Scaling, universality, and renormalization: Three pillars of modern critical phenomena

H. Eugene Stanley

Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215

This brief overview is designed to introduce some of the advances that have occurred in our understanding of phase transitions and critical phenomena. The presentation is organized around three simple questions: (i) What are the basic phenomena under consideration? (ii) Why do we care? (iii) What do we actually do? To answer the third question, the author shall briefly review scaling, universality, and renormalization, three of the many important themes which have served to provide the framework of much of our current understanding of critical phenomena. The style is that of a colloquium, not that of a mini-review article. [S0034-6861(99)02902-5]

I. THE FIRST QUESTION: ''WHAT ARE CRITICAL PHENOMENA?''

Suppose we have a simple bar magnet. We know it is a ferromagnet because it is capable of picking up thumbtacks, the number of which is called the order parameter M. As we heat this system, M decreases and eventually, at a certain critical temperature T_c, it reaches zero: no more thumbtacks remain! In fact, the transition is remarkably sharp, since M approaches zero at T_c with infinite slope. Such singular behavior is an example of a ''critical phenomenon.''

Critical phenomena are by no means limited to the order parameter. For example, the response-functions constant-field specific heat C_H and isothermal susceptibility χ_T both become infinite at the critical point.

II. THE SECOND QUESTION: ''WHY DO WE CARE?''

One reason for interest in any field is that, simply put, we do not fully understand the basic phenomena. For example, for even the simplest three-dimensional system we cannot make exact predictions of all the relevant quantities from any realistic *microscopic* model at our disposal. Of the models that can be solved in closed form, most make the same predictions for behavior near the critical point as the classical mean-field model, in which one assumes that each magnetic moment interacts with all other magnetic moments in the entire system with equal strength (see, e.g., the review of Domb, 1996). The mean-field model predicts that both M^2 and χ_T^{-1} approach zero *linearly* as $T\to T_c$, and that C_H does not diverge at all. In fact, the mean-field theory cannot locate the value of T_c to better than typically about 40%.

A second reason for our interest is the striking similarity in behavior near the critical point among systems that are otherwise quite different in nature. A celebrated example is the ''lattice-gas'' analogy between the behavior of a single-axis ferromagnet and a simple fluid, near their respective critical points (Lee and Yang, 1952). Even the numerical values of the critical-point exponents describing the quantitative nature of the singularities are identical for large groups of apparently diverse physical systems.

A third reason is awe. We wonder how it is that spins ''know'' to align so suddenly as $T\to T_c^+$. How can the spins propagate their correlations so extensively throughout the entire system that $M\neq 0$ and $\chi_T\to\infty$?

III. THE THIRD QUESTION: ''WHAT DO WE DO?''

The answer to this question will occupy the remainder of this brief overview. The recent past of the field of critical phenomena has been characterized by several important conceptual advances, three of which are scaling, universality, and renormalization.

A. Scaling

The scaling hypothesis was independently developed by several workers, including Widom, Domb and Hunter, Kadanoff, Patashinskii and Pokrovskii, and Fisher (authoritative reviews include Fisher, 1967 and Kadanoff, 1967). The scaling hypothesis has two categories of predictions, both of which have been remarkably well verified by a wealth of experimental data on diverse systems. The first category is a set of relations, called *scaling laws*, that serve to relate the various critical-point exponents. For example, the exponents α, 2β, and γ describing the three functions C_H, M^2, and χ_T are related by the simple scaling law $\alpha+2\beta+\gamma=2$. Here the exponents are defined by $C_H\sim\epsilon^{-\alpha}$, $M^2\sim\epsilon^{2\beta}$, and $\chi_T\sim\epsilon^{-\gamma}$, where $\epsilon\equiv(T-T_c)/T_c$ is the reduced temperature.

The second category is a sort of *data collapse*, which is perhaps best explained in terms of our simple example of a uniaxial ferromagnet. We may write the equation of state as a functional relationship of the form $M=M(H,\epsilon)$, where M is the order parameter and H is the magnetic field. Since $M(H,\epsilon)$ is a function of two variables, it can be represented graphically as M vs ϵ for a sequence of different values of H. The scaling hypothesis predicts that all the curves of this family can be ''collapsed'' onto a single curve provided one plots not M vs ϵ but rather a *scaled* M (M divided by H to some

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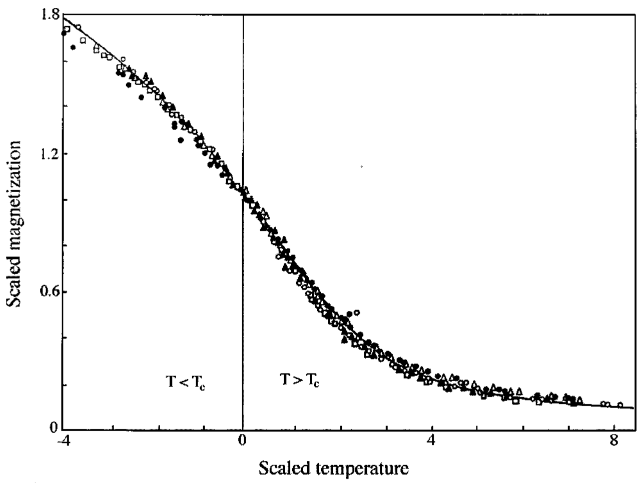

FIG. 1. Experimental *MHT* data on five different magnetic materials plotted in scaled form. The five materials are $CrBr_3$, EuO, Ni, YIG, and Pd_3Fe. None of these materials is an idealized ferromagnet: $CrBr_3$ has considerable lattice anisotropy, EuO has significant second-neighbor interactions. Ni is an itinerant-electron ferromagnet, YIG is a ferrimagnet, and Pd_3Fe is a ferromagnetic alloy. Nonetheless, the data for all materials collapse onto a single scaling function, which is that calculated for the $d=3$ Heisenberg model [after Milošević and Stanley (1976)].

power) vs a *scaled* ϵ (ϵ divided by H to some different power).

The predictions of the scaling hypothesis are supported by a wide range of experimental work, and also by numerous calculations on model systems such as the n-vector model. Moreover, the general principles of scale invariance used here have proved useful in interpreting a number of other phenomena, ranging from elementary-particle physics (Jackiw, 1972) to galaxy structure (Peebles, 1980).

B. Universality

The second theme goes by the rather pretentious name "universality." It was found empirically that one could form an analog of the Mendeleev table if one partitions all critical systems into "universality classes." The concept of universality classes of critical behavior was first clearly put forth by Kadanoff, at the 1970 Enrico Fermi Summer School, based on earlier work of a large number of workers including Griffiths, Jasnow and Wortis, Fisher, Stanley, and others.

Consider, e.g., experimental *M-H-T* data on five diverse magnetic materials near their respective critical points (Fig. 1). The fact that data for each collapse onto a scaling function supports the scaling hypotheses, while the fact that the scaling function is the *same* (apart from two material-dependent scale factors) for all five diverse materials is truly remarkable. This apparent universality of critical behavior motivates the following question: *"Which features of this microscopic interparticle force are important for determining critical-point exponents and scaling functions, and which are unimportant?"*

Two systems with the same values of critical-point exponents and scaling functions are said to belong to the same universality class. Thus the fact that the exponents and scaling functions in Fig. 1 are the same for all five materials implies they all belong to the same universality class.

C. Renormalization

The third theme stems from Wilson's essential idea that the critical point can be mapped onto a fixed point of a suitably chosen transformation on the system's Hamiltonian (see the recent reviews: Goldenfeld, 1994; Cardy, 1996; Lesne, 1998). This resulting "renormalization group" description has (i) provided a foundation for understanding the themes of scaling and universality, (ii) provided a calculational tool permitting one to obtain numerical estimates for the various critical-point exponents, and (iii) provided us with altogether new concepts not anticipated previously.

One altogether new concept that has emerged from renormalization is the idea of upper and lower marginal dimensionalities d_+ and d_- (see the review of Als-Nielsen and Birgeneau, 1977). For $d>d_+$, the classical theory provides an adequate description of critical-point exponents and scaling functions, whereas for $d<d_+$, the classical theory breaks down in the immediate vicinity of the critical point because statistical fluctuations neglected in the classical theory become important. The case $d=d_+$ must be treated with great care; usually, the classical theory "almost" holds, and the modifications take the form of weakly singular corrections.

For $d<d_-$, fluctuations are so strong that the system cannot sustain long-range order for any $T>0$. For $d_- <d<d_+$, we do not know exactly the properties of systems (in most cases) except when n approaches infinity, where n will be introduced below as the spin dimension. One can, however, develop expansions in terms of the parameters (d_+-d), $(d-d_-)$, and $1/n$ (see, e.g., the reviews of Fisher, 1974; and Brézin and Wadia, 1993).

In the remainder of this brief overview, we shall attempt to define somewhat more precisely the concepts underlying the three themes of scaling, universality, and renormalization without sacrificing the stated purpose, that of a colloquium-level presentation.

IV. WHAT IS SCALING?

I offer here a very brief introduction to the spirit and scope of the scaling approach to phase transitions and critical phenomena using, for the sake of concreteness, a simple system: the Ising magnet. Further, we discuss only the simplest static property, the order parameter, and the two response functions C_H and χ_T. The rich subject of dynamic scaling is beyond our scope here (see, e.g., the authoritative review of Hohenberg and Halperin, 1977).

A. The scaling hypothesis

The scaling hypothesis for thermodynamic functions is made in the form of a statement about one particular thermodynamic potential, generally chosen to be the Gibbs potential per spin, $G(H,T)=G(H,\epsilon)$. One form of the hypothesis is the statement (see, e.g., Stanley, 1971) that asymptotically close to the critical point, $G_s(H,\epsilon)$, the singular part of $G(H,\epsilon)$, is a generalized homogeneous function (GHF). Thus the scaling hypothesis may be expressed as a relatively compact statement that asymptotically close to the critical point, there exist two numbers, a_H and a_T (termed the field and temperature scaling powers) such that for all positive λ, $G_s(H,\epsilon)$ obeys the functional equation:

$$G_s(\lambda^{a_H}H,\lambda^{a_T}\epsilon)=\lambda G_s(H,\epsilon). \tag{1}$$

B. Exponent relations: The scaling laws

The predictions of the scaling hypothesis are simply the properties of GHFs: (i) Legendre transforms of GHFs are also GHFs, so all thermodynamic potentials are GHFs. (ii) Derivatives of GHFs are also GHFs. Since *every* thermodynamic function is expressible as some derivative of some thermodynamic potential, it follows that the singular part of every thermodynamic function is asymptotically a GHF.

Two useful facts are worth noting:

(a) The critical-point exponent for any function is simply given by the ratio of the scaling power of the function to the scaling power of the path variable along which the critical point is approached:

$$\text{arbitrary exponent}=\frac{a_{\text{function}}}{a_{\text{path}}}. \tag{2}$$

Thus one can "write down by inspection" expressions for any critical-point exponent. Equation (2) holds generally, and proves useful in practice. For the special case of a uniaxial ferromagnet, we have

$$a_{\text{path}}=\begin{cases} a_H & \text{strong path } [T=T_c, H\to 0], \\ a_T & \text{weak path } [H=0^{\pm}, T\to T_c^{\pm}]. \end{cases} \tag{3}$$

From property (ii), it follows that

$$a_{\text{function}}=\begin{cases} 1-a_H & \text{for } \overline{M}\propto(\partial G/\partial H)_T, \\ 1-a_T & \text{for } \overline{S}\propto(\partial G/\partial T)_H. \end{cases} \tag{4a}$$

Similarly, from the definitions for the susceptibility and specific heat, we have

$$a_{\text{function}}=\begin{cases} 1-2a_H & \text{for } \overline{\chi_T}\propto(\partial^2 G/\partial H^2)_T, \\ 1-2a_T & \text{for } \overline{C_H}\propto(\partial^2 G/\partial T^2)_H. \end{cases} \tag{4b}$$

(b) Since each critical-point exponent is directly expressible in terms of a_H and a_T, it follows that one can eliminate these two unknown scaling powers from the expressions for three different exponents, and thereby obtain a family of equalities called *scaling laws.*

To illustrate the utility of facts (a) and (b), we note from Eqs. (3) and (4b) that

$$-\alpha'=\frac{1-2a_T}{a_T}, \tag{5a}$$

$$\beta=\frac{1-a_H}{a_T}, \tag{5b}$$

and

$$-\gamma'=\frac{1-2a_H}{a_T}. \tag{5c}$$

We thus have three equations and two unknowns. Eliminating a_H and a_T, we find

$$\alpha'+2\beta+\gamma'=2, \tag{6}$$

which is the Rushbrooke inequality $\alpha'+2\beta+\gamma\geqslant 2$ in the form of an equality. Defining δ through $M\sim H^{\delta}$, it follows that

$$\delta^{-1}=\frac{a_M}{a_H}=\frac{1-a_H}{a_H}. \tag{7}$$

Eliminating a_H and a_T from Eqs. (5a), (5b), and (7), we obtain the Griffiths equality

$$\alpha'+\beta(\delta+1)=2. \tag{8}$$

Similarly, Eqs. (5b), (5c), and (7) give the Widom equality

$$\gamma'=\beta(\delta-1). \tag{9}$$

Thus one hallmark of the scaling approach is a family of three-exponent equalities—called *scaling laws*—of which Eqs. (6), (8), and (9) are but examples. In general, it suffices to determine two exponents since these will in general fix the scaling powers a_H and a_T, which in turn may be used to obtain the exponents for *any* thermodynamic function.

C. Equation of state and scaling functions

Next we discuss a second hallmark of the scaling approach, the equation of state. The scaling hypothesis of Eq. (1) constrains the form of a thermodynamic potential, near the critical point, so this constraint must have implications for quantities derived from that potential, such as the equation of state.

Consider, for example, the $M(H,T)$ equation of state of a uniaxial ferromagnet near the critical point [$H=0,T=T_c$]. On differentiating Eq. (1) with respect to H, we find

$$M(\lambda^{a_H}H,\lambda^{a_T}\epsilon)=\lambda^{1-a_H}M(H,\epsilon). \tag{10}$$

Since Eq. (10) is valid for all positive values of λ, it must certainly hold for the particular choice $\lambda=H^{-1/a_H}$. Hence

$$M_H=M(1,\epsilon_H)=\mathcal{F}^{(1)}(\epsilon_H), \tag{11a}$$

where

$$M_H \equiv \frac{M}{H^{(1-a_H)/a_H}} = \frac{M}{H^{1/\delta}}, \tag{11b}$$

and

$$\epsilon_H \equiv \frac{\epsilon}{H^{a_T/a_H}} = \frac{\epsilon}{H^{1/\Delta}} \tag{11c}$$

are termed the *scaled magnetization* and *scaled temperature*, while the function $\mathcal{F}^{(1)}(x) = M(1,x)$ defined in Eq. (11a) is called a *scaling function*.

In Fig. 1, the scaled magnetization M_H is plotted against the scaled temperature ϵ_H, and the entire family of $M(H=\text{const},T)$ curves "collapse" onto a single function. This scaling function $\mathcal{F}^{(1)}(H) = M(1,\epsilon_H)$ evidently is the magnetization function in fixed nonzero magnetic field.

V. WHAT IS UNIVERSALITY?

Empirically, one finds that all systems in nature belong to one of a comparatively small number of such universality classes. Two specific microscopic interaction Hamiltonians appear almost sufficient to encompass the universality classes necessary for static critical phenomena.

The first of these is the Q-state Potts model (Potts, 1952; Wu, 1982). One assumes that each spin i can be in one of Q possible discrete orientations ζ_i ($\zeta_i = 1,2,\ldots,Q$). If two neighboring spins i and j are in the same orientation, then they contribute an amount $-J$ to the total energy of a configuration. If i and j are in different orientations, they contribute nothing. Thus the interaction Hamiltonian is [Fig. 2(a)]

$$\mathcal{H}(d,s) = -J \sum_{\langle ij \rangle} \delta(\zeta_i,\zeta_j), \tag{12a}$$

where $\delta(\zeta_i,\zeta_j)=1$ if $\zeta_i=\zeta_j$, and is zero otherwise. The angular brackets in Eq. (12a) indicate that the summation is over all pairs of nearest-neighbor sites $\langle ij \rangle$. The interaction energy of a pair of neighboring parallel spins is $-J$, so that if $J>0$, the system should order ferromagnetically at $T=0$.

The second such model is the n-vector model (Stanley, 1968), characterized by spins capable of taking on a continuum of states [Fig. 2(b)]. The Hamiltonian for the n-vector model is

$$\mathcal{H}(d,n) = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j . \tag{12b}$$

Here, the spin $\vec{S}_i \equiv (S_{i1},S_{i2},\ldots,S_{in})$ is an n-dimensional unit vector with $\Sigma_{\alpha=1}^{n} S_{i\alpha}^2 = 1$, and $\vec{S}_i$ interacts isotropically with spin $\vec{S}_j$ localized on site j. Two parameters in the n-vector model are the system dimensionality d and the spin dimensionality n. The parameter n is sometimes called the order-parameter symmetry number; both d and n determine the universality class of a system for static exponents.

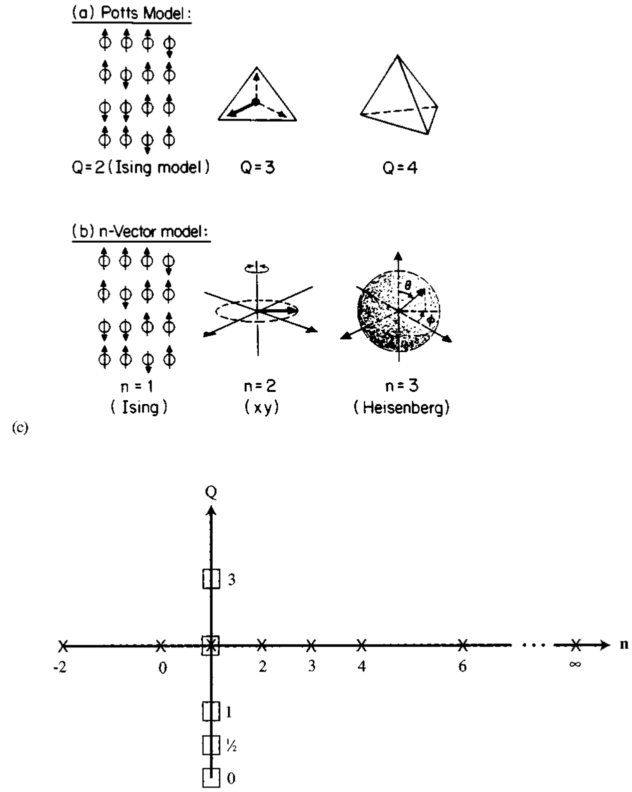

FIG. 2. Schematic illustrations of the possible orientations of the spins in (a) the s-state Potts model, and (b) the n-vector model. Note that the two models coincide when $Q=2$ and $n=1$. (c) North-south and east-west "Metro lines."

Both the Potts and n-vector hierarchies are generalization of the simple Ising model of a uniaxial ferromagnet. This is indicated schematically in Fig. 2(c), in which the Potts hierarchy is depicted as a north-south "Metro line," while the n-vector hierarchy appears as an east-west line. The various stops along the respective Metro lines are labeled by the appropriate value of s and n. The two Metro lines have a *correspondence* at the Ising model, where $Q=2$ and $n=1$.

Along the north-south Metro line (the Q-state hierarchy), Kasteleyn and Fortuin showed that the limit $Q=1$ reduces to the random percolation problem, which may be relevant to the onset of gelation (Stauffer and Aharony, 1992; Bunde and Havlin, 1996). Stephen demonstrated that the limit $Q=0$ corresponds to a type of treelike percolation, while Aharony and Müller showed that the case $Q=3$ has been demonstrated to be of relevance in interpreting experimental data on structural phase transitions and on absorbed monolayer systems.

The east-west Metro line, though newer, has probably been studied more extensively than the north-south line; hence we shall discuss the east-west line first. For $n=1$, the spins S_i are one-dimensional unit vectors which take on the values ± 1. Equation (12b), $\mathcal{H}(d,1)$, is the Ising Hamiltonian, which has proved extremely useful in interpreting the properties of the liquid-gas critical point (Levelt Sengers *et al.*, 1977). This case also corresponds to the uniaxial ferromagnet introduced previously.

Other values of n correspond to other systems of interest. For example, the case $n=2$ describes a set of isotropically interacting classical spins whose motion is confined to a plane. The Hamiltonian $\mathcal{H}(d,2)$ is sometimes called the plane-rotator model or the XY model. It is relevant to the description of a magnet with an easy plane of anisotropy such that the moments prefer to lie in a given plane. The case $n=2$ is also useful in interpreting experimental data on the λ-transition in ^{4}He.

For the case $n=3$, the spins are isotropically interacting unit vectors free to point anywhere in three-dimensional space. Indeed, $\mathcal{H}(d,3)$ is the classical Heisenberg model, which has been used for some time to interpret the properties of many isotropic magnetic materials near their critical points.

Two particular "Metro stops" are more difficult to see yet nevertheless have played important roles in the development of current understanding of phase transitions and critical phenomena. The first of these is the limiting case $n\rightarrow\infty$, which Stanley showed (in a paper reprinted as Chapter 1 of Brézin and Wadia, 1993) corresponds to the Berlin-Kac spherical model of magnetism, and is in the same universality class as the ideal Bose gas. The second limiting case $n=0$ de Gennes showed has the same statistics of a d-dimensional self-avoiding random walk, which in turn models a system of dilute polymer molecules (see, e.g., de Gennes, 1979 and references therein). The case $n=-2$ corresponds, as Balian and Toulouse demonstrated, to random walks, while Mukamel and co-workers showed that the cases $n=4,6,8,\ldots$ may correspond to certain antiferromagnetic orderings.

VI. WHAT IS RENORMALIZATION?

This is the second most-often-asked question. In one sense this question is easier to answer than "what is scaling," because to some degree renormalization concepts lead to a well-defined prescription for obtaining numerical values of critical exponents, unlike the scaling hypothesis which leads only to relations among exponents. Answering the question can involve considerable mathematics, so we concentrate here not on momentum-space renormalization but rather on the simpler position space. Instead of treating thermal phenomena we treat a different class of critical phenomena, the purely geometric connectivity phenomena generally called "percolation." The example we give requires such simple mathematics that one could imagine that renormalization could have been invented by the Greek geometers.

A. The percolation problem

We begin by defining the percolation problem. This is a phase-transition model that was formulated only in comparatively recent times. Recent reviews describing the wealth of current research on percolation include Stauffer and Aharony (1992), and Bunde and Havlin (1996).

Suppose a fraction p of the sites of an infinite d-dimensional lattice are occupied. For p small, most of the occupied sites are surrounded by vacant neighboring sites. However as p increases, many of the neighboring sites become occupied, and the sites are said to form clusters (sites i and j belong to the same cluster if there exists a path joining nearest-neighbor pairs of occupied sites leading from site i to site j). One can describe the clusters by various functions, such as their characteristic linear dimension $\xi(p)$. As p increases, $\xi(p)$ increases monotonically, and at a critical value of p—denoted p_c—it diverges:

$$\xi(p)\sim|p-p_c|^{-\nu}. \tag{13}$$

For $p\geqslant p_c$ there appears, in addition to the finite clusters, a cluster that is infinite in extent.

The number p_c is referred to as the *connectivity threshold* because of the fact that for $p<p_c$ the connectivity is not sufficient to give rise to an infinite cluster, while for $p>p_c$ it is. Indeed, we shall see that the role of $\epsilon\equiv(T-T_c)/T_c$ is played by $(p_c-p)/p_c$. The numerical value of p_c depends upon both the dimensionality d of the lattice and on the lattice type; however percolation exponents depend only on d.

B. Kadanoff cells and the renormalization transformation

Percolation functions can be calculated in closed form for $d=1$ by Reynolds and co-workers (see, e.g, the review Stanley, 1982). In particular, one finds that $p_c=1$, and that $\nu=1$. It is instructive to illustrate some aspects of the position-space renormalization approach on this exactly-soluble system (Stanley, 1982). The treatment presented below is intended to illustrate—in terms of a simple example—some of the features of the position-space renormalization approach.

The starting point of our illustrative example is the Kadanoff-cell transformation (Kadanoff, 1967). This is illustrated for one-dimensional percolation in Fig. 3(a), which shows b^d-site Kadanoff cells with $b=2$ and $d=1$. Just as each *site* in the lattice is described by a parameter p, its probability of being occupied, so each *cell* is described by a parameter p', which we may regard as being the "cell occupation probability" [Fig. 3(b)]. The essential step in the renormalization-group approach is the construction of a functional relation between the original parameter p and the "renormalized" parameter p',

$$p'=R_b(p). \tag{14}$$

The function $R_b(p)$ is termed a renormalization transformation.

The transformation $R_b(p)$ is particularly simple for one-dimension percolation. Since the percolation threshold is a connectivity phase transition, it is reasonable to say that a cell is "occupied" only if all the sites in the cell are occupied (for if a single site were empty, then the connectivity would be lost). If the probability of a single site being occupied is p, then the probability of all b sites in the cell being occupied is p^b. Hence $R_b(p)=p^b$, and Eq. (14) becomes

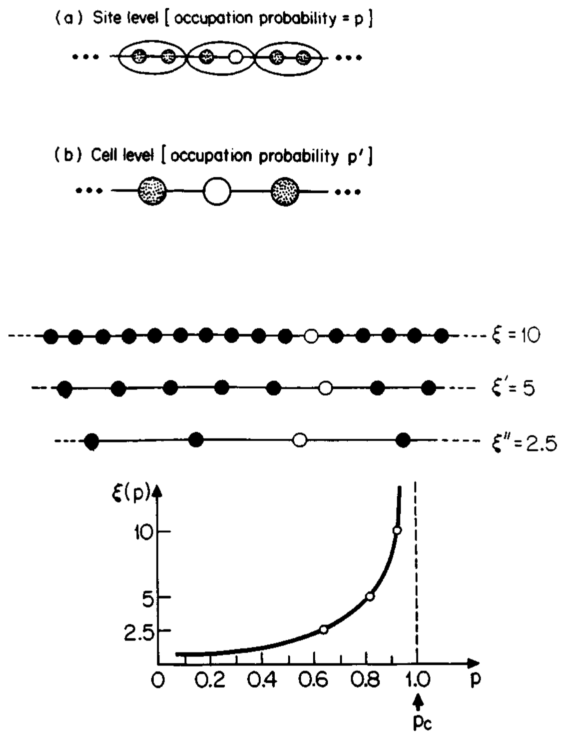

FIG. 3. The Kadanoff-cell transformation applied to the example of one-dimensional percolation. The site level in (a) is characterized by a single parameter p—the probability of a *site* being occupied. The cell level in (b) is characterized by the parameter p'—the probability of a *cell* being occupied. The relation between the two parameters, p and p', is given by the renormalization transformation $R(p)$ of Eqs. (14) and (15). Also shown are successive Kadanoff-cell transformations. After each transformation, the correlation length $\xi(p)$ is halved. The corresponding value of occupation probability is reduced to $p'=p^b=p^2$, thus taking the system "farther away" from the critical point $p=p_c=1$. Occupied sites and cells are shown solid, while empty sites and cells are open.

$$p'=p^b. \tag{15}$$

C. Fixed points of the renormalization transformation

The actual choice of the function $R_b(p)$ varies, of course, from one problem to the other. However the remaining steps to be followed after selecting a suitable $R_b(p)$ are essentially the same for all problems. First, we note [Fig. 3(c)] that on carrying out the renormalization transformation, the new correlation length $\xi'(p')$ is smaller than the original correlation length $\xi(p)$ by a factor of b:

$$\xi'(p')=b^{-1}\xi(p). \tag{16}$$

Next we consider the effect of carrying out *successive* Kadanoff-cell transformations with our one-dimensional example. Suppose the system starts out at an initial parameter value $p=p_0=0.9$, as shown schematically in Fig. 4. After a single renormalization transformation, the value of p becomes $p_0'=R_b(p_0)=0.81$ by Eq. (15). The transformed system is *farther* from the critical point, and

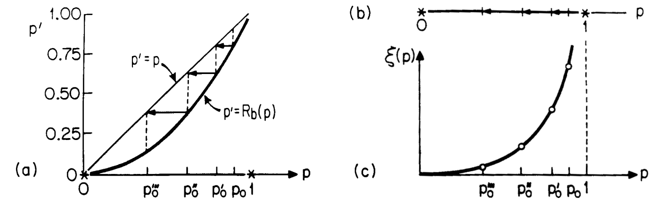

FIG. 4. Generic idea of a flow diagram, illustrated here for the pedagogical example of one dimension. (a) Two curves, $p'=p$ and $p'=R_2(p)=p^2$. The fixed points $p^*=0,1$ are given by the intersection of these two curves; the "thermal" scaling power a_T is related to the slope of $R_b(p)$ at the unstable fixed point $p^*=1$. Also shown is the effect of successive Kadanoff-cell transformations, Eq. (15), on a system whose initial value of the parameter p is $p_0=0.9$. This information is capsulized in the one-dimensional *flow diagram* of part (b), which illustrates the result of Eq. (16)—that each renormalization serves to halve the correlation length ξ.

hence $\xi'(p')$ is smaller—just as we noted in Fig. 3(c). If we now perform a renormalization transformation on the transformed system, we have $p_0''=R_b[R_b(p_0)]=(p_0')^2=0.64$. The doubly-transformed system is now farther still from the critical point.

Thus the effect of successive Kadanoff-cell transformations for the example at hand is to take the system *away* from its critical point. An important exception to this statement is the following: if a system is initially *at* its critical point (e.g., if $p_0=p_c=1$), then $\xi=\infty$ and hence ξ', by Eq. (16), is also infinite. A necessary but not sufficient condition that this occur is for p' to equal p. The values of p for which $p'=p$ are termed the *fixed points* p^* of the transformation $R_b(p)$,

$$R_b(p^*)=p^*. \tag{17}$$

Thus, by obtaining all the fixed points of a given renormalization transformation $R_b(p)$, we should be able to obtain the critical point. For the example of one-dimension percolation, $R_b(p)=p^b$ and there are two fixed points. One is $p^*=0$ and the other is $p^*=1$. Indeed, we recognize the critical point, $p_c=1$, as one of the two fixed points.

Now if the system is initially at a value $p=p_0$, which is close to the $p^*=1$ fixed point, then under the renormalization transformation it is carried to a value of p_0', which is farther from that fixed point. We may say a fixed point is *unstable* for the "relevant" scaling field $u=(p-p_c)$. Conversely, if p_0 is close to the $p^*=0$ fixed point, then it is carried to a value p_0' that is still closer to that fixed point; we term such a fixed point *stable*. Thus for the example at hand, there is one unstable fixed point, $p^*=1$, and one stable fixed point, $p^*=0$.

We often indicate the results of successive renormalization transformations schematically by means of a simple *flow* diagram, as is shown in Fig. 4(b). The arrows in the flow diagram indicate the effect of successive renormalization on the system's parameters. Note that the "flow" under successive transformations is from the unstable fixed point toward the stable fixed point. In the example treated here, there is only one parameter p and hence the flow diagram is one dimensional; in general, there can be many parameters, and the flow diagram is multidimensional.

D. Calculations of the "thermal" scaling power

We can also obtain numerical values for the scaling powers once we have a renormalization transformation. The "thermal" scaling power can be calculated for the basic reason that knowledge of $R_b(p)$ near p^* provides information on how $\xi(p)$ behaves for p near p^*. Perhaps the simplest and most straightforward fashion of demonstrating this fact is to expand $R_b(p)$ about $p=p^*$:

$$R_b(p)=R_b(p^*)+\lambda_T(b)(p-p^*)+\mathcal{O}(p-p^*)^2. \quad (18)$$

Here we use the symbol $\lambda_T(b)$ to denote the first derivative of the renormalization function evaluated at the fixed point p^*. From Eq. (15), we find

$$\lambda_T(b)=\left(\frac{dR_b}{dp}\right)_{p=p^*}=b. \quad (19)$$

If we now substitute Eqs. (14) and Eq. (17) into (18), and if we neglect terms of order $(p-p^*)^2$, then we obtain simply

$$p'-p^*=\lambda_T(b)(p-p^*). \quad (20)$$

Equation (20) expresses the deviation of p' from the fixed point in the transformed system in terms of the deviation of p from the fixed-point value in the original system.

As we noted above, the effect of the renormalization transformation on $\xi(p)$ is given by Eq. (16). If we regard Eq. (16) as a functional equation valid for all values of p, p', and b, then we can set $b=1$ and conclude that

$$\xi'(p)=\xi(p), \quad (21)$$

where the equality $p'=p$ follows from Eqs. (19) and (20). Thus ξ' and ξ are the same functions, so that if $\xi(p)$ has a power-law dependence near the critical point—given by Eq. (13)—then it follows from Eq. (21) that

$$\xi'(p')=|p'-p_c|^{-\nu}. \quad (22a)$$

Substituting Eqs. (13) and (22a) into Eq. (16), we have

$$|p'-p_c|^{-\nu}\sim b^{-1}|p-p_c|^{-\nu}. \quad (22b)$$

Since p_c is the value of p at which ξ diverges, we set $p^*=p_c$ in Eq. (20). Hence

$$|p'-p_c|^{-\nu}=[\lambda_T(b)]^{-\nu}|p-p_c|^{-\nu}. \quad (22c)$$

Comparing Eqs. (22b) and (22c), we can express ν in terms of the scale change b and the "derivative" $\lambda_T(b)$,

$$\nu=\frac{\ln b}{\ln \lambda_T(b)}. \quad (22d)$$

The argument thus far is valid generally. Returning to the example of one-dimensional percolation, we note from Eq. (19) that $\lambda_T(b)=b$. Hence from Eq. (22d) $\nu=1$, which is the exact result.

The renormalization approach to critical phenomena leads to scaling (see, e.g., the discussion in Nelson and Fisher, 1975 and Fisher, 1998). As a result of scaling, knowledge of ν is sufficient to determine the value of a_T, the "thermal" scaling power for the weak direction, since

$$a_T=\frac{1}{d\nu}. \quad (23a)$$

It is becoming customary to normalize scaling powers by a factor of d, the system dimensionality. Thus one defines $y_T\equiv da_T$ and finds from Eqs. (22d) and (23a) that

$$y_T=\frac{\ln \lambda_T(b)}{\ln b}. \quad (23b)$$

VII. DO WE UNDERSTAND THE CRITICAL POINT?

About half of the physicists I know feel the critical point is not understood, while the other half seem to feel that it is. It all depends on what we mean by the word "understood." For some, the term means that one can solve a model in closed form and calculate all the exponents. Then the situation is like Schubert's unfinished symphony—albeit perhaps not finished, it is nonetheless very beautiful. And, like Schubert's symphony, what is not finished will never be since even the "simple" Ising model is believed hopelessly insoluble except for the case of $d=1,2$. Even the $d=2$ case is hopeless to solve in nonzero magnetic field, so do not expect exact calculations of scaling functions and all the field-dependent exponents. In three dimensions, no models are solved in closed form, with a few notable exceptions such as the $n\rightarrow\infty$ limit of the n-vector model, and some initial terms for the $1/n$ expansion (Brézin and Wadia, 1993).

If we relax our standards of rigor and consider the scaling hypothesis, then we can make some concrete predictions for all dimensions, but not for the exponent values or the threshold values. While not rigorous, the various "handwaving" arguments to justify scaling and renormalization are sufficient to convince a reasonable person—but not a stubborn one (to paraphrase the critical-phenomena pioneer Marc Kac). But even the handwaving arguments do not explain why in some systems scaling holds for only 1–2 % away from the critical point and in other systems it holds for 30–40 % away. Moreover, no modern theory makes exact predictions for experimentally interesting critical parameters such as T_c, which varies from one material to the next by as much as six orders of magnitude.

Despite this "unfinished" situation, the conceptual framework of critical phenomena is increasingly finding application in other fields, ranging from chemistry and biology on the one hand to econophysics (Mantegna and Stanley, 1999) and even liquid water (Stanley *et al.*, 1997; Mishima and Stanley, 1998). Why is this? One possible answer concerns the way in which correlations spread throughout a system comprised of subunits. Like the economy, ‘‘everything depends on everything else.’’ But how can these interdependencies give rise not to exponential functions, but rather to the power laws characteristic of critical phenomena?

The paradox is simply stated. The probability that a spin at the origin 0 is aligned with a spin a distance r away, $(1+\langle s_0 s_r\rangle)/2$, is unity only at $T=0$. For $T>0$, our intuition tells us that the spin correlation function $C(r)\equiv\langle s_0 s_r\rangle-\langle s_0\rangle\langle s_r\rangle$ must decay exponentially with r—for the same reason that the value of money stored in a mattress decays exponentially with time (each year it loses a constant fraction of its worth). Thus we might expect that $C(r)\sim e^{-r/\xi}$, where ξ, the correlation length, is the characteristic length scale above which the correlation function is negligibly small. Experiments and also calculations on mathematical models confirm that correlations do indeed decay exponentially, but if the system is at its critical point, then the rapid exponential decay magically turns into a long-range power-law decay of the form $C(r)\sim 1/r^{d-2+\eta}$.

So then how can correlations actually propagate an infinite distance, without requiring a series of amplification stations all along the way? We can understand such ‘‘infinite-range propagation’’ as arising from the huge multiplicity of interaction paths that connect two spins if $d>1$ (if $d=1$, there is no multipicity of interaction paths, and spins order only at $T=0$). Enumeration algorithms take into account exactly the contributions of such interaction paths of length ℓ—up to a maximum length that depends on the strength of the computer used. Remarkably accurate quantitative results are obtained if this hierarchy of exact results for successive finite values of ℓ is then extrapolated to $\ell=\infty$.

For any $T>T_c$, the correlation between two spins along each of the interaction paths that connect them *decreases* exponentially with the length of the path. On the other hand, the number of such interaction paths *increases* exponentially, with a characteristic length that is temperature independent, depending primarily on the lattice dimension. This exponential increase is multiplied by a ‘‘gently decaying’’ power law that is negligible except for one special circumstance which we will come to.

Consider a fixed temperature T_1 far above the critical point, so that ξ is small, and consider two spins separated by a distance r which is larger than ξ. The exponentially decaying correlations along each interaction path connecting these two spins is so severe that it cannot be overcome by the exponentially growing number of interaction paths between the two spins. Hence at T_1 the exponential decrease in correlation along each path wins the competition between the two exponentials, and we anticipate that $\langle s_0 s_r\rangle$ falls off exponentially with the distance r. Consider now the same two spins at a fixed temperature T_2 far below the critical point. Now the exponentially decaying correlation along each interaction path connecting these two spins is insufficiently severe to overcome the exponentially growing number of interaction paths between the two spins. Thus at T_2 the exponential increase in the number of interaction paths wins the competition. Clearly there must exist some intermediate temperature in between T_1 and T_2 where the the two exponentials just balance, and this temperature is the critical temperature T_c. Right at the critical point, the gently decaying power-law correction factor in the number of interaction paths, previously negligible, emerges as the victor in this stand-off between the two warring exponential effects. As a result, two spins are well correlated even at arbitrarily large separation.

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