

# Scaling of a random walk on a supercritical contact process

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**Abstract.** We prove a strong law of large numbers for a one-dimensional random walk in a dynamic random environment given by a supercritical contact process in equilibrium. The proof uses a coupling argument based on the observation that the random walk eventually gets trapped inside the union of space–time cones contained in the infection clusters generated by single infections. In the case where the local drifts of the random walk are smaller than the speed at which infection clusters grow, the random walk eventually gets trapped inside a single cone. This in turn leads to the existence of regeneration times at which the random walk forgets its past. The latter are used to prove a functional central limit theorem and a large deviation principle under the annealed law.

The qualitative dependence of the asymptotic speed and the volatility on the infection parameter is investigated, and some open problems are mentioned.

**Résumé.** Nous prouvons une loi forte des grands nombres pour une marche aléatoire dans un milieu aléatoire dynamique donné par un processus de contact sur-critique unidimensionnel en équilibre. La preuve utilise un argument de couplage basé sur l'observation que la marche est finalement confinée dans l'union de cônes spatio-temporels inclus dans les clusters d'infection générés par des infections individuelles. Si les taux locaux de saut de la marche sont plus petits que la vitesse de propagation de l'infection, la marche est finalement confinée dans un seul cône, ce qui entraîne l'existence de temps de régénération en lesquels la marche oublie son passé. Ces temps de régénération sont utilisés pour prouver un théorème central limite fonctionnel et un principe de grandes déviations sous la loi “annealed.”

La dépendance de la vitesse et de la variance asymptotiques par rapport au paramètre d'infection est étudiée, et quelques problèmes ouverts sont mentionnés.

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*Keywords:* Random walk; Dynamic random environment; Contact process; Strong law of large numbers; Functional central limit theorem; Large deviation principle; Space–time cones; Clusters of infections; Coupling; Regeneration times

## 1. Introduction

### 1.1. Background, motivation and outline

#### *Background*

A random walk in a dynamic random environment on  $\mathbb{Z}^d$ ,  $d \geq 1$ , is a random process where a “particle” makes random jumps with transition rates that depend on its location and themselves evolve with time. A typical example is when the dynamic random environment is given by an interacting particle system

$$\xi = (\xi_t)_{t \geq 0} \quad \text{with } \xi_t = \{\xi_t(x) : x \in \mathbb{Z}^d\} \in \Omega, \quad (1.1)$$

where  $\Omega$  is the configuration space, and  $\xi_0$  is typically drawn from equilibrium. In the case where  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ , the configurations can be thought of as consisting of “particles” and “holes.” Given  $\xi$ , run a random walk  $W = (W_t)_{t \geq 0}$

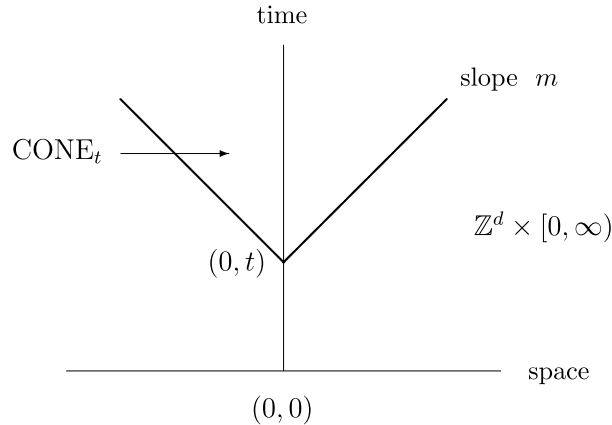


Fig. 1. The cone defined in (1.2).

on  $\mathbb{Z}^d$  that jumps at a fixed rate, but uses different transition kernels on a particle and on a hole. The key question is: What are the scaling properties of  $W$  and how do these properties depend on the law of  $\xi$ ?

The literature on random walks in dynamic random environments is still modest (for a recent overview, see Avena [1], Chapter 1). In Avena, den Hollander and Redig [4] a strong law of large numbers (SLLN) was proved for a class of interacting particle systems satisfying a mild space–time mixing condition, called *cone-mixing*. Roughly speaking, this is the requirement that for every  $m > 0$  all states inside the space–time cone (see Figure 1)

$$\text{CONE}_t := \{(x, s) \in \mathbb{Z}^d \times [t, \infty) : \|x\| \leq m(s - t)\}, \quad (1.2)$$

are conditionally independent of the states at time zero in the limit as  $t \rightarrow \infty$ . The proof of the SLLN uses a *regeneration-time* argument. Under a cone-mixing condition involving multiple cones, a functional central limit theorem (FCLT) can be derived as well, and under monotonicity conditions also a large deviation principle (LDP).

Many interacting particle systems are cone-mixing, including spin-flip systems with spin-flip rates that are weakly dependent on the configuration, e.g. the stochastic Ising model above the critical temperature. However, also many interacting particle systems are not cone-mixing, including independent simple random walks, the exclusion process, the contact process and the voter model. Indeed, these systems have *slowly decaying space–time correlations*. For instance, in the exclusion process particles are conserved and cannot sit on top of each other. Therefore, if at time zero there are particles everywhere in the box  $[-t^2, t^2] \cap \mathbb{Z}^d$ , then these particles form a “large traffic jam around the origin.” This traffic jam will survive up to time  $t$  with a probability tending to 1 as  $t \rightarrow \infty$ , and will therefore affect the states near the tip of  $\text{CONE}_t$ . Similarly, in the contact process, if at time zero there are no infections in the box  $[-t^2, t^2] \cap \mathbb{Z}^d$ , then no infections will be seen near the tip of  $\text{CONE}_t$  as well.

### Motivation

Several attempts have been made to extend the SLLN to interacting particle systems that are not cone-mixing, with partial success. Examples include: independent simple random walks (den Hollander, Kesten and Sidoravicius [8]) and the exclusion process (Avena, dos Santos and Völlering [5], Avena [2]). The present paper considers the *supercritical contact process*. We exploit the graphical representation, which allows us to simultaneously couple all realizations of the contact process starting from different initial configurations. This coupling in turn allows us to first prove the SLLN when the initial configuration is “all infected” (with the help of a subadditivity argument), and then show that the same result holds when the initial configuration is drawn from equilibrium. The main idea is to use the coupling to show that configurations agree in large space–time cones containing the infection clusters generated by single infections and that the random walk eventually gets trapped inside the *union* of these cones.

Under the assumption that the local drifts of the random walk are smaller than the speed at which infection clusters grow, the random walk eventually gets trapped inside a *single* cone. We show that this implies the existence of *regeneration times* at which the random walk “forgets its past.” The latter in turn allow us to prove the FCLT and the LDP.

It is typically difficult to obtain information about the speed in the SLLN, the volatility in the FCLT and the rate function in the LDP. In general, these are non-trivial functions of the parameters in the model, a situation that is well known from the literature on random walks in static random environments (for overviews, see Sznitman [15] and Zeitouni [17]). The reason is that these quantities depend on the *environment process* (i.e., the process of environments as seen from the location of the walk), which is typically hard to analyze. For the supercritical contact process we are able to derive a few qualitative properties as a function of the infection parameter, but it remains a challenge to obtain a full quantitative description.

A model of a random walk on the infinite cluster of supercritical oriented percolation (the discrete-time analogue of the contact process) is treated in Birkner, Černý, Depperschmidt and Gantert [7], where a SLLN and a quenched and annealed CLT are obtained. This model can be viewed as a random walk in a dynamic random environment, but it has non-elliptic transition probabilities different from the ones we consider here, because the random walk is confined to the infinite cluster.

### Outline

In Section 1.2 we define the model. In Section 1.3 we state our main results: two theorems claiming the SLLN, the FCLT and the LDP under appropriate conditions on the model parameters. In Section 1.4 we mention some open problems. The proofs of the theorems are given in Sections 3, 5, and 6. Sections 2 and 4 contain preparatory work.

### 1.2. Model

In this paper we consider the case where the dynamic random environment is the one-dimensional linear *contact process*  $\xi = (\xi_t)_{t \geq 0}$ , i.e., the spin-flip system on  $\Omega := \{0, 1\}^{\mathbb{Z}}$  with local transition rates given by

$$\eta \rightarrow \eta^x \quad \text{with rate} \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda\{\eta(x-1) + \eta(x+1)\} & \text{if } \eta(x) = 0, \end{cases} \quad (1.3)$$

where  $\lambda \in (0, \infty)$  and  $\eta^x$  is defined by  $\eta^x(y) := \eta(y)$  for  $y \neq x$ ,  $\eta^x(x) := 1 - \eta(x)$ . We call a site *infected* when its state is 1, and *healthy* when its state is 0. See Liggett [11], Chapter VI, for proper definitions.

The empty configuration  $\mathbf{0} \in \Omega$ , given by  $\mathbf{0}(x) = 0$  for all  $x \in \mathbb{Z}$ , is an absorbing state for  $\xi$ , while the full configuration  $\mathbf{1} \in \Omega$ , given by  $\mathbf{1}(x) = 1$  for all  $x \in \mathbb{Z}$ , evolves towards an equilibrium measure  $\nu_\lambda$ , called the “upper invariant measure,” that is stationary and ergodic under space-shifts. All equilibrium measures are convex combinations of  $\delta_{\mathbf{0}}$  and  $\nu_\lambda$ , and there is a critical threshold  $\lambda_c \in (0, \infty)$  such that: (1) for  $\lambda \in (0, \lambda_c]$ ,  $\nu_\lambda = \delta_{\mathbf{0}}$ ; (2) for  $\lambda \in (\lambda_c, \infty)$ ,  $\rho_\lambda := \nu_\lambda(\eta(0) = 1) > 0$ . It is known that  $\nu_\lambda$  has exponentially decaying correlations, and that  $\lambda \mapsto \rho_\lambda$  is continuous and non-decreasing with  $\lim_{\lambda \rightarrow \infty} \rho_\lambda = 1$ .

For a fixed realization of  $\xi$ , we define the random walk  $W := (W_t)_{t \geq 0}$  as the time-inhomogeneous Markov process on  $\mathbb{Z}$  that, given  $W_t = x$ , jumps to

$$\begin{aligned} x+1 & \quad \text{at rate } \alpha_1 \xi_t(x) + \alpha_0 [1 - \xi_t(x)], \\ x-1 & \quad \text{at rate } \beta_1 \xi_t(x) + \beta_0 [1 - \xi_t(x)], \end{aligned} \quad (1.4)$$

where  $\alpha_i, \beta_i \in (0, \infty)$ ,  $i = 0, 1$ . Letting

$$\begin{aligned} v_+ & := \alpha_1 \vee \alpha_0 - \beta_1 \wedge \beta_0, \\ v_- & := \alpha_1 \wedge \alpha_0 - \beta_1 \vee \beta_0, \end{aligned} \quad (1.5)$$

one can see that  $W$  may be coupled to two homogeneous random walks  $W^{(\pm)}$  with respective drifts  $v_\pm$  in such a way that  $W_t^{(-)} \leq W_t \leq W_t^{(+)}$  for all  $t \geq 0$ .

### 1.3. Theorems

For a probability measure  $\mu$  on  $\Omega$ , let  $\mathbb{P}_\mu$  denote the joint law of  $W$  and  $\xi$  when the latter is started from  $\mu$ . For our first theorem, we will make the following two assumptions on the jump rates:

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1, \quad (1.6)$$

and

$$v_1 > v_0 \quad \text{where } v_i := \alpha_i - \beta_i, i \in \{0, 1\}, \tag{1.7}$$

i.e., we assume that the total jump rate is constant, while the drift to the right is larger on infected sites than on healthy sites. The latter is made without loss of generality: since the contact process is invariant under reflection in the origin,  $-W$  has the same law as  $W$  with inverted jump rates. Observe that, under (1.6)–(1.7),  $v_1 = v_+$  and  $v_0 = v_-$ .

Our SLLN reads as follows.

**Theorem 1.1.** *Suppose that (1.6)–(1.7) hold.*

- (a) *For every  $\lambda \in (\lambda_c, \infty)$  there exists a  $v(\lambda) \in [v_0, v_1]$  such that, for any probability measure  $\mu$  on  $\Omega$  that is stochastically larger than a non-trivial shift-invariant and ergodic probability measure,*

$$\lim_{t \rightarrow \infty} t^{-1} W_t = v(\lambda) \quad \mathbb{P}_\mu\text{-a.s. and in } L^p, p \geq 1. \tag{1.8}$$

*In particular, (1.8) holds for  $\mu = \nu_\lambda$ .*

- (b) *The function  $\lambda \mapsto v(\lambda)$  is non-decreasing and right-continuous on  $(\lambda_c, \infty)$ , with  $v(\lambda) \in (v_0, v_1)$  for all  $\lambda \in (\lambda_c, \infty)$  and  $\lim_{\lambda \rightarrow \infty} v(\lambda) = v_1$ .*

By “non-trivial probability measure” we mean one different from  $\delta_0$ .

We note in passing that if  $\lambda \in (0, \lambda_c)$ , then  $\xi_t$  agrees with  $\mathbf{0}$  on an interval that grows exponentially fast in  $t$  irrespective of the initial configuration (Liggett [11], Chapter VI), and so it is trivial to deduce that  $W$  satisfies the SLLN with  $v(\lambda) = v_0$ .

For our second theorem, we will impose a different condition involving the jump rates and the infection parameter, which also implies the SLLN and, additionally, a FCLT and an LDP. This condition is  $\lambda \in (\lambda_W, \infty)$  with

$$\lambda_W := \inf\{\lambda \in (\lambda_c, \infty) : \iota(\lambda) > |v_-| \vee |v_+|\}. \tag{1.9}$$

Here,  $\lambda \mapsto \iota(\lambda)$  is the infection propagation speed (see (2.4) in Section 2.1), which is known to be continuous, strictly positive and strictly increasing on  $(\lambda_c, \infty)$ , with  $\lim_{\lambda \downarrow \lambda_c} \iota(\lambda) = 0$  and  $\lim_{\lambda \rightarrow \infty} \iota(\lambda) = \infty$ .

**Theorem 1.2.**

- (a) *For every  $\lambda \in (\lambda_W, \infty)$  there exists a  $v(\lambda) \in [v_-, v_+]$  such that (1.8) holds under  $\mathbb{P}_{\nu_\lambda}$  and a  $\sigma(\lambda) \in (0, \infty)$  such that*

$$\left( \frac{W_{nt} - v(\lambda)nt}{\sigma(\lambda)\sqrt{n}} \right)_{t \geq 0} \implies (B_t)_{t \geq 0} \quad \text{as } n \rightarrow \infty, \tag{1.10}$$

*where  $B$  is standard Brownian motion and  $\implies$  denotes weak convergence under  $\mathbb{P}_{\nu_\lambda}$  in the Skorohod topology.*

- (b) *The functions  $\lambda \mapsto v(\lambda)$  and  $\lambda \mapsto \sigma(\lambda)$  are continuous on  $(\lambda_W, \infty)$ .*
- (c) *If (1.6)–(1.7) hold, then for every  $\lambda \in (\lambda_W, \infty)$ , the laws of  $(t^{-1}W_t)_{t > 0}$  satisfy under  $\mathbb{P}_{\nu_\lambda}$  the large deviation principle on  $\mathbb{R}$  with a finite and convex rate function that has a unique zero at  $v(\lambda)$ .*

The intuitive reason why the rate function has a unique zero is that deviations of the empirical speed in the: (i) upward direction require a density of infected sites larger than  $\rho_\lambda$ , which is costly because infections become healthy independently of the states at the other sites; (ii) downward direction require a density of infected sites smaller than  $\rho_\lambda$ , which is costly because infection clusters grow at a linear speed and rapidly fill up healthy intervals everywhere.

#### 1.4. Discussion

1. Under (1.6)–(1.7), it is natural to expect that  $\lambda \mapsto v(\lambda)$  is continuous and strictly increasing on  $(\lambda_c, \infty)$  with  $\lim_{\lambda \downarrow \lambda_c} v(\lambda) = v_0$ . Figure 2 shows a qualitative plot of the speed in that setting. If  $0 \in (v_0, v_1)$ , then there is a critical threshold  $\lambda^* \in (\lambda_c, \infty)$  at which the speed changes sign. It is natural to ask whether  $\lambda \mapsto v(\lambda)$  is concave on  $(\lambda_c, \infty)$  and Lipschitz at  $\lambda_c$ .

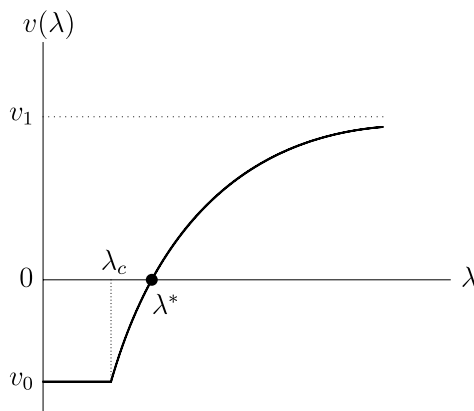


Fig. 2. Qualitative plot of  $\lambda \mapsto v(\lambda)$  when  $0 \in (v_0, v_1)$ .

2. We know that  $W$  is transient when  $v(\lambda) \neq 0$ . Is  $W$  recurrent when  $v(\lambda) = 0$ ?
3. We expect the condition  $\lambda > \lambda_W$  to be redundant. Moreover, we expect that for every  $\lambda \in (\lambda_c, \infty)$  the *environment process* (i.e., the process of environments as seen from the location of the random walk) has a unique and non-trivial equilibrium measure that is absolutely continuous with respect to  $\nu_\lambda$ .
4. Theorem 1.1 can be extended to arbitrary initial configurations that have a “strictly positive lower density” (see Remark 3.7 in Section 3.2 below). Also, Theorem 1.1 remains valid for  $\mu$  stochastically larger than  $\nu_\lambda$  even when some of the jump rates  $\alpha_i, \beta_i, i \in \{0, 1\}$ , are equal to zero (see Remark 3.4 in Section 3.2 below).
5. Theorem 1.2(a) can be extended (with the same mean and variance) to arbitrary initial configurations containing infinitely many infections, while Theorem 1.2(c) can be extended (with a different rate function) to any initial measure that has positive correlations and is stochastically larger than a non-trivial Bernoulli product measure (see Remark 6.3 in Section 6.1 below).
6. Theorems 1.1–1.2 can presumably be extended to  $\mathbb{Z}^d$  with  $d \geq 2$ . Also in higher dimensions single infections create infection clusters that grow at a linear speed (i.e., asymptotically form a ball with a linearly growing radius). The construction of the regeneration times when  $\lambda \in (\lambda_W, \infty)$ , with  $\lambda_W$  the analogue of (1.9), appears to be possible.
7. It would be interesting to extend Theorems 1.1–1.2 to *multi-type contact processes*. On each type  $i$  the random walk has transition rates  $\alpha_i, \beta_i$  such that  $\alpha_i + \beta_i = \gamma$  for all  $i$ . As long as the dynamics is monotone and  $i \mapsto v_i$  is non-decreasing, many of the arguments in the present paper carry over.

## 2. Construction

In Section 2.1 we construct the contact process, in Section 2.2 the random walk on top of the contact process.

### 2.1. Contact process

A càdlàg version of the contact process can be constructed from a graphical representation in the following fashion. Let  $:= (H(x))_{x \in \mathbb{Z}}$  and  $I := (I(x))_{x \in \mathbb{Z}}$  be two independent collections of i.i.d. Poisson processes with rates 1 and  $\lambda$ , respectively. On  $\mathbb{Z} \times [0, \infty)$ , draw the events of  $H(x)$  as crosses over  $x$  and the events of  $I(x)$  as two-sided arrows between  $x$  and  $x + 1$  (see Figure 3).

(The standard graphical representation uses Poisson processes of one-sided arrows to the right and to the left on every time line, each with rate  $\lambda$ . This gives the same dynamics.)

For  $x, y \in \mathbb{Z}$  and  $0 \leq s \leq t$ , we say that  $(x, s)$  and  $(y, t)$  are *connected*, written  $(x, s) \leftrightarrow (y, t)$ , if and only if there exists a nearest-neighbor path in  $\mathbb{Z} \times [0, \infty)$  starting at  $(x, s)$  and ending at  $(y, t)$ , going either upwards in time or sideways in space across arrows without hitting crosses. For  $x \in \mathbb{Z}$ , we define the cluster of  $x$  at time  $t$  by

$$C_t(x) := \{y \in \mathbb{Z}: (x, 0) \leftrightarrow (y, t)\}. \quad (2.1)$$

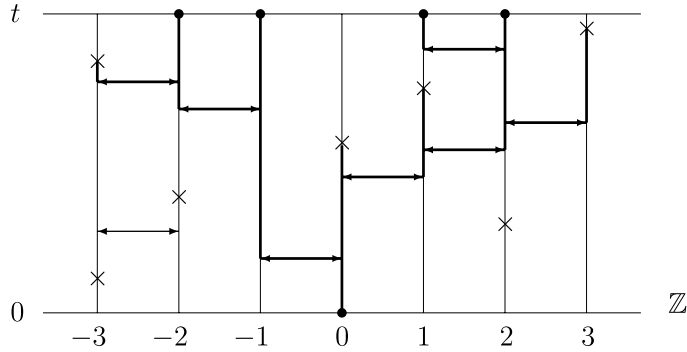


Fig. 3. Graphical representation. The crosses are events of  $H$  and the arrows are events of  $I$ . The thick lines cover the region that is infected when the initial configuration has a single infection at the origin.

For example, in Figure 3,  $C_t(0) = \{-2, -1, 1, 2\}$  and  $C_t(2) = \emptyset$ . Note that  $C_t(x)$  is a function of  $H$  and  $I$ .

For a fixed initial configuration  $\eta$ , we declare  $\xi_t(y) = 1$  if there exists an  $x$  such that  $y \in C_t(x)$  and  $\eta(x) = 1$ , and we declare  $\xi_t(y) = 0$  otherwise. Then  $\xi$  is adapted to the filtration

$$\mathcal{F}_t := \sigma(\xi_0, (H_s, I_s)_{s \in [0, t]}). \tag{2.2}$$

This construction allows us to *simultaneously couple* copies of the contact process starting from *all* configurations  $\eta \in \Omega$ . In the following we will write  $\xi(\eta)$  and  $\xi_t(\eta)(x)$  when we want to exhibit that the initial configuration is  $\eta$ .

We note two consequences of the graphical construction, stated in Lemmas 2.1–2.3 below. The first is the monotonicity of  $\eta \mapsto \xi(\eta)$ , the second concerns the state of the sites surrounded by the cluster of an infected site. The notation  $\eta \leq \eta'$  stands for  $\eta(x) \leq \eta'(x)$  for all  $x \in \mathbb{Z}$ .

**Lemma 2.1.** *If  $\eta \leq \eta'$ , then  $\xi_t(\eta) \leq \xi_t(\eta')$  for all  $t \geq 0$ .*

**Proof.** Immediate from the definition of  $\xi_t$  in terms of  $\eta$  and  $(C_t(x))_{x \in \mathbb{Z}}$ . □

For  $x \in \mathbb{Z}$ , define the left-most and the right-most site influenced by site  $x$  at time  $t$  as

$$\begin{aligned} L_t(x) &:= \inf C_t(x), \\ R_t(x) &:= \sup C_t(x), \end{aligned} \tag{2.3}$$

where  $\inf \emptyset = \infty$  and  $\sup \emptyset = -\infty$ . By symmetry, for any  $t \geq 0$ ,  $R_t(x) - x$  and  $x - L_t(x)$  have the same distribution, independently of  $x$ .

**Lemma 2.2.** *Fix  $x \in \mathbb{Z}$  and  $t \geq 0$ . If  $C_t(x) \neq \emptyset$  and  $y \in [L_t(x), R_t(x)] \cap \mathbb{Z}$ , then  $\eta \mapsto \xi_t(\eta)(y)$  is constant on  $\{\eta \in \Omega: \eta(x) = 1\}$ .*

**Proof.** It suffices to show that, under the conditions stated,  $\xi_t(\eta)(y) = 1$  if and only if  $y \in C_t(x)$ . The “if” part is obvious. For the “only if” part, note that if there is a  $z \neq x$  such that  $(z, 0) \leftrightarrow (y, t)$ , then any path realizing the connection must cross a path connecting  $(x, 0)$  to either  $(R_t(x), t)$  or  $(L_t(x), t)$ , so that  $(x, 0) \leftrightarrow (y, t)$  as well. □

If  $\xi_0 = \mathbb{1}_x$ , then  $R_t(x)$  and  $L_t(x)$  are, respectively, the right-most and the left-most infections present at time  $t$ . In particular, in this case the infection survives for all times if and only if  $R_t(x) - L_t(x) \geq 0$  for all  $t \geq 0$ . For  $\lambda \in (\lambda_c, \infty)$  it is well known that, given  $\xi_0 = \mathbb{1}_0$ , the infection survives with positive probability and there exists a constant  $\iota = \iota(\lambda) > 0$  such that, conditionally on survival,

$$\lim_{t \rightarrow \infty} t^{-1} R_t(0) = \iota, \quad \xi\text{-a.s.} \tag{2.4}$$

## 2.2. Random walk on top of contact process

The random walk  $W$  can be constructed as follows. Put  $\gamma := (\alpha_1 + \beta_1) \vee (\alpha_0 + \beta_0)$  and let  $N := (N_t)_{t \geq 0}$  be a Poisson process with rate  $\gamma$ . Denote by  $J := (J_k)_{k \in \mathbb{N}_0}$  its generalized inverse, i.e.,  $J_0 = 0$  and  $(J_{k+1} - J_k)_{k \in \mathbb{N}_0}$  are i.i.d.  $\text{EXP}(\gamma)$  random variables. Let  $U := (U_k)_{k \in \mathbb{N}}$  be an i.i.d. sequence of  $\text{UNIF}([0, 1])$  random variables, independent of  $N$ . Set  $S_0 := 0$  and, recursively for  $k \in \mathbb{N}_0$ ,

$$S_{k+1} := S_k + (1 - \xi_{J_{k+1}}(S_k))(\mathbb{1}_{\{0 \leq U_{k+1} \leq \alpha_0/\gamma\}} - \mathbb{1}_{\{\alpha_0/\gamma < U_{k+1} \leq (\alpha_0 + \beta_0)/\gamma\}}) + \xi_{J_{k+1}}(S_k)(\mathbb{1}_{\{0 \leq U_{k+1} \leq \alpha_1/\gamma\}} - \mathbb{1}_{\{\alpha_1/\gamma < U_{k+1} \leq (\alpha_1 + \beta_1)/\gamma\}}), \quad (2.5)$$

i.e.,  $S_{k+1} = S_k + 1$  with probability  $\alpha_i/\gamma$ ,  $S_{k+1} = S_k - 1$  with probability  $\beta_i/\gamma$  and  $S_{k+1} = S_k$  with probability  $1 - (\alpha_i + \beta_i)/\gamma$  when  $\xi_{J_{k+1}}(S_k) = i$ , for  $i = 0, 1$ . Setting

$$W_t := S_{N_t}, \quad (2.6)$$

we can use the right-continuity of  $\xi$  to verify that  $W$  indeed is a Markov process with the correct jump rates.

Under (1.6)–(1.7), the above construction has the useful property of being monotone in the environment, in the following sense. For two dynamic random environments  $\xi$  and  $\xi'$ , we say that  $\xi \leq \xi'$  when  $\xi_t \leq \xi'_t$  for all  $t \geq 0$ . Writing  $W = W(\xi)$  in the previous construction (i.e., exhibiting  $W$  as a function of  $\xi$ ), it is easy to verify using (2.5) that, under (1.6)–(1.7),

$$\xi \leq \xi' \implies W_t(\xi) \leq W_t(\xi') \quad \forall t \geq 0. \quad (2.7)$$

We denote by

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma((N_s)_{s \in [0, t]}, (U_k)_{1 \leq k \leq N_t}) \quad (2.8)$$

the filtration generated by all the random variables that are used to define the contact process  $\xi$  and the random walk  $W$ .

## 3. The strong law of large numbers

Theorem 1.1(a) is proved in two steps. In Section 3.1 we use subadditivity to prove the SLLN when  $\xi$  starts from  $\delta_{\mathbf{1}}$ . In Section 3.2 we show that, under the hypotheses stated about  $\mu$  in Theorem 1.1(a), we can couple two copies of  $\xi$  starting from  $\mu$  and  $\delta_{\mathbf{1}}$  so as to transfer the SLLN, with the same speed.

In the following, for a random process  $X = (X_t)_{t \in \mathcal{I}}$  with  $\mathcal{I} = \mathbb{R}$  or  $\mathcal{I} = \mathbb{Z}$ , we write

$$X_{[0, t]} := (X_s)_{s \in [0, t] \cap \mathcal{I}}. \quad (3.1)$$

### 3.1. Starting from the full configuration: Subadditivity

Since  $\eta \leq \mathbf{1}$  for all  $\eta \in \Omega$ , it follows from (2.7) and Lemma 2.1 that  $W_t(\xi(\eta)) \leq W_t(\xi(\mathbf{1}))$  for all  $t \geq 0$ . Therefore, if in the graphical construction we replace  $\xi_s$  by  $\mathbf{1}$  at any given time  $s$ , then the new increments after time  $s$  lie to the right of the old increments after time  $s$ , and are independent of the increments before time  $s$ . This leads us to a subadditivity argument, which we now formalize.

For  $n \in \mathbb{N}_0$ , let

$$\begin{aligned} H^{(n)} &= (H_t^{(n)}(x))_{t \geq 0, x \in \mathbb{Z}} := (H_{t+n}(x + W_n) - H_n(x + W_n))_{t \geq 0, x \in \mathbb{Z}}, \\ I^{(n)} &= (I_t^{(n)}(x))_{t \geq 0, x \in \mathbb{Z}} := (I_{t+n}(x + W_n) - I_n(x + W_n))_{t \geq 0, x \in \mathbb{Z}}, \\ N^{(n)} &= (N_t^{(n)})_{t \geq 0} := (N_{t+n} - N_n)_{t \geq 0}, \\ U^{(n)} &= (U_k^{(n)})_{k \in \mathbb{N}} := (U_{k+N_n})_{k \in \mathbb{N}}. \end{aligned} \quad (3.2)$$

Then, for any  $n \in \mathbb{N}_0$ ,  $(H^{(n)}, I^{(n)}, N^{(n)}, U^{(n)})$  has the same distribution as  $(H, I, N, U)$  and is independent of

$$H_{[0, n-j]}^{(j)}, I_{[0, n-j]}^{(j)}, N_{[0, n-j]}^{(j)}, U_{[1, N_{n-j}^{(j)}]}^{(j)}, \quad 0 \leq j \leq n-1. \quad (3.3)$$

Abbreviate  $\xi = \xi(\eta, H, I)$  and  $W = W(\xi, N, U)$ . For  $n \in \mathbb{N}_0$ , let

$$\begin{aligned} \xi^{(n)} &:= \xi(\mathbf{1}, H^{(n)}, I^{(n)}), \\ W^{(n)} &:= W(\xi^{(n)}, N^{(n)}, U^{(n)}), \end{aligned} \quad (3.4)$$

and define the double-indexed sequence

$$X_{m,n} := W_{n-m}^{(m)}, \quad n, m \in \mathbb{N}_0, n \geq m. \quad (3.5)$$

**Lemma 3.1.** *The following properties hold:*

- (i) For all  $n, m \in \mathbb{N}_0, n \geq m$ :  $X_{0,n} \leq X_{0,m} + X_{m,n}$ .
- (ii) For all  $n \in \mathbb{N}_0$ :  $(X_{n,n+k})_{k \in \mathbb{N}_0}$  has the same distribution as  $(X_{0,k})_{k \in \mathbb{N}_0}$ .
- (iii) For all  $k \in \mathbb{N}$ :  $(X_{nk, (n+1)k})_{n \in \mathbb{N}_0}$  is i.i.d.
- (iv)  $\sup_{n \in \mathbb{N}} \mathbb{E}_{\delta_1} [n^{-1} |X_{0,n}|] < \infty$ .

**Proof.** (i) Fix  $n, m \in \mathbb{N}_0, n \geq m$  and define  $\hat{\xi} := \xi(\hat{\eta}, H^{(m)}, I^{(m)})$ , where  $\hat{\eta}(x) = \xi_m(x + W_m)$ . This is the contact process after time  $m$  as seen from  $W_m$ . Note that  $X_{0,n} - X_{0,m} = W_n - W_m = W_{n-m}(\hat{\xi}, N^{(m)}, U^{(m)})$ . Since  $\hat{\eta} \leq \mathbf{1}$ , it follows from (2.7) and Lemma 2.1 that the latter is  $\leq W_{n-m}(\xi^{(m)}, N^{(m)}, U^{(m)}) = W_{n-m}^{(m)}$ .

(ii) Immediate from the construction.

(iii) By definition,  $X_{nk, (n+1)k} = W_k(\xi^{(nk)}, N^{(nk)}, U^{(nk)})$ . By construction, for each  $t \geq 0$ ,  $W_t(\xi, N, U)$  is a function of  $N_{[0,t]}$ ,  $U_{[1, N_t]}$  and  $\xi_{[0,t]}$ , which in turn is a function of  $H_{[0,t]}$ ,  $I_{[0,t]}$  and  $\eta$ . Therefore  $X_{nk, (n+1)k}$  is equal to a (fixed) function of

$$H_{[0,k]}^{(nk)}, I_{[0,k]}^{(nk)}, N_{[0,k]}^{(nk)}, U_{[1, N_{(n+1)k}^{(nk)}]}^{(nk)}, \quad (3.6)$$

which are jointly i.i.d. in  $n$  (when  $k$  is fixed).

(iv) This follows from the fact that  $|W_t| \leq N_t$ . □

Lemma 3.1 allows us to prove the SLLN when  $\xi$  starts from  $\delta_1$ .

**Proposition 3.2.** *Let*

$$v(\lambda) := \inf_{n \in \mathbb{N}} \mathbb{E}_{\delta_1} [n^{-1} W_n], \quad \lambda \in [0, \infty). \quad (3.7)$$

Then

$$\lim_{t \rightarrow \infty} t^{-1} W_t = v(\lambda), \quad \mathbb{P}_{\delta_1}\text{-a.s. and in } L^p, p \geq 1. \quad (3.8)$$

**Proof.** Conditions (i)–(iv) in Lemma 3.1 allow us to apply the subadditive ergodic theorem of Liggett [12] (see also Liggett [11], Theorem VI.2.6) to the sequence  $(X_{0,n})_{n \in \mathbb{N}_0} = (W_n)_{n \in \mathbb{N}_0}$ , which gives  $\lim_{n \rightarrow \infty} n^{-1} W_n = v$   $\mathbb{P}_{\delta_1}$ -a.s. Via a standard argument this can subsequently be extended to  $(t^{-1} W_t)_{t \geq 0}$  by using that, for any  $n \in \mathbb{N}_0$ ,

$$\sup_{s \in [0,1]} |W_{n+s} - W_n| \leq N_{n+1} - N_n, \quad (3.9)$$

which implies that  $\lim_{t \rightarrow \infty} t^{-1} |W_t - W_{\lfloor t \rfloor}| = 0$  a.s. w.r.t.  $\mathbb{P}_{\delta_1}$ . Since  $|W_t| \leq N_t$ , we see that  $(t^{-p} |W_t|^p)_{t \geq 1}$  is uniformly integrable for any  $p \geq 1$ , so the convergence also holds in  $L^p$ . □



### 3.2. Other initial measures: Coupling

In this section, we show that whenever two copies of the contact process starting from  $\mu$  and  $\delta_{\mathbf{1}}$  can be coupled so as to agree with large probability at large times inside a space–time cone, the LLN holds also under  $\mathbb{P}_\mu$  with the same velocity  $v(\lambda)$ . We subsequently show that such a coupling is possible when  $\mu$  is stochastically larger than a non-trivial shift-invariant and ergodic measure. Some remarks regarding extensions are made after the corresponding results.

For  $m > 0$ , let

$$V_m := \{(x, s) \in \mathbb{Z} \times [0, \infty) : |x| \leq ms\} \tag{3.10}$$

be a cone of inclination  $m$  opening upwards in space–time.

**Proposition 3.3.** *Fix  $\lambda \in (0, \infty)$ , and let  $\xi^{(\mu)}$  and  $\xi^{(\mathbf{1})}$  denote the contact process started from  $\mu$  and  $\mathbf{1}$ , respectively. Suppose that there exists a coupling measure  $\mathbb{P}$  of  $\xi^{(\mu)}$  and  $\xi^{(\mathbf{1})}$  such that, for some  $m > |v_0| \vee |v_1|$ ,*

$$\lim_{T \rightarrow \infty} \mathbb{P}(\exists(x, t) \in V_m \cap \mathbb{Z} \times [T, \infty) : \xi_t^{(\mu)}(x) \neq \xi_t^{(\mathbf{1})}(x)) = 0. \tag{3.11}$$

Then

$$\lim_{t \rightarrow \infty} t^{-1} W_t = v(\lambda), \quad \mathbb{P}_\mu\text{-a.s. and in } L^p, p \geq 1, \tag{3.12}$$

where  $v(\lambda)$  is as in (3.7).

**Proof.** Let  $N^{(\mu)}, U^{(\mu)}$  and  $N^{(\mathbf{1})}, U^{(\mathbf{1})}$  be independent copies of  $N, U$  and rename  $\mathbb{P}$  to denote the joint law of  $\xi^{(a)}, N^{(a)}, U^{(a)}, a \in \{\mu, \mathbf{1}\}$ . Then  $W^{(\mu)} := W(\xi^{(\mu)}, N^{(\mu)}, U^{(\mu)})$  under  $\mathbb{P}$  has the same law as  $W$  under  $\mathbb{P}_\mu$ .

Denote by  $\bar{\mathbf{0}}, \bar{\mathbf{1}}$  the elements of  $\Omega^{[0, \infty)}$  that are constant and equal to 0, respectively, 1, i.e.,  $\bar{\mathbf{0}}_t(x) = 0$  for all  $(x, t) \in \mathbb{Z} \times [0, \infty)$  and analogously for  $\bar{\mathbf{1}}$ , and set

$$\bar{W}_s^{(\mu)} := |W_s(\bar{\mathbf{1}}, N^{(\mu)}, U^{(\mu)})| \vee |W_s(\bar{\mathbf{0}}, N^{(\mu)}, U^{(\mu)})|, \quad s \geq 0. \tag{3.13}$$

For  $T > 0$ , let

$$D_T := \{\bar{W}_s^{(\mu)} \leq ms \ \forall s \geq T\} \tag{3.14}$$

and

$$\Gamma_T := D_T \cap \{\xi_s^{(\mu)}(x) = \xi_s^{(\mathbf{1})}(x) \ \forall (x, s) \in V_m \cap \mathbb{Z} \times [T, \infty)\}. \tag{3.15}$$

For  $i \in \{0, 1\}$ ,  $W(\bar{\mathbf{i}}, N^{(\mu)}, U^{(\mu)})$  is a homogeneous random walk with total jump rate  $\gamma$  and drift  $v_i$ . Hence, by (3.11),

$$\lim_{T \rightarrow \infty} \mathbb{P}(\Gamma_T) = 1. \tag{3.16}$$

Therefore it suffices to prove that

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} t^{-1} W_t^{(\mu)} = v | \Gamma_T\right) = 1 \quad \forall T > 0. \tag{3.17}$$

In order to prove (3.17), we couple  $W^{(\mu)}$  with a random walk  $\widehat{W}$  distributed as  $W$  under  $\mathbb{P}_{\delta_{\mathbf{1}}}$ , as follows. Fix  $T > 0$  and let  $\widehat{N} = (\widehat{N}_s)_{s \geq 0}, \widehat{U} = (\widehat{U}_n)_{n \in \mathbb{N}}$  be defined by

$$\widehat{N}_t := \begin{cases} N_t^{(\mathbf{1})} & \text{if } t \leq T, \\ N_T^{(\mathbf{1})} + N_t^{(\mu)} - N_T^{(\mu)} & \text{otherwise,} \end{cases} \tag{3.18}$$

and

$$\widehat{U}_n := \begin{cases} U_n^{(1)} & \text{if } n \leq N_T^{(1)}, \\ U_n^{(\mu)} & \text{otherwise.} \end{cases} \tag{3.19}$$

Then it is clear that  $\widehat{W} := W(\xi^{(1)}, \widehat{N}, \widehat{U})$  has the correct law, and that  $\widehat{N}_{[0,T]}$ ,  $\widehat{U}_{[1,\widehat{N}_T]}$  are independent of  $W^{(\mu)}$ . Moreover, we claim that

$$\text{on } \Gamma_T: \quad W_T^{(\mu)} = \widehat{W}_T \implies W_s^{(\mu)} = \widehat{W}_s \quad \forall s \geq T. \tag{3.20}$$

To see (3.20), note that, by monotonicity,

$$W_s(\bar{\mathbf{0}}, N^{(\mu)}, U^{(\mu)}) \leq W_s^{(\mu)} \leq W_s(\bar{\mathbf{1}}, N^{(\mu)}, U^{(\mu)}) \quad \forall s \geq 0, \tag{3.21}$$

so that, on  $D_T$ ,  $(W_s^{(\mu)}, s) \in V_m$  for all  $s \geq T$ . Since  $W^{(\mu)}$  and  $\widehat{W}$  use the same jump decisions after time  $T$ , if they are equal at time  $T$  and  $\Gamma_T$  occurs, then they will see forever the same random environment, and will thus remain equal for all subsequent times.

With this observation, we are now ready to prove (3.17) by showing that

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} t^{-1} W_t^{(\mu)} = v \mid \Gamma_T, W_T^{(\mu)} = x\right) = 1 \tag{3.22}$$

for each  $x \in \mathbb{Z} \cap [-mT, mT]$ . To that end, first note that, for each fixed  $x$ , there exists an event  $B_x \in \sigma(\widehat{N}_{[0,T]}, \widehat{U}_{[1,\widehat{N}_T]})$  with positive probability such that  $\widehat{W}_T = x$  on  $B_x$ . Indeed, since all the jump rates  $\alpha_i, \beta_i, i \in \{0, 1\}$  are strictly positive, we can fix the number and direction of jumps of  $\widehat{W}$  on  $[0, T]$  by imposing restrictions on  $\widehat{N}_{[0,T]}, \widehat{U}_{[1,\widehat{N}_T]}$ . To conclude (3.22), we write

$$\begin{aligned} \mathbb{P}\left(\lim_{t \rightarrow \infty} t^{-1} W_t^{(\mu)} = v \mid \Gamma_T, W_T^{(\mu)} = x\right) &= \mathbb{P}\left(\lim_{t \rightarrow \infty} t^{-1} W_t^{(\mu)} = v \mid \Gamma_T, W_T^{(\mu)} = x, B_x\right) \\ &= \mathbb{P}\left(\lim_{t \rightarrow \infty} t^{-1} W_t^{(1)} = v \mid \Gamma_T, W_T^{(\mu)} = x, B_x\right) \\ &= 1, \end{aligned} \tag{3.23}$$

where for the last step we use Proposition 3.2. □

**Remark 3.4.** In the case  $\mu = \nu_\lambda$  (for which (3.11) holds by Proposition 3.5 below), the conclusion of Proposition 3.3 is true even when some of the jump rates  $\alpha_i, \beta_i, i \in \{0, 1\}$  are equal to zero (note that the proof of Proposition 3.2 does not need all rates to be strictly positive). To adapt the proof, replace the conditioning on  $W_T^{(\mu)} = x$  in (3.22) by the event  $\{N_T^{(\mu)} = 0\}$ , which implies  $W_T^{(\mu)} = 0$ . Then  $(W_{t+T}^{(\mu)} - W_T^{(\mu)})_{t \geq 0}$  under  $\mathbb{P}(\cdot \mid N_T^{(\mu)} = 0)$  has the same distribution as  $W$  under  $\mathbb{P}_{\nu_\lambda}$ . Since  $\widehat{N}_T = 0$  implies  $\widehat{W}_T = 0$  and has positive probability, the claim follows as before.

We next show that (3.11) is satisfied whenever  $\mu$  contains a non-trivial ergodic measure. This together with Proposition 3.3 finishes the proof of Theorem 1.1(a).

**Proposition 3.5.** *If  $\mu$  is stochastically larger than a non-trivial probability measure  $\bar{\mu}$  that is shift-invariant and ergodic, then (3.11) holds under the coupling given by the graphical representation discussed in Section 2.1.*

**Proof.** Let  $\xi^{(\mu)}, \xi^{(\bar{\mu})}$  and  $\xi^{(1)}$  be copies of the contact process started from the corresponding initial measures, constructed with the same graphical representation given by  $H, I$  and such that  $\xi_0^{(\bar{\mu})} \leq \xi_0^{(\mu)}$ . Since this coupling preserves the ordering, we have  $\xi_t^{(\bar{\mu})} \leq \xi_t^{(\mu)} \leq \xi_t^{(1)}$  for all  $t \geq 0$ , and so we may assume that  $\mu$  is non-trivial, shift-invariant and ergodic.

Denote by  $P$  the joint law of  $\xi_0^{(\mu)}$ ,  $H$ ,  $I$ . Regarding  $P$  as a law on the product space

$$(\{0, 1\} \times D(\mathbb{N}_0, [0, \infty))^{\mathbb{Z}} = \{0, 1\}^{\mathbb{Z}} \times (D(\mathbb{N}_0, [0, \infty))^2)^{\mathbb{Z}}, \tag{3.24}$$

where  $D(\mathbb{N}_0, [0, \infty))$  is the space of càdlàg functions from  $[0, \infty)$  to  $\mathbb{N}_0$ , we see that  $P$  is shift-ergodic because it is the product of probability measures that are shift-ergodic, namely,  $\mu$  and the distributions of  $H$  and  $I$ . Let

$$\Lambda_x := \{ \eta(x) = 1, (x - L_t(x)) \wedge (R_t(x) - x) \geq \lfloor (\iota/2)t \rfloor \forall t \geq 0 \}, \tag{3.25}$$

i.e., the event that  $x$  generates a “wide-spread infection” (moving at speed at least half the typical asymptotic speed  $\iota$ ). Since  $\Lambda_x$  is a translation of  $\Lambda_0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^n \mathbb{1}_{\Lambda_x} = P(\Lambda_0) = P(\xi_0(0) = 1)P(\Lambda_0 | \xi_0(0) = 1) := \varrho > 0, \quad P\text{-a.s.}, \tag{3.26}$$

where the last inequality is justified as follows:  $P(\xi_0(0) = 1) > 0$  since  $\mu$  is assumed to be non-trivial, and  $P(\Lambda_0 | \xi_0(0) = 1) > 0$  by (2.4) and local modifications of the graphical representation.

Next, for  $n \in \mathbb{N}$ , define  $Z_n \in \mathbb{N}$  by the equation

$$\sum_{x=1}^{Z_n} \mathbb{1}_{\Lambda_x} = n. \tag{3.27}$$

Then we also have

$$\lim_{n \rightarrow \infty} \frac{Z_n}{n} = \varrho^{-1}, \quad P\text{-a.s.} \tag{3.28}$$

$(Z_n)_{n \in \mathbb{N}}$  marks the positions of wide-spread infections to the right of the origin, i.e.,  $x > 0$  such that  $\Lambda_x$  occurs. Equation (3.28) means that these wide-spread infections are not too far apart. Extending the definition of  $Z_n$  to the negative integers, we obtain analogously that  $\lim_{n \rightarrow \infty} n^{-1}(-Z_{-n}) = \varrho^{-1}$   $P$ -a.s. Let  $\mathcal{Z} := \bigcup_{n \in \mathbb{N}} \{Z_n, Z_{-n}\}$  and

$$\mathcal{S} := \{(y, t) \in \mathbb{Z} \times [2/\iota, \infty) : \exists x \in \mathcal{Z} \text{ such that } |y - x| \leq (\iota/2)t - 1\}. \tag{3.29}$$

Then  $\mathcal{S}$  is the union of cones of inclination angle  $\iota/2$  with tips at  $(2/\iota, z)$  with  $z \in \mathcal{Z}$  (see Figure 4). We call  $\mathcal{S}$  the *safe region*. This is justified by the following fact, whose proof is a direct consequence of Lemma 2.2.

**Lemma 3.6.** *If  $(x, t) \in \mathcal{S}$ , then  $\xi_t^{(\mu)}(x) = \xi_t^{(1)}(x)$ .*

By Lemma 3.6, it is enough to prove that, for any  $m > 0$ ,

$$V_m \cap \mathcal{S}^c \text{ is a bounded subset of } \mathbb{Z} \times [0, \infty), \quad P\text{-a.s.} \tag{3.30}$$

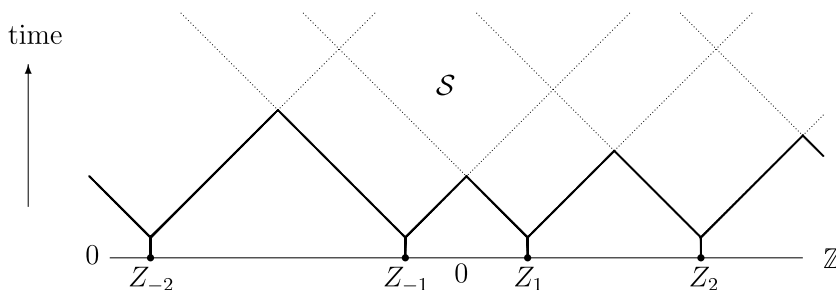


Fig. 4. Cones have inclination angle  $\iota/2$ . The safe region  $\mathcal{S}$  lies above the thick lines.

To that end, note that  $\mathcal{S}^c$  is contained in the union of space–time “houses” (unions of triangles and rectangles) with base at time 0. The tips of the houses to the right of 0 form a sequence with spatial coordinates  $\frac{1}{2}(Z_{n+1} + Z_n)$  and temporal coordinates  $(Z_{n+1} - Z_n + 2)/t$ ,  $n \in \mathbb{N}$ . By (3.28), the ratio of temporal/spatial coordinates tends to 0 as  $n \rightarrow \infty$ , so that only finitely many tips can be inside  $V_m$ . The same is true for the tips of the houses to the left of 0. Therefore  $V_m$  touches only finitely many houses, which proves (3.30).  $\square$

**Remark 3.7.** *It is possible to show that (3.11) holds for any initial configuration that has a positive lower density of infections to the right and to the left of the origin. We will not pursue this extension here, and content ourselves with giving a sketch of a proof strategy that uses the techniques from Section 4 below. Let*

$$\begin{aligned} \Lambda_x^* &:= \left\{ \text{there exist two paths } \pi_s^-, \pi_s^+ \text{ such that } (x, 0) \leftrightarrow (\pi_s^\pm, s) \right. \\ &\quad \left. \text{and } \lfloor t/2s \rfloor \leq \pm(\pi_s^\pm - x) \leq 2ts \ \forall s \geq 0 \right\}. \end{aligned} \quad (3.31)$$

Fix  $x$  with  $\xi_0(x) = 1$ . With the help of the methods used in the proof of Lemma 4.3 below, we may show that if  $\Lambda_x^*$  does not occur, then there is a positive random variable  $L_x$  with a (uniform) exponential moment such that, for any  $k \in \mathbb{N}$ ,  $\Lambda_{x+L_x+k}^*$  is independent of  $\Lambda_x^*$ ,  $L_x$  and distributed as  $\Lambda_0^*$ . Therefore, after a geometric number of trials we find a point  $y > x$  such that  $\xi_0(y) = 1$  and  $\Lambda_y^*$  occurs. Next, we use Lemma 4.1 and the FKG-inequality (see [6]) to forget all information gathered so far and start afresh at the next point  $z > y$  such that  $\xi_0(z) = 1$ . In this way we prove that the configuration  $\eta^*(x) := \xi_0(x)\mathbb{1}_{\Lambda_x^*}$  has a positive lower density, and from this point on we may continue as in the proof of Proposition 3.5.

#### 4. More on the contact process

In this section we collect some additional facts about the contact process on  $\mathbb{Z}$  that will be needed in the remainder of the paper. The proofs rely on geometric observations that will also illuminate the proof strategies developed in Sections 5–6.

In the following we will use the notation

$$\mathbb{Z}_{\leq x} := \mathbb{Z} \cap (-\infty, x] \quad (4.1)$$

and analogously for  $\mathbb{Z}_{\geq x}$ .

##### Stochastic domination

We start with a useful alternative construction of the equilibrium  $\nu_\lambda$ . Let  $\eta(x) := \mathbb{1}_{\{C_t(x) \neq \emptyset \ \forall t \geq 0\}}$ . Then, by the graphical representation,  $\eta$  has distribution  $\nu_\lambda$ . This follows from duality (see Liggett [11], Chapter VI). We can also graphically construct the contact process starting from  $\nu_\lambda$ : extend the graphical representation to negative times, and declare  $\xi_t(x) = 1$  if and only if for all  $s \in (-\infty, t)$  there exists a  $y$  such that  $(y, s) \leftrightarrow (x, t)$ , i.e., if and only if there exists an infinite infection path going backwards in time from  $(x, t)$ .

Let  $\bar{\nu}_\lambda$  denote the restriction of  $\nu_\lambda$  to  $\mathbb{Z}_{\leq -1}$ . Abusing notation, we will write the same symbol to denote the measure on  $\Omega$  that is the product of  $\bar{\nu}_\lambda$  with the measure concentrated on all sites healthy to the right of  $-1$ . Using the alternative construction above, we can prove that the restriction of  $\nu_\lambda(\cdot | \eta(0) = 1)$  to  $\mathbb{Z}_{\leq -1}$  is stochastically larger than  $\bar{\nu}_\lambda$ . In the following, we will focus on a similar result for the distribution of  $\xi_t$  to the left of certain infection paths.

For  $\varpi_{[0,t]}$  a nearest-neighbor càdlàg path with values in  $\mathbb{Z}$ , let

$$\begin{aligned} \bar{\mathcal{R}}_t^\varpi &:= \sigma \left\{ \left( \xi_0(x) \right)_{x \geq \varpi_0}, \right. \\ &\quad \left. \left( H_v(x) - H_u(x), I_v(x) - I_u(x) \right)_{\{(x,u,v) \in \mathbb{Z} \times [0,t]^2 : u \leq v, x \geq \sup_{s \in [u,v]} \varpi_s\}} \right\}. \end{aligned} \quad (4.2)$$

Suppose that  $\pi_{[0,t]}$  is a random path of the same type, with the following properties:

(p1)  $\xi_0(\pi_0) = 1$  a.s. and  $(\pi_s, s) \leftrightarrow (\pi_u, u)$  for all  $s, u \in [0, t]$ .

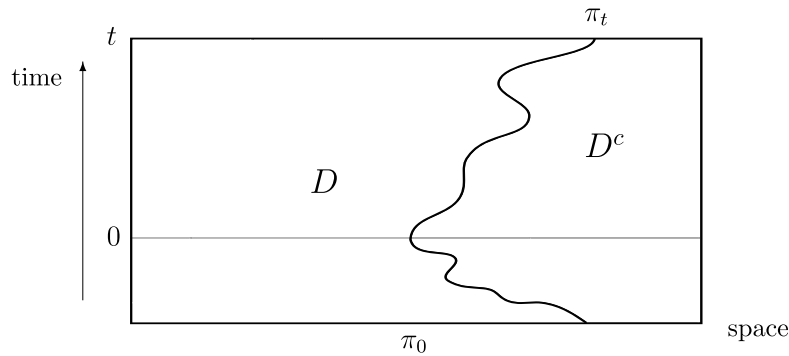


Fig. 5. The thick line represents the random infection path  $\pi$ . The dashed lines represent other infection paths.

(p2)  $\pi$  is  $\mathcal{F}$ -adapted and  $\{\pi_s \geq \varpi_s \ \forall s \in [0, t]\} \in \bar{\mathcal{R}}_t^\varpi$  for all deterministic paths  $\varpi$ .

We call  $\pi$  a *random infection path* (see Figure 5), a name that is justified by (p1). Property (p2) means that  $\pi$  is causal and that, when we discover it, we leave the graphical representation to its left untouched. For such  $\pi$ , let

$$\mathcal{R}_t^\pi := \left\{ A \in \mathcal{F}_\infty : A \cap \left\{ \pi_s \geq \varpi_s \ \forall s \in [0, t] \right\} \in \bar{\mathcal{R}}_t^\varpi \text{ for every deterministic nearest-neighbor càdlàg path } \varpi_{[0,t]} \right\}. \tag{4.3}$$

Note that, since  $\pi$  is an infection path, also  $(\xi_s(x))_{x \geq \pi_s} \in \mathcal{R}_t^\pi$  for each  $s \in [0, t]$  (see the proof of Lemma 2.2). We have the following stochastic domination result.

**Lemma 4.1.** *For any random infection path  $\pi_{[0,t]}$  as above, the law of  $\xi_t(\cdot + \pi_t + 1)$  under  $\mathbb{P}_{\bar{v}_\lambda}(\cdot | \mathcal{R}_t^\pi)$  is stochastically larger than  $\bar{v}_\lambda$ .*

**Proof.** Construct  $\mathbb{P}_{\bar{v}_\lambda}$  from a graphical representation on  $\mathbb{Z} \times \mathbb{R}$  as outlined above by adding healing events on  $(x, 0)$  for each  $x \in \mathbb{Z}_{\geq 0}$ . Extend  $\pi$  to negative times by making it equal to the right-most infinite infection path going backwards in time from  $(\pi_0, 0)$ . (Such a path exists because  $\xi_0(\pi_0) = 1$ .) We may check that the resulting path still has properties (p1) and (p2). Extend also  $\mathcal{R}_t^\pi$  to include negative times.

Next, regard  $H$  and  $I$  as Poisson point processes on subsets of  $\mathbb{Z} \times \mathbb{R}$ . Let (see Figure 5)

$$D := \left\{ (x, s) \in \mathbb{Z} \times \mathbb{R} : s > t \text{ or } \pi_s > x \right\}. \tag{4.4}$$

Given  $\mathcal{R}_t^\pi$ , by (p2)  $H$  and  $I$  are still Poisson point processes with the same densities on  $D$ . This can be justified first for  $\pi$  taking values in a countable set and then for general  $\pi$  using right-continuity.

With this observation we can couple  $\mathbb{P}_{\bar{v}_\lambda}$  to  $\mathbb{P}_{\bar{v}_\lambda}(\cdot | \mathcal{R}_t^\pi)$  in the following way. Draw independent Poisson point processes  $\hat{H}, \hat{I}$  on  $D^c$ . Take  $\hat{\xi}$  to be the contact process obtained by using  $H, I$  on  $D$  and  $\hat{H}, \hat{I}$  on  $D^c$ . Then  $\hat{\xi}$  is distributed as the contact process under  $\mathbb{P}_{\bar{v}_\lambda}$ , and is independent of  $\mathcal{R}_t^\pi$ . Furthermore,  $\xi_t(x) \geq \hat{\xi}_t(x)$  for all  $x < \pi_t$ . Indeed, if  $\hat{\xi}_t(x) = 1$ , then infinite infection paths going backwards in time must either stay inside  $D$  or cross  $\pi$ , so that, by (p1),  $\xi_t(x) = 1$  as well.  $\square$

**Remark 4.2.** *In Lemma 4.1, we may replace  $t$  by a finite stopping time  $\mathcal{T}$  w.r.t. the filtration  $\mathcal{F}$ , as long as the event in (p2) is replaced by  $\{\mathcal{T} \leq t, \pi_s \geq \varpi_s \ \forall s \in [0, \mathcal{T}]\}$  and we add  $\mathcal{T}$  to  $\mathcal{R}_t^\pi$ . We may also enlarge all filtrations by adding information that is independent of  $\xi_0, H, I$ , in particular,  $N_{[0,t]}$  and  $U_{[1, N_t]}$  (recall Section 2.2).*

*Infection range*

Lemma 4.3 below concerns the positions of wide-spread infections. For  $\delta \in (0, \iota)$  and  $x \in \mathbb{Z}$ , let  $\mathcal{W}_x^\delta := \{(z, t) \in \mathbb{Z} \times [0, \infty) : (\iota - \delta)t - 1 < z - x \leq (\iota + \delta)t\}$  be a wedge between two lines of inclination  $\iota - \delta$  and  $\iota + \delta$ . Set  $C_t^\delta(x) := \{y \in \mathbb{Z} : (y, t) \leftrightarrow (x, 0) \text{ via a path contained in } \mathcal{W}_x^\delta\}$ , and

$$Z_\delta(x) := \sup\{z \in \mathbb{Z}_{<x} : \xi_0(z) = 1, C_t^\delta(z) \neq \emptyset \forall t \geq 0\}, \tag{4.5}$$

i.e., the first infected site to the left of  $x$  that spreads its infection forever inside a wedge.

**Lemma 4.3.** *If  $\lambda \in (\lambda_c, \infty)$  then  $|Z_\delta(x) - x|$  has exponential moments under  $\mathbb{P}_{v_\lambda}$  for every  $\delta \in (0, \iota)$ , uniformly in  $x \in \mathbb{Z}_{\leq 0}$ .*

**Proof.** We will use the fact that, for any  $\lambda \in (\lambda_c, \infty)$ ,  $v_\lambda$  stochastically dominates a non-trivial Bernoulli product measure  $\mu_\lambda$ . This follows from Liggett and Steif [14], Theorem 1.2, Durrett and Schonmann [9], Theorem 1, and van den Berg, Häggström and Kahn [16], Theorem 3.5. Since  $Z_\delta(x)$  is monotone in  $\xi_0$ , it is therefore enough to prove the statement under  $\mathbb{P}_{\mu_\lambda}$ . We may also assume  $x = 0$ , as  $Z_\delta(x)$  does not depend on  $(\xi_0(z))_{z \geq x}$ .

Construct a sequence of pairs  $(Z_n, T_n)_{n \in \mathbb{N}_0}$  as follows. Set  $Z_0 = T_0 := 0$  and, recursively for  $n \in \mathbb{N}_0$ ,

$$\begin{aligned} Z_{n+1} &:= \begin{cases} Z_n & \text{if } T_n = \infty, \\ \sup\{z < Z_n - \lceil(\iota + \delta)T_n\rceil : \xi_0(z) = 1\} & \text{otherwise,} \end{cases} \\ T_{n+1} &:= \begin{cases} \infty & \text{if } T_n = \infty, \\ \inf\{t > 0 : C_t^\delta(Z_{n+1}) = \emptyset\} & \text{otherwise.} \end{cases} \end{aligned} \tag{4.6}$$

Conditionally on  $T_n < \infty$ ,  $\Delta_{n+1} := Z_{n+1} - Z_n + \lceil(\iota + \delta)T_n\rceil$  and  $T_{n+1}$  are independent of  $(Z_k, T_k)_{k=1}^n$  and distributed as  $(Z_1, T_1)$ . This is because the region of the graphical representation plus initial configuration on which  $T_{n+1}$  and  $\Delta_{n+1}$  depend is disjoint from the region on which the previous random variables depend. Since  $\mu_\lambda$  is a non-trivial product measure,  $|Z_1|$  has exponential moments. Noting that  $T_1$  is independent of  $Z_1$  we conclude, using standard facts about the contact process (see Liggett [11], Chapter VI, Theorem 2.2, Corollary 3.22 and Theorem 3.23), that  $\mathbb{P}_{\mu_\lambda}(T_1 = \infty) > 0$  and that, conditionally on  $T_1 < \infty$ ,  $T_1$  has exponential moments. Defining the random index

$$K := \inf\{n \in \mathbb{N} : T_n = \infty\} \tag{4.7}$$

whose distribution is  $\text{GEO}(\mathbb{P}_{\mu_\lambda}(T_1 = \infty))$ , we see that  $|Z_\delta(0)| \leq |Z_K|$ . Taking  $a > 0$  such that  $\mathbb{E}_{\mu_\lambda}[e^{a(|Z_1| + \lceil(\iota + \delta)T_1\rceil)} | T_1 < \infty] < 1/\mathbb{P}_\mu(T_1 < \infty)$ , we get after a short calculation that  $\mathbb{E}_{\mu_\lambda}[\mathbb{1}_{\{K=n\}} e^{a|Z_n|}]$  decays exponentially in  $n$ .  $\square$

**5. Properties of the speed**

In this section we prove Theorem 1.1(b).

Take  $\lambda_\infty, \lambda_k \in (\lambda_c, \infty)$  such that  $\lim_{k \rightarrow \infty} \lambda_k = \lambda_\infty$ , and write  $\mathbb{P}_\mu^{\lambda_\infty}, \mathbb{P}_\mu^{\lambda_k}$  for the measure described in Section 2 with the corresponding parameter and initial measure  $\mu$ . Fix  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , and take  $L_{n,\varepsilon} > 0$  such that  $\mathbb{E}_{\delta_1^{\lambda_k}}[N_n \mathbb{1}_{\{N_n > L_{n,\varepsilon}\}}] \leq \varepsilon$  uniformly in  $k \in \mathbb{N} \cup \{\infty\}$ . On the event  $\{N_n \leq L_{n,\varepsilon}\}$ ,  $W_n$  depends on  $\xi$  only inside the finite space-time region  $\mathbb{Z} \cap [-L_{n,\varepsilon}, L_{n,\varepsilon}] \times [0, n]$ . Therefore  $\mathbb{E}_{\delta_1^{\lambda_k}}[n^{-1} W_n \mathbb{1}_{\{N_n \leq L_{n,\varepsilon}\}}]$  converges as  $k \rightarrow \infty$  to the same quantity with parameter  $\lambda_\infty$  (see Liggett [13], Part I). Since  $|W_n| \leq N_n$ , it follows that

$$\begin{aligned} |\mathbb{E}_{\delta_1^{\lambda_k}}[n^{-1} W_n] - \mathbb{E}_{\delta_1^{\lambda_\infty}}[n^{-1} W_n]| &\leq |\mathbb{E}_{\delta_1^{\lambda_k}}[n^{-1} W_n \mathbb{1}_{\{N_n \leq L_{n,\varepsilon}\}}] - \mathbb{E}_{\delta_1^{\lambda_\infty}}[n^{-1} W_n \mathbb{1}_{\{N_n \leq L_{n,\varepsilon}\}}]| \\ &\quad + 2\varepsilon. \end{aligned} \tag{5.1}$$

Taking the lim sup as  $k \rightarrow \infty$  of (5.1) followed by  $\varepsilon \downarrow 0$ , we get that  $\lambda \mapsto \mathbb{E}_{\delta_1} [n^{-1} W_n]$  is continuous. By monotonicity, the latter is also non-decreasing, so it follows from (3.7) that  $\lambda \mapsto v(\lambda)$  is right-continuous and non-decreasing.

It remains to show that  $v(\lambda) \in (v_0, v_1)$  and  $\lim_{\lambda \rightarrow \infty} v(\lambda) = v_1$ . This will be done in Sections 5.1–5.2 below. These properties come from the fact that the random walk spends positive fractions of its time on top of infected sites and on

top of healthy sites. To keep track of this, define  $N_t^i := \#\{n \in \mathbb{N} : \xi_{J_n}(W_{J_{n-1}}) = i\}$ ,  $i \in \{0, 1\}$ . Recalling the construction of  $W$  in Section 2.2, we may write

$$W_t = S_{N_t^0}^0 + S_{N_t^1}^1, \tag{5.2}$$

where  $S_n^i$ ,  $i = 0, 1$ , are discrete-time homogeneous random walks that jump to the right with probability  $\alpha_i/\gamma$  and to the left with probability  $\beta_i/\gamma$ . From this representation we immediately get the following.

**Lemma 5.1.**

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-1} W_t &= v_0 + (v_1 - v_0) \liminf_{t \rightarrow \infty} (\gamma t)^{-1} N_t^1, \\ \limsup_{t \rightarrow \infty} t^{-1} W_t &= v_1 - (v_1 - v_0) \liminf_{t \rightarrow \infty} (\gamma t)^{-1} N_t^0. \end{aligned} \tag{5.3}$$

Lemma 5.1 is valid for any dynamic random environment, even without a SLLN for  $W$ . But (5.3) shows that a SLLN for  $W$  holds with speed  $v$  if and only if a SLLN holds for  $N^1$  with limit  $\gamma \rho_{\text{eff}}$ , where  $\rho_{\text{eff}} := (v - v_0)/(v_1 - v_0)$  is the effective density of 1’s seen by  $W$ . Thus,  $v > v_0$  and  $v < v_1$  are equivalent to, respectively,  $\rho_{\text{eff}} > 0$  and  $\rho_{\text{eff}} < 1$ .

5.1. *Proof of  $v(\lambda) < v_1$*

In the contact process, infected sites heal spontaneously. Therefore it is easier to find 0’s than 1’s. For this reason, it is easier to prove that  $W$  often jumps from healthy sites than from infected sites.

**Proof.** For  $k \in \mathbb{N}$ , let  $Y_k := \xi_{J_k}(W_{J_{k-1}})$ , and note that  $\{Y_{k+1} = 0\}$  contains all configurations that between times  $J_k$  and  $J_{k+1}$  have a cross at site  $W_{J_k}$  and no arrows between  $W_{J_k}$  and its nearest-neighbors, i.e., such that the events  $H_{J_{k+1}}(W_{J_k}) - H_{J_k}(W_{J_k}) \geq 1$  and  $I_{J_{k+1}}(W_{J_k}) - I_{J_k}(W_{J_k}) = I_{J_{k+1}}(W_{J_k} - 1) - I_{J_k}(W_{J_k} - 1) = 0$  occur. The probability of the latter events given  $\sigma\{(J_k, \xi_s, W_s)_{0 \leq s \leq J_k}\}$  is constant in  $k$  and equal to  $p := \gamma/(\gamma + 2\lambda)(1 + \gamma + 2\lambda)$ . Therefore the sequence  $(Y_k)_{k \in \mathbb{N}}$  is stochastically dominated by a sequence of i.i.d. BERN( $1 - p$ ) random variables, which implies that  $\liminf_{t \rightarrow \infty} t^{-1} N_t^0 \geq \gamma p > 0$ , so that  $v(\lambda) < v_1$  by Lemma 5.1. □

5.2. *Proof of  $v(\lambda) > v_0$  and  $\lim_{\lambda \rightarrow \infty} v(\lambda) = v_1$*

This is the harder part of the proof. We will need results from Section 4. In the following we will assume that  $v_0 \leq 0$ . The case  $v_0 > 0$  can be treated analogously.

Let us start with an informal description of the argument. The idea is that there are “waves of infection” coming from  $\pm\infty$  from which the random walk cannot escape. When  $v_0 \leq 0$ , we can concentrate on the waves coming from the left, represented schematically in Figure 6. Each time the random walk hits a new wave, there is an infection path starting from its current location and going backwards in time entirely to the left of the random walk path. By Lemma 4.1, at this time the law of  $\xi$  to the left of the random walk has an appreciable density, which means that there are new waves coming in from locations not very far to the left. On the other hand, any infections to the right of the

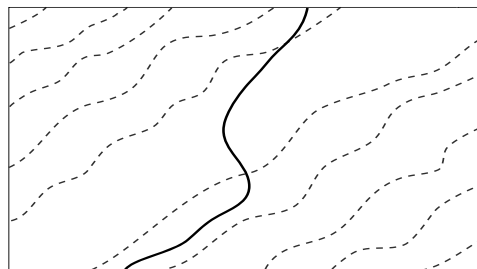


Fig. 6. The dashed lines represent infection waves. The thick line represents the path of  $W$ .

random walk can be ignored, since they only push it to the right. But doing so makes the random walk behave as a homogeneous random walk with a non-positive drift, meaning that it does not take the random walk long to hit the next infection wave. Since at each collision there is a fixed probability for the random walk to jump while sitting on an infection,  $v(\lambda) > v_0$  will follow from Lemma 5.1. With some care in the computations we also get the limit for large  $\lambda$ .

**Proof.** Using the graphical representation, we will construct, on a larger probability space, a second random walk  $\hat{W}$  coupled to  $W$  in such a way that  $\hat{W}_t \leq W_t$  for all  $t \geq 0$  and that  $\hat{W}$  has a speed with the desired properties. Let

$$V_1 := \inf\{t > 0: \xi_t(W_t) = 1\}. \quad (5.4)$$

Note that  $V_1$  has exponential moments under  $\mathbb{P}_{\bar{v}_\lambda}$  by Lemma 4.3 and the fact that  $v_0 \leq 0$ . Let

$$\tau_1 := \inf\{t > V_1: W_t \neq W_{V_1} \text{ or } H_t(W_{V_1}) > H_{V_1}(W_{V_1})\}, \quad (5.5)$$

i.e.,  $\tau_1$  is the first time after time  $V_1$  at which either  $W$  jumps or there is a healing event at the position of the random walk. Note that  $\tau_1$  is a stopping time w.r.t. the filtration  $\mathcal{G}$  and that, given  $\mathcal{G}_{V_1}$ ,  $\tau_1 - V_1$  has distribution  $\text{EXP}(1 + \gamma)$ .

We will construct a sequence  $(W^{(n)}, \tau_n)_{n \in \mathbb{N}}$  with the following properties:

- (A1)  $W_t^{(n+1)} \leq W_{\tau_n+t}^{(n)} - W_{\tau_n}^{(n)}$  for all  $t \geq 0$ ;
- (A2)  $(W^{(n)}, \tau_n)$  is distributed as  $(W, \tau_1)$  under  $\mathbb{P}_{\bar{v}_\lambda}$ ;
- (A3)  $(W_{[0, \tau_n]}^{(n)}, \tau_n)_{n \in \mathbb{N}}$  is i.i.d.;
- (A4) If  $\hat{v}(\lambda) := \mathbb{E}_{\bar{v}_\lambda}[W_{\tau_1}]/\mathbb{E}_{\bar{v}_\lambda}[\tau_1]$ , then  $\hat{v}(\lambda) > v_0$  and  $\lim_{\lambda \rightarrow \infty} \hat{v}(\lambda) = v_1$ .

Once we have this sequence, we can put  $T_0 := 0$ ,  $T_n := \sum_{k=1}^n \tau_k$  for  $n \in \mathbb{N}$ , and

$$\hat{W}_t := \sum_{k=1}^n W_{\tau_k}^{(k)} + W_{t-T_n}^{(n+1)} \quad \text{for } T_n \leq t < T_{n+1}. \quad (5.6)$$

By (A1),  $\hat{W}_t \leq W_t^{(1)}$  for all  $t \geq 0$ . By (A2), the latter is distributed as  $W$  under  $\mathbb{P}_{\bar{v}_\lambda}$ , which by monotonicity is stochastically smaller than  $W$  under  $\mathbb{P}_{v_\lambda}$ . By (A3),  $\lim_{n \rightarrow \infty} T_n^{-1} \hat{W}_{T_n} = \hat{v}(\lambda)$ , and so the claim follows from (A4). Thus, it remains to construct the sequence  $(W^{(n)}, \tau_n)_{n \in \mathbb{N}}$  with properties (A1)–(A4).

To do so, we draw  $\xi_0$  from  $\bar{v}_\lambda$ , let  $\xi^{(1)} := \xi$ ,  $W^{(1)} := W$ , define  $\tau_1$  as above, and note the following.

**Lemma 5.2.** *Under  $\mathbb{P}_{\bar{v}_\lambda}(\cdot | \tau_1, W_{[0, \tau_1]})$ , the law of  $\xi_{\tau_1}(\cdot + W_{\tau_1})$  is stochastically larger than  $\bar{v}_\lambda$ .*

**Proof.** Since  $\xi_{V_1}(W_{V_1}) = 1$ , there exists a right-most infection path  $\pi_{[0, V_1]}$  connecting  $(W_{V_1}, V_1)$  to  $\mathbb{Z}_{\leq -1} \times \{0\}$ . Extend  $\pi$  to  $[V_1, \tau_1]$  by making it constant and equal to  $W_{V_1}$  on this time interval. Since  $\pi_s \leq W_s$  for all  $0 \leq s < \tau_1$ , we have  $(\tau_1, W_{[0, \tau_1]}) \in \mathcal{R}_{\tau_1}^x \vee \sigma(N_{[0, \tau_1]}, U_{[1, N_{\tau_1}]})$ . Note that  $\pi$  is not an infection path, but only because of a possible healing event at time  $\tau_1$ , which does not affect  $(\xi_{\tau_1}(x + W_{V_1}))_{x \leq -1}$ . Therefore, by Lemma 4.1, the distribution of  $\eta_1(\cdot) := \xi_{\tau_1}(\cdot + W_{V_1})$  given  $(\tau_1, W_{[0, \tau_1]})$  is stochastically larger than  $\bar{v}_\lambda$ . Moreover,  $(\eta_1(x))_{x < W_{V_1}}$  is independent of  $W_{\tau_1} - W_{V_1}$ , and so we need only verify that the distribution of  $\eta_2 := \xi_{\tau_1}(\cdot + W_{\tau_1}) = \theta_{W_{\tau_1} - W_{V_1}} \eta_1$  is stochastically larger than  $\bar{v}_\lambda$  for each possible outcome of  $W_{\tau_1} - W_{V_1} \in \{0, \pm 1\}$ . Since, by the definitions of  $V_1$  and  $\tau_1$ ,  $W_{\tau_1} \neq W_{V_1}$  if and only if  $\xi_{\tau_1}(W_{V_1}) = 1$ , the possible choices for  $\eta_2$  are:  $\eta_1$  if  $W_{\tau_1} = W_{V_1}$ ,  $\theta_{-1}\eta_1$  if  $W_{\tau_1} = W_{V_1} - 1$ , or  $\theta_1\eta_1$  if  $W_{\tau_1} = W_{V_1} + 1$ . In the latter case,  $\eta_2(-1) = 1$ , and therefore all three possibilities are indeed stochastically larger than  $\bar{v}_\lambda$  as claimed.  $\square$

By Lemma 5.2, there exists a configuration  $\xi_0^{(2)}$  distributed as  $\bar{v}_\lambda$ , independent of  $(\tau_1, W_{[0, \tau_1]})$  and smaller than  $\xi_{\tau_1}^{(1)}(\cdot + W_{\tau_1})$ . We may now define  $\xi^{(2)}$  by using the events of the graphical representation that lie above time  $\tau_1$  with the origin shifted to  $W_{\tau_1}$ , using  $\xi_0^{(2)}$  as starting configuration. We may then define  $W^{(2)}$  and  $\tau_2$  from  $\xi^{(2)}$ ,  $(N_{t+\tau_1} - N_{\tau_1})_{t \geq 0}$  and  $(U_k)_{k > N_{\tau_1}}$ . With this coupling, clearly  $W_t^{(2)} \leq W_{\tau_1+t}^{(1)} - W_{\tau_1}^{(1)}$  for all  $t \geq 0$ . Furthermore, since  $\xi_0^{(2)}$  is independent of  $(\tau_1, W_{[0, \tau_1]})$ , the distribution of  $\xi_{\tau_2}^{(2)}(\cdot + W_{\tau_2}^{(2)})$  given  $(W_{[0, \tau_1]}^{(i)}, \tau_i)_{i=1,2}$  depends only on the random variables with  $i = 2$  and hence, by Lemma 5.2, is again stochastically larger than  $\bar{v}_\lambda$ .



We may therefore repeat the argument. More precisely, suppose by induction that we have defined  $\xi^{(k)}$ ,  $W^{(k)}$  and  $\tau_k$  for  $k = 1, \dots, n$  and  $n \geq 2$ , in such a way that:

- (B1)  $W_t^{(k+1)} \leq W_{\tau_n+t}^{(k)} - W_{\tau_n}^{(n)}$  for all  $t \geq 0$  and  $k = 1, \dots, n - 1$ ;
- (B2)  $(W^{(k)}, \tau_k)$  is distributed as  $(W, \tau_1)$  under  $\mathbb{P}_{\bar{v}_\lambda}$  for all  $k = 1, \dots, n$ ;
- (B3)  $(W_{[0, \tau_k]}^{(k)}, \tau_k)_{k=1}^n$  is i.i.d.;
- (B4) The law of  $\xi^{(n)}(\cdot + W_{\tau_n}^{(n)})$  given  $(W_{[0, \tau_k]}^{(k)}, \tau_k)_{k=1}^n$  is stochastically larger than  $\bar{v}_\lambda$ .

Then we proceed as before: there exists a configuration  $\xi_0^{(n+1)}$  distributed as  $\bar{v}_\lambda$ , smaller than  $\xi^{(n)}(\cdot + W_{\tau_n}^{(n)})$  and independent of  $(W_{[0, \tau_k]}^{(k)}, \tau_k)_{k=1}^n$ , from which we obtain  $\xi^{(n+1)}$ ,  $W^{(n+1)}$  and  $\tau_{n+1}$ , and we prove (B1)–(B4) as in the case  $n = 2$ . This settles the existence of the sequence  $(W^{(n)}, \tau_n)_{n \in \mathbb{N}}$ . All that is left to show is that  $\hat{v}(\lambda) > v_0$  and  $\lim_{\lambda \rightarrow \infty} \hat{v}(\lambda) = v_1$ .

Note that Lemma 5.1 is valid also for  $\hat{W}$ , and write  $\hat{N}_t^1$  to denote the number of jumps that  $\hat{W}$  takes on infected sites. Then  $\hat{N}_{T_n}^1$  has distribution  $\text{BINOM}(n, \gamma/(1 + \gamma))$ , and by standard arguments we obtain

$$\lim_{t \rightarrow \infty} t^{-1} \hat{N}_t^1 = \frac{\gamma}{(1 + \gamma) \mathbb{E}_{\bar{v}_\lambda}[\tau_1]} > 0, \tag{5.7}$$

which proves  $\hat{v}(\lambda) > v_0$ . Furthermore, we claim that  $\lim_{\lambda \rightarrow \infty} \mathbb{E}_{\bar{v}_\lambda}[V_1] = 0$ . Indeed,  $V_1$  is nonincreasing in  $\lambda$  and, since  $\lim_{\lambda \rightarrow \infty} \rho_\lambda = 1$  (recall Section 1.2), it is not hard to see that  $V_1$  converges in probability to zero as  $\lambda \rightarrow \infty$ . Therefore  $\lim_{\lambda \rightarrow \infty} \mathbb{E}_{\bar{v}_\lambda}[\tau_1] = 1/(1 + \gamma)$ , and so  $\lim_{\lambda \rightarrow \infty} \hat{v}(\lambda) = v_1$ . □

### 6. Regeneration, functional CLT and LDP

The proof of Theorem 1.2 depends on the construction of *regeneration times*, i.e., times at which the random walk forgets its past. This construction is carried out in Section 6.1 and yields two propositions (Propositions 6.1–6.2 below), which are proved in Sections 6.2–6.3. At the end of Section 6.1 we will see that these propositions imply Theorem 1.2(a), (c). The proof of Theorem 1.2(b) is deferred to Section 6.4.

#### 6.1. Regeneration times

If the infection propagation speed  $\iota = \iota(\lambda)$  is larger than  $|v_-| \vee |v_+|$ , the maximum absolute speed at which the random walk can move, then at each time  $W$  finds itself on an infected site it can become “trapped” forever in an infection cluster generated by this site alone. In that case, by Lemma 2.2, the future increments of  $W$  become independent of its past. The issue is therefore to find enough moments when  $W$  sits on an infection. This can be dealt with in a way similar to what was done in the proof of  $v(\lambda) > v_0$  in Section 5.2.

##### Hitting, failure and trial times

In order to build the regeneration structure, we first need to extend some definitions related to clusters and right-most infections. For  $s \geq t$  and  $x \in \mathbb{Z}$ , let

$$C_s(x, t) := \{y \in \mathbb{Z}: (x, t) \leftrightarrow (y, s)\} \tag{6.1}$$

and

$$R_s(x, t) := \sup C_s(x, t), \quad L_s(x, t) := \inf C_s(x, t). \tag{6.2}$$

Furthermore, let

$$r_s(x, t) := \sup_{\substack{y < x \\ \xi_t(y)=1}} R_s(y, t), \tag{6.3}$$

i.e.,  $r_s(x, t)$  is the right-most infection at time  $s$  that comes from  $\mathbb{Z}_{\leq x-1} \times \{t\}$ .

For  $t \geq 0$  and  $z \in \mathbb{Z}$ , let

$$V_t(z) := \inf\{s > t : W_s = r_s(z, t)\} \tag{6.4}$$

be the first time after time  $t$  at which  $W$  meets the right-most infection coming from  $\mathbb{Z}_{\leq z-1} \times \{t\}$ . We will call this the  $z$ -wave hitting time after  $t$ . It is not hard to see that  $V_t(z) < \infty$   $\mathbb{P}_{v_\lambda}$ -a.s. for any  $t$  and  $z \leq W_t$ . Indeed, at any time  $t$  there is an infected site  $x < z$  whose infection survives forever, and in this case  $\lim_{s \rightarrow \infty} s^{-1} R_s(x, t) = t > |v_-| \vee |v_+|$ . Therefore there must be an  $s > t$  for which  $R_s(x, t) = W_s$ . By right-continuity,  $\mathbb{P}_{v_\lambda}(V_t(z) < \infty \forall z \leq W_t, t \geq 0) = 1$  as well.

Now define the first failure time after time  $t$  by (see Figure 8)

$$F_t := \inf\{s > t : W_s \notin [L_s(W_t, t), R_s(W_t, t)]\}, \tag{6.5}$$

i.e., the first time after time  $t$  when  $W$  exits the region surrounded by the cluster of  $(W_t, t)$ . To keep track of the space-time region on which the failure time depends, define, for  $t \geq 0$  and  $x \in \mathbb{Z}$ ,

$$(Y_s(x, t))_{s \geq t} \tag{6.6}$$

as the process with values in  $\mathbb{Z}$  that starts at time  $t$  at site  $x$  and jumps down by following the infection arrows to the left in the graphical representation (see Figure 6.1). Then, given  $\mathcal{G}_t$ ,  $(x - Y_{t+s}(x, t))_{s \geq 0}$  is a Poisson process with rate  $\lambda$ .

With the above observations we can define the trial time after a failure time (see Figure 8):

$$T_t := \begin{cases} \infty & \text{if } F_t = \infty, \\ V_{F_t}(Y_{F_t}(W_t, t)) & \text{otherwise,} \end{cases} \tag{6.7}$$

i.e.,  $T_t$  is the  $Y_{F_t}(W_t, t)$ -wave time after time  $F_t$  when the latter is finite. This wave ensures “good conditions” at the trial time, meaning an appreciable density of infections to the left of  $W$ .

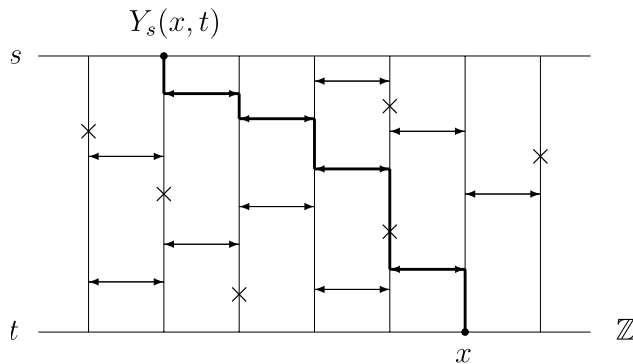


Fig. 7.  $Y_s(x, t)$  starts at  $x$  and goes upwards and to the left across the arrows of the graphical representation.

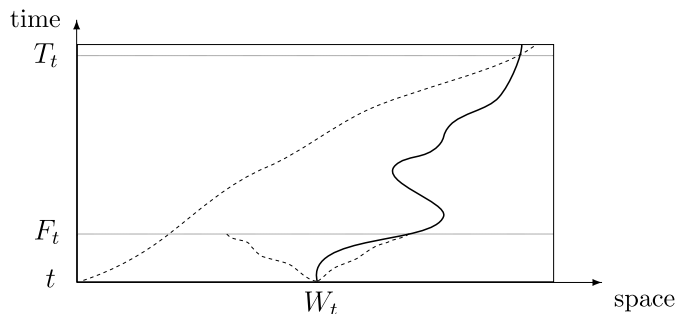


Fig. 8. A failure time  $F_t$  and a trial time  $T_t$  after time  $t$ . The dashed lines represent infection paths. The thick line represents the path of  $W$ .

*Regeneration times*

We can now define our regeneration time  $\tau$ . First let

$$\mathcal{T}_1 := V_0(0) \tag{6.8}$$

and, under the assumption that  $\mathcal{T}_1, \dots, \mathcal{T}_k, k \in \mathbb{N}$ , are all defined, let

$$\mathcal{T}_{k+1} := \begin{cases} \infty & \text{if } \mathcal{T}_k = \infty, \\ T_{\mathcal{T}_k} & \text{otherwise.} \end{cases} \tag{6.9}$$

Note that the  $\mathcal{T}_k$ 's are stopping times w.r.t. the filtration  $\mathcal{G}$ . Finally, put

$$K := \inf\{k \in \mathbb{N}: \mathcal{T}_k < \infty, \mathcal{T}_{k+1} = \infty\}, \tag{6.10}$$

and let

$$\tau := \mathcal{T}_K. \tag{6.11}$$

Note that  $K < \infty$  a.s. since, at any trial time, the probability for the next failure time to be infinite is uniformly bounded from below. We will prove in Sections 6.2–6.3 that  $\tau$  is a regeneration time and has exponential moments. This is stated in the following two propositions.

**Proposition 6.1.** *The distribution of  $(W_{t+\tau} - W_\tau)_{t \geq 0}$  under both  $\mathbb{P}_{v_\lambda}(\cdot | \tau, W_{[0, \tau]})$  and  $\mathbb{P}_{v_\lambda}(\cdot | \Gamma, \tau, W_{[0, \tau]})$  is the same as that of  $W$  under  $\mathbb{P}_{v_\lambda}(\cdot | \Gamma)$ , where*

$$\Gamma := \{\xi_0(0) = 1, F_0 = \infty\}. \tag{6.12}$$

**Proposition 6.2.**  *$\tau$  and  $|W_\tau|$  have exponential moments under both  $\mathbb{P}_{v_\lambda}$  and  $\mathbb{P}_{v_\lambda}(\cdot | \Gamma)$ , uniformly in  $\lambda \in [\lambda_-, \lambda_+]$  for any fixed  $\lambda_-, \lambda_+ \in (\lambda_W, \infty)$ .*

These two propositions imply Theorem 1.2(a), with

$$v(\lambda) = \frac{\mathbb{E}_{v_\lambda}[|W_\tau| \Gamma]}{\mathbb{E}_{v_\lambda}[\tau | \Gamma]}, \quad \sigma(\lambda)^2 = \frac{\mathbb{E}_{v_\lambda}[(W_\tau - v(\lambda)\tau)^2 | \Gamma]}{\mathbb{E}_{v_\lambda}[\tau | \Gamma]}. \tag{6.13}$$

They also imply that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}_{v_\lambda}(t^{-1} W_t \notin (v - \varepsilon, v + \varepsilon)) < 0 \quad \forall \varepsilon > 0. \tag{6.14}$$

For a proof of these facts, the reader can follow word-by-word the arguments given in Avena, dos Santos and Völlering [5], Theorem 3.8 and Section 4.1.

Theorem 1.2(c) follows from (6.14) and the partial LDP proven in Avena, den Hollander and Redig [3] for attractive spin-flip systems (including the contact process). Here, partial means that the LDP is shown to hold outside a possible interval where the rate function is zero. However, (6.14) precisely precludes the presence of such an interval (see Glynn and Whitt [10], Theorem 3, for more details). We note that the LDP in [3] was proved for the particular case when  $\alpha_1 = \beta_0, \alpha_0 = \beta_1$ , but the same proof goes through under (1.6)–(1.7).

The proof of Theorem 1.2(b) is deferred to Section 6.4.

**Remark 6.3.** *It is easy to check that, if  $\lambda > \lambda_W$ , then  $\tau < \infty$   $\mathbb{P}_\eta$ -a.s. whenever  $\eta$  contains infinitely many infections. Hence the SLLN and the FCLT also hold under  $\mathbb{P}_\eta$  for any such  $\eta$ . Moreover, the result in [3] concerning the partial LDP holds whenever the initial measure  $\mu$  has positive correlations. If additionally  $\mu$  is stochastically larger than a non-trivial Bernoulli product measure, then it is possible to show that  $\tau$  has exponential moments under  $\mathbb{P}_\mu$ , which implies that the LDP also holds under  $\mathbb{P}_\mu$ , with a possibly different rate function that however still has a unique zero at  $v$ . A proof that  $\tau$  has exponential moments in this situation does not follow directly from the proof of Theorem 6.2 given below, but can be obtained with the help of the method used in the proof of Lemma 4.3.*

## 6.2. Proof of Proposition 6.1

We first show that the regeneration strategy indeed makes sense.

**Lemma 6.4.** For all  $t \geq 0$ ,

$$\mathbb{P}_{v_\lambda}(F_t = \infty, (W_{s+t} - W_t)_{s \geq 0} \in \cdot | \mathcal{G}_t) = \mathbb{P}_{\mathbb{1}_0}(\Gamma_0, W \in \cdot) \quad \text{a.s. on } \{\xi_t(W_t) = 1\}, \quad (6.15)$$

where  $\Gamma_0 := \{F_0 = \infty\}$ . The same is true for a finite stopping time w.r.t.  $\mathcal{G}$  replacing  $t$ .

**Proof.** First note that  $\mathbb{P}_\eta(\Gamma_0, W \in \cdot) = \mathbb{P}_{\mathbb{1}_0}(\Gamma_0, W \in \cdot)$  for any  $\eta$  with  $\eta(0) = 1$ . This follows from Lemma 2.2 because, on  $\Gamma_0$ ,  $W$  depends on  $\xi$  only through  $\{\xi_t(x) : t \geq 0, x \in [L_t(0), R_t(0)]\}$ , and  $\Gamma_0$  does not depend on  $\xi_0$ . Now, letting  $\hat{\xi}_t(\cdot) := \xi_t(\cdot + W_t)$ , we can write (recall (6.5))

$$\begin{aligned} & \mathbb{P}_{v_\lambda}(\xi_t(W_t) = 1, F_t = \infty, (W_{s+t} - W_t)_{s \geq 0} \in \cdot | \mathcal{G}_t) \\ &= \mathbb{E}_{v_\lambda}[\xi_t(W_t) \mathbb{P}_{\hat{\xi}_t}(\Gamma_0, W \in \cdot) | \mathcal{G}_t] = \xi_t(W_t) \mathbb{P}_{\mathbb{1}_0}(\Gamma_0, W \in \cdot), \end{aligned} \quad (6.16)$$

where the first equality is justified by the Markov property and the translation invariance of the graphical representation. To extend the result to stopping times we can use the strong Markov property of  $(\xi, W)$ .  $\square$

With the help of Lemma 6.4 we are ready to prove Proposition 6.1.

**Proof of Proposition 6.1.** We will closely follow the proof of Theorem 3.4 in [5]. Let  $\mathcal{G}_\tau$  be the  $\sigma$ -algebra of all events  $B$  such that, for all  $n \in \mathbb{N}_0$ , there exists a  $B_n \in \mathcal{G}_{\mathcal{T}_n}$  such that  $B \cap \{K = n\} = B_n \cap \{K = n\}$ . Note that  $\tau$  and  $W_{[0, \tau]}$  are in  $\mathcal{G}_\tau$ .

In the following, we abbreviate  $W^{(\tau)} := (W_{s+t} - W_t)_{s \geq 0}$ . Pick  $f$  bounded and measurable,  $B \in \mathcal{G}_\tau$ , and write (recall (6.9))

$$\begin{aligned} \mathbb{E}_{v_\lambda}[\mathbb{1}_B f(W^{(\tau)})] &= \sum_{n \in \mathbb{N}_0} \mathbb{E}_{v_\lambda}[\mathbb{1}_{B_n} \mathbb{1}_{\{K=n\}} f(W^{(\mathcal{T}_n)})] \\ &= \sum_{n \in \mathbb{N}_0} \mathbb{E}_{v_\lambda}[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}] \mathbb{E}_{v_\lambda}[\mathbb{1}_{\{F_{\mathcal{T}_n} = \infty\}} f(W^{(\mathcal{T}_n)}) | \mathcal{G}_{\mathcal{T}_n}]. \end{aligned} \quad (6.17)$$

Since  $\xi_{\mathcal{T}_n}(W_{\mathcal{T}_n}) = 1$  on  $\{\mathcal{T}_n < \infty\}$ , by Lemma 6.4 the last line of (6.17) equals

$$\begin{aligned} & \mathbb{E}_{\mathbb{1}_0}[f(W) \mathbb{1}_{\Gamma_0}] \sum_{n \in \mathbb{N}_0} \mathbb{E}_{v_\lambda}[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}] \\ &= \mathbb{E}_{\mathbb{1}_0}[f(W) | \Gamma_0] \sum_{n \in \mathbb{N}_0} \mathbb{E}_{v_\lambda}[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}] \mathbb{P}_{\mathbb{1}_0}(\Gamma_0), \end{aligned} \quad (6.18)$$

which, again by Lemma 6.4, equals

$$\begin{aligned} & \mathbb{E}_{\mathbb{1}_0}[f(W) | \Gamma_0] \sum_{n \in \mathbb{N}_0} \mathbb{E}_{v_\lambda}[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}] \mathbb{P}_{v_\lambda}(F_{\mathcal{T}_n} = \infty | \mathcal{G}_{\mathcal{T}_n}) \\ &= \mathbb{E}_{\mathbb{1}_0}[f(W) | \Gamma_0] \sum_{n \in \mathbb{N}_0} \mathbb{P}_{v_\lambda}(B_n, K = n) \\ &= \mathbb{E}_{\mathbb{1}_0}[f(W) | \Gamma_0] \mathbb{P}_{v_\lambda}(B) \\ &= \mathbb{E}_{v_\lambda}[f(W) | \Gamma] \mathbb{P}_{v_\lambda}(B), \end{aligned} \quad (6.19)$$

where the last equality is, one more time, justified by Lemma 6.4. This proves the claim under  $\mathbb{P}_{v_\lambda}$ .

To extend the claim to  $\mathbb{P}_{v_\lambda}(\cdot|\Gamma)$ , note that  $\Gamma \in \mathcal{G}_\tau$  since

$$\Gamma \cap \{K = n\} = \{\xi_0(0) = 1, W_s \in [L_s(0), R_s(0)] \forall s \in [0, T_n]\} \cap \{K = n\}, \tag{6.20}$$

and apply (6.19) to  $B \cap \Gamma$  instead of  $B$ . □

### 6.3. Proof of Proposition 6.2

#### Exponential moments

We first show that  $T_0$  has exponential moments when it is finite, uniformly for  $\lambda$  in compact sets. Fix  $\lambda_-, \lambda_+ \in (\lambda_W, \infty)$ ,  $\lambda_- \leq \lambda_+$ .

**Lemma 6.5.** *For every  $\varepsilon > 0$  there exists an  $a = a(\lambda_-, \lambda_+, \varepsilon) > 0$  such that, for every  $\lambda \in [\lambda_-, \lambda_+]$  and any probability measure  $\mu$  stochastically larger than  $\bar{v}_\lambda$ ,*

$$\begin{aligned} \text{(a)} \quad & \mathbb{E}_\mu[\mathbb{1}_{\{T_0 < \infty\}} e^{aT_0}] \leq 1 + \varepsilon, \\ \text{(b)} \quad & \mathbb{E}_\mu[e^{aV_0(0)}] \leq 1 + \varepsilon. \end{aligned} \tag{6.21}$$

**Proof.** We couple systems with infection rates  $\lambda_-, \lambda$  and  $\lambda_+$  starting, respectively, from  $\bar{v}_{\lambda_-}, \mu$  and  $\mathbf{1}$ , by coupling their initial configurations and their infection events monotonically. Denote their joint law by  $\mathbb{P}$ . In what follows, we will refer to these systems by their rates and we will use a superscript to indicate on which system a random variable depends.

We will bound  $T_0 \mathbb{1}_{\{T_0 < \infty\}} = T_0 \mathbb{1}_{\{F_0 < \infty\}}$  by a time  $D_0$  that depends only on systems  $\lambda_\pm$  and has exponential moments under  $\mathbb{P}$ . We start by bounding  $F_0 \mathbb{1}_{\{F_0 < \infty\}}$  by a variable  $D_1$  depending only on system  $\lambda_-$ . Let

$$r_t := \sup_{x \in \mathbb{Z}_{\leq 0}} R_t(x), \quad l_t := \inf_{x \in \mathbb{Z}_{\geq 0}} L_t(x). \tag{6.22}$$

Then  $r_t$  is the same as  $r_t(1, 0)$  in (6.3) when all sites in  $\mathbb{Z}_{\leq 0}$  are infected, and analogously for  $l_t$ . Furthermore,  $R_t(0), L_t(0)$  are equal to  $r_t, l_t$  while  $C_t(0) \neq \emptyset$ : this can be seen by using the graphical representation (see e.g. Liggett [11], Chapter VI, Theorem 2.2). Therefore

$$F_0 = \inf\{t \geq 0: r_t < W_t \text{ or } l_t > W_t\}. \tag{6.23}$$

Let  $m := \frac{1}{2}(\iota(\lambda_-) + |v_-| \vee |v_+|)$ . Take homogeneous random walks  $X^{(\pm)}$  with respective drifts  $v_\pm$ , independent of  $\xi$  and coupled to  $W$  in such a way that  $X_t^{(-)} \leq W_t \leq X_t^{(+)}$  for all  $t \geq 0$ . Set

$$\begin{aligned} D_{1a} &:= \sup\{t \geq 0: l_t^{\lambda_-} \geq -mt \text{ or } r_t^{\lambda_-} \leq mt\}, \\ D_{1b} &:= \sup\{t \geq 0: |X_t^{(-)}| \vee |X_t^{(+)}| > mt\}. \end{aligned} \tag{6.24}$$

Then  $D_{1a}$  depends only on system  $\lambda_-$  and has exponential moments by known large deviation bounds for  $r_t$  (see Liggett [11], Chapter VI, Corollary 3.22), while  $D_{1b}$  is independent of  $\xi$  and has exponential moments by standard large deviation bounds for  $X^{(\pm)}$ . Noting that  $r_t$  and  $l_t$  are monotone, we can take  $D_1 := D_{1a} \vee D_{1b}$ , which does not depend on the initial configuration.

Set  $\delta := \frac{1}{2}(\iota(\lambda_-) - m)$ ,  $x_0 := Y_{D_1}^{\lambda_+}(0, 0) - \lceil (\iota(\lambda_+) + \delta)D_1 \rceil$  and note, using the graphical representation, that  $\Delta_0 := x_0 - Z_\delta^{\lambda_-}(x_0)$  is independent of  $x_0$ , where  $Z_\delta(x)$  is as in (4.5). Then

$$D_0 := \frac{\Delta_0 + |x_0| + 1}{\iota(\lambda_-) - \delta - m} = 4 \frac{\Delta_0 + |x_0| + 1}{\iota(\lambda_-) - |v_-| \vee |v_+|} \tag{6.25}$$

depends only on  $\lambda_-, \lambda_+$  and has exponential moments under  $\mathbb{P}$  by Lemma 4.3. It is easy to check that  $D_0$  is the intersection time of the line of inclination  $\iota(\lambda_-) - \delta$  passing through  $(Z_\delta^{\lambda_-}(x_0) - 1, 0)$  and the line of inclination  $m$  passing through the origin. Since the system  $\lambda$  has more infections than the system  $\lambda_-$  and  $D_0 \geq D_1$ , we have  $T_0 \mathbb{1}_{\{T_0 < \infty\}} \leq D_0$ , which proves (a). For (b), we can bound  $V_0(0)$  analogously, taking  $x_0 = 0$  instead. □

*Infections at trial times*

We next show that at trial times there are more infections to the left of the random walk than under  $\bar{v}_\lambda$ .

**Lemma 6.6.** *For all  $n \in \mathbb{N}$ , a.s. on the event  $\{\mathcal{T}_n < \infty\}$ , the law of  $\xi_{\mathcal{T}_n}(\cdot + W_{\mathcal{T}_n})$  under  $\mathbb{P}_{v_\lambda}(\cdot | \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]})$  is stochastically larger than  $\bar{v}_\lambda$ .*

**Proof.** Suppose that  $n \geq 2$  (the case  $n = 1$  is simpler). Using the definition of  $\mathcal{T}_n$ , we can show by induction that, if  $\mathcal{T}_n < \infty$ , then there exist infection paths connecting  $(W_{\mathcal{T}_n}, \mathcal{T}_n)$  to  $\mathbb{Z}_{\leq -1} \times \{0\}$  and never touching the paths  $Y_{[\mathcal{T}_k, F_{\mathcal{T}_k}]}(W_{\mathcal{T}_k}, \mathcal{T}_k)$ ,  $k = 1, \dots, n-1$ , or the region to the right of  $W$ . Take  $\pi$  to be the right-most of these infection paths. Then  $\pi$  is a random infection path with properties (p1) and (p2), and

$$(\mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}) \in \mathcal{R}_{\mathcal{T}_n}^\pi \vee \sigma(N_{[0,\mathcal{T}_n]}, U_{[1,N_{\mathcal{T}_n}]}). \quad (6.26)$$

Therefore the result follows from Lemma 4.1.  $\square$

*Conclusion*

We are now ready to prove Proposition 6.2.

**Proof of Proposition 6.2.** Let

$$\kappa := \mathbb{P}_{\mathbb{1}_0}(I_0). \quad (6.27)$$

By Lemma 6.4,  $\mathbb{P}_{v_\lambda}(I) = \kappa \rho_\lambda \geq \kappa \rho_{\lambda_-}$  by monotonicity (recall the definition of  $\rho_\lambda$  from Section 1.2). Also, there exists a  $\kappa_- > 0$  such that  $\kappa \geq \kappa_-$  for any  $\lambda \geq \lambda_-$ : we can take  $\kappa_-$  to be the probability that  $X^{(\pm)}$  as in the proof of Lemma 6.5 never cross  $L(0)$  or  $R(0)$  in system  $\lambda_-$ . Therefore it is enough to prove the claim for  $\mathbb{P}_{v_\lambda}$ . Since  $|W_t|$  is dominated by  $N_t$ , which is a Poisson process independent of  $\xi$ , we only need to worry about  $\tau$ .

For  $\varepsilon > 0$  such that  $(1 + \varepsilon)(1 - \kappa_-) < 1$ , take  $a > 0$  as in Lemma 6.5. On the event  $\{\mathcal{T}_n < \infty\}$ , let  $\hat{\xi}_n := \xi_{\mathcal{T}_n}(\cdot + W_{\mathcal{T}_n})$  and note that, given  $\mathcal{G}_{\mathcal{T}_n}$ ,  $\mathcal{T}_{n+1} - \mathcal{T}_n$  is distributed as  $T_0$  under  $\mathbb{P}_{\hat{\xi}_n}$ . By Lemma 6.6, the law of  $\hat{\xi}_n$  under  $\mathbb{P}_{v_\lambda}(\cdot | \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]})$  is stochastically larger than  $\bar{v}_\lambda$ , and we get from Lemma 6.5 that

$$\begin{aligned} & \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a(\mathcal{T}_{n+1} - \mathcal{T}_n)} | \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}] \\ &= \mathbb{E}_{v_\lambda} [\mathbb{E}_{\hat{\xi}_n} [\mathbb{1}_{\{T_0 < \infty\}} e^{aT_0}] | \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}] \leq 1 + \varepsilon. \end{aligned} \quad (6.28)$$

Using this bound, estimate

$$\begin{aligned} \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a\mathcal{T}_{n+1}}] &= \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n} \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a(\mathcal{T}_{n+1} - \mathcal{T}_n)} | \mathcal{T}_n]] \\ &\leq (1 + \varepsilon) \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}], \end{aligned} \quad (6.29)$$

so that, by induction,

$$\mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}] \leq (1 + \varepsilon)^n. \quad (6.30)$$

Using Lemma 6.4, write, for  $n \in \mathbb{N}$ ,

$$\mathbb{P}_{v_\lambda}(K \geq n + 1) = \mathbb{P}_{v_\lambda}(\mathcal{T}_n < \infty, F_{\mathcal{T}_n} < \infty) = (1 - \kappa) \mathbb{P}_{v_\lambda}(K \geq n) \quad (6.31)$$

to note that  $K$  has distribution  $\text{GEO}(\kappa)$ . To conclude, use (6.30)–(6.31) to write

$$\begin{aligned} \mathbb{E}_{v_\lambda} [e^{(a/2)\tau}] &= \sum_{n \in \mathbb{N}} \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{K=n\}} e^{(a/2)\mathcal{T}_n}] = \sum_{n \in \mathbb{N}} \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{K=n\}} \mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{(a/2)\mathcal{T}_n}] \\ &\leq \sum_{n \in \mathbb{N}} \mathbb{P}_{v_\lambda}(K = n)^{1/2} \mathbb{E}_{v_\lambda} [\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}]^{1/2} \\ &\leq (1 - \kappa_-)^{-1/2} \sum_{n \in \mathbb{N}} (\sqrt{(1 - \kappa_-)(1 + \varepsilon)})^n < \infty, \end{aligned} \quad (6.32)$$

where in the second line we use the Cauchy–Schwarz inequality. □

6.4. *Continuity of the speed and the volatility*

Given  $\lambda_- \leq \lambda_+$  in  $(\lambda_W, \infty)$  and  $(\lambda_n)_{n \in \mathbb{N}}, \lambda_* \in [\lambda_-, \lambda_+]$  such that either  $\lambda_n \uparrow \lambda_*$  or  $\lambda_n \downarrow \lambda_*$  as  $n \rightarrow \infty$ , we can simultaneously construct systems with infection rates  $(\lambda_n)_{n \in \mathbb{N}}, \lambda_*$  and  $\lambda_{\pm}$ , starting from equilibrium, with a single graphical representation in the standard fashion, taking a monotone sequence of Poisson processes for infection events and coupling the initial configurations monotonically. For  $n \in \mathbb{N} \cup \{*, +, -\}$ , denote by  $\Lambda^n := (\xi_0^n, H, I^n, N, U)$  the system with infection rate  $\lambda_n$ , and by  $\mathbb{P}$  their joint law. In the following, we will use a superscript  $n$  to indicate functionals of  $\Lambda^n$ .

In view of (6.13) and Proposition 6.2, in order to prove convergence of  $v(\lambda_n)$  and  $\sigma(\lambda_n)$  it is enough to prove convergence in distribution of  $\Gamma^n$  and of  $(W_\tau^n, \tau^n) \mathbb{1}_{\Gamma^n}$ . The main step to achieve this will be to approximate relevant random variables with uniformly large probability by random variables depending on bounded regions of the graphical representation.

Note that, by monotonicity and continuity of  $\lambda \mapsto \rho_\lambda$  (see Liggett [11], Chapter VI, Theorem 1.6),

$$\lim_{n \rightarrow \infty} \xi_0^n(x) = \xi_0^*(x) \quad \forall x \in \mathbb{Z}, \mathbb{P}\text{-a.s.} \tag{6.33}$$

Recall the definitions of  $F_0, \mathcal{T}_k$  and  $K$  in (6.5), (6.8)–(6.9) and (6.10), respectively. For  $n \in \mathbb{N} \cup \{*\}$  and  $k \in \mathbb{N}$ , let

$$\Gamma_k^n := \{ \xi_0^n(0) = 1, W_s^n \in [L_s^n(0), R_s^n(0)] \forall s \in [0, \mathcal{T}_k^n] \cap \mathbb{R} \}, \tag{6.34}$$

so that  $\Gamma^n = \Gamma_k^n$  on  $\{K^n = k\}$  as in (6.20).

**Proposition 6.7.** *For every  $k \in \mathbb{N}$ ,  $(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n, \mathbb{1}_{\Gamma_k^n}) \mathbb{1}_{\{\mathcal{T}_k^n < \infty\}}, \mathbb{1}_{\{\mathcal{T}_k^n < \infty\}}$  and  $\mathbb{1}_{\{F_0^n < \infty\}}$  converge in probability as  $n \rightarrow \infty$  to the corresponding functionals of  $\Lambda^*$ .*

**Proof.** We first show that, for every fixed  $T \in (0, \infty)$ ,

$$(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n, \mathbb{1}_{\Gamma_k^n}) \mathbb{1}_{\{\mathcal{T}_k^n \leq T\}}, \quad \mathbb{1}_{\{\mathcal{T}_k^n \leq T\}}, \quad \mathbb{1}_{\{F_0^n \leq T\}}, \tag{6.35}$$

converge a.s. as  $n \rightarrow \infty$  to the corresponding functionals of  $\Lambda^*$ . To that end, let  $\bar{Y}_s(x, t)$  be the increasing analogue of  $Y_s(x, t)$  in (6.6), starting from  $x$  but jumping across the arrows of  $I$  to the right. Let  $\bar{Z}_\delta(x)$ , analogously to  $Z_\delta(x)$  in (4.5), be the first infected site to the right of  $x$  whose infection spreads inside a wedge between lines of inclination  $-(\iota + \delta)$  and  $-(\iota - \delta)$ . Take  $\delta := \iota(\lambda_-)/2$ , set  $\underline{y} := Y_T^+(-N_T, 0)$  and  $\underline{z} := Z_\delta^-(\underline{y} - \lceil (\iota(\lambda_-) + \delta)T \rceil)$ . Analogously, put  $\bar{y} := \bar{Y}_T^+(N_T, 0)$  and  $\bar{z} := \bar{Z}_\delta^-(\bar{y} + \lceil (\iota(\lambda_-) + \delta)T \rceil)$ .

Now observe that, for any  $n \in \mathbb{N} \cup \{*\}$ , all random variables in (6.35) depend on  $\Lambda^n$  only in the space–time box  $\mathcal{B} := [\underline{z}, \bar{z}] \times [0, T]$ . Indeed, for any  $0 \leq t \leq s \leq T$ , we have  $L_s^n(W_t^n, t) \geq Y_s^n(W_t^n, t) \geq \underline{y}^-$  and  $R_s^n(W_t^n, t) \leq \bar{y}^+$ , so that  $\{F_t^n \leq s\}$  depends on  $\Lambda^n$  only inside  $[\underline{y}, \bar{y}] \times [0, T]$ . Also, there are infection paths from time 0 to time  $T$  inside  $[\underline{z}, \underline{y})$  and  $(\bar{y}, \bar{z}]$ . Therefore the states of  $\xi^n$  inside  $[\underline{y}, \bar{y}] \times [0, T]$  depend on  $\Lambda^n$  only in  $\mathcal{B}$  (see the proof of Lemma 2.2). The same is true for  $\{T_t^n \leq s\}$ , since any infection path needed to discover  $T_t^n$  can be taken inside  $\mathcal{B}$ . Therefore, by (6.33) (and since the graphical representation is a.s. eventually constant inside bounded space–time regions), the claim after (6.35) follows.

To conclude note that, because  $\mathcal{T}_k \mathbb{1}_{\{\mathcal{T}_k < \infty\}} \leq \tau$  and  $F_0 \mathbb{1}_{\{F_0 < \infty\}} \leq T_0 \mathbb{1}_{\{T_0 < \infty\}}$ ,

$$\lim_{T \rightarrow \infty} \sup_{n \in \mathbb{N} \cup \{*\}} \mathbb{P}(T < \mathcal{T}_k^n < \infty \text{ or } T < F_0^n < \infty) = 0 \tag{6.36}$$

by Proposition 6.2 and Lemma 6.5, which implies that, for large  $T$ , the random variables in the statement are equal to the ones in (6.35) with uniformly large probability. □

**Corollary 6.8.** *Let  $\kappa^n$  be as in (6.27). Then  $\lim_{n \rightarrow \infty} \kappa^n = \kappa^*$  and  $K^n$  converges in distribution to  $K^*$ .*

**Proof.** This follows directly from Proposition 6.7 and the definition of  $\kappa$  since, by (6.31),  $K^n$  is a geometric random variable with parameter  $\kappa^n$ .  $\square$

With these results we can conclude the proof of Theorem 1.2(c).

**Proof of Theorem 1.2(c).** Let  $f$  be a bounded measurable function. For  $k \in \mathbb{N}$ , write

$$\begin{aligned} \mathbb{E}[f(W_{\tau^n}^n, \tau^n) \mathbb{1}_{\Gamma^n} \mathbb{1}_{\{K^n=k\}}] &= \mathbb{E}[f(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n) \mathbb{1}_{\Gamma_k^n} \mathbb{1}_{\{\mathcal{T}_k^n < \infty, F_{\mathcal{T}_k^n} = \infty\}}] \\ &= \kappa^n \mathbb{E}[f(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n) \mathbb{1}_{\Gamma_k^n} \mathbb{1}_{\{\mathcal{T}_k^n < \infty\}}] \\ &\xrightarrow{n \rightarrow \infty} \kappa^* \mathbb{E}[f(W_{\mathcal{T}_k^*}^*, \mathcal{T}_k^*) \mathbb{1}_{\Gamma_k^*} \mathbb{1}_{\{\mathcal{T}_k^* < \infty\}}] \\ &= \mathbb{E}[f(W_{\tau^*}^*, \tau^*) \mathbb{1}_{\Gamma^*} \mathbb{1}_{\{K^*=k\}}], \end{aligned} \tag{6.37}$$

where for the second and the third equality we use Lemma 6.4 and the strong Markov property, and for the convergence we use Proposition 6.7 and Corollary 6.8. Therefore

$$\begin{aligned} &|\mathbb{E}[f(W_{\tau^n}^n, \tau^n) \mathbb{1}_{\Gamma^n}] - \mathbb{E}[f(W_{\tau^*}^*, \tau^*) \mathbb{1}_{\Gamma^*}]| \\ &\leq \|f\|_{\infty} \{ \mathbb{P}(K^n > M) + \mathbb{P}(K^* > M) \} \\ &\quad + \sum_{k=1}^M |\mathbb{E}[f(W_{\tau^n}^n, \tau^n) \mathbb{1}_{\Gamma^n} \mathbb{1}_{\{K^n=k\}}] - \mathbb{E}[f(W_{\tau^*}^*, \tau^*) \mathbb{1}_{\Gamma^*} \mathbb{1}_{\{K^*=k\}}]|, \end{aligned} \tag{6.38}$$

and we conclude by taking  $n \rightarrow \infty$ , using Corollary 6.8 and (6.37), and taking  $M \rightarrow \infty$ .  $\square$

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