Scaling of entanglement close to a quantum phase transition

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Classical phase transitions occur when a physical system reaches a state below a critical temperature characterized by macroscopic order[1]. Quantum phase transitions occur at absolute zero; they are induced by the change of an external parameter or coupling constant[2], and are driven by quantum fluctuations. Examples include transitions in quantum Hall systems[3], localization in Si-MOSFETs (metal oxide silicon field-effect transistors; ref. 4) and the superconductor–insulator transition in two-dimensional systems[5,6]. Both classical and quantum critical points are governed by a diverging correlation length, although quantum systems possess additional correlations that do not have a classical counterpart. This phenomenon, known as entanglement, is the resource that enables quantum computation and communication[8]. The role of entanglement at a phase transition is not captured by statistical mechanics—a complete classification of the critical many-body state requires the introduction of concepts from quantum information theory[9]. Here we connect the theory of critical phenomena with quantum information by exploring the entangling resources of a system close to its quantum critical point. We demonstrate, for a class of one-dimensional magnetic systems, that entanglement shows scaling behaviour in the vicinity of the transition point.

There are various questions that emerge in the study of this problem. Because the ground-state wavefunction undergoes qualitative changes at a quantum phase transition, it is important to understand how its genuine quantum aspects evolve throughout the transition. Will entanglement between distant subsystems be extended over macroscopic regions, as correlations are? Will it carry distinct features of the transition itself and show scaling behaviour? Answering these questions is important for a deeper understanding of quantum phase transitions, and also from the perspective of quantum information theory. So results that bridge these two areas of research are of great relevance.

We study a set of localized spins coupled through exchange interaction and subjected to an external magnetic field (we consider only spin-1/2 particles), a model central both to condensed-matter and information theory and subject to intense study[10]. In the Heisenberg chain, the maximization of the entanglement at zero temperature is related to the energy minimization[11]. It is known that Werner states[12] can be generated in a one dimensional XY model[13], and that temperature and magnetic field can increase the entanglement of the systems, as shown for the Ising and Heisenberg models[14,15]. Finally, we mention the study of the role of the entanglement in the density matrix renormalization group flow and the introduction of entanglement-preserving renormalization schemes[16]. Here we address the problem of the relation between macroscopic order, classical correlations and quantum correlations. Therefore we analyse the entanglement near the critical point of the XY model in a transverse magnetic field. Because of the universality principle—the critical behaviour depends only on the dimension of the system and the symmetry of the order parameter—our results have much broader validity. We find that in the vicinity of a quantum phase transition the entanglement obeys scaling behaviour. On the other hand, this analysis provides a clear distinction between the role of entanglement and correlations in quantum systems close to a critical point. (We have been made aware that similar work to that reported here is being performed by T. Osborne and M. Nielsen; T. Osborne, personal communication.)

The system under consideration is a spin-1/2 ferromagnetic chain with an exchange coupling J in a transverse magnetic field of

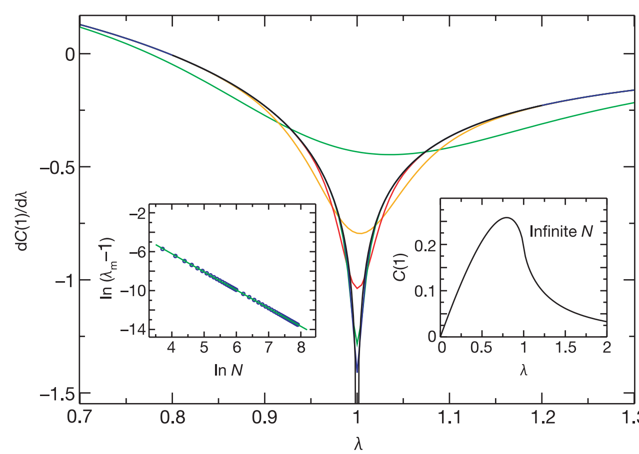

Figure 1 The change in the ground-state wavefunction in the critical region is analysed considering $dC(1)/d\lambda$ as a function of the reduced coupling strength λ. The curves correspond to different lattice sizes $N = 11, 41, 101, 251, 401, \infty$. We choose N odd to avoid the subtleties connected with boundary terms[16]. On increasing the system size, the minimum gets more pronounced. Also the position of the minimum changes and tends as $N^{-1.87}$ (left inset) towards the critical point $\lambda_c = 1$ where for an infinite system a logarithmic divergence is present (see equation (3)). The right inset shows the behaviour of the concurrence $C(1)$ itself for an infinite system. The maximum that occurs below λ_c is not related to the critical properties of the Ising model. As explained in the text, it is the change in the ground state and not the wavefunction itself that is a good indicator of the transition. The structure of the reduced density matrix, necessary to calculate the concurrence, follows from the symmetry properties of the hamiltonian. Reality and parity conservation of H together with translational invariance already fix the structure of ρ to be real symmetric with ρ_{11}, $\rho_{22} = \rho_{33}$, ρ_{23}, ρ_{14}, ρ_{44} as the only non-zero entries.

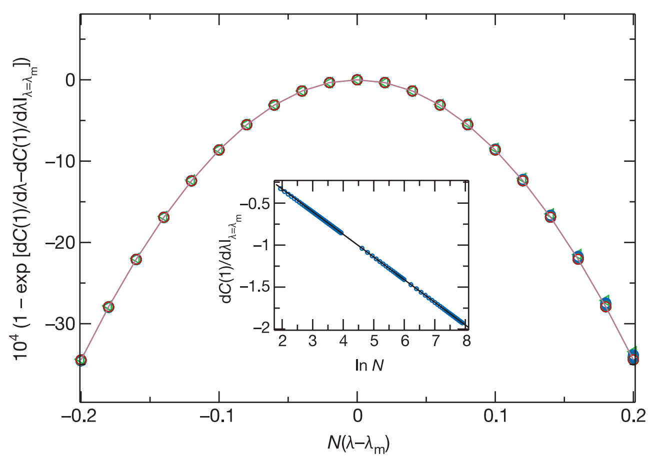

Figure 2 The finite size scaling is performed for the case of logarithmic divergences[22]. The concurrence, considered as a function of the system size and the coupling constant is a function of $N^{1/\nu}(\lambda - \lambda_m)$ only, and in the case of logarithmic divergence it behaves as $[dC(1)/d\lambda]_{N,\lambda} - [dC(1)/d\lambda]_{N,\lambda_0} \sim Q[N^{1/\nu}(\lambda - \lambda_m)] - Q[N^{1/\nu}(\lambda_0 - \lambda_m)]$ where λ_0 is a non-critical value and $Q(x) \sim Q(\infty) \ln x$ (for large x). All the data from $N = 41$ up to $N = 2{,}701$ collapse on a single curve. The critical exponent is $\nu = 1$, as expected for the Ising model. The inset shows the divergence of the value at the minimum as the system size increases.

strength h. The hamiltonian is

$$H = -\frac{J}{2}(1+\gamma)\sum_{i=1}^{N}\sigma_i^x\sigma_{i+1}^x - \frac{J}{2}(1-\gamma)\sum_{i=1}^{N}\sigma_i^y\sigma_{i+1}^y - h\sum_{i=1}^{N}\sigma_i^z \quad (1)$$

where σ^a are the Pauli matrices ($a = x, y, z$) and N is the number of sites. We assume periodic boundary conditions. It is convenient, for later purposes, to define a dimensionless coupling constant $\lambda = J/2h$. For $\gamma = 1$ equation (1) reduces to the Ising model, whereas for $\gamma = 0$ it is the XY model. For all the interval $0 < \gamma \leq 1$ the models belong to the Ising universality class and for $N = \infty$ they undergo a quantum phase transition at the critical value $\lambda_c = 1$. The magnetization $\langle\sigma^x\rangle$ is different from zero for $\lambda > 1$ and it vanishes at the transition. On the contrary the magnetization along the z direction $\langle\sigma^z\rangle$ is different from zero for any value of λ. At the phase transition the correlation length ξ diverges as $\xi \sim |\lambda - \lambda_c|^{-\nu}$ with $\nu = 1$ (refs 17 and 18).

We confine our interest to the entanglement between two spins, of position i and j, in the chain. All the information needed is contained in the reduced density matrix $\rho(i,j)$ obtained from the ground-state wavefunction after all the spins except those at positions i and j have been traced out. The resulting $\rho(i,j)$ represents a mixed state of a bipartite system; a good deal of work has been devoted to quantifying the entanglement in this case[19–22]. We use the concurrence[22] between sites i and j, related to the "entanglement of formation"[21], and defined as

$$C(i,j) = \max\{r_1(i,j) - r_2(i,j) - r_3(i,j) - r_4(i,j), 0\} \quad (2)$$

In equation (2), $r_\alpha(i,j)$ are the square roots of the eigenvalues of the product matrix $R = \rho(i,j)\tilde{\rho}(i,j)$ in descending order; the spin flipped matrix is defined as $\tilde{\rho} = \sigma^y \otimes \sigma^y \rho^* \sigma^y \otimes \sigma^y$. In the previous definition, the eigenstates of σ^z $\{|\uparrow\rangle, |\downarrow\rangle\}$ should be used. Translation invariance implies that $C(i,j) = C(|i-j|)$. The concurrence will be evaluated as a function of the relative position $|i-j|$ between the spins and the distance $|\lambda - \lambda_c|$ from the critical point. The structure of the reduced density matrix is obtained by exploiting symmetries of the model (see Fig. 1 legend). The non-zero entries of ρ can then be related to the various correlation functions, and the concurrence of the ground state is evaluated exactly starting from the results in refs 17, 18 and 23.

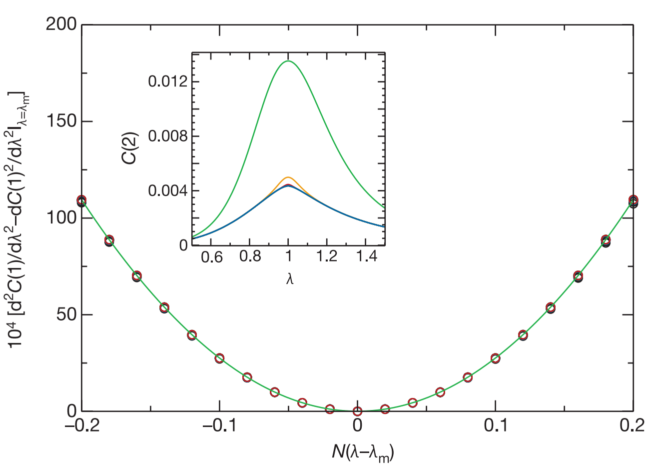

Figure 3 As in the case of the nearest-neighbour concurrence, data collapse is also obtained for the next-nearest-neighbour concurrence $C(2)$. In the figure, data for system size from $N = 41$ to $N = 401$ are plotted. The inset shows a peculiarity of the Ising model: $C(2)$ has its maximum at the critical point for arbitrary system size (note that the maximum decreases as the size increases). Therefore we consider the second derivative to perform the scaling analysis. It can also be seen that $C(2)$ is two orders of magnitude smaller than $C(1)$. For the smallest system sizes, the concurrence is different from zero for $|i-j| = 3$ and $\lambda > 1.05$ (for $N = 7$; for $N \approx 9$ $C(3) = 0$ for all λ). In contrast, the correlation functions are long-ranged at the critical point.

First, we look at the Ising model ($\gamma = 1$). The first question we consider is the range of the entanglement ξ_E, that is, the maximum distance between two spins at which the concurrence is different from zero. The result is surprising: even at the critical point, where spin–spin correlations extend over a long range (the correlation length is diverging for an infinite system), the concurrence vanishes unless the two sites are at most next-nearest neighbours. The truly non-local quantum part of the two-point correlations is nonetheless very short-ranged.

In order to quantify the change of the many-body wavefunction when the system crosses the critical point, we look at the derivatives of the concurrence as a function of λ. In this case we need to consider only the nearest-neighbour and next-nearest-neighbour concurrence. We first discuss the behaviour of the nearest-neighbour concurrence. The results for systems of different size (including the thermodynamic limit) are presented in Fig. 1. For the infinite chain $\partial_\lambda C(1)$ diverges on approaching the critical value as:

$$\frac{\mathrm{d}C(1)}{\mathrm{d}\lambda} = \frac{8}{3\pi^2}\ln|\lambda - \lambda_c| + \text{const.} \quad (3)$$

Equation (3) quantifies non-local correlations in the critical region. One aspect of this system, particularly relevant for quantum information, is the study of the precursors of the critical behaviour in finite samples. This study is known as finite size scaling[24]. In Fig. 1 the derivative of $C(1)$ with respect to λ is considered for different system sizes. As expected, there is no divergence for finite N, but there are clear anomalies. The position of the minimum λ_m scales as $\lambda_m \sim \lambda_c + N^{-1.86}$ and its value diverges logarithmically with increasing system size as:

$$\left.\frac{\mathrm{d}C(1)}{\mathrm{d}\lambda}\right|_{\lambda_m} = -0.2702\ln N + \text{const.} \quad (4)$$

According to the scaling ansatz[24] in the case of logarithmic singularities, the ratio between the two prefactors of the logarithm in equations (3) and (4) is the exponent that governs the divergence of the correlation length ν. In this case ($8/3\pi^2 \approx 0.2702$) it follows that $\nu = 1$, as it is known from the solution of the Ising model[17]. By proper scaling[24] and taking into account the distance of the minimum of $C(1)$ from the critical point, it is possible to make all the data from different N collapse onto a single curve (Fig. 2). This figure contains the data for the lattice size ranging from $N = 41$ up to $N = 2{,}701$. These results show that all the key ingredients of the finite size scaling are present in the concurrence. We note that

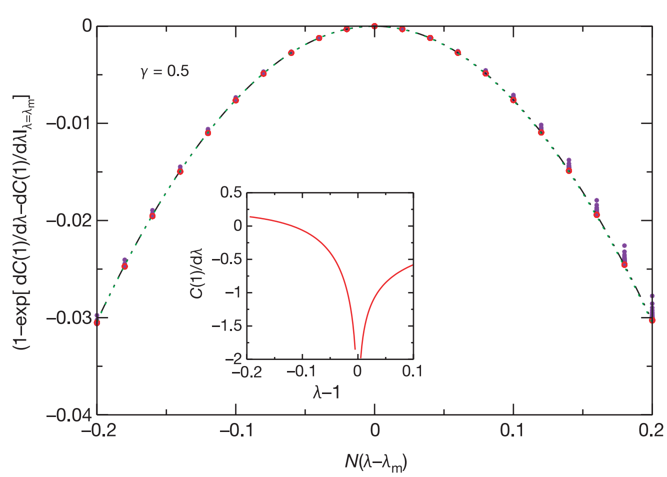

Figure 4 The universality hypothesis for the entanglement is checked by considering the model hamiltonian, defined in equation (1), for a different value of γ. In this case we chose $\gamma = 0.5$ and N ranging from 41 up to 401. Data collapse, shown here for $C(1)$, is obtained for $\nu = 1$, consistent with the model being in the universally class of the Ising model. In the inset is shown the divergence at the critical point for the infinite system.

finite size scaling is fulfilled over a very broad range of values of N which are of interest in several protocols in quantum information.

A similar analysis can be carried on for the next-nearest-neighbour concurrence $C(2)$. Since $\partial_\lambda C(2)|_{\lambda_c} = 0$, the logarithmic singularity here appears first in the second derivative with respect to λ (see Fig. 3 legend). In the thermodynamic limit $\partial^2_\lambda C(2) = 0.108 \ln|\lambda - \lambda_c| + \text{const}$. In this case also, the data collapse and finite size scaling (Fig. 3) agree with the expected scaling behaviour, $\nu = 1$. This completes the analysis of entanglement for the one-dimensional Ising model.

A cornerstone of the theory of critical phenomena is the concept of universality—that is, the critical properties depend only on the dimensionality of the system and the broken symmetry in the ordered phase. Universality in the critical properties of entanglement was verified by considering the properties of the family of models defined in equation (1) with $\gamma \neq 1$.

Second, we consider the case for $\gamma \neq 1$. The range of entanglement ξ_E is not universal. The maximum possible distance between entangled pairs increases and tends to infinity as γ tends to zero. From the asymptotic behaviour of the reduced density matrix[18] we find that ξ_E goes as γ^{-1}. This however has no dramatic consequences; the "total concurrence" $\Sigma_n C(n)$ stored in the chain is an increasing function of γ (for $0 < \gamma \leq 1$, $0 < \Sigma_n C(n) < 0.2$). More interesting is the critical behaviour of the concurrence. To be specific we consider $C(1)$ in the case $\gamma = 0.5$, shown in Fig. 4. As it was obtained for the hamiltonian of equation (1), scaling is fulfilled with the critical exponent $\nu = 1$ in agreement with the universality hypothesis.

The analysis of the 'resource' entanglement for a condensed-matter system close to a quantum critical point allows us to characterize both quantitatively and qualitatively the change in the wavefunction of the ground state on passing the phase transition. A notable feature which emerges is that though the entanglement itself is not an indicator of the phase transition, an intimate connection exists between entanglement, scaling and universality. In a way, this analysis allows us to discern what is genuinely quantum in a zero-temperature phase transition. The results presented here might be tested by measuring different correlation functions, for example with neutron scattering, and extracting from these the entanglement properties of the ground state close to the critical point[14].

We finally discuss this work in the context of quantum computation. First, system sizes considered here ($\sim 10^3$) could be those of a realistic quantum computer. Second, the scaling behaviour found could be a powerful tool to evaluate (and hence to use) entanglement in systems having different numbers of qubits. In particular, close to the critical point, the entanglement depends strongly on the field—so it could be tuned, realizing an 'entanglement switch'. Last, long-range correlations, typical of the critical region, might be of great importance in stabilizing the system against errors due to imperfections. □

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Competing interests statement

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Spontaneous breaking of time-reversal symmetry in the pseudogap state of a high-T_c superconductor

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A change in 'symmetry' is often observed when matter undergoes a phase transition—the symmetry is said to be spontaneously broken. The transition made by underdoped high-transition-temperature (high-T_c) superconductors is unusual, in that it is not a mean-field transition as seen in other superconductors. Rather, there is a region in the phase diagram above the superconducting transition temperature T_c (where phase coherence and superconductivity begin) but below a characteristic temperature T^* where a 'pseudogap' appears in the spectrum of electronic excitations[1,2]. It is therefore important to establish if

