International Journal of Fracture **83:** 19–40, 1997.

Scaling of quasibrittle fracture: asymptotic analysis

ZDENĚK P. BAŽANT
Walter P. Murphy Professor of Civil Engineering and Materials Science, Northwestern University, Evanston, Illinois 60208, USA

Received 10 April 1996; accepted in revised form 5 September 1996

Abstract. Fracture of quasibrittle materials such as concrete, rock, ice, tough ceramics and various fibrous or particulate composites, exhibits complex size effects. An asymptotic theory of scaling governing these size effects is presented, while its extension to fractal cracks is left to a companion paper [1] which follows. The energy release from the structure is assumed to depend on its size D, on the crack length, and on the material length c_f governing the fracture process zone size. Based on the condition of energy balance during fracture propagation and the condition of stability limit under load control, the large-size and small-size asymptotic expansions of the size effect on the nominal strength of structure containing large cracks or notches are derived. It is shown that the form of the approximate size effect law previously deduced [2] by other arguments can be obtained from these expansions by asymptotic matching. This law represents a smooth transition from the case of no size effect, corresponding to plasticity, to the power law size effect of linear elastic fracture mechanics. The analysis is further extended to deduce the asymptotic expansion of the size effect for crack initiation in the boundary layer from a smooth surface of structure. Finally, a universal size effect law which approximately describes both failures at large cracks (or notches) and failures at crack initiation from a smooth surface is derived by matching the aforementioned three asymptotic expansions.

Key words: Scaling, Size effect, Structural strength, Load capacity, Nonlinear fracture mechanics, quasibrittle fracture, asymptotic methods, stable load growth, crack initiation, energy release.

1. Introduction

Scaling, that is, the change of response due to similarity preserving changes of the size of a physical system, is the most fundamental aspect of every physical theory. If scaling is not understood, the phenomenon itself is not understood. While scaling has played a central role in many branches of physics, especially fluid mechanics, and important theories have been developed [3–5], in solid mechanics the problem of scaling has until recently been largely neglected. The reason for this is that the theories of structural failure that have prevailed for a long time exhibit no deterministic size effect [6]. These are:

(1) plasticity and other theories based on the concept of critical stress (strength) or critical strain, and
(2) fracture mechanics applied to a critical flaw (crack) whose size at incipient failure is independent of the structure size D and negligible in comparison to D, as is typical of most metal structures embrittled by fatigue.

Therefore, the experimentally observed size effects were generally attributed to the randomness on material strength, as proposed by Weibull [7]. However, even though this explanation is realistic for metallic and other structures that fail before the crack reaches macroscopic dimensions, it does not work for quasibrittle structures [8, 9], as will be explained in the subsequent companion paper [1] in this issue.

This paper deals with structures made of quasibrittle materials, such as concrete, rock, ice, tough ceramics and some composites. As a result of their heterogeneity and development of a large fracture process zone, these materials typically fail only after a large crack has grown in a stable manner. The size effect is understood to be the change of σ_N nominal strength as a function of D in geometrically similar structures with similar cracks (the effect of deviation of cracks from similarity is an effect of shape, which must be described separately from the size effect). σ_N of such structures exhibits a complex size effect, which has been explained by the release of stored energy caused by fracture. While the energy dissipated in geometrically similar structures with similar cracks is proportional to the crack length and thus to the structure size D, the energy release caused by fracture in structures under the same nominal stress grows with D faster than proportionally – hence the size effect. But because of the large size of the fracture process zone, which releases additional energy in proportion to the fracture length, the overall energy release grows with D less than quadratically. This causes the size effect to deviate from the power law size effect $(\sigma \propto D^{-(1/2)})$ of linear elastic fracture mechanics (LEFM).

The deviation of size effect from LEFM was experimentally observed on geometrically similar notched concrete specimens by Walsh [10, 11]. Subsequently, an approximate size effect law describing such observations was derived by simple energy release arguments and by dimensional analysis with similitude arguments [12, 2, 6, 9]. This simple law represents a smooth transition from the case of plasticity, for which there is no size effect, to the case of LEFM, for which the size effect is the strongest possible. This law has been verified by numerous test data, and corroborated by extensive non-local finite element solutions and by discrete element (random particle) simulations [13, 14]. It has been generalized to a form involving material fracture parameters [15]. This generalization made it possible to determine the fracture energy G_f of the material and the effective length c_f of the fracture process zone by measuring only the maximum loads of geometrically similar notched specimens of different sizes.

The size effect law proposed by Bažant applies only to failures after large stable crack growth. Such failures are typical of reinforced concrete structures as well as some plain concrete structures such as dams, in which continuous cracks, typically extending over 50 percent to 90 percent of the cross section, grow in a stable manner before the maximum load is reached.

A different size effect occurs for failures at crack initiation in the boundary layer at a smooth surface, as known from the tests of the modulus of rupture and demonstrated by finite element analysis based on the cohesive (or fictitious) crack model [16, 17]. By analysis of the stress redistribution during the formation of a finite-size fracture process zone at a smooth surface, a simple law for this type of size effect has also been derived [18, 19].

The problem of scaling of quasibrittle failure is very difficult for normal structural dimensions. But it becomes easy in the asymptotic sense for very large structures as well as very small structures. For such problems of physics, a powerful approximate approach is the asymptotic matching [3, 4], that is, interpolation between opposite asymptotic expansions. The purpose of the present paper (whose main contents were summarized at three recent conferences [20–22]) is to derive the general large-size and small-size asymptotic expansions of the size effect and then deduce the approximation for normal structure sizes by asymptotic matching. New size effect laws for quasibrittle failure at crack initiation from a smooth surface and for the transition to failure after large stable crack growth will be derived.

The companion paper which follows in this issue [1] (whose main contents were also summarized in [20–22]) will explore the question whether the observed size effects can be explained by fractality, either by fractal nature of a continuous crack surface or by fractal

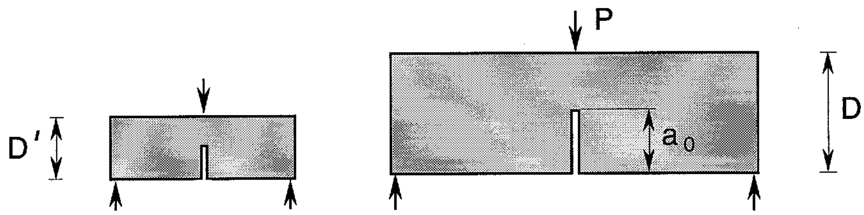

Figure 1. Geometrically similar structures of different sizes, with large similar cracks of notches.

distribution of microcracks. The latter is in a certain sense related to the statistical Weibull-type theory of size effect, and in that context it will also be explained why that theory does not apply to quasibrittle failures.

2. Energy release and dimensionless parameters

To characterize the size effect in geometrically similar structures of different sizes D (Figure 1), we introduce, as usual, the nominal stress

$$\sigma_N = \frac{P}{bD}, \tag{1}$$

where D = characteristic size (dimension) of the structure, P = load applied on the structure (or load parameter), and b = structure thickness in the third dimension. Our analysis will be restricted to two-dimensional similarity, although generalization to three-dimensional similarity would be easy and would not change the basic nature of the conclusions. When $P = P_{\max}$ = maximum load, σ_N is called the nominal strength of the structure, and σ_N is then just a convenient parameter of the maximum load. The load is considered to be a dead load (i.e., a load independent of displacement).

For a quasibrittle material, i.e., a brittle material with a large fracture process zone, the fracture energy G_f (with metric dimension J/m^2) is defined as the rate of energy dissipation of a statically propagating crack per unit area of the crack surface in a sufficiently large structure, such that the fracture process zone is negligible compared to the structure dimensions. (A large enough structure is required to ensure that the asymptotic LEFM stress field surround the process zone; because this field is independent of the boundary geometry and loading arrangement, the same is true for the fracture energy so defined.) For a quasibrittle material, one must also introduce a second essential fracture characteristic, defined as the effective length c_f of the fracture process zone (or cohesive zone) at fracture front in a sufficiently large structure. The actual fracture process zone size varies depending on the structure geometry and characteristic structure size (dimension) D. The special case of LEFM will, of course, be included in the analysis as the limiting case for $c_f \to 0$.

We have three basic parameters influencing σ_N – the current crack length a, the initial traction-free crack length a_0 (the notch length), and c_f, all having the dimension of length. They must appear in the energy release expression nondimensionally. The dimensionless variables may be chosen as

$$\alpha_0 = a_0/D, \qquad \alpha = a/D, \qquad \theta = c_f/D. \tag{2}$$

The energy stored in the structure is characterized by the complementary energy Π^* (representing, under isothermal conditions, Gibbs' free energy of the structure). Π^* must be expressed as a function of θ, α and α_0

$$\Pi^* = \frac{\sigma_N^2}{E}\, bD^2 f(\alpha_0, \alpha, \theta), \tag{3}$$

where f is a dimensionless continuous function. This function characterizes the geometry of the structure (including the initial notch depth, boundary geometry and arrangement of loads), but is independent of D. That Π^* can be expressed in the form of (3) follows from Buckingham's Π-theorem of dimensional analysis (e.g. [5, 3]), which states that any physical relationship can be expressed as a dimensionless function of a dimensionless combination of governing parameters, whose number is the total number of governing parameters minus the number of governing parameters of different dimensions.

The energy, R, dissipated by the crack per unit area of the crack surface depends on the size, shape and other characteristics of the fracture process zone which, in quasibrittle materials, vary during crack propagation. Therefore, same as Π^*, R must in general also depend on α_0, α and θ, and so we may write

$$R = G_f r(\alpha_0, \alpha, \theta), \tag{4}$$

where r is a continuous function such that $r \to 1$ or $R \to G_f$ for $\alpha \gg \theta$ or $a \gg c_f$ and $D \gg c_f$. Usually it is assumed that R depends only on $c = a - a_0$, in which case the function $R(c)$ is called the resistance curve or briefly the R-curve. But, in general, R also depends on the structure geometry, as indicated in (4).

Instead of the effective length of the fracture process zone, we may consider c_f to represent, more generally, any material length governing failure. In continuum damage mechanics, originated by Kachanov, material failure is characterized in terms of a critical damage energy release rate W_d per unit volume of material (e.g. [23]). Plasticity, or any failure theory with a yield surface in terms of stresses or strains, can also be cast in a form in which the size of the yield surface is proportional to a certain critical energy stored per unit volume (with von Mises plasticity as the simplest example). In fracture mechanics, by contrast, the material failure is characterized by critical energy dissipation G_f per unit surface area. A quasibrittle material possesses both characteristics. So, instead of directly postulating the existence of material length c_f, one might prefer to assume that failure is governed by both G_f and W_d. However, such an assumption is equivalent. Because various kinds of dimensionless combinations of the governing parameters can be used as the basic parameters, one may, instead of (2), define the dimensionless parameter θ as

$$\theta = \frac{1}{D} \frac{G_f}{W_d}. \tag{5}$$

But this can again be written as $\theta = c_f/D$, where $c_f = G_f/W_d$. Thus, a material length emerges in the formulation anyway.

Similarly, consider nonlinear fracture mechanics, in which the crack tip is surrounded by a finite fracture process zone. Instead of directly postulating the existence of material length, one makes the hypothesis that failure is governed not only by fracture energy G_f but also by tensile strength f_t. But this hypothesis is again equivalent. Instead of (2), the dimensionless parameter may be defined as

$$\theta = \frac{1}{D}\frac{EG_f}{f_t^2}. \tag{6}$$

This can again be written as $\theta = c_f/D$, where $c_f = EG_f/f_t^2$ (which coincides with Irwin's characteristic size of the fracture process zone [24, 25]).

3. Energy conditions for crack propagation at maximum load

The crack can propagate if its energy release rate $\mathcal{G}$ becomes equal to R. Introducing the well-known expression for the energy release rate in terms of Π^*, we have the condition

$$\mathcal{G} = \frac{1}{b}\left[\frac{\partial\Pi^*}{\partial a}\right]_{\sigma_N} = R, \tag{7}$$

where b is the structure thickness in the third dimension at the place of the fracture process zone, and subscript σ_N means that the partial derivative must be calculated at constant σ_N, or at constant load P. Substituting (3) and differentiating, we obtain the equation

$$\mathcal{G} = \frac{\sigma_N^2}{E} D\tilde{g}(\alpha_0, \alpha, \theta) = G_f r(\alpha_0, \alpha, \theta), \tag{8}$$

in which we introduced the notation

$$\tilde{g}(\alpha_0, \alpha, \theta) = \frac{\partial f(\alpha_0, \alpha, \theta)}{\partial \alpha}. \tag{9}$$

Function $\tilde{g}$ represents the generalized dimensionless energy release rate (generalized, because the usual energy release rate is considered to be a function of only one variable, a or α). For the special case $c_f/D \to 0$, function $\tilde{g}$ coincides with that in LEFM. Same as f, function $\tilde{g}$ reflects the geometry of the structure, crack and load, but is independent of D.

The condition of maximum load, as is well known (e.g. [26], Sect. 12.3), is the condition that the energy release rate curve at constant load (or constant σ_N) must be tangent to the R-curve, that is, $[\partial\mathcal{G}/\partial\alpha]_{\sigma_N} = \partial R/\partial\alpha$. It is convenient to divide this equation by $\mathcal{G} = R$, which yields the maximum load condition in the form

$$\frac{1}{\mathcal{G}}\left[\frac{\partial\mathcal{G}}{\partial\alpha}\right]_{\sigma_N} = \frac{1}{R}\frac{\partial R}{\partial\alpha}, \tag{10}$$

which may also be written as $[\partial \ln \mathcal{G}/\partial\alpha]_{\sigma_N} = \partial \ln R/\partial\alpha$. Expressing $\mathcal{G}$ and R according to (8) and (4), we acquire the equation

$$\frac{\partial \ln \tilde{g}(\alpha_0, \alpha, \theta)}{\partial\alpha} = \frac{\partial \ln r(\alpha_0, \alpha, \theta)}{\partial\alpha}, \tag{11}$$

in which σ_N, E, D, G_f and $\mathcal{G}$ do not appear. This equation can, in principle, be solved for α. The solution, representing the value $\alpha = \alpha_m$ for a crack propagating at maximum load, depends on α_0 and θ and may be written as

$$\alpha = \alpha_m(\alpha_0, \theta). \tag{12}$$

At constant α_0, the fracture process zone size at maximum load, $c_m = D\alpha_m$, increases with D (i.e., decreases with θ). But it is logical to expect that c_m has a finite asymptotic value for $D \to \infty$, which is, by definition

$$\lim_{D\to\infty} c_m = \lim_{D\to\infty} \left\{ D \left[\alpha_m \left(\alpha_0, \frac{c_f}{D} \right) - \alpha_0 \right] \right\} = c_f. \tag{13}$$

This means that, in the limit case of an infinitely large body, the fracture process zone occupies a negligible fraction of the structure volume. In that limit case, the fracture process zone is subjected at its boundary to the LEFM near-tip asymptotic stress and displacement fields. These fields are not influenced by the geometry of the structure and loading. Therefore, if the crack is critical (i.e., propagates), these fields are in the limit case the same for any structure (note that we do not need to consider only the first term of the LEFM asymptotic expansion with the inverse square-root singularity in the radial coordinate; the nonsingular second term and a finite number of the higher-order terms of the expansion are the same as well if the structure is infinitely large). Therefore, the state of the fracture process zone in an infinitely large body is the same, too, and so the effective length c_f must also be independent of structure geometry. Due to the limit $D \to \infty$, c_f thus defined is also independent of D, and so it must be a material constant [32, 33], as assumed.

It must be admitted, though, that the finiteness of the limit c_f is not an inevitable mathematical consequence of the preceding equations alone but is also inferred from the physical nature of fracture, particularly the finiteness of the binding forces in the solid (and the fact that highly homogeneous brittle materials, with only microscopic inhomogeneities, have a microscopic fracture process zone). From the strictly mathematical viewpoint, it cannot for example be ruled out that c_m might increase as $\log D$, in which case also $\lim(c_m/D) = 0$ for $D \to \infty$ (such behavior might manifest itself at the intermediate scale, complicating interpretation of experiments).

Substituting (12) into (8), and solving the equation for σ_N, we finally get the result

$$\sigma_N = \sqrt{\frac{EG_f}{D\hat{g}(\alpha_0, \theta)}}, \tag{14}$$

in which function $\hat{g}$ is defined as

$$\hat{g}(\alpha_0, \theta) = \frac{\tilde{g}[\alpha_0, \alpha_m(\alpha_0, \theta), \theta]}{r[\alpha_0, \alpha_m(\alpha_0, \theta), \theta]}. \tag{15}$$

Equation (14) has the same form as the LEFM equation for σ_N, except that it includes function $\hat{g}(\alpha_0, \theta)$ instead of the usual LEFM dimensionless energy release rate $g(\alpha)$. This means that function $\hat{g}(\alpha_0, \theta)$ may be regarded as the effective dimensionless energy release rate.

The argument leading from (10) to (14) was proposed to the writer by J. Planas. The essential equation in this argument is (11). As indicated in Appendix I, this equation can also be obtained from the cohesive crack model, which is more general but not as simple as Planas' argument.

4. Simple geometrical explanation

Equation (14), which is fundamental for the present analysis, allows an intuitive geometrical justification for some simple structures such as the center-cracked panel in Figure 2. In that

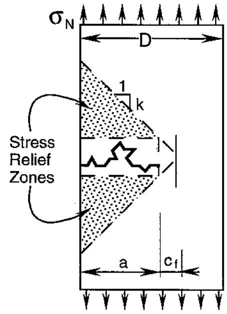

Figure 2. Interpretation of energy release from a simple specimen by stress-relief zones.

Figure, $c = c_f =$ the effective length of the fracture process zone, and $a = a_0 + c$ represents the length of the equivalent crack according to LEFM. According to the shape of the principal stress trajectories, it is clear that formation of the crack causes the strain energy to be relieved from the shaded triangular zones in Figure 2, limited by the so-called ‘stress-diffusion’ lines of slope k. The area of the two triangular shaded zones is $k(a_0+c)^2$. The strain energy density before fracture is $\sigma_N^2/2E$, and so the strain energy content of the shaded zone before fracture is $\Pi^* = bk(a_0 + c)^2\sigma_N^2/2E$. Now we may set $\partial\Pi^*/\partial c = 2bG_f$ for crack tip dissipation because the R-curve behavior (variation of effective fracture energy) is taken into account by (12). Setting also $c = c_m$ for the maximum load, we get $bk(a_0 + c_m)\sigma_N^2/E = bG_f$. Finally, solving for σ_N, we obtain (14) in which $\hat{g}(\alpha_0, \theta) = k[\alpha_0 + (c_m/D)]$ or

$$\hat{g}(\alpha_0, \theta) = k\alpha_m(\alpha_0, \theta). \tag{16}$$

The slope k is empirical but it is known that, for different sizes of geometrically similar specimens, the k value giving the correct energy release is exactly the same. For an infinitely large body with a finite crack, the value $k = \pi/2$ yields the exact result. Similar geometric interpretations of (14) are possible for a few other simple specimen geometries (e.g. [26], Section 12.1).

5. Large-size asymptotic expansion and approximate size effect law

Because $\hat{g}(\alpha_0, \theta)$ ought to be a smooth function, we may expand it into Taylor series about the point $(\alpha, \theta) \equiv (\alpha_0, 0)$. Equation (14) thus yields

$$\sigma_N = \sqrt{\frac{EG_f}{D}}\left[\hat{g}(\alpha_0, 0) + \hat{g}_1(\alpha_0, 0)\frac{c_f}{D} + \frac{1}{2!}\hat{g}_2(\alpha_0, 0)\left(\frac{c_f}{D}\right)^2 + \cdots\right]^{-(1/2)} \tag{17}$$

$$= \frac{Bf_t'}{\sqrt{D}}(D_0^{-1} + D^{-1} + \kappa_2 D^{-2} + \kappa_3 D^{-3} + \cdots)^{-(1/2)}, \tag{18}$$

where all the following constants are introduced: $\hat{g}_1(\alpha_0, 0) = \partial\hat{g}(\alpha_0, \theta)/\partial\theta$, $\hat{g}_2(\alpha_0, 0) = \partial^2\hat{g}(\alpha_0, \theta)/\partial\theta^2, \ldots$, all evaluated at $\theta = 0$, $\kappa_2 = \hat{g}_2(\alpha_0, 0)/c_f\hat{g}_1(\alpha_0, 0)$, $\kappa_3 = \hat{g}_3(\alpha_0, 0)/c_f\hat{g}_1(\alpha_0, 0), \ldots$, and

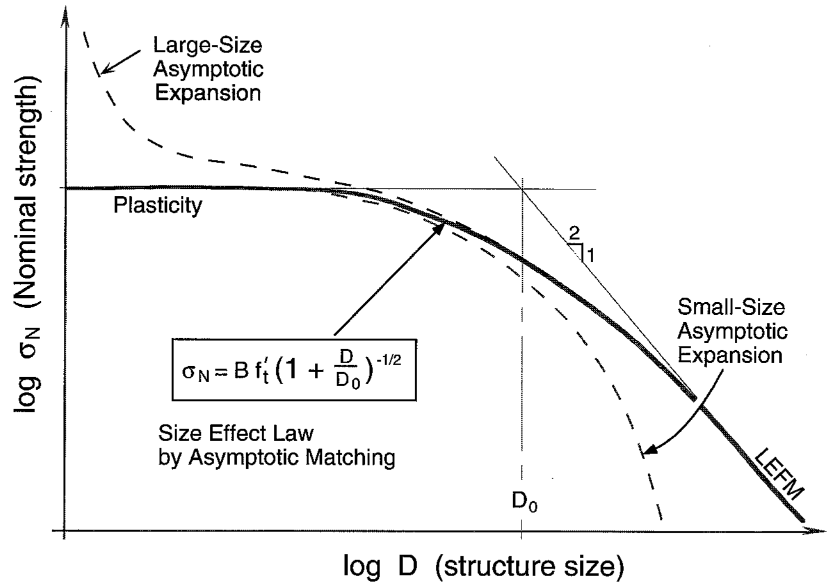

Figure 3. Size effect law (solid curve), and large-size and small-size asymptotic series expansions of size effect (dashed curves).

$$D_0 = c_f \frac{\hat{g}_1(\alpha_0, 0)}{\hat{g}(\alpha_0, 0)}, \tag{19}$$

$$Bf'_t = \sqrt{\frac{EG_f}{c_f \hat{g}_1(\alpha_0, 0)}}. \tag{20}$$

With (17) or (18), we have gained a large-size asymptotic series expansion because the terms of nonzero powers vanish as $D \to \infty$ and the truncation error tends to zero. The first-order asymptotic approximation is obtained by truncating the series after the linear term

$$\sigma_N = \frac{Bf'_t}{\sqrt{1+\beta}}, \qquad \beta = \frac{D}{D_0}. \tag{21}$$

This equation represents the approximate size effect law [12, 2] proposed for quasibrittle fracture on the basis of several different arguments. The relative structure size β is also called the brittleness number because $\beta \to \infty$ represents a fully brittle behavior, and $\beta = 0$ represents a fully nonbrittle, or plastic, behavior.

The solid curve in Figure 3 shows the size effect plot of $\log \sigma_N$ versus $\log D$ at constant α_0, obtained according to the size effect law (21). The size effect curve is seen to represent the transition from a horizontal asymptote, corresponding to a power law of exponent 0 (characterizing the strength theory), to a descending asymptote, corresponding to a power law of exponent $-\frac{1}{2}$ (characterizing LEFM). The transitional size $D = D_0$ represents the point of intersection of the left-side and right-side asymptotes, that is, the center of the transition from one power law to another. On the microscale, i.e. for $D \ll D_0$, the energy release from the structure is negligible, and on the macroscale, i.e., for $D \gg D_0$, the energy release is dominant. The special case of LEFM, for which no material length matters, is obtained as the limit for $c_f/D \to 0$. In this limit case, the size effect curve is a power curve coinciding in the log-log plot in Figure 3 with the straight line of slope $-\frac{1}{2}$.

To be able to apply (21), the α_0-value at maximum load must be known and the same for all sizes D. In fracture test specimens (of positive geometry) α_0 is ensured to be constant by cutting similar notches because, at maximum load, $c = a - a_0 \ll a_0$ where a_0 is the length of the notch. In geometrically similar specimens the value of α_0 is constant (independent of D). For brittle failures of geometrically similar reinforced concrete structures without notches, such as diagonal shear of beams, punching shear of slabs, torsion, anchor pullout or bar pullout, extensive laboratory evidence as well as finite element solutions [6, 9] indicate that the failure modes are approximately similar and $\alpha_0 \approx$ constant for a broad enough range of D. However, geometric similarity of failure mode is violated for some cases if a very broad size range is considered, e.g., for the Brazilian split-cylinder tests of a size range exceeding about 1:8 [9].

It must be stressed, however, that the nominal strength $\sigma_P = Bf_t' = \lim \sigma_N$ for $\beta \to 0$ cannot be predicted by plastic limit analysis on the basis of f_t' as a fixed material yield strength. The reason is that the zero size limit in (21) is an extrapolation of fracture mechanics. The value of f_t' for which plastic limit analysis gives the correct σ_P for (21) depends on structure shape, which includes a dependence on α_0. It can be easily shown [19] that this value is $f_t' = \sqrt{EG_f/c_f g'(\alpha_0)}/B_p(\alpha_0)$ where $B_p(\alpha_0)$ is the nominal strength calculated by plastic limit analysis for a unit value of the yield strength of material.

6. Size effect law in terms of material fracture parameters

Let the dimensionless LEFM energy release rate function be defined as $g(\alpha) = \mathcal{G}Eb^2D/P^2 = K_{\mathrm{I}}^2 b^2 D/P^2$ (e.g. [26], Section 12.2); K_{I} = mode I stress intensity factor, which is given for many geometries in textbooks [25, 27, 28] and handbooks such as [29]. By contrast to the general function $g(\alpha, \theta)$, the LEFM function $g(\alpha)$ has only one argument. In LEFM, $\sigma_N = \sqrt{EG_f/[Dg(\alpha)]}$. The similarity of this expression with (14) indicates that, for a LEFM approximation, the function $\hat{g}(\alpha_0, \theta)$ in this expression ought to be approximated as $g(\alpha_0 + \theta)$. This provides for the nominal strength the LEFM approximation

$$\sigma_N \approx \sqrt{\frac{EG_f}{Dg(\alpha_0 + \theta)}}. \tag{22}$$

The error of this approximation approaches 0 faster than θ when $\theta \to 0$, as shown for both the R-curve model and the cohesive crack model by J. Planas (private communication, 1995).

According to this approximation, $g(\alpha_0, 0)$ reduces to $g(\alpha_0)$, $\partial/\partial\theta = d/d\alpha$, and $\hat{g}_1(\alpha_0, 0)$ takes the meaning of $g'(\alpha_0) = dg(\alpha)/d\alpha$ at $\alpha = \alpha_0$, with the prime denoting derivatives (or rates) with respect to α. Equations (19), (20) and (21) thus take the form

$$D_0 = c_f \frac{g'(\alpha_0)}{g(\alpha_0)}, \tag{23}$$

$$Bf_t' = \sqrt{\frac{EG_f}{c_f g'(\alpha_0)}}, \tag{24}$$

and the size effect law in (21) may be written in the form

$$\sigma_N = \sqrt{\frac{EG_f}{g'(\alpha_0)c_f + g(\alpha_0)D}}\,. \tag{25}$$

This form involves the material fracture parameters and also gives, through function g, the effect of structure shape, thus allowing comparison of dissimilar structures.

The form of size effect law in (25) was derived in a different manner by Bažant and Kazemi [15] (see also 12.2.11 in [26]) and was amply justified experimentally, particularly by the success of the size effect method for determining G_f and c_f, based on this equation.

Defining $\bar{D} = Dg(\alpha_0)/g'(\alpha_0) = \beta c_f$ = intrinsic structure size and $\tau_N = \sigma_N\sqrt{g'(\alpha_0)}$ = intrinsic stress, one may rewrite (25) in terms of the material parameters only [30, 15]

$$\tau_N = \sqrt{\frac{EG_f}{c_f + \bar{D}}}. \tag{26}$$

Equation (25) or (26) serves as the basis of the size effect method – a simple and reliable method for measuring G_f and c_f. To this end, the equation may be reduced to a straight line plot of σ_N^{-2} versus D. The material fracture parameters may then be determined by linear regression from the measured maximum loads of specimens of a sufficiently broad range of brittleness numbers $\beta = D/D_0$. The specimens need not be geometrically similar because the effect of differences in geometry is taken into account by different values of $g(\alpha_0)$ and $g'(\alpha_0)$. However, because function $g(\alpha)$ is only an approximation of function $\hat{g}(\alpha,\theta)$, the accuracy may be expected to be somewhat better when the test specimens adhere to geometric similarity. Knowing G_f and c_f, one can further easily evaluate the critical crack tip opening displacement, $\delta_{\text{CTOD}} = (\sqrt{8G_f c_f/E})/\pi$ [38, 39]. In relation to the cohesive crack model for concrete, G_f represents approximately the area under the initial tangent of the curve of bridging stress versus crack opening, and the total area under this curve is $G_F \approx 2G_f$.

7. General large-size asymptotic series expansion

There is an alternative, albeit equivalent, crack propagation criterion in terms of the stress intensity factor K_{I}, which reads $K_{\text{I}} = K_{\text{Ic}} = \sqrt{EG_f}$, where K_{Ic} is a constant called the fracture toughness. This suggests rewriting (14) in the form $\sigma_N = K_{\text{I}}/h(\alpha)\sqrt{D}$, where $h(\alpha) = \sqrt{g(\alpha)}$ = dimensionless function. Now, expanding function $h(\alpha)$, instead of $g(\alpha)$, about the point α_0 leads to a size effect formula different from (21).

In view of this fact, consider now more general dimensionless variables and functions

$$\xi = \theta^r = \left(\frac{c_f}{D}\right)^r, \qquad h(\alpha_0,\xi) = [\hat{g}(\alpha_0,\theta]^r, \tag{27}$$

with any $r > 0$, and let function $h(\alpha_0,\xi)$ be expanded into Taylor series with respect to ξ, instead of function $g(\alpha,\theta)$ with respect to θ. This leads to a more general asymptotic series expansion [31, 32]

$$\sigma_N = \sqrt{\frac{EG_f}{D}}\left[h(\alpha_0,0) + \frac{\partial h(\alpha_0,0)}{\partial\xi}\xi + \frac{1}{2!}\frac{\partial^2 h(\alpha_0,0)}{\partial\xi^2}\xi^2 + \cdots\right]^{-(1/2r)}. \tag{28}$$

Rearranging, we obtain the large-size asymptotic series expansion [31, 32]

$$\sigma_N = \sigma_P \left[\left(\frac{D}{D_0} \right)^r + 1 + \kappa_1 \left(\frac{D}{D_0} \right)^{-r} + \kappa_2 \left(\frac{D}{D_0} \right)^{-2r} + \kappa_3 \left(\frac{D}{D_0} \right)^{-3r} + \cdots \right]^{-(1/2r)}, \quad (29)$$

where $\kappa_1, \kappa_2, \ldots$ are certain constants and the following notations are made

$$\sigma_P = \sqrt{\frac{EG_f}{c_f}} \left[\frac{\partial h(\alpha_0, 0)}{\partial \xi} \right]^{-(1/2r)}, \qquad D_0 = \frac{c_f}{g(\alpha_0, 0)} \left[\frac{\partial h(\alpha_0, 0)}{\partial \xi} \right]^{1/r}. \quad (30)$$

Previous study of the size effect formula given by the first two terms of this asymptotic series expansion, however, indicated that the optimum fit of the test data for concrete is obtained for $r \approx 1$ [33].

8. Small-size asymptotic series expansion and strength theory limit

The error of the truncated asymptotic series expansion in (28) or (29) increases with decreasing D/D_0. Retaining more terms of the expansion, its accuracy gets extended to smaller values of D/D_0. The large-size asymptotic expansion, however, can never establish the asymptotic behavior for the opposite limit of zero size D. The property that the simple size effect law (21) approaches for $D \to 0$ the size effect of strength theory (or plasticity), for which there is no size effect, has not been justified by our preceding calculations. The approach to the strength theory for $D \to 0$ was a fortuitous result of truncating the series after the linear term. Justification of this property must be sought in a different argument.

One argument that the strength theory, which implies no size effect [6], must be approached for $D \to 0$ is that the fracture process zone occupies the entire body, that there is no distinct fracture at maximum load and no stress concentrations, and that no definable energy flow mechanism exists. Another argument is that, according to finite element results, the nonlocal theory of distributed damage tends, for $D \to 0$, to the strength theory.

In this light it is not surprising that although the retention of quadratic and higher-order terms in the asymptotic series can increase the accuracy for large and intermediate sizes D, it would actually give an incorrect size effect for $D \to 0$, disagreeing with the size effect of strength theory. Consequently, as far as the overall approximate description of the entire size range is concerned, it is in fact better to truncate the Taylor series expansion after the linear term.

Let us now propose a better argument for the strength theory limit, also providing the opposite, small-size, asymptotic expansion. We introduce a new variable and a new function

$$\eta = \theta^{-r} = \left(\frac{D}{c_f} \right)^r, \qquad \psi(\alpha_0, \eta) = \left(\frac{\hat{g}(\alpha_0, \theta)}{\theta} \right)^r, \quad (31)$$

where r is any positive constant. Substituting $D = c_f \eta^{1/r}$ and $\hat{g}(\alpha_0, \theta) = [\psi(\alpha_0, \eta)]^{1/r} c_f / D$ into (14), we obtain for the nominal strength the expression

$$\sigma_N = \sqrt{\frac{EG_f}{c_f}} [\psi(\alpha_0, \eta)]^{-(1/2r)}. \quad (32)$$

Function $\psi(\alpha_0, \eta)$ ought to be sufficiently smooth to permit being expanded into Taylor series about point $(\alpha_0, 0)$, which now corresponds to the zero-size limit rather than the infinite size limit. The expansion furnishes

$$\sigma_N = \sqrt{\frac{EG_f}{c_f}} \left[\psi(\alpha_0, 0) + \frac{\partial\psi(\alpha_0, 0)}{\partial\eta}\eta + \frac{1}{2!}\frac{\partial^2\psi(\alpha_0, 0)}{\partial\eta^2}\eta^2 + \cdots\right]^{-(1/2r)}, \tag{33}$$

which may be rewritten in the form

$$\sigma_N = \sigma_P \left[1 + \left(\frac{D}{D_0}\right)^r + b_2\left(\frac{D}{D_0}\right)^{2r} + b_3\left(\frac{D}{D_0}\right)^{3r} + \cdots\right]^{-(1/2r)}. \tag{34}$$

Here $b_2, b_3, \ldots$ are certain constants and the following notations are made

$$\sigma_P = \sqrt{\frac{EG_f}{c_f[\psi(\alpha_0, 0)]^r}}; \qquad D_0 = c_f \left[\frac{1}{\psi(\alpha_0, 0)}\frac{\partial\psi(\alpha_0, 0)}{\partial\eta}\right]^{-(1/r)}. \tag{35}$$

With (34) we acquired the small-size asymptotic series expansion because, for $D \to 0$, the terms of nonzero power become negligible and the truncation error tends to zero. This expansion of course cannot yield the asymptotic limit for $D \to \infty$. The expansion proves that the small-size asymptotic limit must be the strength theory. So we have established what we accepted in the preceding analysis on the basis of more extraneous arguments.

Note that, in defining η (and likewise in defining ξ in (27)), exponent r in (34) cannot be replaced by some exponent s different from r. It would cause the second term of the series expansion to be either zero or infinite.

9. Intermediate approximation by asymptotic matching

The large-size and small-size asymptotic series expansions in (17) or (29) and in (34) are different (Figure 3). However, if both series are truncated after the linear term, they yield formulae of the same form

$$\sigma_N = \sigma_P(1 + \beta^r)^{-(1/2r)}, \qquad (\beta = D/D_0), \tag{36}$$

where σ_P, D_0 and r are constants. This is the generalized size effect law proposed in [31]. Because this law, including its special case for $r = 1$, is anchored to the asymptotic cases on both sides, it represents an intermediate approximation of uniform applicability. This kind of approximation is akin to the theory of intermediate asymptotics or matched asymptotics, which has been enormously successful in many branches of physics, for example, the boundary layer theory of fluid mechanics [4, 3].

We must be aware, however, of one limitation of the asymptotic matching as just performed. Our way of arguing does not guarantee that the values of σ_P and D_0 obtained from the two asymptotic series in (29) and in (34) match each other. In fact, the values of σ_P and D_0 obtained from these two series would surely be different if the former were determined by fitting only large-size data and the latter by fitting only small-size data. In practice, however, such fitting would hardly suffice, because of scatter. Normally one cannot determine these values a priori but can do so only by fitting the matching formula in (36) to the full range of data. Our asymptotic matching only established the form of the size effect law but not the parameter values, and this is the sense in which the asymptotic matching is performed here.

By fitting of test data, the optimum value of exponent r in (36) was found to be close to $r = 1$ [33]. However, the optimum was weak and the scatter of the available test data for concrete or rock was generally too high to make any definitive conclusion about the value of r on the basis of tests.

By numerical calculations, e.g. with the cohesive crack model, it is possible to generate numerical results ranging over several orders of magnitude of D. Such results can be fitted into (36) very closely. However, it is found that the optimum value of exponent r strongly depends on the specimen geometry. For three-point bend fracture specimens, the optimum value of r in (36) for describing finite element results with the cohesive crack model for a size range of several orders of magnitude of D [34] has been found to be between 0.42 and 0.46. On the other hand, for a large panel with a small central crack, the optimum r-value exceeds 1.5. The optimum r-value also depends on the softening stress-displacement laws.

From the physical viewpoint, a value $r < 1$ has some questionable implications. Since the large-size asymptotic form of (36) is $\sigma_N = \sigma_P/[1+(\beta^r/2r)]$, the extrapolation to infinite size implies the effective fracture process zone length c_f to be infinite when $r < 1$. Also, when the R-curve is calculated from (36) as the envelope of fracture equilibrium curves for all sizes (e.g. [26]), the final value is not reached for a finite crack length but is approached asymptotically. This would in turn imply the softening stress-displacement curve in the cohesive (or fictitious) crack model to have an infinite tail (i.e., the crack-bridging stress could not be reduced to 0). These features do not seem realistic, which indicates that exponent r ought to be considered as 1. (In practice, though, a value $r \neq 1$ might work best at the intermediate scale. A size range of many orders of magnitude would hardly be tractable because of the lack of the test data needed for calibration.)

The asymptotic approach is appropriate even if the transition to much larger sizes involves a change of failure mechanism, as in the Brazilian split-cylinder test. What should be extrapolated to infinite and zero sizes is the theory applicable to the size range considered, not the real structure such as a reinforced concrete beam. The intermediate approximate solution must be supported on the asymptotic cases of the theory for that range, not on a different theory that may be required for a different range.

The fracture mechanism of failure may change at large size to some ductile mechanism. This might happen, for instance, in compression tests and, probably, in split-cylinder (Brazilian) tests. In that case, one of the following two generalizations of (36) with nonzero residual nominal strength σ_r may be appropriate [32]

$$\sigma_N = \sigma_P(1+\beta^r)^{-(1/2r)} + \sigma_r \quad \text{or} \quad \sigma_N = \text{Max}[\sigma_P(1+\beta^r)^{-(1/2r)}, \sigma_r]. \tag{37a,b}$$

In the special case that the existence of residual stress σ_r in the size effect law is caused by transmission of residual cohesive normal stress σ_r across the crack, rather than by a transition to some completely different ductile failure mechanism, the following formulas result if σ_r is assumed to be uniform along the crack

$$\sigma_N = \sqrt{\frac{\sigma_P^2}{1+(D/D_0)} + \sigma_r^2} \quad \text{or} \quad \sigma_N = \sqrt{\frac{EG_f + [\gamma'(\alpha_0)c_f + \gamma(\alpha_0)D]\sigma_y^2}{g'(\alpha_0)c_f + g(\alpha_0)D}}. \tag{38a,b}$$

Function $\gamma(\alpha)$ is analogous to $g(\alpha)$ and defines the energy release rate $\mathcal{G}_y(a) = (\sigma_y^2/E)D\gamma(\alpha)$ which corresponds to a uniform closing pressure σ_r applied along the entire crack surface

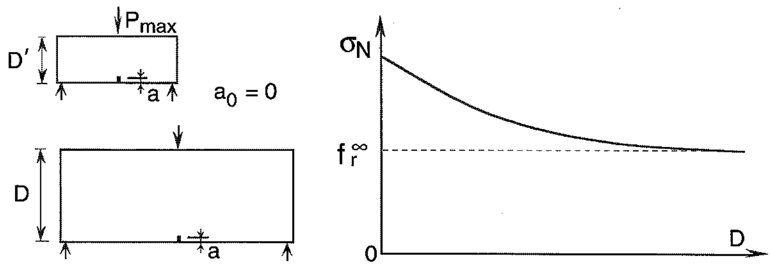

Figure 4. Similar structures with initiating cohesive cracks of length a and law of size effect for crack initiation from a smooth surface (or for modulus of rupture).

up to the tip $a = a_0$. Equation (38a) is derived by considering that the crack strip in Figure 2 transmits a nonzero residual normal stress σ_r and that, consequently, the strain energy density in the shaded triangular areas is reduced from $\sigma_N^2/2E$ to $\sigma_r^2/2E$ rather than to 0. Equation (38b) is derived similarly to our preceding derivation; the only difference is that the energy release rate available to drive the fracture is $\mathcal{G}(a) - \mathcal{G}_y(a)$, which must be equal to R, and that $\sigma_y = \sigma_r\sqrt{g(\alpha_0)/\gamma(\alpha_0)}$.

In the size effect plot of $\log \sigma_N$ versus $\log D$, (37a) and (38a, b) approach a horizontal asymptote ($\sigma_N = \sigma_r$) for $D \to \infty$. They also exhibit a positive curvature for larger D/D_0 values. It appears that in some tests the value of D_0 is so small that only such a positive curvature is seen in the test results (Carpinteri fitted such data wih his MFSL law, but they can be equally closely fitted with (37) or (38), as shown in Figure 4(c) of [1].

10. Size effect for failures at crack initiation from smooth surface

Consider now the failure of quasibrittle structures that have no notches and reach the maximum load when a cohesive crack initiates from a smooth surface (Figure 4). This happens, for example, when a plain concrete beam is used to test the flexural strength, called the modulus of rupture, f_r). One might at first think of applying the preceding solution in (25) with $\alpha_0 \to 0$. But this is not possible because $g(\alpha_0, 0)$ vanishes as the initial crack length a_0 vanishes, i.e. $\alpha_0 \to 0$. However, the total (cohesive) crack length a at failure, of course, is not zero; $a \approx c_f$. A continuous crack can form and start to propagate only after a microcracked boundary layer grows to a certain depth, approximately equal to c_f [19].

To treat failures at crack initiation, one might first try to center the Taylor series expansion about the point (α_0, θ). However, then the effect of size D, embodied in θ, would not get separated. Therefore, the Taylor series expansion in (17) must be centered about point $(\alpha_0, 0)$ or $(0, 0)$. But then the series cannot be truncated after the second term because the first term vanishes. So, let us truncate the large-size asymptotic series after the third (quadratic) term. Then, considering that $r = 1$, and noting that $g(\alpha_0, 0) = 0$ or $g(\alpha_0) = 0$, we obtain, instead of (25)

$$\sigma_N = \sqrt{\frac{EG_f}{g'(0)c_f + \frac{1}{2}g''(0)c_f^2 D^{-1}}} = f_r^\infty \left(1 - \frac{2D_b}{D}\right)^{-(1/2)} \approx f_r^\infty \left(1 + \frac{D_b}{D}\right), \qquad (39)$$

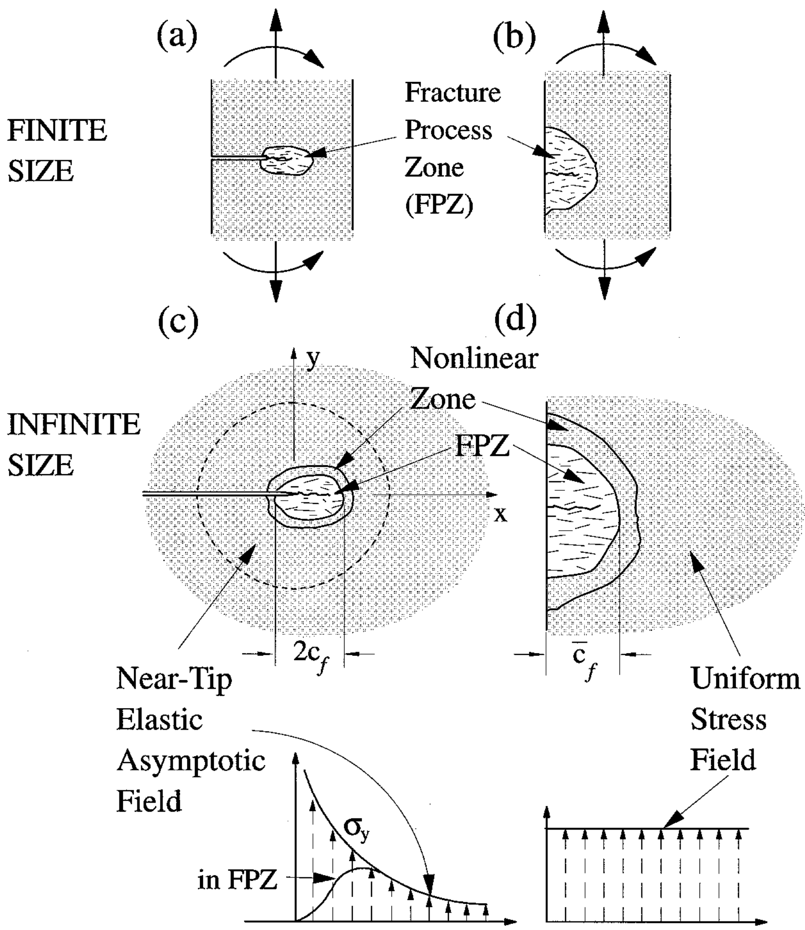

Figure 5. Fracture process zones at infinite structure size for cracked (or notched) structure and for crack initiation from a smooth surface.

in which f_r^∞ and D_b denote the following constants

$$f_r^\infty = \sqrt{\frac{EG_f}{g'(0)c_f}}, \qquad D_b = \frac{\langle -g''(0)\rangle}{4g'(0)}\bar{c}_f, \qquad \bar{c}_f = \kappa c_f. \tag{40}$$

Subscript b refers to the thin boundary layer, in which the crack tip is located for failures at crack initiation. The operator $\langle .. \rangle$ is the Macauley bracket, denoting the positive part of the argument, i.e. $\langle x \rangle = \mathrm{Max}(x, 0)$. The reason for introducing this operator is that $g''(0)$ can be negative, in which case fracture cannot initiate at the surface. Without taking only the positive part of $g''(0)$, (39) for small enough D could give for σ_N an imaginary value, which would be meaningless. The symbol $\approx$ in (39) refers to the asymptotic approximation for $\delta = D_b/D \ll 1$, obtained upon noting that, up to the first two terms of the Taylor series expansion, $(1 - 2\delta)^{-(1/2)} \approx 1 + \delta$.

The c_f value in (40) has been modified by the empirical constant factor $\kappa > 1$ (but close to 1), in order to reflect the fact that, for crack initiation, the size of the fracture process zone at maximum load and for $D \to \infty$ must be expected to be larger than it is for notched or precracked structures. This fact is clarified by Figure 5, which shows the body boundaries relevant to the shape of the fracture process zone at $D \to \infty$, along with the stress fields acting on the fracture process zone at its boundary. Obviously, for $D \to 0$, only the limiting shape of the boundary infinitely close to the crack front controls the size (and shape) of the fracture process zone. This limiting shape is the same for all notched or pre-cracked structures but different for uncracked structures (Figure 5).

Equation (39) is only second-order accurate in D^{-1}. It can be rearranged to another form which is also second-order accurate, sharing the first two terms of the large-size asymptotic series expansion, but has a more desirable behavior for $D \to 0$, giving a finite rather than infinite value for σ_N. To this end, we may apply an alternative second-order approximation $(1-2\zeta)^{-(1/2)} = 1 + \zeta + 1.5\zeta^2 + \cdots \approx 1 + [\zeta/(1+\eta\zeta)]$ with an error of the order of ζ^2 and notation $\zeta = D_b/D$ where η is a positive empirical constant (which may be imagined to control the size of initial fracture process zone at smooth surface). This approximation yields the following preferable alternative to (39)

$$\sigma_N = f_r^\infty \left(1 + \frac{D_b}{D + \eta D_b}\right). \tag{41}$$

In view of the dependence of the energy release rate function $g(\alpha)$ on the structure shape, (40) and (41) seem, at first sight, to violate the obvious requirement that f_r^∞, representing the material strength in a very large structure (or the limiting value of the modulus of rupture, f_r), must be independent of the shape of the structure. Not so, however. The limiting value $g'(0)$ is shape-independent, provided the crack initiates from a smooth surface (and not from a sharp corner tip). This can be inferred from Saint-Venant's principle and can also be checked from various LEFM solutions in handbooks; $g'(0) = 1.12^2\pi$.

Equation (39), or (41) for $\eta = 0$, identical to the formula in [19], was also derived in a different manner [1] and was extensively validated by the available test data for the modulus of rupture of concrete beams of various depths D. Equation (41) for $\eta = 0$ can be written as a linear regression equation $\sigma_N = A + CX$ with $X = 1/D$ and $f_r^\infty = A$, $D_b = C/A$, and so the values of f_r^∞ and D_b can be easily identified by linear regression of test data on the modulus of rupture for various sizes D.

As for the value of η, the available test data exhibit too much random scatter for permitting a meaningful determination of η. For the sake of simplicity, one may use $\eta = 0$. However, a non-zero value of η makes the strength σ_N for $D \to 0$ finite, which seems more reasonable. Nevertheless, it must be admitted that the values of σ_N have no physical meaning for structure sizes D that are too small (e.g., smaller than the aggregate size in concrete or the grain size in rock). So, strictly speaking, an infinite limit value of σ_N for $D \to 0$ (or $\eta = 0$) cannot be declared as unacceptable.

Instead of the second-order approximation that led to (41), we could have more generally used a whole range of approximations $(1-2\zeta)^{-(1/2)} \approx (1 + n\zeta)^{1/n}$, with any $n > 0$. Again, however, the proper value of n can hardly be clarified by experiments, because of their high scatter. A clarification may nevertheless be found in the analysis according to the strength theory. Aside from fracture mechanics, this theory (or plasticity) should apply as the opposite asymptotic limit for $a \to 0$ because, at crack initiation, no crack exists as yet and the energy dissipation rate is still vanishing. The solution of the size effect on f_r according to the strength theory has been carried out [19], and it agrees with the present result in (41) if and only if $n = 1$. This lends support to the value $n = 1$ implied in (41).

The small-size asymptotic expansion can also be introduced for the case of failures at crack initiation from smooth surface ($\alpha_0 \approx 0$). In that case, ψ and $\partial\psi/\partial\eta$ are positive. The first two terms of the asymptotic series expansion (33) yield the initial trend $\sigma_N \propto 1 - (D/2c_f)$. This expression, which can of course be valid only for very small D, can be made tangent to (41) at the point $D = 0$. So we may observe that formula (41) represents asymptotic matching of the large-size and small-size asymptotic expansions for failures at crack initiation from a smooth surface. This makes it preferable to formula (39) which does not have this property, i.e., does not match the aforementioned initial trend of the small size expansion.

11. Universal size effect law for cracked and uncracked structures

The simple size effect law in (21) [2] is obtained by truncating the large-size asymptotic expansion (17) after the linear term, provided that $g(\alpha_0) > 0$. This restriction excludes the case of notchless specimens, for which $\alpha_0 = 0$ or $g(\alpha_0) = 0$. To obtain a universal size effect law valid for both cracked and uncracked structures, it is, in general, necessary to keep the terms up to the quadratic term with θ^2 [20–22]. However, the quadratic term would engender for unnotched specimens incorrect small-size asymptotic behavior at $D \to 0$. In other words, it would deprive (21) of its asymptotic matching character. Therefore, the quadratic term in (17) must be modified so as to preserve the existence of a finite asymptotic value for $D \to 0$. To this end, we first rearrange (17) as follows

$$\sigma_N = \sigma_0 \left(\frac{D}{D_0} + 1 - \frac{2D_b}{D} \right)^{-(1/2)} = \sigma_0 \left(1 + \frac{D}{D_0} \right)^{-(1/2)} (1 - 2\gamma)^{-(1/2)}, \tag{42}$$

where

$$\gamma = \frac{D_b}{D} \left(1 + \frac{D}{D_0} \right)^{-1}, \qquad \sigma_0 = \sqrt{\frac{EG_f}{c_f g'}}, \qquad D_0 = \frac{c_f g'}{g}, \qquad D_b = \bar{c}_f \frac{\langle -g'' \rangle}{4g'}. \tag{43}$$

For the sake of brevity only, we have made here the notations $g = g(\alpha_0)$, $g' = g'(\alpha_0)$, and $g'' = g''(\alpha_0)$. Further we must ensure that

$$\bar{c}_f = c_f \quad \text{for} \quad \alpha_0 \geqslant c_f, \qquad \bar{c}_f = \kappa c_f \quad \text{for} \quad \alpha_0 = 0. \tag{44}$$

To achieve a correct small-size asymptotic limit for notched specimens while preserving the first three terms of the large-size asymptotic expansion in (17), we may introduce into (42) the same approximation as before, namely $(1 - 2\gamma)^{-(1/2)} = 1 + \gamma + 1.5\gamma^2 + \cdots \approx 1 + \gamma$. In this manner, (42) may be brought to the form

$$\sigma_N = \sigma_0 \left(1 + \frac{D}{D_0} \right) \left\{ 1 + \left[\frac{D}{D_b} \left(1 + \frac{D}{D_0} \right) \right]^{-1} \right\}. \tag{45}$$

Furthermore, to obtain a finite limiting value of σ_N for $D \to 0$, it is again desirable that D_b/D in the expression for γ in (43) be replaced with $D_b/(D + \eta D_b)$ where η is an empirical constant. This replacement is admissible because it does not alter the first three terms of the large-size asymptotic series expansion in (17). For the sake of generality, one may further insert here parameter r similar to (36), whose value has no effect on the first three terms of the asymptotic series expansion. Moreover, empirical parameter $\bar{\eta} = 1 + \eta$ may be introduced

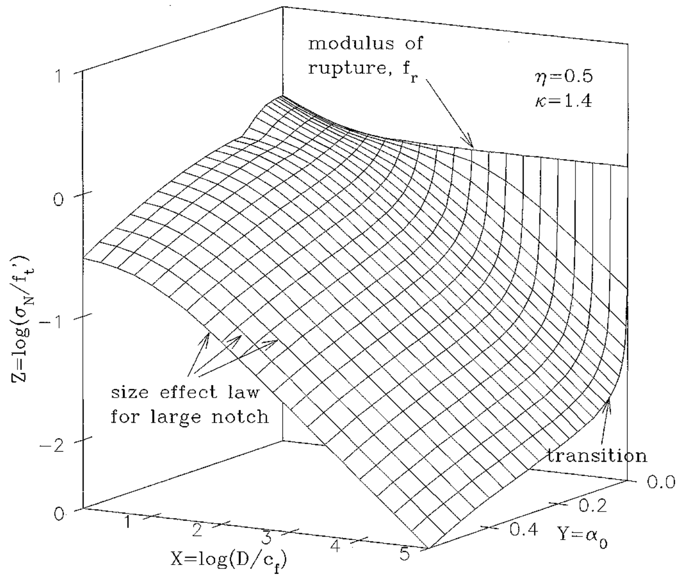

Figure 6. Surface of universal size effect law amalgamating the size effects for structures with large cracks (or notches) and structures failing at crack initiation from a smooth surface.

for the same reasons as pointed out for crack initiation. Also, another parameter s may be introduced, for the sake of generality, in a manner that has no effect on the first three terms of the asymptotic series expansion. With these adjustments, one gets

$$\sigma_N = \sigma_0 \left[1 + \left(\frac{D}{D_0}\right)^r\right]^{-(1/2r)} \left\{1 + s\left[\left(\bar{\eta} + \frac{D}{D_b}\right)\left(1 + \frac{D}{D_0}\right)\right]^{-1}\right\}^{1/s}. \tag{46}$$

However, same as for r, the value $s = 1$ seems to be most appropriate.

Equation (46) represents the universal size effect law, applicable to both notched and notchless structures. A three-dimensional plot of this equation for a typical three-point-bend fracture specimen is shown in Figure 6. The preceding derivation guarantees that the asymptotic behaviors for:

(1) $D \gg D_0$ and $\alpha_0 > 0$,
(2) $D \ll D_0$ and $\alpha_0 > 0$, and
(3) $D > D_b$ and $\alpha_0 = 0$

are all correct. This means that (46) is a matched asymptotic for all the three relevant asymptotic cases.

By expanding (46) into a Taylor series of powers of $1/D$, one can directly prove that (46) agrees with the first three terms of the asymptotic expansion (17), and that truncation after the second term of the expansion yields the original Bažant's size effect law in (21). Furthermore, by expanding (46) into a Taylor series of powers of D one can prove that (46) agrees with the first two terms of the asymptotic expansion for $D \ll D_0$ and $\alpha_0 > 0$. If there is no notch ($\alpha_0 = 0$, which implies $g = 0$ or $D_0 \to \infty$), then (46) (with $\sigma_N = f_r$ = modulus of rupture) reduces to $f_r = f_r^\infty\{1 + [D_b/(D + \eta D_b)]\}$ which coincides with (41) and agrees with the law of the size effect on the modulus of rupture f_r in (41) up to the second (linear) term of the expansion in $1/D$ as well as the expansion in D.

12. Measurement of material fracture characteristics

The size effect law in (25) serves as the basis of the size effect method for measuring G_f, c_f and other fracture characteristics [15, 35]. For this purpose, (25) is rearranged into a linear plot and the values of G_f and c_f are then easily obtained by linear regression, along with their standard deviations. This method is simple to use because one needs to measure only the maximum loads of notched fracture specimens of sufficiently different brittleness numbers β, which can be achieved by using sufficiently different sizes (one does not need to measure the crack tip location, notoriously ambiguous in materials such as concrete, nor to use a very stiff testing frame with fast-feedback closed-loop control of displacement, as required for testing post-peak softening response).

Recently, a more convenient version of the size effect method in which it suffices to test notched specimens of only one size has been developed [19, 21, 22]. A sufficient range of the brittleness number is achieved by methods of two kinds.

The first kind of method supplements the σ_N value of a notched specimen by the zero-size limit of $\sigma_N = \sigma_P$ calculated for a geometrically similar specimen of zero-size ($D \to 0$) according to plastic limit analysis (using, in the case of concrete, the Mohr–Coulomb yield criterion). However, the proper value of the material tensile strength (yield limit) f_y to be used in this calculation is not the tensile strength f_t' of the material determined by standard test. Rather, it depends on the specimen shape, particularly the value of α_0. Upon equating the zero-size limit $\sigma_P = Bf_y$ to $[EG_f/c_f g'(\alpha_0)]^{1/2}$ where $B = B(\alpha_0)$ = nominal strength calculated according to plasticity for a unit value of material yield strength, one finds that $f_y = [EG_f/c_f g'(\alpha_0)]^{1/2}/B(\alpha_0)$; in more detail, see [19].

The second kind of method supplements the σ_N value of a notched specimen by the σ_N value tested on an unnotched specimen of the same size (as in the modulus of rupture test). For this kind of method, the universal size effect law, (46) is needed. Fitting it to the measured σ_N values yields the values of G_f and c_f; in detail, see again [19].

13. Conclusions and general observations

The scaling problem for structures made of quasibrittle materials such as concrete, rock, ice, tough ceramics and composites is a typical example of a problem that is very hard to solve in the size range of practical interest, but becomes easier to solve for the asymptotic cases of very small and very large structures which fall into the realm of either plasticity or LEFM. It is known from physics (for example, the boundary layer theory in fluid mechanics), that the most effective approach to such problems is to solve the simple asymptotic cases first, and then find an intermediate approximation that matches these asymptotic cases (i.e., interpolates between them). This is the approach of matched asymptotics, which generally leads to simple solutions. Even if extremely large and extremely small structures are not of practical interest, the knowledge of their solution is useful for the intermediate range that is of practical interest.

The approximate size effect law proposed by Bažant [6, 2] can be obtained as the matched asymptotic on the basis of large-size and small-size asymptotic expansions. The size effect law for failures at crack initiation in the boundary layer can be derived by asymptotic analysis from the same energy expression as that used for large cracks. An approximate universal size effect law applicable to both initially cracked and uncracked structures can be obtained by matching the large-size and small-size asymptotic expansions for structures with large cracks and with no cracks.

Appendix I. Crack length at maximum load from cohesive crack model

Equation (11) which is crucial for obtaining the asymptotic expansion can be derived more generally from the cohesive crack model. The necessary condition of static crack propagation in the cohesive crack model reduces to the condition that the total stress intensity factor for the tip of the cohesive crack must be zero. Assuming a linear softening relation between the crack-bridging (cohesive) stress and the opening displacement, and using equation 9 of [37], one can reduce this condition to the following homogeneous Fredholm integral equation

$$\frac{1}{\theta}\int_{\alpha_0}^{\alpha_m} C(\xi',\xi)v(\xi')\,\mathrm{d}\xi' = 2v(\xi). \tag{47}$$

Here $\theta = c_f/D$ and c_f is defined according to (6); $c_f = EG_f/(f_t')^2$; $C(\xi',\xi)$ is the dimensionless crack compliance function, which is independent of structure size; it characterizes the structure geometry and represents the opening displacement at ξ caused by a unit load applied on the crack faces at ξ' (with $\xi = x/D$, $x =$ coordinate of the crack line).

Equation (11) represents an eigenvalue problem, in which $1/\theta$ is the eigenvalue and $v(\xi)$ (having the meaning of the derivative of cohesive stress with respect to the crack length) is the eigenfunction. The solution of this eigenvalue problem yields the θ-value as a function of α_0 and α_m. Inversion of this function then shows that α_m is a function of α_0 and θ, as written in (11). This is what we wanted to prove.

Appendix II. Second-order size effect law

The size effect law in (21) corresponds to the first two terms of the large-size or small-size asymptotic series expansions. Including the third, quadratic terms of the large-size expansion improves accuracy for large sizes but gives infinite σ_N for $D \to 0$. However, the following formula proposed by Bažant and Kazemi [36] gives finite σ_N for zero size D (Figure 3) and its large-size asymptotic expansion can be made to coincide with that in (17) up to the quadratic term

$$\tau_N = \sqrt{\frac{EG_f}{\bar{c}+\bar{D}}}, \qquad \bar{c} = \frac{\bar{D}+c_g}{\bar{D}+c_f}, \tag{48}$$

where c_g is an additional constant. Rearrangement of this equation can give

$$\begin{aligned}\frac{EG_f}{\sigma_N^2} &= g'(\alpha_0)c_f\left(1+\frac{c_g}{\bar{D}}\right)\left(1+\frac{c_f}{\bar{D}}\right)^{-1} + g(\alpha_0)D \\ &= g'(\alpha_0)c_f\left(1+\frac{c_g}{\bar{D}}\right)\left(1-\frac{c_f}{\bar{D}}+\cdots\right) + g(\alpha_0)D.\end{aligned} \tag{49}$$

Comparing the terms of powers D^1, D^0 and D^{-1} with those in (17), we conclude that coincidence with the first three terms of the large-size asymptotic expansion is obtained for

$$c_g = c_f\left(1+\frac{g(\alpha_0)g''(\alpha_0)}{2g'(\alpha_0)^2}\right). \tag{50}$$

However, this formulation gives no size effect for $g(\alpha_0) = 0$. Therefore, unlike the universal size effect law, it cannot be used for crack initiation from the surface.

Acknowledgment

The financial support for the present asymptotic analysis and its applications to sea ice and fiber composites was obtained under ONR Grant N00014-91-J-1109 to Northwestern University (monitored by Dr. Y. Rajapakse). Supplementary funds were obtained for the final phase of the asymptotic analysis under a contract with Sandia Laboratories (monitored by Dr. E.P.T. Chen), and for the related experimental studies of size effect in concrete under a grant from the Center for Advanced Cement-Based Materials at Northwestern University. Thanks are due to Professor Jaime Planas of the Technical University, Madrid, for his stimulating discussions and especially for suggesting the R-curve argument leading to (12).

References

1. Z.P. Bažant, Scaling of quasibrittle fracture: The fractal hypothesis, its critique and Weibull connection, *International Journal of Fracture*, 83 (1996) 41–65.
2. Z.P. Bažant, Size effect in blunt fracture: Concrete, rock, metal. *Journal of Engineering Mechanics, ASCE*, 110 (1984) 518–535.
3. G.I. Barenblatt, *Similarity, Self-Similarity and Intermediate Asymptotics*. Consultants Bureau, New York, N.Y. (1979).
4. M.C. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. McGraw Hill, New York (Chapters 9–11) (1978).
5. L.I. Sedov, *Similarity and Dimensional Methods in Mechanics*. Academic Press, New York (1959).
6. Z.P. Bažant, Scaling laws in mechanics of failure. *ASCE Journal of Engineering Mechanics* 119(9) (1993) 1828–1844.
7. W. Weibull, Phenomenon of rupture in solids. *Ingenioersvetenskaps Akad. Handl.* 153 (1939) 1–55.
8. Z.P. Bažant, and Y. Xi, Statistical size effect in quasi-brittle structures: II. Nonlocal theory. *ASCE Journal of Engineering Mechanics* 117 (11) (1991) 2623–2640.
9. Z.P. Bažant, J. Ožbolt and R. Eligehausen, Fracture size effect: review of evidence for concrete structures. *ASCE Journal of Structural Engineering* 120 (8) (1994) 2377–2398.
10. P.F. Walsh, Fracture of plain concrete. *Indian Concrete Journal* 46 (11) (1972).
11. P.F. Walsh, Crack initiation in plain concrete. *Magazine of Concrete Research* 28 (1976) 37–41.
12. Z.P. Bažant, Fracture in concrete and reinforced concrete, *Mechanics of Geomaterials: Rocks, Concretes, Soils* (Preprints, IUTAM Prager Symposium held at Northwestern University), ed. by Z.P. Bažant, Evanston, Illinois, (1983) 281–317.
13. Z.P. Bažant, M.R. Tabbara, M.T. Kazemi and G. Pijaudier-Cabot, Random particle model for fracture of aggregate or fiber composites. *ASCE Journal of Engineering Mechanics* 116 (8) (1990) 1686–1705.
14. M. Jirásek and Z.P. Bažant, Macroscopic fracture characteristics of random particle systems. *International Journal of Fracture* 69 (1994) 201–228.
15. Z.P. Bažant and M.T. Kazemi, Size effect on diagonal shear failure of beams without stirrups, *ACI Structural Journal* 88 (1991) 268–276.
16. A. Hillerborg, M. Modéer and P.E. Petersson, Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Research* 6 (1976) 773–782.
17. P.E. Petersson, *Crack growth and development of fracture zones in plain concrete and similar materials* (Report TVBM-1006), Division of Building Materials, Lund Institute of Technology, Lund, Sweden (1991).
18. Z.P. Bažant and Z. Li, Modulus of rupture: size effect due to fracture initiation in boundary layer. *Journal of Structural Engineering ASCE* 121 (4) (1995) 739–746.
19. Z.P. Bažant and Z. Li, Zero-brittleness size-effect method for one-size fracture test of concrete. *ASCE Journal of Engineering Mechanics* 122(5) 458–468.
20. Z.P. Bažant and Y.-N. Li, Scaling of cohesive fracture (with ramification to fractal cracks). In *Size-Scale Effects in the Failure Mechanics of Materials and Structures* (Proc., IUTAM Symp., held at Politecnico di Torino, 1994), ed. by A. Carpinteri, E & FN Spon, London (1996) 274–289.
21. Z.P. Bažant, Scaling theories for quasibrittle fracture: Recent advances and new directions. In *Fracture Mechanics of Concrete Structures*, 1, (Proc., 2nd Int. Conf. on Fracture Mech. of Concrete Structures (FraMCoS-2), held at ETH, Zürich), ed. by F.H. Wittmann, Aedificatio Publishers, Freiburg, Germany (1995) 515–534.
22. Z.P. Bažant, Scaling of quasibrittle fracture and the fractal question. *ASME Journal of Materials and Technology* 117 (Oct. 1995) 361–367 (Materials Division Special 75th Anniversary Issue).

23. J. Lemaitre and J.-L. Chaboche, *Mechanics of Solid Materials.* Cambridge University Press, Cambridge, U.K. (1990).
24. G.R. Irwin, Fracture, in *Handbuch der Physik*, VI, ed. by W. Flügge, Springer Verlag, Berlin (1958) 551–590.
25. D. Broek, *Elementary Engineering Fracture Mechanics*, 4th ed., Martinus Nijhoff, Dordrecht, Netherlands (1986).
26. Z.P. Bažant and L. Cedolin, *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York (1991).
27. J.F. Knott, *Fundamentals of Fracture Mechanics*, Butterworth, London (1973).
28. M.F. Kanninen and C.H. Popelar, *Advanced Fracture Mechanics*, Oxford University Press, New York (1985).
29. H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, 2nd ed., Paris Productions, St. Louis (1985).
30. Z.P. Bažant, Advances in material modeling of concrete, Transactions, *Tenth International Conference on Structural Mechanics in Reactor Technology* (SMiRT10, held in Anaheim, CA, August 1989), A, Principal Division Lectures, ed. by A.H. Hadjian, 301–330.
31. Z.P. Bažant, Fracture mechanics and strain-softening in concrete. Preprints, *U.S.- Japan Seminar on Finite Element Analysis of Reinforced Concrete Structures*, Tokyo, 1 (1985) 47–69.
32. Z.P. Bažant, Fracture energy of heterogeneous material and similitude. Preprints, *SEM-RILEM International Conference on Fracture of Concrete and Rock* (held in Houston, Texas, June 1987), ed. by S.P. Shah and S.E. Swartz, published by SEM (Soc. for Exper. Mech.) (1987) 390–402.
33. Z.P. Bažant and P.A. Pfeiffer, Determination of fracture energy from size effect and brittleness number, *ACI Materials Journal* 84 (1987) 463–480.
34. Y.N. Li and Z.P. Bažant, Eigenvalue analysis of size effect for cohesive crack model. *International Journal of Fracture* 66 (1984) 213–224.
35. Z.P. Bažant, Editor, *Fracture Mechanics of Concrete Structures* (Part I) (Proc., First Int. Conf. on Fracture Mech. of Concrete Structures (FraMCoS-1), held in Breckenridge, Colorado), Elsevier, London (1991).
36. Z.P. Bažant and M.T. Kazemi, Determination of fracture energy, process zone length and brittleness number from size effect, with application to rock and concrete. *International Journal of Fracture* 44 (1990) 111–131.
37. Z.P. Bažant and Y.-N. Li, Stability of cohesive crack model: Part II – Eigenvalue analysis of size effect on strength and ductility of structures. *Trans. ASME, Journal of Applied Mechanics* 62 (1995) 965–969.
38. Z.P. Bažant, Size effect aspects of measurement of fracture characteristics of quasibrittle material. In *Fracture Mechanics of Concrete Structures*, 3, (Proc., 2nd Int. Conf. on Fracture Mech. of Concrete Structures (FraMCoS-2), held at ETH, Zürich), ed. by F.H. Wittmann, Aedificatio Publishers, Freiburg, Germany (1996) 1749–1772; reprinted in *Journal of Advanced Cement-Based Materials* 4 (3/4) 128–137.
39. Z.P. Bažant, R. Gettu and M.T. Kazemi, Identification of nonlinear fracture properties from size-effect tests and structural analysis based on geometry-dependent R-curves. *International Journal of Rock Mechanics and Mining Sciences* 28(1) (1991) 43–51.