap}(P)}{2\pi}\cot(\pi z)\right|\left|h_\nu(x-z) - h_\nu(x)\right|\mathrm{d}z \\
&\qquad + \operatorname{lcap}(P)^{-1}\int_{-1/2}^{-cd}\left|\tilde{\gamma}_P(z) - \frac{\operatorname{lcap}(P)}{2\pi}\cot(\pi z)\right|\left|h_\nu(x-z) - h_\nu(x)\right|\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{lcap}(P)^{-1} \int_{-cd}^{cd} \frac{\operatorname{lcap}(P)}{2\pi} \left|\cot(\pi z)\right| \left|h_\nu(x-z) - h_\nu(x)\right| \mathrm{d}z \\
&\le \|h_\nu\|_\infty \left| \operatorname{lcap}(P)^{-1} \int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z - c_0 \right| + \left\|h_\nu'\right\|_\infty c' d \operatorname{lcap}(P)^{-1} (cd)^2 \\
&+ \frac{\|h_\nu'\|_\infty cd}{\pi^3} \left(\left|\log \sin(\pi cd)\right| + \pi cd \cot(\pi cd) \right) \\
&+ \frac{cd \|h_\nu'\|_\infty}{\pi} \sup_{z \in (-cd, cd)} \left| z \cot(\pi z) \right| \\
&\to 0
\end{aligned}
$$

as $d \to 0$. To obtain the above inequalities we have used (15), the inequality $|h_\nu(x-z) - h_\nu(x)| \le \|h_\nu'\|_\infty |z|$, the periodicity of $\tilde{\gamma}_P$ and h_ν, and the integral identity

$$
\int_x^{1/2} \frac{z}{\sin^2(\pi z)}\,\mathrm{d}z = \frac{|\log \sin(\pi x)|}{\pi^2} + \frac{x \cot(\pi x)}{\pi}
$$

for $0 < x \le 1/2$. □

Note that $\int_0^1 \cot(\pi z)(h_\nu(x-z) - h_\nu(x))\,\mathrm{d}z$ is the Hilbert transform of h_ν, as defined in Section 2.1. In particular, this implies that $b(x)$ is constant only when h_ν is the uniform density on the circle. It is for this reason that the behaviour in the uniform case is very different to the non-uniform case.

Define $\phi \in D^\circ$ to be the solution to the ordinary differential equation

$$
\dot{\phi}_{(s,t]}(x) = b\big(\phi_{(s,t]}(x)\big) \quad \text{for } t \ge s, \qquad \phi_{(s,s]}(x) = x. \tag{16}
$$

We shall prove that the boundary flow converges to the flow determined by (16). Note that away from the support of h_ν, this equation coincides with the lifted Loewner ODE

$$
\partial_t \gamma_t(x) = \frac{1}{2\pi} \int_0^1 \cot\big(\pi\big(\gamma_t(x) - z\big)\big) h_\nu(z)\,\mathrm{d}z.
$$

However, on the support of ν, where we have to interpret the integral as a principal value, we get an additional drift term $c_0 h_\nu(\gamma_t)$ in the right-hand side. In the case of symmetric particles, the drift vanishes everywhere, and the resulting flow is governed by the extended Loewner flow given by

$$
\partial_t \gamma_t(x) = H[\nu]\big(\gamma_t(x)\big).
$$

Proposition 3. *For all $T > s$,*

$$
\mathbb{E}\Big(\Big(\sup_{s<t<T} |M_{ts}|\Big)^2\Big) \le 4\|h_\nu\|_\infty \operatorname{lcap}(P)^{-1} \rho(P)^{-1}(T-s).
$$

Hence, for all $\varepsilon > 0$,

$$
\mathbb{P}\Big(\sup_{s<t<T} |M_{ts}| > \varepsilon\Big) \to 0
$$

as $d \to 0$.

Proof. Since, for any fixed $(s,x) \in [0,\infty) \times \mathbb{R}$, the processes M_{ts} are martingales, by Doob's L^2 inequality, for all $T > s$,

$$
\mathbb{E}\Big(\Big(\sup_{s<t<T} |M_{ts}|\Big)^2\Big) \le 4\mathbb{E}\big(|M_{Ts}|^2\big)
$$

$$
\begin{aligned}
&= 4\int_s^T \int_0^1 \mathbb{E}\big(\tilde{\gamma}_P(X_r - z)^2\big)\,\mathrm{lcap}(P)^{-1} h_\nu(z)\,\mathrm{d}z\,\mathrm{d}r \\
&\le 4\,\mathrm{lcap}(P)^{-1}\|h_\nu\|_\infty (T-s)\int_0^1 \tilde{\gamma}_P(z)^2\,\mathrm{d}z \\
&= 4\|h_\nu\|_\infty\,\mathrm{lcap}(P)^{-1}\rho(P)^{-1}(T-s).
\end{aligned}
$$

The first equality is from, for example, [25], Proposition 19.5, and the second inequality follows from Markov's inequality and the assymptotic behaviour of $\rho(P)$ and $\mathrm{lcap}(P)$. □

Recall the definition of ϕ as the solution of (16).

Theorem 3. *As $d \to 0$,*

$$
d_D(X, \phi) \to 0
$$

in probability.

Proof. Given $\varepsilon > 0$, for fixed $e = (s, x) \in [0, \infty) \times \mathbb{R}$ and $T > s$, choose $d_0 > 0$ sufficiently small that $\|\,\mathrm{lcap}(P)^{-1}\beta_\nu - b\|_\infty < \varepsilon \mathrm{e}^{-\|b'\|_\infty T}/2(T-s)$ for all $d \le d_0$, and set

$$
\Omega_{T,d} = \Big\{ \sup_{s<t\le T} |M_{ts}| \le \varepsilon \mathrm{e}^{-\|b'\|_\infty T}/2 \Big\}.
$$

Then if $d \le d_0$, on the set $\Omega_{T,d}$,

$$
\begin{aligned}
&\sup_{s<t\le T}\big|X_t - \phi_{(s,t]}(x)\big| \\
&\quad\le \sup_{s<t\le T}|M_{ts}| + \sup_{s<t\le T}\int_s^t \big|\mathrm{lcap}(P)^{-1}\beta_\nu(X_r) - b(X_r)\big|\,\mathrm{d}r \\
&\qquad + \sup_{s<t\le T}\int_s^t \big|b(X_r) - b\big(\phi_{(s,r]}(x)\big)\big|\,\mathrm{d}r \\
&\quad\le \varepsilon \mathrm{e}^{-\|b'\|_\infty T} + \big\|b'\big\|_\infty \int_s^T \sup_{s<t\le r}\big|X_t - \phi_{(s,t]}(x)\big|\,\mathrm{d}r.
\end{aligned}
$$

Hence, by Grönwall's Lemma,

$$
\sup_{s<t\le T}\big|X_t - \phi_{(s,t]}(x)\big| \le \varepsilon.
$$

Therefore, by Proposition 3,

$$
\limsup_{d\to 0}\mathbb{P}\Big(\sup_{s<t\le T}\big|X_t - \phi_{(s,t]}(x)\big| > \varepsilon\Big) \le \limsup_{d\to 0}\mathbb{P}\big(\Omega_{T,d}^c\big) = 0.
$$

For any countable dense set $E \subset [0, \infty) \times \mathbb{R}$, $(X_t)_{(s,x)\in E} \to (\phi_{(s,t]}(x))_{(s,x)\in E}$ in D_E, in probability as $d \to 0$. Therefore, by Proposition 10.11 in [20], $d_D(X, \phi) \to 0$ in probability as $d \to 0$. □

Corollary 4. *Let $x_1, \dots, x_n$ be a positively oriented set of points in $\mathbb{R}/\mathbb{Z}$ and set $x_0 = x_n$. Set $K_t = K_{\lfloor \mathrm{lcap}(P)^{-1} t\rfloor}$. For $k = 1, \dots, n$, write ω_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k. Then, in the limit $d \to 0$, $(\omega_t^1, \dots, \omega_t^n)_{t\ge 0}$ converges weakly in $D([0,\infty), [0,1]^n)$ to $(\phi_{(0,t]}(x_1) - \phi_{(0,t]}(x_0), \dots, \phi_{(0,t]}(x_n) - \phi_{(0,t]}(x_{n-1}))_{t\ge 0}$.*

A geometric consequence of this result is that the number of infinite fingers of the cluster converges to the number of stable equilibria of the ordinary differential equation $\dot{x}_t = b(x_t)$, and the positions at which these fingers are rooted to the unit disk converge to the unstable equilibria of the ODE.

4.1. *Fluctuations*

In this section, suppose that P is chosen so that

$$d^{-1/2}\left|\operatorname{lcap}(P)^{-1}\int_0^1 \tilde{\gamma}_P(z)\,\mathrm{d}z - c_0\right| \to 0$$

as $d \to 0$. For fixed $(s,x) \in [0,\infty) \times \mathbb{R}$, define

$$Z_t^P = \big(\operatorname{lcap}(P)\rho(P)\big)^{1/2}\big(X_{(s,t]}(x) - \phi_{(s,t]}(x)\big)$$

and let Z_t be the solution to the linear stochastic differential equation

$$\mathrm{d}Z_t = \sqrt{h_\nu\big(\phi_{(s,t]}(x)\big)}\,\mathrm{d}B_t + b'\big(\phi_{(s,t]}(x)\big)Z_t\,\mathrm{d}t, \quad t \geq s,$$

starting from $Z_s = 0$, where B_t is a standard Brownian motion.

Note that if $\phi_{(s,t]}(x)$ stays off the support of h_ν, then $Z_t = 0$ for all $t \geq s$. Also observe that in the case where ν is the uniform measure on the unit circle, $(\operatorname{lcap}(P)\rho(P))^{1/2}(X_{(s,t]}(x) - x - c_0(t-s))_{t\geq s}$ converges to standard Brownian motion, starting from 0 at time s.

Lemma 5. *For fixed x and $s < T < \infty$ there exists some constant C, dependent only on T, h_ν and b such that*

$$\mathbb{E}\Big(\sup_{s\leq t\leq T}\big|Z_t^P\big|^2\Big) \leq C$$

and, for all $s \leq t_1 < t_2 \leq T$,

$$\mathbb{E}\Big(\sup_{t_1\leq t\leq t_2}\big|Z_t^P - Z_{t_1}^P\big|^2\Big) \leq C(t_2 - t_1).$$

Therefore (see, for example [3], p. 143) the family of processes $(Z_t^P)_{t\geq s}$ is tight with respect to parameter d.

Proof. Since $\operatorname{lcap}(P)\rho(P) \asymp d^{-1}$ there exists $C' > 0$ such that

$$\big(\operatorname{lcap}(P)\rho(P)\big)^{1/2}\big|\operatorname{lcap}(P)^{-1}\beta_\nu(x) - b(x)\big| < C'.$$

Then,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\leq t\leq T}\big|Z_t^P\big|^2\Big)\\
&\quad\leq 3\operatorname{lcap}(P)\rho(P)\mathbb{E}\Big(\sup_{s\leq t\leq T}|M_{ts}|^2\Big)\\
&\qquad + 3\operatorname{lcap}(P)\rho(P)\int_s^T \mathbb{E}\Big(\big|\operatorname{lcap}(P)^{-1}\beta_\nu(X_r) - b(X_r)\big|^2\Big)\,\mathrm{d}r\\
&\qquad + 3\operatorname{lcap}(P)\rho(P)\int_s^T \mathbb{E}\Big(\sup_{s\leq t\leq r}\big|b(X_t) - b\big(\phi_{(s,t]}(x)\big)\big|^2\Big)\,\mathrm{d}r\\
&\quad\leq \big(12\|h_\nu\|_\infty + 3\big(C'\big)^2\big)(T-s) + 3\big\|b'\big\|_\infty^2\int_s^T \mathbb{E}\Big(\sup_{s\leq t\leq r}\big|Z_t^P\big|^2\Big)\,\mathrm{d}r.
\end{aligned}$$

The result follows by Grönwall's Lemma. Similarly

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_1\le t\le t_2}\big|Z_t^P - Z_{t_1}^P\big|^2\Big)\\
&\quad\le \big(12\|h_\nu\|_\infty + 3\big(C'\big)^2\big)(t_2-t_1)\\
&\qquad + 3\big\|b'\big\|_\infty^2\int_{t_1}^{t_2}\mathbb{E}\big(\big|Z_{t_1}^P\big|^2\big)\,\mathrm{d}r\\
&\qquad + 3\big\|b'\big\|_\infty^2\int_{t_1}^{t_2}\mathbb{E}\Big(\sup_{t_1\le t\le r}\big|Z_t^P - Z_{t_1}^P\big|^2\Big)\,\mathrm{d}r\\
&\quad\le \big(12\|h_\nu\|_\infty + 3\big(C'\big)^2 + 3\big\|b'\big\|_\infty^2\mathbb{E}\big(\big|Z_{t_1}^P\big|^2\big)\big)(t_2-t_1)\\
&\qquad + 3\big\|b'\big\|_\infty^2\int_{t_1}^{t_2}\mathbb{E}\Big(\sup_{t_1\le t\le r}\big|Z_t^P - Z_{t_1}^P\big|^2\Big)\,\mathrm{d}r.
\end{aligned}$$

Again, the result follows by Grönwall's Lemma. □

Theorem 6. *As $d \to 0$, the processes $Z_t^P \to Z_t$ in distribution.*

Proof. For simplicity, let $s = 0$, and $x_t = \phi_{(0,t]}(x)$.

Define ψ_t to be the solution to the linear ordinary differential equation

$$\dot{\psi}_t = -b'(x_t)\psi_t, \quad \psi_0 = 1.$$

We shall show that $(\psi_t Z_t^P)_{t\ge s} \to (\psi_t Z_t)_{t\ge s}$ in distribution as $d \to 0$. Since the processes $(Z_t^P)_{t\ge s}$ are tight, and $(\psi_t)_{t\ge s}$ is bounded away from 0 on compact intervals, the result will follow.

By Itô's formula,

$$\psi_t Z_t = \int_0^t \psi_s\sqrt{h_\nu(x_s)}\,\mathrm{d}B_s \sim N\biggl(0, \int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\biggr).$$

Hence Z_t is a Gaussian process. Similarly

$$\psi_t Z_t^P = \big(\mathrm{lcap}(P)\rho(P)\big)^{1/2}\int_0^t \psi_s\,\mathrm{d}M_s + \int_0^t R_s^P\,\mathrm{d}s,$$

where

$$R_t^P = \big(\mathrm{lcap}(P)\rho(P)\big)^{1/2}\psi_t\big(\mathrm{lcap}(P)^{-1}\beta_\nu(X_t) - b(x_t) - b'(x_t)(X_t - x_t)\big).$$

Using the bounds on Z_t^P established above, it is straightforward to show that

$$\int_0^t R_s^P\,\mathrm{d}s \to 0$$

in probability. Therefore it suffices to show that

$$\big(\mathrm{lcap}(P)\rho(P)\big)^{1/2}\int_0^t \psi_s\,\mathrm{d}M_s \to N\biggl(0, \int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\biggr)$$

in distribution.

In order to show that the characteristic function

$$\chi(\eta) = \mathbb{E}\biggl(\exp\biggl(\mathrm{i}\eta\big(\mathrm{lcap}(P)\rho(P)\big)^{1/2}\int_0^t \psi_s\,\mathrm{d}M_s\biggr)\biggr)$$

converges to

$$\exp\biggl(-\frac{1}{2}\eta^2\int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\biggr),$$

it is helpful to define

$$\begin{aligned}\zeta(\theta,x) &= \int_0^1 \bigl(\mathrm{e}^{\mathrm{i}\theta\tilde{\gamma}_P(x-z)} - 1 - \mathrm{i}\theta\tilde{\gamma}_P(x-z)\bigr)h_\nu(z)\operatorname{lcap}(P)^{-1}\,\mathrm{d}z \\ &= -\frac{\theta^2}{\operatorname{lcap}(P)}\int_0^1\int_0^1 \tilde{\gamma}_P(x-z)^2(1-r)\mathrm{e}^{\mathrm{i}r\theta\tilde{\gamma}_P(x-z)}h_\nu(z)\,\mathrm{d}r\,\mathrm{d}z \\ &= -\frac{\theta^2}{2\operatorname{lcap}(P)\rho(P)}\rho(P)\int_0^1 \tilde{\gamma}_P(x-z)^2 h_\nu(z)\,\mathrm{d}z \\ &\quad - \frac{\theta^2}{\operatorname{lcap}(P)}\int_0^1 (1-r)\int_0^1 \tilde{\gamma}_P(x-z)^2\bigl(\mathrm{e}^{\mathrm{i}r\theta\tilde{\gamma}_P(x-z)}-1\bigr)h_\nu(z)\,\mathrm{d}z\,\mathrm{d}r.\end{aligned}$$

By an extension of Itô's formula (see, for example, [16], p. 521), the process

$$N_t(\theta) = \exp\biggl(\mathrm{i}\theta\int_0^t \psi_s\,\mathrm{d}M_s - \int_0^t \zeta(\theta\psi_s, X_s)\,\mathrm{d}s\biggr), \quad t \ge 0,$$

is a martingale. Furthermore, again using (15) and the same kind of arguments as in the proof of Proposition 2, we find that

$$\begin{aligned}&\biggl|\rho(P)\int_0^1 \tilde{\gamma}_P(x-z)^2 h_\nu(z)\,\mathrm{d}z - h_\nu(x)\biggr| \\ &\quad\le \rho(P)\int_{-1/2}^{1/2} \tilde{\gamma}_P(x-z)^2\bigl|h_\nu(z)-h_\nu(x)\bigr|\,\mathrm{d}z \\ &\quad\le \|h_\nu'\|_\infty\rho(P)\int_{-1/2}^{1/2} \tilde{\gamma}_P(x-z)^2|x-z|\,\mathrm{d}z \\ &\quad= \|h_\nu'\|_\infty\rho(P)\biggl(\int_{-cd}^{cd} \tilde{\gamma}_P(z)^2|z|\,\mathrm{d}z + \int_{cd}^{1/2}\tilde{\gamma}_P(z)^2 z\,\mathrm{d}z + \int_{-1/2}^{-cd}\tilde{\gamma}_P(z)^2 z\,\mathrm{d}z\biggr) \\ &\quad\le \|h_\nu'\|_\infty\rho(P)\biggl(2(cd)^4 + \frac{2\operatorname{lcap}(P)^2}{\pi^4}\bigl(\pi cd\cot(\pi cd) + \bigl|\log\sin(\pi cd)\bigr|\bigr)\biggr) \\ &\quad\to 0\end{aligned}$$

as $d \to 0$, and

$$\biggl|\int_0^1 \tilde{\gamma}_P(x-z)^2\bigl(\mathrm{e}^{\mathrm{i}r\theta\tilde{\gamma}_P(x-z)}-1\bigr)h_\nu(z)\,\mathrm{d}z\biggr| \le \|h_\nu\|_\infty|\theta|\|\tilde{\gamma}_P\|_\infty\rho(P)^{-1} \le \|h_\nu\|_\infty|\theta|c'd\rho(P)^{-1}.$$

Hence

$$\zeta\bigl(\eta\bigl(\operatorname{lcap}(P)\rho(P)\bigr)^{1/2}\psi_s, X_s\bigr) \to -\frac{1}{2}\eta^2\psi_s^2 h_\nu(x_s)$$

in probability as $d \to 0$. Therefore, $N_t((\operatorname{lcap}(P)\rho(P))^{1/2}\eta)$ is bounded as $d \to 0$ and integrates to 1. Hence

$$\chi(\eta) - \exp\biggl(-\frac{1}{2}\eta^2\int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\biggr)$$

$$= \mathbb{E}\left(N_t\left(\left(\mathrm{lcap}(P)\rho(P)\right)^{1/2}\eta\right)\left(\exp\int_0^t \zeta\left(\eta\left(\mathrm{lcap}(P)\rho(P)\right)^{1/2}\psi_s, X_s\right)\mathrm{d}s - \exp\left(-\frac{1}{2}\eta^2\int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\right)\right)\right)$$
$$\to 0,$$

and so

$$\left(\mathrm{lcap}(P)\rho(P)\right)^{1/2}\int_0^t \psi_s\,\mathrm{d}M_s \to N\left(0, \int_0^t \psi_s^2 h_\nu(x_s)\,\mathrm{d}s\right)$$

in distribution, as required.

Since the processes $((\mathrm{lcap}(P)\rho(P))^{1/2}\int_0^t \psi_s\,\mathrm{d}M_s)_{t\geq 0}$ and $(\psi_t Z_t)_{t\geq 0}$ have independent increments, the finite dimensional distributions of $(\psi_t Z_t^P)_{t\geq 0}$ converge to those of $(\psi_t Z_t)_{t\geq 0}$. The result follows by the tightness of the processes $(\psi_t Z_t^P)_{t\geq 0}$, which is a consequence of the tightness of $(Z_t^P)_{t\geq 0}$. □

4.2. *The uniform case*

In the case of non-uniform ν, the behaviour of the boundary flow $(X_t)_{t\geq s}$ is dominated by non-trivial deterministic drift behaviour, and the random fluctuations only contribute as lower order perturbations. In the case when ν is the uniform measure on $[0, 1)$, however, the drift vanishes and the random fluctations describe the highest order behaviour. This case is explored in detail in [20], where it is shown that under suitable scaling, the boundary flow converges to the coalescing Brownian flow described in Section 2.2.

A similar result holds for HL(0) clusters constructed with random diameters. The proofs are straightforward adaptations of those in [20] and are therefore omitted. We summarize the main theorems below.

For law σ with finite third moment σ_3, define $\rho(\sigma)$ by

$$\rho(\sigma)\int_0^\infty\int_0^1 \tilde{\gamma}_{P(d)}(x)^2\,\mathrm{d}x\,\mathrm{d}\sigma(d) = 1.$$

Note that $\rho(\sigma)$ is well defined and $\rho(\sigma) \asymp \sigma_3^{-1}$.

Recall the construction of the flow $(\Gamma_I\colon I \subseteq [0,\infty))$ from Section 2.2 (with the drift compensated for), but constructed from particles with random diameters with law σ, and with rate $\rho(P)$ replaced by $\rho(\sigma)$. Let $X \in D^\circ$ be a lifting of Γ onto the real line. Then for fixed $e = (s, x) \in [0,\infty) \times \mathbb{R}$, $X_t = X_{(s,t]}(x)$ satisfies the integral equation

$$X_t = x + \int_{(s,t]\times(0,\infty)\times[0,1)} \tilde{\gamma}_{P(d)}(X_{r-} - z)\mu(\mathrm{d}r, \mathrm{d}d, \mathrm{d}z), \quad t \geq s,$$

where μ is a Poisson random measure of intensity $\rho(\sigma)h_\nu(z)\,\mathrm{d}z\,\mathrm{d}\sigma(d)\,\mathrm{d}r$.

Write μ_e^σ for the distribution of $(X_t)_{t\geq s}$ on the Skorokhod space $D_e = D_x([s,\infty),\mathbb{R})$ of cadlag paths starting from x at time s. Write μ_e for the distribution on D_e of a standard Brownian motion starting from e.

By a straightforward adaptation of Theorem 6, $\mu_e^\sigma \to \mu_e$ weakly on D_e as $\sigma_3 \to 0$.

Recall the definitions of E, D_E, μ_E from Section 2.2.

Proposition 4. *We have $\mu_E^\sigma \to \mu_E$ weakly on D_E as $\sigma_3 \to 0$.*

As observed above, X is a D°-valued random variable. Let μ_A^σ denote the law of X on the Borel σ-algebra of D°. Then, as in [20], the following results hold.

Theorem 7. *We have $\mu_A^\sigma \to \mu_A$ weakly on D° as $\sigma_3 \to 0$.*

Corollary 8. *Let $x_1, \ldots, x_n$ be a positively oriented set of points in $\mathbb{R}/\mathbb{Z}$ and set $x_0 = x_n$. Set $K_t = K_{\lfloor\rho(\sigma)t\rfloor}$. For $k = 1, \ldots, n$, write ω_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k. Let $(B_t^1, \ldots, B_t^n)_{t\geq 0}$ be a family of coalescing Brownian motions in $\mathbb{R}/\mathbb{Z}$ starting from $(x_1, \ldots, x_n)$. Then, in the limit $\sigma_3 \to 0$, $(\omega_t^1, \ldots, \omega_t^n)_{t\geq 0}$ converges weakly in $D([0,\infty), [0,1]^n)$ to $(B_t^1 - B_t^0, \ldots, B_t^n - B_t^{n-1})_{t\geq 0}$.*

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