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Scaling with a Parameter in Spin Systems near the Critical Point. I*)

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The Hamiltonian of the form $\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1$ is discussed, where λ is a parameter to change *symmetry*, *dimensionality*, or *potential range*. Scaling with the parameter λ is studied for thermodynamic quantities such as the free energy. By assuming the scaled form $F(\varepsilon,\lambda)=\varepsilon^{2-\alpha}F(\lambda/\varepsilon^{\phi})$ for the singular part of the free energy near the critical point $T_c\,(\varepsilon=(T-T_c)/T_c)$, the expression (or explicit value) of the critical exponent ϕ is obtained in each case of change of symmetry, dimensionality and potential range. In particular, the universal relation $\phi=\gamma$ is found for change of dimensionality, where γ is the critical exponent of the susceptibility in the unperturbed Hamiltonian $\mathcal{H}_0$. An extension to dynamical critical phenomena is also discussed briefly, particularly in connection with the critical slowing down. A possibility is suggested to derive this generalized scaling with the parameter λ by applying the usual scaling law to each term of the perturbational expansion with respect to the parameter λ.

§ 1. Introduction

At present the scaling law[1] seems to be useful in discussing critical phenomena. In the usual static scaling law, the free energy F of a system is scaled with respect to an external force, from which we can derive scaling relations[1] among critical exponents.[2]

Quite recently, Riedel and Wegner,[2] and other several authors[4),5] have discussed the generalized scaling law with respect to parameters concerning *symmetry*, *dimensionality* and *potential range*, which are all assumed to be parameters linear in the Hamiltonian of a system. In connection with this, the Fisher-Griffiths proposition[2),6),7] may be of use, which roughly states that the critical exponents as functions of these parameters remain constant except possibly at points where symmetry, dimensionality and potential range change abruptly. Thus, critical behavior near these "symmetry points" in a generalized sense is most fascinating, because we can expect anomalies of thermodynamic quantities with respect to these parameters near symmetry points.

The purposes of this series of notes are (1) to derive the generalized scaling with parameters, by assuming the usual scaling law, and (2) to find expressions

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(or values) of critical exponents appearing in such generalized scaling. In this first note, a formal perturbation theory is discussed in connection with this generalized scaling, and expressions of critical exponents for such scaling are given explicitly, by assuming the generalized scaling, rather than by deriving it, which will be discussed in a separate paper.

Now, we consider the Hamiltonian of the form

$$\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1\,, \tag{1·1}$$

where λ is a parameter such as

(a) to change symmetry,
(b) to change dimensionality,
(c) to change potential range.

Although in the case (c) we can discuss a parameter nonlinear in the Hamiltonian, it seems difficult to treat this nonlinear parameter in a systematic way as in a linear parameter.

The partition function of a system described by the Hamiltonian (1·1) is written as

$$Z=\mathrm{Tr}[e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}]; \qquad \beta=1/k_BT\,, \tag{1·2}$$

which may be expanded with respect to the parameter λ:

$$\ln Z=\ln Z_0-\lambda\beta\langle O\mathcal{H}_1\rangle_c+(\lambda^2\beta^2/2\,!)\langle O\mathcal{H}_1{}^2\rangle_c+\cdots\,, \tag{1·3}$$

where $\langle O\mathcal{H}_1{}^n\rangle_c$ is a generalized cumulant[8] defined by

$$\begin{aligned}\langle O\mathcal{H}_1{}^n\rangle_c&=\frac{1}{\beta^n}\int_0^\beta dt_1\cdots\int_0^\beta dt_n\langle O\mathcal{H}_1(t_1)\cdots\mathcal{H}_1(t_n)\rangle_c\\&=\frac{n\,!}{\beta^n}\int_0^\beta dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\langle\mathcal{H}_1(t_1)\mathcal{H}_1(t_2)\cdots\mathcal{H}_1(t_n)\rangle_c\,,\end{aligned} \tag{1·4}$$

with

$$\mathcal{H}_1(t)=e^{t\mathcal{H}_0}\mathcal{H}_1e^{-t\mathcal{H}_0}, \tag{1·5}$$

and

$$\langle Q\rangle=\mathrm{Tr}\,Qe^{-\beta\mathcal{H}_0}/\mathrm{Tr}\,e^{-\beta\mathcal{H}_0}. \tag{1·6}$$

The above expansion will be used in the following sections.

§ 2. Critical exponent of generalized scaling and the singularities of the first few terms in the perturbation expansion

In most cases, we may be able to derive the asymptotic behavior of the form

$$\langle O\mathcal{H}_1{}^n\rangle_c\sim\varepsilon^{-\psi_n}; \qquad \psi_n=\psi_1+(n-1)\phi\,, \tag{2·1}$$

near the critical point with $\varepsilon=(T-T_c)/T_c$, by using the usual scaling law, following Kawasaki.[9] (Detailed discussions will be given in a separate paper.) Thus, one obtains the generalized scaling with a parameter:

$$F(\varepsilon, \lambda) \simeq \varepsilon^{2-\alpha} F(\lambda/\varepsilon^{\phi}), \tag{2·2}$$

or in more general in the presence of a reduced external field h,

$$F(\varepsilon, h, \lambda) \simeq \varepsilon^{2-\alpha} \tilde{F}(h/\varepsilon^{\Delta/2}, \lambda/\varepsilon^{\phi}), \tag{2·3}$$

with the relation

$$\phi = 2-\alpha+\psi_1 . \tag{2·4}$$

This relation (2·4) has been pointed out by Jasnow and Fisher.[5] Here, note that the generalized cumulant of the first order can be always reduced to the average itself:

$$\langle O\mathcal{H}_1\rangle_c = \langle \mathcal{H}_1\rangle . \tag{2·5}$$

When $\langle \mathcal{H}_1\rangle \equiv 0$ by symmetry, as occurs in many cases, we have to discuss the term of the second order in Eq. (1·3), and we obtain the relation

$$\phi = \tfrac{1}{2}(2-\alpha+\psi_2), \tag{2·6}$$

where the critical exponents α and ψ_2 are defined by

$$\ln Z_0 \equiv \ln \mathrm{Tr}\, e^{-\beta\mathcal{H}_0} \sim \varepsilon^{2-\alpha}, \tag{2·7}$$

and

$$\langle O\mathcal{H}_1^{\,2}\rangle_c \sim \varepsilon^{-\psi_2}. \tag{2·8}$$

Here, the symbol $A\sim B$ indicates that the most singular parts of A and B agree with each other, apart from their prefactors and non-singular parts (such as constants). The term of the second order $\langle O\mathcal{H}_1^{\,2}\rangle_c$ is, in general, reduced to the following canonical correlation:

$$\langle O\mathcal{H}_1^{\,2}\rangle_c = (\delta\mathcal{H}_1, \delta\mathcal{H}_1), \tag{2·9}$$

where $\delta\mathcal{H}_1 = \mathcal{H}_1 - \langle\mathcal{H}_1\rangle$ and the canonical correlation (B, A) is defined by[10]

$$(B, A) = \frac{1}{\beta}\int_0^{\beta} \langle e^{\lambda\mathcal{H}_0} B e^{-\lambda\mathcal{H}_0} A\rangle d\lambda . \tag{2·10}$$

The quantity (2·9) is the static response function (or a generalized susceptibility) with respect to the parameter λ. Here, it is useful to remember the following inequalities[11),12)] concerning the canonical and direct correlations:

$$\frac{1-e^{-\beta\bar{\omega}}}{\beta\bar{\omega}} \leq \frac{(\delta A, \delta A)}{\langle(\delta A)^2\rangle} \leq 1 , \tag{2·11}$$

for an hermitian quantity A, where $\bar{\omega}$ is the first moment defined by

$$\bar{\omega} = \tfrac{1}{4}\langle[\delta A, [\mathcal{H}_0, \delta A]]\rangle / \langle(\delta A)^2\rangle . \tag{2·12}$$

As $\bar{\omega}$ is finite in usual situations, one may expect that the canonical and direct correlations (i.e., the susceptibility and the fluctuation) should diverge in the same manner:[12)]

$$\langle O\mathcal{H}_1^2\rangle_c = (\delta\mathcal{H}_1, \delta\mathcal{H}_1) \sim \langle(\delta\mathcal{H}_1)^2\rangle \tag{2·13}$$

except possibly at $T=0$. Thus, we may study the singularity of the fluctuation $\langle(\delta\mathcal{H}_1)^2\rangle$ instead of the susceptibility $\langle O\mathcal{H}_1^2\rangle_c$, in order to discuss the critical exponents ψ_2 and ϕ.

In a similar way, one may have the scaled susceptibility

$$\chi_0 \sim \varepsilon^{-\gamma} G(\lambda\varepsilon^{-\phi}). \tag{2·14}$$

As discussed already by several authors,[3)~5)] the critical temperature $T_c(\lambda)$ of the system described by the total Hamiltonian $\mathcal{H}$ is given by

$$T_c(\lambda) = T_c(1 + a\lambda^{1/\phi}) \tag{2·15}$$

for the small λ. If we use the notation

$$\dot{\varepsilon} = \{T - T_c(\lambda)\}/T_c\,, \tag{2·16}$$

following Jasnow and Fisher,[5)] then one has[3)~5)] the response functions χ_0 and C_0 scaled in the reduced temperature $\dot{\varepsilon}$ as follows:

$$\chi_0 \sim \lambda^{(\dot{\gamma}-\gamma)/\phi}\dot{\varepsilon}^{-\dot{\gamma}} \tag{2·17}$$

and

$$C_0 \sim \lambda^{(\dot{\alpha}-\alpha)/\phi}\dot{\varepsilon}^{-\dot{\alpha}}, \tag{2·18}$$

where $\dot{\alpha}$ and $\dot{\gamma}$ indicate the critical exponents of the total Hamiltonian $\mathcal{H}$, which correspond to α and γ of the unperturbed Hamiltonian $\mathcal{H}_0$, respectively. The crossover temperature $\varepsilon^\times$ is given by[3),5)]

$$\varepsilon^\times \sim \lambda^{1/\phi}. \tag{2·19}$$

The critical behavior with the critical exponents $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, $\cdots$ is observed only in the temperature range $\dot{\varepsilon} < \varepsilon^\times$ and the critical property with the critical exponents α, β, γ, $\cdots$ is observed for the temperature range $\dot{\varepsilon} > \varepsilon^\times$. Thus, the critical exponent ϕ is a fundamental index in generalized scaling with a parameter.

The main purpose of this note is to give an explicit expression (or value) of the critical exponent ϕ according to the classification of the parameter λ as discussed in § 1.

§ 3. Change of symmetry

1. *The Heisenberg-Ising model*

The most interesting case may be the Heisenberg-Ising model *near the symmetry point*. The Hamiltonian of this system is given by

$$\mathcal{H} \equiv \mathcal{H}_0 + \lambda \mathcal{H}_1$$
$$= -\sum_{ij} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - \lambda \sum_{ij} J_{ij} S_i^z S_j^z . \tag{3·1}$$

From the symmetry property of the Heisenberg model $\mathcal{H}_0$, we have

$$\langle O\mathcal{H}_1 \rangle_c = \langle \mathcal{H}_1 \rangle = -\sum_{ij} J_{ij} \langle S_i^z S_j^z \rangle$$
$$= -\tfrac{1}{3} \sum_{ij} J_{ij} \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle = \tfrac{1}{3} \langle \mathcal{H}_0 \rangle \sim \varepsilon^{1-\alpha} \sim \varepsilon^{-\psi_1}. \tag{3·2}$$

Thus, fortunately we can obtain the rigorous value of the critical exponent ϕ:

$$\phi = 2 - \alpha + \psi_1 = 1 , \tag{3·3}$$

where we have used the result

$$\psi_1 = \alpha - 1 , \tag{3·4}$$

which is derived from Eq. (3·2). This disproves a relation $\phi = \gamma$ conjectured by Riedel and Wegner[3] for the anisotropic Heisenberg model. In general, we may have

$$\langle O\mathcal{H}_1^n \rangle_c \sim \langle (\delta \mathcal{H}_0)^n \rangle \sim \varepsilon^{2-\alpha-n}, \tag{3·5}$$

with $\delta \mathcal{H}_0 = \mathcal{H}_0 - \langle \mathcal{H}_0 \rangle$. Consequently, the scaled form of the free energy (2·2) is obtained together with $\phi = 1$.

2. *The anisotropic XY-model*

The Hamiltonian to consider here is given by

$$\mathcal{H} \equiv \mathcal{H}_0 + \lambda \mathcal{H}_1 = \sum J_{ij} (S_i^x S_j^x + S_i^y S_j^y) + \lambda \sum J_{ij} (S_i^x S_j^x - S_i^y S_j^y). \tag{3·6}$$

From the symmetry of the Hamiltonian $\mathcal{H}_0$ in the x-y plane, the first term of the expansion (1·3) is always vanishing:

$$\langle O\mathcal{H}_1 \rangle_c = \langle \mathcal{H}_1 \rangle = \sum_{ij} J_{ij} (\langle S_i^x S_j^x \rangle - \langle S_i^y S_j^y \rangle) = 0 . \tag{3·7}$$

Thus, we have to study the singularity of the next term $\langle O\mathcal{H}_1^2 \rangle_c$, which is expected to diverge in the same manner as the correlation $\langle \mathcal{H}_1^2 \rangle$ does, due to the discussion given in § 2. Then, one may have

$$\langle O\mathcal{H}_1^2 \rangle_c \sim \langle (\delta \mathcal{H}_1)^2 \rangle \sim \langle (\delta \mathcal{H}_0)^2 \rangle \tag{3·8}$$
$$\sim C_0 \sim \varepsilon^{-\alpha} \sim \varepsilon^{-\psi_2},$$

where the second relation of Eq. (3·8) is only plausible. That is,

$$\psi_2 = \alpha . \tag{3·9}$$

Consequently, we have

$$\phi = \tfrac{1}{2} (2 - \alpha + \psi_2) = 1 . \tag{3·10}$$

3. *The Ising model with XY-interaction*

Here, we consider the following Hamiltonian:

$$\mathcal{H} \equiv \mathcal{H}_0 + \lambda \mathcal{H}_1 = -\sum_{i<j} J_{ij} S_i^z S_j^z - \lambda \sum_{i>j} J_{ij} (S_i^x S_j^x - S_i^y S_j^y). \tag{3·11}$$

The symmetry of the Hamiltonian $\mathcal{H}_0$ yields the vanishing of the first term $\langle O\mathcal{H}_1 \rangle_c$:

$$\langle O\mathcal{H}_1 \rangle_c = \langle \mathcal{H}_1 \rangle = -\sum_{i>j} J_{ij} (\langle S_i^x S_j^x \rangle - \langle S_i^y S_j^y \rangle) = 0 . \tag{3·12}$$

Then, we have to analyze the next term $\langle O\mathcal{H}_1^2 \rangle_c$, which may be approximated by the fluctuation $\langle \mathcal{H}_1^2 \rangle$ as discussed in § 2:

$$\langle O\mathcal{H}_1^2 \rangle_c \sim \langle \mathcal{H}_1^2 \rangle . \tag{3·13}$$

From the symmetry of $\mathcal{H}_0$, one easily obtains

$$\langle \mathcal{H}_1^2 \rangle = \sum_{i>j} J_{ij}^2 \{ \langle (S_i^x)^2 (S_j^x)^2 \rangle + \langle (S_i^y)^2 (S_j^y)^2 \rangle - \langle S_i^x S_i^y S_j^x S_j^y \rangle - \langle S_i^y S_i^x S_j^y S_j^x \rangle \}, \tag{3·14}$$

where the following symmetry properties have been used:

$$\langle S_i^x S_j^x S_k^x S_l^x \rangle = 0 , \quad \text{etc.,} \qquad \text{for } i \neq j,\ k \text{ and } l . \tag{3·15}$$

For brevity, we consider the case of spin $S=\frac{1}{2}$. Then, we have a simple expression for the fluctuation $\langle \mathcal{H}_1^2 \rangle$ in the form

$$\langle \mathcal{H}_1^2 \rangle = \frac{1}{2} \sum_{i>j} J_{ij}^2 \langle S_i^z S_j^z \rangle + \frac{1}{8} \sum_{i>j} J_{ij}^2 . \tag{3·16}$$

In particular, for the nearest neighbor interaction, this is reduced to the energy of the system $\mathcal{H}_0$:

$$\langle \mathcal{H}_1^2 \rangle = -\frac{1}{2} J \langle \mathcal{H}_0 \rangle + \text{constant} \sim \varepsilon^{1-\alpha} \sim \varepsilon^{-\phi_2}. \tag{3·17}$$

Therefore, the critical exponent ϕ takes the value

$$\phi = \frac{1}{2} (2 - \alpha + \phi_2) = \frac{1}{2} . \tag{3·18}$$

Here, by accepting the belief[13] that the critical behavior does not depend upon the range of interaction *while it is of short range*, the above result (3·18) can be extended to the more general interaction J_{ij} *of short range*. Furthermore, the above value of ϕ should be valid for general spin S, from generalized Griffiths' argument[6),14] that the ferromagnets with spin $S > \frac{1}{2}$; i.e., $S=1, \frac{3}{2}, 2$, etc., but with $S < \infty$ should have the same critical exponents as $S=\frac{1}{2}$ on the same lattice.[15]

The result $\phi = \frac{1}{2}$ for general interaction J_{ij} of short range and for higher spin S may be again argued directly from the expression (3·14), in which the right-hand side of Eq. (3·14) is essentially a short range correlation, and consequently it is expected to have the same singularity of the energy.

4. *The Ising model with a transverse field*

The Hamiltonian to discuss here is given by

$$\mathcal{H}\equiv\mathcal{H}_0+\lambda\mathcal{H}_1=-\sum_{i>j}J_{ij}S_i^zS_j^z-\lambda\sum_j S_j^x. \tag{3·19}$$

Note that

$$\langle\mathcal{H}_1\rangle=0 \tag{3·20}$$

and

$$\langle O\mathcal{H}_1^2\rangle_c=\chi_\perp=(M^x, M^x) \tag{3·21}$$

with $M^x=\sum S_j^x$. By extending Fisher's arguments[16] on the perpendicular susceptibility $\chi_\perp$ in the two-dimensional Ising model with nearest neighbor interaction, this perpendicular susceptibility can be expressed,[17),18)] in general, by short range correlations for systems *with short range interaction*. These short range correlations should have the same critical exponent as the energy of the system $\mathcal{H}_0$. Thus, we have again

$$\langle O\mathcal{H}_1^2\rangle_c=\chi_\perp\sim\varepsilon^{1-\alpha}\sim\varepsilon^{-\phi_2}, \tag{3·22}$$

and consequently

$$\phi=\tfrac{1}{2}\,. \tag{3·23}$$

5. *The XYZ-model*

Here, we summarize and generalize the previous discussions on particular cases of the Hamiltonian. The most general anisotropic Heisenberg model is called the XYZ-model,[19] the Hamiltonian of which is expressed by

$$\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1\,, \tag{3·24}$$

where

$$\mathcal{H}_0=-\sum_{i>j}J_{ij}\{(S_i^xS_j^x+S_i^yS_j^y)+\varDelta_zS_i^zS_j^z\},$$

and

$$\mathcal{H}_1=-\sum_{i>j}J_{ij}(S_i^xS_j^x-S_i^yS_j^y). \tag{3·25}$$

Now, we have always

$$\langle O\mathcal{H}_1\rangle_c=0\,. \tag{3·26}$$

According to the value of the anisotropy parameter $\varDelta_z$, we classify the problem into the following three cases:

5–a) For $0\leq\varDelta_z<1$, one should have the same critical exponent as that of the example 2 (the anisotropic XY-model): i.e.,

$$\phi=1 \tag{3·27}$$

from the Fisher-Griffiths postulation discussed in § 1, which implies in this case that the critical behavior of the system should change only at the "symmetry points" $\Delta_z=0$ and $\Delta_z=1$ (where the symmetry of the Hamiltonian changes abruptly).

5-b) For $\Delta_z=1$, see the example 1: i.e., $\phi=1$.

5-c) For $\Delta_z>1$, the critical exponent ϕ should take the same value as that of the example 3: i.e.,

$$\phi=\tfrac{1}{2} \tag{3·28}$$

with the similar reasoning as in 5-a).

6. *The Heisenberg model with an anisotropy term*

Here, we consider a different type of anisotropy of the form

$$\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1=-\sum_{i>j} \boldsymbol{S}_i\cdot\boldsymbol{S}_j-\lambda\sum_j(S_{jx}^2+S_{jy}^2+S_{jz}^2). \tag{3·29}$$

It seems difficult to evaluate the first term of the form

$$\langle\mathcal{H}_1\rangle=-3\sum_j\langle S_{jz}^2\rangle\,. \tag{3·30}$$

However, for the next term, we may expect that

$$\langle O\mathcal{H}_1{}^2\rangle_c\sim\langle(\delta\mathcal{H}_1)^2\rangle\sim\sum_{jk}\langle(\delta S_{jz}^2)(\delta S_{kz}^2)\rangle$$

$$\sim\chi_0\sim\varepsilon^{-\gamma}\sim\varepsilon^{-\psi_2}. \tag{3·31}$$

Thus, we may have

$$\phi=\tfrac{1}{2}(2-\alpha+\psi_2)=\tfrac{1}{2}(2-\alpha+\gamma). \tag{3·32}$$

(The usage of the same notation both for the critical exponent β of the spontaneous magnetization and for $\beta=1/k_BT$ will not be confusing to the reader.)

7. *The Schrödinger-Heisenberg model*

The Hamiltonian to study here is

$$\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1\,, \tag{3·33}$$

where $\mathcal{H}_0$ is the Hamiltonian for the Schrödinger exchange-interaction model of ferromagnetism[20)~22)] defined by

$$\mathcal{H}_0=-\sum J_{ij}P_{ij}\,;\qquad P_{ij}=\sum_{n=0}^{2S}A_n(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^n. \tag{3·34}$$

Here, the coefficients A_n are determined so that the exchange operator P_{ij} may satisfy the relation

$$P_{ij}|m\rangle_i|m'\rangle_j=|m'\rangle_i|m\rangle_j \tag{3·35}$$

for any state $|m\rangle_i$; $S_i^z|m\rangle_i=m|m\rangle_i$. The degeneracy of the ferromagnetic ground state of this exchange model is $2S+1$, though that of the Heisenberg model is double. As discussed in a previous paper,[21] there are many order parameters which commute with the Hamiltonian of this exchange model, besides the total magnetization $M=\sum_j S_j^z$. (For example, $\Psi\equiv\sum_j(S_{jz}^2-\langle S_{jz}^2\rangle)$ is one of order parameters.) Therefore, the symmetry of this model is higher than that of the Heisenberg model. If we add the perturbational Hamiltonian $\mathcal{H}_1$ of the Heisenberg interaction, or in more general, of the form

$$\mathcal{H}_1=-\sum_{ij} J_{ij}\mathcal{H}_{ij}\,, \tag{3·36}$$

$\mathcal{H}_{ij}$ being a linear combination of $(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)$, $(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^2$, $\cdots$, and $(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^{2S}$, then the symmetry of the total Hamiltonian $\mathcal{H}$ becomes lower and it will be the same as the symmetry of the Heisenberg model. Thus, we may call the Hamiltonian (3·33) with Eqs. (3·34) and (3·36) the Schrödinger-Heisenberg model as for the Heisenberg-Ising model.

Now, the first term of the expansion with respect to the parameter λ may be written

$$\begin{aligned}\langle O\mathcal{H}_1\rangle_c&\sim\langle\mathcal{H}_1\rangle=-\sum_{ij} J_{ij}\langle\mathcal{H}_{ij}\rangle\\&\sim\varepsilon^{1-\alpha}\sim\varepsilon^{-\phi_1}.\end{aligned} \tag{3·37}$$

Consequently, one may expect that

$$\phi=2-\alpha+\phi_1=1\,. \tag{3·38}$$

8. *Broken symmetry by a magnetic field*

The last case is concerning the usual scaling with a magneitc field $\lambda=H$. The Hamiltonian is expressed by

$$\mathcal{H}=\mathcal{H}_0+\lambda\sum_j S_j^z. \tag{3·39}$$

Usually we have

$$\langle\mathcal{H}_1\rangle=\sum_j\langle S_j^z\rangle=0\,, \tag{3·40}$$

from the symmetry of $\mathcal{H}_0$. The next term is just the usual susceptibility:

$$\langle O\mathcal{H}_1^2\rangle_c=\chi_0\sim\varepsilon^{-\gamma}\sim\varepsilon^{-\phi_2}. \tag{3·41}$$

That is,

$$\phi=\frac{1}{2}(2-\alpha+\gamma)=\frac{\Delta}{2}\,. \tag{3·42}$$

This is the well-known result in the usual scaling law.[1),5)] All the previous arguments from 1 to 7 are nothing but an extension of this discussion on the usual scaling law.

§ 4. Change of dimensionality

The most interesting change of dimensionality may be the transition from two dimensions to three dimensions, because one-dimensional systems with short-range interaction show, in general, no phase transition.[23] This change of dimensionality is expressed by the following Hamiltonian[4),24)]

$$\mathcal{H}=\mathcal{H}_0+\lambda\mathcal{H}_1=-\sum_n\sum_{\langle j,k\rangle}(S_{n,j};S_{n,k})-\lambda\sum_{n,j}(S_{n,j};S_{n+1,j}), \qquad (4\cdot1)$$

where

$$(S_p;S_q)=J_xS_p^xS_q^x+J_yS_p^yS_q^y+J_zS_p^zS_q^z, \qquad (4\cdot2)$$

$\langle j,k\rangle$ indicates the sum over the nearest neighbors, and without loss of generality, we can assume that

$$|J_x|\leq|J_y|\leq|J_z|\,. \qquad (4\cdot3)$$

Here, $\{S^x_{n,j}\}$ indicate spin operators at the j-th lattice point of the n-th layer. First, note that

$$\langle O\mathcal{H}_1\rangle_c=\langle\mathcal{H}_1\rangle=0\,. \qquad (4\cdot4)$$

Then, as discussed in § 2, we may replace the generalized cumulant (or the generalized susceptibility) by the correlation

$$\langle O\mathcal{H}_1^2\rangle\sim\langle\mathcal{H}_1^2\rangle\,. \qquad (4\cdot5)$$

This correlation $\langle\mathcal{H}_1^2\rangle$ may be approximated again by the main contribution of the z-z correlations as

$$\langle\mathcal{H}_1^2\rangle\sim NJ_z^2\sum_{ij}\langle S_i^zS_j^z\rangle^2, \qquad (4\cdot6)$$

under relation (4·3) where N is the number of layers. Relations (4·5) and (4·6) are exact in the Ising model ($J_x=J_y=0$). Here, we assume the asymptotic behavior of the form

$$\langle S_i^zS_{i+R}^z\rangle\sim\frac{e^{-\kappa R}}{R^{d-2+\eta}}; \qquad \kappa\sim\varepsilon^\nu \qquad (4\cdot7)$$

for the spin correlation, where κ is the inverse correlation length and d is the dimensionality of the system. Then we have

$$\sum_{ij}\langle S_i^zS_j^z\rangle^2\sim\int_a^\infty\frac{e^{-2\kappa R}}{R^{2(d-2+\eta)}}\cdot R^{d-1}dR$$

$$\sim\kappa^{d-2(2-\eta)}\sim\varepsilon^{d\nu-2\nu(2-\eta)}, \qquad (4\cdot8)$$

where a is the lattice spacing. By using the scaling relations

$$\gamma=(2-\eta)\nu \qquad \text{and} \qquad 2-\alpha=d\nu\,, \qquad (4\cdot9)$$

we arrive at

$$\langle O\mathcal{H}_1^2\rangle_c \sim \varepsilon^{-\psi_2}; \qquad \psi_2 = 2\gamma - d\nu\,, \tag{4·10}$$

and consequently we obtain

$$\phi = \tfrac{1}{2}(2\gamma + 2 - \alpha - d\nu) = \gamma\,. \tag{4·11}$$

This relation has been obtained first by Abe[4] in the three-dimensional anisotropic Ising model.

This argument can be easily extended to the more general short-range interaction of the form

$$(S_p; S_q) = J^x_{p,q} S_p^x S_q^x + J^y_{p,q} S_p^y S_q^y + J^z_{p,q} S_p^z S_q^z. \tag{4·12}$$

Under the condition $|J^z_{p,q}| \geq |J^y_{p,q}| \geq |J^x_{p,q}|$, the term of the second order may be written as

$$\langle O\mathcal{H}_1^2\rangle_c \sim \langle \mathcal{H}_1^2\rangle \sim \sum J^z_{nj,mk} J^z_{n'j',m'k'} \langle S^z_{nj} S^z_{mk} S^z_{n'j'} S^z_{m'k'}\rangle$$
$$= \sum_{R,R'} \widehat{G}(R-R') \langle S_0^z S_R^z\rangle \langle S_0^z S_{R'}^z\rangle\,, \tag{4·13}$$

where

$$\widehat{G}(r) = \sum_{nj} J(n,j) J(n,j+r) = \text{of short-range} \tag{4·14}$$

and

$$J(n-m, j-k) = J^z_{nj,mk}\,. \tag{4·15}$$

In a similar way as before, we obtain the critical exponent relation

$$\phi = \gamma\,. \tag{4·16}$$

The expression (4·13) for $\langle O\mathcal{H}_1^2\rangle_c$ is exact in the case of the Ising model ($J^x = J^y = 0$). Consequently, the result (4·16) is rigorous for the Ising model with general short-range interaction. This is just the generalization of Abe's argument[4] on the Ising model with nearest neighbor interaction.

A similar argument will be given in § 6 on dynamical aspects of scaling with a parameter.

§ 5. Change of potential range

The simplest example of long-range interaction may be expressed by the following Hamiltonian,[6),25),26)]

$$\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{H}_1 = -\sum_{i>j} (J/r_{ij}^s + \lambda/r_{ij}^\sigma) S_i^z S_j^z \tag{5·1}$$

for $d+1 > s > \sigma > d$, and $J > 0$. Now, we have

$$\langle O\mathcal{H}_1\rangle_c = \langle \mathcal{H}_1\rangle = -\sum_{i>j} r_{ij}^{-\sigma} \langle S_i^z S_j^z\rangle\,. \tag{5·2}$$

By using the asymptotic form of the correlation function (4·7), one obtains the singularity

$$\langle \mathcal{H}_1 \rangle \simeq - \int_a^\infty \frac{e^{-\kappa R} R^{d-1} dR}{R^{\sigma+d+\eta-2}} \sim \kappa^{\sigma+\eta-2} \sim \varepsilon^{\nu(\sigma+\eta-2)}. \tag{5·3}$$

Therefore, the following critical exponents result:

$$\phi_1 = \nu(2-\sigma-\eta)$$

and

$$\phi = 2-\alpha+\phi_1 = \nu(s-\sigma)+1\,. \tag{5·4}$$

Thus, the critical exponent ϕ of scaling with the parameter λ depends continuously upon the powers of potential range, s and σ. This is easily expected from the beginning, because the critical behavior of the Hamiltonian $\mathcal{H}_0$ itself depends upon the value of the power s of the potential range.[5),26)]

§ 6. Dynamical aspects of scaling with a parameter

Quite recently Riedel and Wegner[27)] have applied the generalized scaling with a parameter to discussing dynamical critical phenomena in the anisotropic Heisenberg model near the symmetry point. Our method shown in the previous sections can be extended to these dynamical problems. Detailed arguments will be given for general aspects in a separate paper. In this section, we report only preliminary results, particularly focusing our attension on the critical slowing down.[28),29)]

As discussed in a previous paper,[29)] the relaxation time of any quantity A may be defined, in general, by

$$\tau_A = i\langle A\mathcal{L}^{-1}A\rangle_{\mathcal{H}} \langle A^2\rangle_{\mathcal{H}}^{-1}, \tag{6·1}$$

where $\mathcal{L}$ is the Liouville operator defined by

$$\mathcal{L}A = [\mathcal{H}, A]; \qquad \hbar = 1\,, \tag{6·2}$$

and $\langle\cdots\rangle_{\mathcal{H}}$ indicates the canonical ensemble average over the total Hamiltonian. Here, we have assumed $\langle A\rangle_{\mathcal{H}}=0$ for simplicity. The total Hamiltonian is separated into two parts:

$$\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{H}_1\,. \tag{6·3}$$

According to this separation, the Liouville operator is also expressed by the following two terms:

$$\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_1\,, \tag{6·4}$$

where

$$\mathcal{L}_0 A = [\mathcal{H}_0, A] \qquad \text{and} \qquad \mathcal{L}_1 A = [\mathcal{H}_1, A]. \tag{6·5}$$

In this dynamical problem, the perturbational expansion of the relaxation time (6·1) is rather complicated, and the first few terms of it are given by

$$\tau_A(\lambda)=\tau_A(0)+\lambda i(A^2)^{-1}[\langle A\mathcal{L}_0^{-1}\mathcal{L}_1\mathcal{L}_0^{-1}A\rangle$$
$$-\int_0^\beta ds\{\langle A\mathcal{L}_0^{-1}A\mathcal{H}_1(s)\rangle+\langle A\mathcal{L}_0^{-1}A\rangle\langle A^2\rangle^{-1}\langle A^2\mathcal{H}_1(s)\rangle$$
$$-\langle A\mathcal{L}_0^{-1}A\rangle\langle\mathcal{H}_1\rangle\}]+\cdots, \quad (6\cdot6)$$

where $\langle\cdots\rangle$ indicates the canonical average over the unperturbed Hamiltonian $\mathcal{H}_0$. By inspecting systematically the singularity of each term in the above perturbational expansion (6·6), one may argue the following scaled form of the relaxation time

$$\tau_A(\lambda)\sim\varepsilon^{-\Delta_A}f(\lambda/\varepsilon^\phi) \quad (6\cdot7)$$

near the critical point, where Δ_A is the critical exponent of slowing down for the physical quantity A in the system described by the unperturbed Hamiltonian $\mathcal{H}_0$. In the same way as for the static case ,[3)~5)] the λ-dependence of the relaxation time is shown to be

$$\tau_A(\lambda)\sim\lambda^{(\dot{\Delta}_A-\Delta_A)/\phi}\varepsilon^{-\dot{\Delta}_A}, \quad (6\cdot8)$$

where $\dot{\Delta}_A$ indicates the exponent of critical slowing down in the total system $\mathcal{H}$. The crossover temperature is given by

$$\varepsilon^\times\sim\lambda^{1/\phi}. \quad (6\cdot9)$$

The definition (6·1) of the relaxation time is particularly useful in discussing the critical slowing down of the stochastic Ising model,[30)~34)] where a temporal development operator L plays a role of the Liouville operator in usual mechanical systems. Replacing $\mathcal{L}$ by $-iL$ in Eq. (6·1), one obtains[31)] an expression for the relaxation time in the form

$$\tau_A=\left\langle A\frac{1}{L}A\right\rangle\Big/\langle A^2\rangle. \quad (6\cdot10)$$

There are two situations to classify in this stochastic Ising model.

1. *Change of symmetry only through the temporal development operator L, while the basic Ising Hamiltonian remains fixed at* $\mathcal{H}_0$.
2. *Change of the basic Ising Hamiltonian and accordingly of the temporal development operator, such as change of dimensionality and of potential range.*

In both situations, the inverse temporal development operator L^{-1} can be expanded as

$$\frac{1}{L}=\frac{1}{L_0+\lambda L_1}=\frac{1}{L_0}-\lambda\frac{1}{L_0}L_1\frac{1}{L_0}+\lambda^2\frac{1}{L_0}L_1\frac{1}{L_0}L_1\frac{1}{L_0}+\cdots. \quad (6\cdot11)$$

In the first situation, the following operator L may be of interest:

$$L = L_0 + \lambda L_1, \tag{6·12}$$

where L_0 *satisfies some conservation laws*[32),33)] such as *spin* and *energy* conservations, and L_1 has lower symmetry than L_0 has (i.e., L_1 lacks some of the conservation laws which L_0 satisfies). An operator with the lowest symmetry is the Glauber operator[30)~33)] which has no conservation law (i.e., it is a single spin operator). It is usually expected even in the stochastic Ising model that the dynamical critical exponents depend upon the symmetry of the temporal development operator. In particular, the relaxation time τ_A is expanded simply as

$$\tau_A = \left\langle A \frac{1}{L} A \right\rangle \Big/ \langle A^2 \rangle$$
$$= \frac{1}{\langle A^2 \rangle} \left(\left\langle A \frac{1}{L_0} A \right\rangle - \lambda \left\langle A \frac{1}{L_0} L_1 \frac{1}{L_0} A \right\rangle + \cdots \right). \tag{6·13}$$

The dynamical susceptibility $\chi(\omega, \varepsilon, \lambda)$ is expressed by[31)]

$$\chi(\omega, \varepsilon, \lambda) = \beta \left\langle M \frac{L_0 + \lambda L_1}{(i\omega + L_0) + \lambda L_1} M \right\rangle \tag{6·14}$$

with the resultant magnetization $M = \sum_j S_j^z$. This can be expanded as

$$\chi(\omega, \varepsilon, \lambda) = \beta - i\omega\beta \sum_{n=0}^{\infty} \left(\frac{\lambda}{i\omega + L_0} L_1 \right)^n \frac{1}{i\omega + L_0} \tag{6·15}$$

or

$$\chi(\omega, \varepsilon, \lambda) = \beta \sum_{n=0}^{\infty} (-i\omega)^n a_n(\varepsilon, \lambda), \tag{6·16}$$

where

$$a_n(\varepsilon, \lambda) = \left\langle M \frac{1}{(L_0 + \lambda L_1)^n} M \right\rangle, \tag{6·17}$$

which may be expanded again with respect to the parameter λ. By investigating the singularity of each term of the above perturbational series, one may be able to confirm the scaled form

$$\chi(\omega, \varepsilon, \lambda) \sim \varepsilon^{-\gamma} g(i\omega/\varepsilon^{\Delta M}, \lambda/\varepsilon^{\phi}). \tag{6·18}$$

In the second situation, we have to consider the perturbational expansions both in the Hamiltonian $\mathcal{H}$ and in the temporal development operator L:

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_1 \quad \text{and} \quad L = L_0 + \lambda L_1. \tag{6·19}$$

The simplest example of this second situation is *change of dimensionality*. In this case, $\mathcal{H}_1$ expresses interaction among layers, while $\mathcal{H}_0$ is the simple sum of independent Ising Hamiltonians corresponding to each layer. At least for nearest neighbor interaction, in the same way as for the static case we can show that the term of the first order, $b_1(\varepsilon)$, in the expansion of the relaxation time defined by (6·10)

$$a_1(\varepsilon, \lambda) = \left\langle M\frac{1}{L}M\right\rangle = \sum_{n=0}^{\infty} \lambda^n b_n(\varepsilon) \tag{6·20}$$

has the singularity of the form

$$b_1(\varepsilon) \sim \chi_0 b_0(\varepsilon) \sim \varepsilon^{-\gamma} b_0(\varepsilon). \tag{6·21}$$

This implies that

$$\phi = \gamma, \tag{6·22}$$

for the scaling of the relaxation time τ_M with the parameter λ. This seems to be a *universal relation in change of dimensionality* both for the static and dynamical problems. Incidentally, in the two-dimensional stochastic Ising model, the exponent of critical slowing down, Δ_M, for the magnetization has been estimated numerically as[31]

$$\Delta_M \simeq 2.00 \pm 0.05\,. \tag{6·23}$$

The critical exponent of the susceptibility, γ, is given by $\gamma = 7/4$. Thus, the λ-dependence of the relaxation time is given by

$$\tau_M(\lambda) \sim \lambda^{(4/7)(\dot{\Delta}_M - 2)} \dot{\varepsilon}^{-\dot{\Delta}_M}, \tag{6·24}$$

in the three-dimensional stochastic Ising model, where $\dot{\Delta}_A$ is the exponent of critical slowing down in three dimensions, and its value seems to be very close[31] (or equal[29]) to $\dot{\gamma} + \dot{\eta}\dot{\nu}$.

§ 7. Summary and discussion

We have obtained expressions for critical exponents of scaling with a parameter in spin systems. In the case of change of dimensionality, the critical exponent ϕ was shown to be, in general, equal to γ. In other cases, the relation between ϕ and static critical exponents depends upon the structure (or symmetry) of the Hamiltonian to consider. Several typical examples have been investigated. In the case of change of symmetry for the XYZ-model, we have obtained $\phi = 1$ for $0 \leq \Delta_z \leq 1$ and $\phi = \frac{1}{2}$ for $1 < \Delta_z$. The value $\phi = \frac{1}{2}$ has also been obtained for the Ising model with a transverse field. The relation $\phi = \frac{1}{2}(2 - \alpha + \gamma)$ was derived plausibly for the Heisenberg model of spin higher than one half with an anisotropy term. Dynamical aspects of scaling with a parameter were discussed in connection with the critical slowing down in the stochastic Ising model. The λ-dependence of the relaxation time $\tau(\lambda)$ is given by

$$\tau(\lambda) \sim \lambda^{(\dot{\Delta} - \Delta)/\phi} \dot{\varepsilon}^{-\dot{\Delta}}. \tag{7·1}$$

For change of dimensionality, the relation

$$\phi = \gamma \tag{7·2}$$

was obtained again in this stochastic Ising model. The relation (7·2) seems to be universal for change of dimensionality.

It is expected that the critical exponent ϕ for the susceptibility should take the same value as for the free energy. This will be discussed in a separate paper.

There is a possibility to detect experimentally the effect of change of "symmetry" near the crossover temperature in real materials[36] such as MnF_2. It will be of great interest to find and study several materials with similar structure and different values of the parameter λ to change symmetry, in order to confirm the generalized scaling with respect to the parameter.

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Note added in proof:

By applying the above results (4·16) and (5·4) to a *general proposition on critical exponents* (M. Suzuki, to be submitted to Prog. Theor. Phys.), we can derive the following inequalities on critical exponents, with use of Griffiths-Kelly-Sherman inequalities. (i) For the ferromagnetic Ising Model with general spin, we have $\gamma(d)\geq\gamma(d+1)$, $\beta(d)\leq\beta(d+1)$, $\delta(d)\geq\delta(d+1)$, and $\nu(d)\geq\nu(d+1)$, where $\gamma(d)$ etc. denote critical exponents in d dimensions. (ii) For the ferromagnetic Ising model with *long-range interaction* described by the Hamiltonian $\mathcal{H}=-J\sum|\boldsymbol{i}-\boldsymbol{j}|^{-s}S_i^zS_j^z$ with $d+1>s>d$, result the inequalities $\gamma(s)\geq\gamma(\sigma)$, $\beta(s)\leq\beta(\sigma)$, $\delta(s)\geq\delta(\sigma)$ and $\nu(s)\geq\nu(\sigma)$ for $s>\sigma>d$.

Reference 5) will appear in "Phase Transitions and Critical Points" edited by C. Domb and M. S. Green; Academic Press (London).