Scattered linear sets and pseudoreguli

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Abstract

In this paper, we show that one can associate a pseudoregulus with every scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$. We construct a scattered linear set having a given pseudoregulus as associated pseudoregulus and prove that there are $q-1$ different scattered linear sets that have the same associated pseudoregulus. Finally, we give a characterisation of reguli and pseudoreguli in $\mathrm{PG}(3, q^3)$.

1 Motivation and preliminaries

1.1 Motivation

Linear sets in projective spaces have gained attention in recent years because of their connection with other geometrical structures (e.g. blocking sets, translation ovoids, ...). For an overview of the use of linear sets in these topics, we refer to [15]. The motivation for the study of the particular linear sets studied in this paper arose from the relation between linear sets and finite semifields.

In [6] it was shown that to any semifield $\mathbb{S}$ of order q^{nt}, with left nucleus containing $\mathbb{F}_{q^t}$ and center containing $\mathbb{F}_q$, there corresponds an $\mathbb{F}_q$-linear set of rank nt in the projective space $\mathrm{PG}(n^2-1, q)$, disjoint from the $(n-2)$-nd secant variety of a Segre variety, and conversely. This result was previously proved for $n = 2$ by Lunardon [12], and is crucial in the classification of semifields with $n = 2$, $t = 2$ obtained in [3]. It was applied again in

[14], where the case $n = 2, t = 3$ is considered, and the authors prove that there exist eight non-isotopic families of such semifields, according to the different configurations of the associated linear sets of $\mathrm{PG}(3, q^3)$. Also, they prove that to any scattered semifield, there is associated an $\mathbb{F}_q$-pseudoregulus of $\mathrm{PG}(3, q^3)$ and they characterise the known examples of scattered semifields in terms of the associated $\mathbb{F}_q$-pseudoregulus. In this paper, we show that one can associate an $\mathbb{F}_q$-pseudoregulus to any scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$. In the case that $n = 3$, this provides a tool to study *symplectic* scattered semifields of order q^9, with left nucleus containing $\mathbb{F}_{q^3}$ and center containing $\mathbb{F}_q$. (See [7], for a study of such semifields when $n = 2$.) For more applications of the connection between linear sets and semifields we refer to [8] and the references contained therein.

1.2 Preliminaries

If V is a vector space, then we denote by $\mathrm{PG}(V)$ the corresponding projective space. If V has dimension n over the finite field $\mathbb{F}_q$ with q elements, then we also write $\mathrm{PG}(n-1, q)$.

Let V be an r-dimensional vector space over a finite field $\mathbb{F}$. A set $\mathcal{L}$ of points of $\mathrm{PG}(V)$ is called a *linear set* (*of rank* t) if there exists a subset U of V that forms a (t-dimensional) $\mathbb{F}_q$-vector space for some $\mathbb{F}_q \subset \mathbb{F}$, such that $\mathcal{L} = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}} \ : \ u \in U \setminus \{0\}\}.$$

If we want to specify the subfield we call $\mathcal{L}$ an $\mathbb{F}_q$-*linear set*.

In other words, if $\mathbb{F} = \mathbb{F}_{q^n}$, we have the following diagram

$$\begin{array}{ccccccc} & & \mathbb{F}_{q^n}^r & \longleftrightarrow & \mathbb{F}_q^{rn} & \supseteq & U \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{B}(U) & \subseteq & \mathrm{PG}(r-1, q^n) & \longleftrightarrow & \mathrm{PG}(rn-1, q) & \supseteq & \mathrm{PG}(U) \end{array}$$

We also use the notation $\mathcal{B}(\pi)$ for the set of points of $\mathrm{PG}(r-1, q^n)$ induced by $\pi = \mathrm{PG}(U)$. Since the points of $\mathrm{PG}(r-1, q^n)$ correspond to 1-dimensional subspaces of $\mathbb{F}_{q^n}^r$, and by field reduction to n-dimensional subspaces of $\mathbb{F}_q^{rn}$, they correspond to a set $\mathcal{D}$ of $(n-1)$-dimensional subspaces of $\mathrm{PG}(rn-1, q)$, which partitions the point set of $\mathrm{PG}(rn-1, q)$. The set $\mathcal{D}$ is called a *Desarguesian spread*, and we have a one-to-one correspondence between the points of $\mathrm{PG}(r-1, q^n)$ and the elements of $\mathcal{D}$. This gives us a more geometric perspective on the notion of a linear set; namely, an $\mathbb{F}_q$-linear set is a set $\mathcal{L}$ of points of $\mathrm{PG}(r-1, q^n)$ for which there exists a subspace π in $\mathrm{PG}(rn-1, q)$ such that the points of $\mathcal{L}$ correspond to the elements of $\mathcal{D}$ that have a non-empty intersection with π. Also in what follows, we will often identify the elements of $\mathcal{D}$ with the points of $\mathrm{PG}(r-1, q^n)$, which allows us to view $\mathcal{B}(\pi)$ as a subset of $\mathcal{D}$. To avoid confusion, we denote subspaces of $\mathrm{PG}(r-1, q^n)$ by capital letter and subspaces of $\mathrm{PG}(rn-1, q)$ by lowercase letters. For more on this approach to linear sets, we refer to [5] and [9].

If the subspace π intersects each spread element in at most a point, then π is called *scattered* with respect to $\mathcal{D}$ (see [5], [2]). In this case we also call the associated linear set

$\mathcal{B}(\pi)$ *scattered*. Note that if π is $(t-1)$-dimensional and scattered, then the associated $\mathbb{F}_q$-linear set $\mathcal{B}(\pi)$ has rank t and has exactly $\frac{q^t-1}{q-1}$ points, and conversely.

In this paper, we will make use of the following bound on the rank of a scattered linear set, which follows from [2, Theorem 4.3].

Theorem 1. *A scattered $\mathbb{F}_q$-linear set in $\mathrm{PG}(r-1,q^t)$ has rank $\leqslant rt/2$.*

Proof. Immediate from the definition and [2, Theorem 4.3]. □

In this paper, we focus on scattered $\mathbb{F}_q$-linear sets of rank $3n$ in $\mathrm{PG}(2n-1,q^3)$. By Theorem 1, these scattered linear sets are *maximum scattered*.

2 Projectively equivalent scattered linear sets

In this section, we show that all scattered $\mathbb{F}_q$-linear sets of rank $3n$ in $\mathrm{PG}(2n-1,q^3)$ are projectively equivalent.

Desarguesian spreads, introduced in the previous section, are well-known and frequently used in finite geometry. We recall another classic construction of a Desarguesian spread based on the following lemma (see e.g. [11, Lemma 1]).

Lemma 2. *A subspace of $\mathrm{PG}(hn-1,q^h)$ of dimension d is fixed by the mapping $x \mapsto x^q$ if and only if it intersects the subgeometry $\mathrm{PG}(hn-1,q)$ in a subspace of dimension d.*

Now, let Π be an $(n-1)$-space, disjoint from the subgeometry $\rho = \mathrm{PG}(hn-1,q)$ of $\mathrm{PG}(hn-1,q^h)$, such that $\dim\langle \Pi, \Pi^q, \ldots, \Pi^{q^{h-1}}\rangle$ is maximal, i.e. spans $\mathrm{PG}(hn-1,q^h)$. Let P be a point of Π and let $\tau(P)$ denote the $(h-1)$-dimensional subspace generated by the conjugates of P, i.e., $\tau(P) = \langle P, P^q, ..., P^{q^{h-1}}\rangle$. Then $\tau(P)$ is fixed by $x \mapsto x^q$ and so, by Lemma 2, it intersects $\mathrm{PG}(hn-1,q)$ in an $(h-1)$-dimensional subspace over $\mathbb{F}_q$. If we do this for every point of Π we obtain a Desarguesian $(h-1)$-spread of $\mathrm{PG}(hn-1,q)$ (see Segre [16]). For future reference, we denote this spread by $\mathcal{D}(\Pi)$. Moreover, every Desarguesian spread can be constructed this way ([16]), and all Desarguesian $(h-1)$-spreads in $\mathrm{PG}(hn-1,q)$ are projectively equivalent (see e.g. [1]).

In order to prove that the Desarguesian spread $\mathcal{D}(\Pi)$ determines the subspace Π up to conjugacy, we need to introduce the following terminology. A set $\mathcal{R}$ of $q+1$ mutually disjoint $(n-1)$-dimensional subspaces of $\mathrm{PG}(2n-1,q)$, such that a line meeting 3 elements of $\mathcal{R}$, meets all elements of $\mathcal{R}$, is called a *regulus* (or $(n-1)$*-regulus*). A line meeting each element of a regulus $\mathcal{R}$ is called a *transversal of* $\mathcal{R}$.

The following theorem is considered as folklore, but, by lack of a reference, we include a proof.

Theorem 3. *If $\mathcal{D}(\Pi_1) = \mathcal{D}(\Pi_2)$, then Π_1 and Π_2 are conjugate.*

Proof. Let Π_1 and Π_2 be two different $(n-1)$-spaces determining the spread $\mathcal{D}$, and suppose Π_2 is not conjugated to Π_1. Then there exist lines L in Π_1 and M in Π_2 such that L and M are not conjugated and they determine the same $(h-1)$-subspread $\mathcal{D}_1 \subset \mathcal{D}$

in a $(2h-1)$-space τ. Let $\bar{X}$ denote the extension of $X \in \mathcal{D}_1$ to a subspace over $\mathbb{F}_{q^h}$. Let m be minimal such that $M \subset \langle L, L^q, \dots, L^{q^{m-1}} \rangle$, and choose an $X \in \mathcal{D}_1$ such that $\{x_i \ : \ i = 0, \dots, m\}$ is a frame where

$$x_i := \bar{X} \cap L^{q^i}, \ i = 0, \dots, m-1 \text{ and } x_m := \bar{X} \cap M.$$

Observe that $x_i = x_0^{q^i}$, for $i > 0$, and $U := \langle x_0, \dots, x_m \rangle$ is the unique $(m-1)$-space through x_m which intersects all lines $L, L^q, \dots, L^{q^{m-1}}$. Now choose a line ℓ in τ disjoint from $\bar{X}$, and let $\mathcal{R}$ denote the associated regulus induced by the elements $R_0, R_1, \dots, R_q$ of $\mathcal{D}_1$ that intersect ℓ. Since $x \mapsto x^q$ preserves the regulus $\mathcal{R}$, it follows that for each $R \in \mathcal{R}$ we have $R^q = R$ (when $R \cap \ell \neq \emptyset$) or $R^q \cap R = \emptyset$ (when $R \cap \ell = \emptyset$). Also, the lines $L, L^q, \dots, L^{q^{m-1}}, M$ are transversals to the regulus, since each such line intersects the elements $R_0, \dots, R_q$. The uniqueness of U implies that $U \subset R$ for some $R \in \mathcal{R}$. But then $x_1 = x_0^q \in R \cap R^q$ and $R = R^q$. This implies that $R \cap \ell \neq \emptyset$, and hence $R = R_j$ for some $j \in \{0, \dots, q\}$. Since $U \subset R_j \cap \bar{X}$, this implies that $\bar{R_j} = \bar{X}$, contradicting $\ell \cap \bar{X} = \emptyset$. $\square$

The next theorem generalises Proposition 2.7 from [14], where the theorem is proved for $n = 2$.

Theorem 4. *All scattered $\mathbb{F}_q$-linear sets of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$, spanning the whole space, are $\mathrm{P\Gamma L}$-equivalent.*

Proof. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two scattered $\mathbb{F}_q$-linear sets of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$, spanning the whole space. By [9, Theorem 2], for $i = 1, 2$, there exist a subgeometry $\rho_i \cong \mathrm{PG}(3n-1, q)$ of $\mathrm{PG}(3n-1, q^3)$, and an $(n-1)$-space Π_i in $\mathrm{PG}(3n-1, q^3)$, with $\Pi_i \cap \rho_i = \emptyset$, such that

$$\alpha_i(\mathcal{L}_i) = \{\langle x, \Pi_i \rangle / \Pi_i \ : \ x \in \rho_i\},$$

for some collineation α_i from $\mathrm{PG}(2n-1, q^3)$ to $\mathrm{PG}(3n-1, q^3)/\Pi_i$. Suppose $\langle \Pi_i, \Pi_i^q, \Pi_i^{q^2} \rangle$ is a space of dimension d. Then projecting the d-dimensional subspace $\rho_i \cap \langle \Pi_i, \Pi_i^q, \Pi_i^{q^2} \rangle$ from Π_i gives rise to a scattered linear set of rank $d+1$ contained in a projective space $\cong \mathrm{PG}(d-n, q^3)$, and hence $d \geqslant 3n-1$ by Theorem 1.

Since all $(3n-1)$-dimensional $\mathbb{F}_q$-subgeometries of $\mathrm{PG}(3n-1, q^3)$ are PGL-equivalent to the canonical subgeometry $\rho = \{\langle (x_0, x_1, \dots, x_{3n-1}) \rangle | x_j \in \mathbb{F}_q\}$, there is, for $i = 1, 2$ an element ϕ_i of $\mathrm{PGL}(3n, q^3)$ such that $\phi_i(\rho_i) = \rho$. The set

$$\{\langle P, P^q, P^{q^2} \rangle \cap \rho | P \in \phi_i(\Pi_i)\}, \ i = 1, 2,$$

is a Desarguesian 2-spread $\mathcal{D}_i$ of ρ. Since all Desarguesian 2-spreads of $\mathrm{PG}(3n-1, q)$ are projectively equivalent, and, by Theorem 3, the spaces $\Pi_i, \Pi_i^q, \Pi_i^{q^2}$ determining $\mathcal{D}_i$ are uniquely determined up to conjugacy, this implies that there is an element ψ of $\mathrm{P\Gamma L}(3n, q^3)$ such that $\psi(\mathcal{D}_1) = \mathcal{D}_2$ and $\psi(\phi_1(\Pi_1)) = \phi_2(\Pi_2)$. Now $\xi = \phi_2^{-1} \circ \psi \circ \phi_1$ is an element of $\mathrm{P\Gamma L}(3n, q^3)$, and

$$\begin{aligned} \xi(\rho_1) &= (\phi_2^{-1} \circ \psi \circ \phi_1)(\rho_1) \\ &= (\phi_2^{-1} \circ \psi)(\rho) \\ &= \phi_2^{-1}(\rho) = \rho_2; \end{aligned}$$

$$
\begin{aligned}
\xi(\Pi_1) &= (\phi_2^{-1} \circ \psi \circ \phi_1)(\Pi_1) \\
&= \phi_2^{-1}(\phi_2(\Pi_1)) = \Pi_2.
\end{aligned}
$$

Now ξ induces a collineation τ from $\mathrm{PG}(3n-1, q^3)/\Pi_1$ to $\mathrm{PG}(3n-1, q^3)/\Pi_2$ defined by

$$\tau \;:\; \langle x, \Pi_1\rangle/\Pi_1 \mapsto \langle \xi(x), \xi(\Pi_1)\rangle/\xi(\Pi_1) = \langle \xi(x), \Pi_2\rangle/\Pi_2,$$

and

$$\tau(\alpha_1(\mathcal{L}_1)) = \{\langle \xi(x), \Pi_2\rangle/\Pi_2 \;:\; x \in \rho_1\} = \{\langle y, \Pi_2\rangle/\Pi_2 \;:\; y \in \rho_2\} = \alpha_2(\mathcal{L}_2).$$

This shows that $\mathcal{L}_1$ and $\mathcal{L}_2$ are PΓL-equivalent. □

3 Scattered linear sets of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$ and the associated pseudoregulus

In this section, we show that we can associate a pseudoregulus to a scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$ and that there exist exactly two transversal spaces to this pseudoregulus.

3.1 The (q^2+q+1)-secants to a scattered linear set

Lemma 5. *Let $\mathcal{L}$ be a scattered $\mathbb{F}_q$-linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$, i.e. $\mathcal{L} = \mathcal{B}(\mu)$, with μ a $(3n-1)$-space of $\mathrm{PG}(6n-1, q)$.*

(i) A line of $\mathrm{PG}(2n-1, q^3)$ meets $\mathcal{L}$ in $0, 1, q+1$ or q^2+q+1 points.

(ii) Every point of $\mathcal{L}$ lies on exactly one (q^2+q+1)-secant to $\mathcal{L}$ and two different (q^2+q+1)-secants to $\mathcal{L}$ are disjoint.

(iii) If $|L \cap \mathcal{L}| = q^2+q+1$ for some line L, then $L = \mathcal{B}(\pi)$, for a unique plane π contained in μ.

Proof. (i) Immediate, since by Theorem 1 every line of $\mathrm{PG}(2n-1, q^3)$ meets a scattered $\mathbb{F}_q$-linear set in a scattered $\mathbb{F}_q$-linear set of rank at most 3.

(ii) By Theorem 1, μ is a maximum scattered space. This implies that if ν is a $3n$-space of $\mathrm{PG}(6n-1, q)$ through μ, then there is at least one line, say ℓ_1, contained in ν such that $\ell_1 \subset \mathcal{B}(p_1)$, for some $p_1 \in \mu$. Now if there is a second line, say ℓ_2, contained in ν and $\mathcal{B}(p_2)$ with $p_2 \in \mu$, then the 3-space $\langle \ell_1, \ell_2\rangle$ is contained in ν and meets μ in a plane π. Hence, by part (i), $\langle \mathcal{B}(\ell_1), \mathcal{B}(\ell_2)\rangle$ meets $\mathcal{B}(\mu)$ in exactly q^2+q+1 points, the set $\mathcal{B}(\pi)$. If we count the number of pairs (ℓ, ν), where ℓ is a line contained in an element of $\mathcal{B}(\mu)$ and ν is a $3n$-space through μ containing ℓ, we get that, on average, such a $3n$-space ν contains $q+1$ such lines ℓ.

Now suppose that there is a $3n$-space ν containing a set $\mathcal{S}$ of more than $q+1$ such lines, say $\mathcal{S} = \{\ell_1, \ell_2, \dots, \ell_s\}$. If the lines of $\mathcal{S}$ span a subspace of dimension at least 5, then this subspace meets μ in a scattered space of dimension at least 4 with respect to a plane-spread in $\mathrm{PG}(8,q)$. By Theorem 1, this is a contradiction. If the lines of $\mathcal{S}$ span a 4-dimensional space, then each line of $\mathcal{S}$ intersects $\langle l_1, l_2\rangle$, and hence $\langle \mathcal{B}(\ell_1), \mathcal{B}(\ell_2), \ldots, \mathcal{B}(\ell_s)\rangle$ corresponds to a line over $\mathbb{F}_{q^3}$ with q^3+q^2+q+1 points of $\mathcal{L}$, a contradiction. Hence, all the lines of $\mathcal{S}$ are contained in the 3-space $\langle \ell_1, \ell_2\rangle$. But then by [9, Lemma 10], there are q^2+1 lines contained in $\langle \ell_1, \ell_2\rangle$ inducing an $\mathbb{F}_{q^2}$-subline of $\langle \mathcal{B}(\ell_1), \mathcal{B}(\ell_2)\rangle$, and we get that $2|3$, again a contradiction. This implies that every $3n$-space through μ contains exactly $q+1$ lines ℓ_i with $\ell_i \in \mathcal{B}(p_i)$ for some $p_i \in \mu$, $i = 1\ldots q+1$.

Now let $P = \mathcal{B}(r)$ be a point of $\mathcal{L} = \mathcal{B}(\mu)$, where $r \in \mu$. Let ℓ_1 be a line through r in $\mathcal{B}(r)$, then the $3n$-space $\langle \mu, \ell_1\rangle$ contains $q+1$ lines ℓ_i with $\ell_i \in \mathcal{B}(p_i)$, p_i in μ. As seen before, this implies that there is a plane through r, contained in $\langle \mathcal{B}(\ell_1), \mathcal{B}(\ell_2)\rangle \cap \mu$, hence $\langle \mathcal{B}(\ell_1), \mathcal{B}(\ell_2)\rangle$ is a (q^2+q+1)-secant to $\mathcal{L}$ through P. This shows that every point of $\mathcal{L}$ lies on at least one (q^2+q+1)-secant.

Suppose that two (q^2+q+1)-secants, M and N, intersect. Then the plane $\langle M, N\rangle$ meets $\mathcal{L}$ in a scattered linear set of rank at least 5, contradicting Theorem 1. This concludes the proof of part (ii).

(iii) This follows from the proof of part (ii) where we have shown that every point of μ lies on a unique plane $\pi \subset \mu$ such that $\mathcal{B}(\pi) = L \cap \mathcal{L}$, where L is a (q^2+q+1)-secant. $\square$

Definition 6. Let $\mathcal{L}$ be a scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$. In the spirit of the pseudoregulus defined by Freeman in [4], and extending the definition in [14], we define the *pseudoregulus* $\mathcal{P}$ associated with $\mathcal{L}$ as the set $\mathcal{P}$ of $\frac{q^{3n}-1}{q^3-1}$ lines meeting $\mathcal{L}$ in q^2+q+1 points. The set of points lying on the lines of $\mathcal{P}$ is denoted by $\tilde{\mathcal{P}}$.

3.2 The transversal spaces to a pseudoregulus

Let $\mathcal{P}$ denote the pseudoregulus associated to a scattered linear set $\mathcal{L} = \mathcal{B}(\mu)$ of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$.

A subspace whose point set is contained in $\tilde{\mathcal{P}}$ and which intersects all lines of $\mathcal{P}$ in at most a point, is called a *transversal space* to the pseudoregulus $\mathcal{P}$. In this section (Theorem 10) we prove that there exist exactly two $(n-1)$-dimensional transversal spaces to $\mathcal{P}$.

Lemma 7. *There exist two disjoint transversal $(n-1)$-spaces to $\mathcal{P}$.*

Proof. Since $\mathcal{L}$ is a scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1, q^3)$, it can be obtained in the quotient geometry over an $(n-1)$-space Π of $\mathrm{PG}(3n-1, q^3)$ by considering an appropriate subgeometry $\Sigma = \mathrm{PG}(3n-1, q)$ disjoint from Π (see [9, Theorem 2]). Since $\mathcal{L}$ is scattered, the space $\langle \Pi, \Pi^q, \Pi^{q^2}\rangle$ is $(3n-1)$-dimensional, as seen in the proof of Theorem 4. For every $P \in \Pi$, the plane $\langle P, P^q, P^{q^2}\rangle$ meets Σ in a subplane $\cong \mathrm{PG}(2,q)$. This implies that the lines $\langle P, P^q, P^{q^2}, \Pi\rangle/\Pi$ are exactly the (q^2+q+1)-secants to $\mathcal{L}$. Moreover, $\Pi_1 := \langle \Pi^q, \Pi\rangle/\Pi$ and $\Pi_2 := \langle \Pi^{q^2}, \Pi\rangle/\Pi$ are two disjoint $(n-1)$-spaces intersecting each of these (q^2+q+1)-secants to $\mathcal{L}$, whose point sets are contained in $\tilde{\mathcal{P}}$. $\square$

In what follows, Π_1 and Π_2 denote the transversal spaces constructed in Lemma 7.

Lemma 8. *If P_1, P_2, P_3 are three collinear points in Π_1, then the intersection points Q_i of the lines of $\mathcal{P}$ through P_i, $i = 1, 2, 3$, with Π_2 are collinear. Moreover, the only points of $\tilde{\mathcal{P}}$, contained in $\langle P_1, P_2, Q_1, Q_2\rangle$, are the $(q^3+1)^2$ points on the lines of $\mathcal{P}$ in $\langle P_1, P_2, Q_1, Q_2\rangle$.*

Proof. Let S_i denote the line of $\mathcal{P}$ through P_i, $i = 1, 2$, and put $T_1 := \langle P_1, P_2\rangle$ and $T_2 := \langle Q_1, Q_2\rangle$. By Lemma 5(iii), the point P_i corresponds to a spread element lying in $\langle \mathcal{B}(\pi_i)\rangle$, with π_i a plane of μ, where $\mathcal{L} = \mathcal{B}(\mu)$. Since the subspace $\langle S_1, S_2, S_3\rangle$ has dimension at most 4 and intersects $\mathcal{L}$ in a scattered linear set, it follows from the upper bound on the dimension of the subspace $\langle \pi_1, \pi_2, \pi_3\rangle$ (Theorem 1), that there exists a line ℓ in μ, meeting π_1, π_2 and π_3. Hence the line $L := \langle \mathcal{B}(\ell)\rangle$ meets S_1, S_2, and S_3, and these lines are contained in the 3-space $\langle T_1, L\rangle$. Since Π_1 and Π_2 are disjoint, $\langle T_1, L\rangle$ meets Π_2 in the line T_2, and hence Q_1, Q_2, and Q_3 are collinear.

Now, suppose that there is a point R of $\mathcal{P}$, lying in the 3-space $\langle T_1, T_2\rangle$, but not on a line of $\mathcal{P}$ in $\langle T_1, T_2\rangle$, then R lies on a line of $\mathcal{P}$ meeting Π_1, resp. Π_2 in a point R_1, resp R_2, not lying on T_1 or T_2. But then the planes $\langle T_1, R_1\rangle$, and $\langle T_2, R_2\rangle$ must intersect since both are contained in the 4-space $\langle T_1, T_2, R_1, R_2\rangle$. This contradicts $\Pi_1 \cap \Pi_2 = \emptyset$. □

Theorem 9. *All transversal lines to $\mathcal{P}$ lie in one of the transversal spaces Π_1 or Π_2.*

Proof. Suppose that there exists a transversal line $L = R_1R_2$ to $\mathcal{P}$, not in Π_1 or Π_2. Let S_i be the line of $\mathcal{P}$ through R_i and let P_i, resp. Q_i, be the intersection of S_i with Π_1, resp. Π_2. It follows from Lemma 8 that R_1R_2 meets the q^3+1 lines of $\mathcal{P}$ that are contained in the 3-space $\rho = \langle P_1, P_2, Q_1, Q_2\rangle$. If R_1, R_2 meets Π_1 or Π_2, the lines of $\mathcal{P}$ in ρ would intersect, a contradiction. Hence, P_1P_2, R_1R_2, Q_1Q_2 are three disjoint lines in ρ, defining a regulus $\mathcal{R}$. By Lemma 5(iii) the q^3+1 lines of $\mathcal{P}$ contained in the 3-dimensional space ρ correspond to q^3+1 two by two disjoint planes contained in a 5-dimensional subspace ζ of μ, i.e. they form a plane spread of ζ. Let $P = \mathcal{B}(r)$ be a point of $\mathcal{L}$ on the line P_1Q_1 with $r \in \zeta$, then connecting r with the q^2+q+1 points of the plane $\pi_2 \subset \zeta$ corresponding to the (q^2+q+1)-secant S_2 shows that there are at least q^2+q+1 lines through P meeting at least $q+1$ lines of the regulus $\mathcal{R}$, a contradiction unless $\mathcal{B}(\zeta)$ is a line, which contradicts Theorem 1. □

Theorem 10. *There are exactly two $(n-1)$-dimensional transversal spaces to $\mathcal{P}$.*

Proof. This follows immediately from Lemma 7 and Theorem 9. □

3.3 The stabiliser of a pseudoregulus

Lemma 11. *The stabiliser in $\mathrm{PGL}(2n, q^3)$ of the pseudoregulus $\mathcal{P}$ in $\mathrm{PG}(2n-1, q^3)$ acts transitively on the points of a line of $\mathcal{P}$ that do not lie on one of the transversal $(n-1)$-spaces to $\mathcal{P}$.*

Proof. Let Π_1 and Π_2 be the transversal spaces to the pseudoregulus $\mathcal{P}$ and let P be a point on one of the lines L of $\mathcal{P}$ but not contained in Π_i, $i = 1, 2$. Let $P_1, \ldots, P_{2n+1}$ be the points of a standard frame of $\mathrm{PG}(2n-1, q^3)$, chosen in such a way that $P_1, \ldots, P_n$ lie in Π_1, $P_{n+1}\ldots,P_{2n}$ lie in Π_2 and $P=P_{2n+1}$. It follows that the intersection point Q_1 of the line L with Π_1 is $\langle e_1+\ldots+e_n\rangle$ and the intersection point Q_2 of the line L with Π_2 is $\langle e_{n+1}+\ldots+e_{2n}\rangle$. If Q is a point on L, different from Q_1,Q_2, then Q has coordinates $\langle e_1+\ldots+e_n+s(e_n+\ldots+e_{2n})\rangle$. It is easy to check that the element ϕ of $\mathrm{PGL}(2n,q^3)$ corresponding to the matrix $A=(a_{ij})$, with $a_{ij}=0$ if $i\neq j$, $a_{ii}=1$ if $1\leqslant i\leqslant n$ and $a_{ii}=s$ if $n+1\leqslant i\leqslant 2n$, stabilises $\mathcal{P}$ and maps P onto Q. □

4 The reconstruction of a linear set having a fixed pseudoregulus

If $\mathcal{L}$ is a scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1,q^3)$, then we have seen in the previous section that there exists a unique associated pseudoregulus $\mathcal{P}$. The aim of this section is to construct a scattered linear set of rank $3n$ having a given pseudoregulus $\mathcal{P}$ as associated pseudoregulus, and show that there are $q-1$ different scattered linear sets of rank $3n$ giving rise to the same pseudoregulus $\mathcal{P}$.

Theorem 12. *Let $\mathcal{L}$ be a scattered linear set of rank $3n$ in $\mathrm{PG}(2n-1,q^3)$.*

(i) A plane meets $\mathcal{L}$ in 0, 1, $q+1$, q^2+q+1 or q^3+q^2+q+1 points.

(ii) A plane Γ meeting $\mathcal{L}$ in q^3+q^2+q+1 points contains exactly one line with q^2+q+1 points of $\mathcal{L}$.

Proof. (i) Immediate, since a plane meets the scattered linear set $\mathcal{L}$ in a scattered linear set of rank at most 4, by Theorem 1.

(ii) In this case, the plane Γ meets $\mathcal{L}$ in a set $\mathcal{B}(\rho)$, where ρ has dimension 3. Since a line of Γ corresponds to a 5-space in $\mathrm{PG}(8,q)$ and a 3-space and 5-space always meet in $\mathrm{PG}(8,q)$, all lines of Γ meet $\mathcal{L}$ in at least one point. If we denote the number of lines in Γ meeting $\mathcal{L}$ in i points by ℓ_i, we get that $\sum_i \ell_i=q^6+q^3+1$, $\sum_i i\ell_i=(q^3+q^2+q+1)(q^3+1)$ and $\sum_i i(i-1)\ell_i=(q^3+q^2+q+1)(q^3+q^2+q)$.

If we suppose that all lines meet in 1 or $q+1$ points, then we obtain that $\sum_i(i-1)(i-(q+1))\ell_i=0$, a contradiction if we use the previously found values for $\sum_i \ell_i$, $\sum_i i\ell_i$ and $\sum_i i(i-1)\ell_i$. Hence, there is a line meeting $\mathcal{L}$ in more than $q+1$ points, which then, by Lemma 5(i), meets $\mathcal{L}$ in q^2+q+1 points. Suppose that L_1 and L_2 are two different lines in Π meeting $\mathcal{L}$ in q^2+q+1 points, then there would be two intersecting (q^2+q+1)-secants to $\mathcal{L}$, a contradiction by Lemma 5(ii). □

Remark 13. In the case that $n=2$, every plane meets $\mathcal{L}$ in q^2+q+1 points or q^3+q^2+q+1 points. This follows also from [2, Theorem 2.4].

Let us fix some more notation. Let $\mathcal{P}$ denote a pseudoregulus in $\mathrm{PG}(2n-1,q^3)$ corresponding to the scattered linear set $\mathcal{L}$ of rank $3n$. Let μ be a $(3n-1)$-space such that $\mathcal{B}(\mu)=\mathcal{L}$. A (q^2+q+1)-secant to $\mathcal{L}$ defines a 5-space in $\mathrm{PG}(6n-1,q)$ meeting μ in a plane. Since every point of $\mathcal{L}$ lies on a unique (q^2+q+1)-secant by Lemma 5(ii), the $(q^{3n}-1)/(q^3-1)$ planes defined in this way determine a spread of μ. Let us denote this spread by Σ.

Lemma 14. *The spread Σ is Desarguesian.*

Proof. As in the proof of Lemma 7, we see that $\mathcal{L}$ is the projection of a subgeometry $\rho = \mathrm{PG}(3n-1,q)$ of $\mathrm{PG}(3n-1,q^3)$ from an $(n-1)$-space Π onto $\mathrm{PG}(2n-1,q^3)$, and the planes $\langle P, P^q, P^{q^2}\rangle$, with P a point from Π form a Desarguesian spread $\mathcal{D}$ in ρ. If we now return to the spread representation, we get that μ is the projection of ρ from the $(3n-1)$-space $\langle \mathcal{B}(\Pi)\rangle$. Every plane $\langle P, P^q, P^{q^2}\rangle$, with P on Π corresponds to an 8-dimensional space, meeting ρ in a plane of $\mathcal{D}$. The projection of this 8-space from $\langle \mathcal{B}(\Pi)\rangle$ is a 5-space λ, meeting μ in a plane. Since λ corresponds to a (q^2+q+1)-secant, this plane is an element of the spread Σ. This shows that Σ is the projection of the Desarguesian spread $\mathcal{D}$, from which the statement follows (see e.g. [5, Theorem 1.5.4]). □

Lemma 15. *If π_1, π_2, π_3 are planes of Σ defining a regulus with elements $\pi_1, \ldots, \pi_{q+1}$, then the 5-spaces $\langle \mathcal{B}(\pi_1)\rangle, \langle \mathcal{B}(\pi_2)\rangle, \langle \mathcal{B}(\pi_3)\rangle$ determine the regulus with elements $\langle \mathcal{B}(\pi_i)\rangle$, $i = 1, \ldots, q+1$.*

Proof. Each plane π_i, $i = 1, \ldots, q+1$, is contained in some element of the regulus defined by $\langle \mathcal{B}(\pi_1)\rangle, \langle \mathcal{B}(\pi_2)\rangle, \langle \mathcal{B}(\pi_3)\rangle$, since a line ℓ through π_1, π_2 and π_3 meets the elements of the regulus defined by π_1, π_2, π_3, say $\ell \cap \pi_i = \{p_i\}$. Now $\mathcal{B}(p_1), \mathcal{B}(p_2)$ and $\mathcal{B}(p_3)$ form a regulus of the Desarguesian spread $\mathcal{D}$, and the other spread elements in this regulus are $\mathcal{B}(p_i)$. Since a line meeting $\mathcal{B}(p_i)$, $i = 1, 2, 3$ meets $\mathcal{B}(p_i)$ for all $i = 1, \ldots, q+1$, $\mathcal{B}(p_i)$ is contained in some element of the regulus defined by $\langle \mathcal{B}(\pi_1)\rangle, \langle \mathcal{B}(\pi_2)\rangle, \langle \mathcal{B}(\pi_3)\rangle$. Since π_i and $\mathcal{B}(p_i)$ meet in a point, $\langle \pi_i, \mathcal{B}(p_i)\rangle$ is contained in an element of this regulus. The same reasoning holds for a different transversal line ℓ', meeting π_i in a point p'_i, and hence $\langle \pi_i, \mathcal{B}(p'_i)\rangle$ is contained in an element of this regulus. This implies that $\langle \mathcal{B}(\pi_i)\rangle$ is an element of the regulus defined by $\langle \mathcal{B}(\pi_1)\rangle, \langle \mathcal{B}(\pi_2)\rangle, \langle \mathcal{B}(\pi_3)\rangle$. □

Lemma 16. *Let $q > 2$. A set of points $\mathcal{S}$ in $\mathrm{PG}(1,q^3)$, $|\mathcal{S}| \geqslant 3$, such that the subline through any 3 of them is contained in $\mathcal{S}$ is either a subline or a full line.*

Proof. Let $\mathcal{D}$ be the Desarguesian 2-spread of $\mathrm{PG}(5,q)$ obtained from $\mathrm{PG}(1,q^3)$. Suppose $\mathcal{S}$ has at least $q+2$ points, and let $\rho_1, \ldots, \rho_{q+1}$ be the regulus corresponding to a $(q+1)$-secant to $\mathcal{S}$ and let ρ_{q+2} be a spread element, not in this regulus, corresponding to a point of $\mathcal{S}$. Let ℓ_1 be the transversal line through the point p_1 of ρ_1 to the regulus $\rho_1, \rho_2, \ldots, \rho_{q+1}$. Let ℓ_2 be the transversal line through p_1 of the regulus through ρ_1, ρ_2 and ρ_{q+2}, then $\mathcal{B}(\ell_2) \subset \mathcal{S}$ by the hypothesis. We will now show that $\mathcal{B}(\langle \ell_1, \ell_2\rangle) \subset \mathcal{S}$. The plane $\langle \ell_1, \ell_2\rangle$ meets ρ_2 in a line m. Now every line n in $\langle \ell_1, \ell_2\rangle$, not through any of the three points $\ell_1 \cap m$, $\ell_2 \cap m$, $\ell_1 \cap \ell_2$, meets ℓ_1, ℓ_2 and m in a point, and hence, $\mathcal{B}(n)$ contains 3 elements of $\mathcal{S}$. This implies that $\mathcal{B}(n) \subset \mathcal{S}$ for all such lines n. Since $q > 2$, all lines through one of the intersection points of m, ℓ_1 and ℓ_2 now contain at least 3 points of $\mathcal{S}$, hence, this argument shows that $\mathcal{B}(\langle \ell_1, \ell_2\rangle) \subset \mathcal{S}$. If $\mathcal{S} = \mathcal{B}(\langle \ell_1, \ell_2\rangle)$, then this linear set is a linear set of size q^2+1 in $\mathrm{PG}(1,q^3)$, which is not isomorphic to $\mathrm{PG}(1,q^2)$. By Corollary 13 of [9], through two points of such a linear set, there is exactly one subline that is completely contained in this linear set, a contradiction by our assumption on $\mathcal{S}$. Hence, there is an element ρ_{q+3} of $\mathcal{S}$, not in $\mathcal{B}(\langle \ell_1, \ell_2\rangle)$. Repeating the same argument with a transversal line ℓ_3 through ρ_1, ρ_2 and ρ_{q+3} and a line of $\langle \ell_1, \ell_2 \rangle$ shows that $\mathcal{B}(\langle \ell_1, \ell_2, \ell_3 \rangle) \subset \mathcal{S}$, hence, $\mathcal{S}$ is a full line. $\square$

Theorem 17. *Let $q > 2$. A line L in $\mathrm{PG}(2n-1, q^3)$ meets the point set $\tilde{\mathcal{P}}$ of a pseudoregulus $\mathcal{P}$ in $0, 1, 2, q+1$ or q^3+1 points. If $|L \cap \tilde{\mathcal{P}}| = q+1$, then L meets $\tilde{P}$ in a subline.*

Proof. Let L be a line meeting 3 points of $\tilde{\mathcal{P}}$, say P_1, P_2, P_3, and suppose that the points P_1, P_2, P_3 are not contained in the same line of $\mathcal{P}$. Let ρ_1, ρ_2, ρ_3 be the corresponding spread elements, then they determine 3 elements of Σ, say π_1, π_2, π_3, and $\rho_i \in \langle \mathcal{B}(\pi_i) \rangle$. A line through ρ_1, ρ_2, ρ_3 meets $\langle \mathcal{B}(\pi_1) \rangle, \langle \mathcal{B}(\pi_2) \rangle$ and $\langle \mathcal{B}(\pi_3) \rangle$, and by Lemma 15, also $\langle \mathcal{B}(\pi_i) \rangle$, $i = 4, \ldots, q+1$. From this, it follows that the line L meets $\tilde{\mathcal{P}}$ in a set of points $\mathcal{K}$ such that the subline through any 3 of them is contained in $\mathcal{K}$. Such a set is either a subline, or a full line by Lemma 16. $\square$

Lemma 18. *Let $q > 2$. Let $\tilde{\mathcal{S}}$ be the point set of a set $\mathcal{S}$ of q^3+1 mutually disjoint lines in $\mathrm{PG}(3, q^3)$ with the property that the subline through 3 collinear points of $\tilde{\mathcal{S}}$ is contained in $\tilde{\mathcal{S}}$. Then a plane Π through a line L of $\mathcal{S}$ contains q^3 points of $\tilde{\mathcal{S}}$, not on L and this set of q^3 points determines a set D of either 1 or q^2+q+1 directions on L. Moreover, $I \cup D = \mathcal{B}(\nu)$, where ν is a 3-space of $\mathrm{PG}(11, q)$.*

Proof. Since the lines of $\mathcal{S}$ are mutually disjoint, the plane Π meets the q^3 lines of $\mathcal{S}$, different from L in a point. Let $I = \{P_1, \ldots, P_{q^3}\}$ this set of q^3 points. Let $D = \{D_1, \ldots, D_d\}$ be the set of directions determined by the set I. We claim that $d = 1$ or $d = q^2+q+1$ and that the set $I \cup D$ is an $\mathbb{F}_q$-linear set of rank 4.

Let ρ_i be the spread element corresponding to P_i. If the q^3 points in I are collinear, say they lie on the line M, then we are in the first case and $\mathcal{B}(\nu) = M = I \cup D$ for all 3-spaces contained in $\langle \rho_1, \rho_2 \rangle$. Otherwise, every line in Π, different from the line L meets $\tilde{\mathcal{S}}$ in $0, 1, 2$ or $q+1$ points by Lemma 16. The line through P_i and P_j, $j \neq i$, meets L, and hence, contains a third point of $\tilde{\mathcal{S}}$, say R_{ij}. It follows that P_iP_j meets $\tilde{\mathcal{S}}$ in $q+1$ points, forming a subline. Let ℓ_i be the transversal line through a point p_1 of ρ_1 to the regulus defined by ρ_1, ρ_i and the spread element corresponding to R_{1i}. We claim that $\mathcal{B}(\langle \ell_2, \ell_3 \rangle) \subset \tilde{\mathcal{S}}$. Each line m in $\langle \ell_2, \ell_3 \rangle$, for which the points $\mathcal{B}(\ell_2 \cap m)$, $\mathcal{B}(\ell_3 \cap m)$ and $\langle \mathcal{B}(m) \rangle \cap L$ are different points of $\tilde{\mathcal{S}}$, induces the subline $\mathcal{B}(m)$ contained in $\tilde{\mathcal{S}}$ and since $q > 2$, repeating this argument for the other lines in $\langle \ell_2, \ell_3 \rangle$ and m implies that $\mathcal{B}(\langle \ell_2, \ell_3 \rangle) \subset \tilde{S}$. Similarly, we get that $\mathcal{B}(\langle \ell_i, \ell_j \rangle) \subset \tilde{\mathcal{S}}$ for all $i \neq j > 1$, hence $\nu := \langle \ell_2, \ell_3, \ell_4, \ldots \rangle \subset \tilde{S}$, and ν is a 3-dimensional space, since $|I| = q^3$. If a spread element ρ would intersect ν in more than a point, every line in Π through the point corresponding to ρ and a point of $\tilde{S}$, would contain more than $q+1$ points of $\tilde{\mathcal{S}}$, a contradiction. From this, it follows that $\mathcal{B}(\nu)$ is scattered, hence, there are q^2+q+1 determined directions. $\square$

Lemma 19. *Let $q > 2$. A plane through a line L of a pseudoregulus $\mathcal{P}$ and a point of $\tilde{\mathcal{P}}$, outside L meets q^3 other lines of $\mathcal{P}$ in a point, and this set of q^3 points determines either 1 or q^2+q+1 directions on L.*

Proof. Let Π be a plane through one of the lines L of $\mathcal{P}$, and the point R of $\tilde{\mathcal{P}}$, not on L. Let M be the line of $\mathcal{P}$ through R. From Lemma 8, we get that there are exactly q^3+1 lines of $\mathcal{P}$ in $\langle L, M\rangle$, and $\langle L, M\rangle$ does not contain other points of $\tilde{\mathcal{P}}$. Hence, Π meets exactly q^3 of the lines of $\mathcal{P}$ in a point. The statement now follows from Lemma 18. $\square$

Lemma 20. *Let $q>2$. If P is a point of $\tilde{\mathcal{P}}$, not on the transversal spaces Π_1 and Π_2, then the number of $(q+1)$-secants to $\tilde{\mathcal{P}}$ through P is $q^2(q^{3n-3}-1)/(q-1)$. Moreover, if $\mathcal{L} \ni P$ is a linear set with $\mathcal{P}$ as associated pseudoregulus, then each $(q+1)$-secant of $\mathcal{P}$ through P is a $(q+1)$-secant to $\mathcal{L}$.*

Proof. By Lemma 11, we may assume that the point P is contained in the linear set $\mathcal{L}$ defining $\tilde{\mathcal{P}}$. Now $|\mathcal{L}| = (q^{3n}-1)/(q-1)$ and P lies on a unique (q^2+q+1)-secant to $\mathcal{L}$, namely the line S_1 of $\mathcal{P}$ through P, hence, there are $q^2(q^{3n-3}-1)/(q-1)$ $(q+1)$-secants through P to $\mathcal{L}$, which are necessarily also $(q+1)$-secants to $\tilde{\mathcal{P}}$ by Theorem 9 and Theorem 17. Suppose now that there is a $(q+1)$-secant M through P to $\tilde{\mathcal{P}}$ which is not a $(q+1)$-secant to $\mathcal{L}$. Then a plane $\langle P, S_2\rangle$, with S_2 a line of $\mathcal{P}$ through a point of M different from P, contains q^3 points of $\mathcal{L}\cap\tilde{\mathcal{P}}$, not on S_2, and q points of M, the plane $\langle P, S_2\rangle$ contains more than q^3+q^2+q+1 points of $\tilde{\mathcal{P}}$, a contradiction by Lemma 19. $\square$

Lemma 21. *Let $q>2$. Let L_1 and L_2 be two $(q+1)$-secants to $\tilde{\mathcal{P}}$ through a point P of $\tilde{\mathcal{P}}$. Then the subplane, defined by the intersection of L_1 and L_2 with $\tilde{\mathcal{P}}$ is contained in $\tilde{\mathcal{P}}$.*

Proof. By Lemma 11, we may assume that the point P is contained in the linear set $\mathcal{L}$ defining $\mathcal{P}$, and from Lemma 20, we get that the $(q+1)$-secants to $\mathcal{L}$ through P are the $(q+1)$-secants to $\tilde{\mathcal{P}}$. Hence, the subplane, defined by the intersection of L_1 and L_2 with $\tilde{\mathcal{P}}$, is the subplane defined by the intersection of L_1 and L_2 with the linear set $\mathcal{L}$. This subplane is entirely contained in $\mathcal{L}$, hence, in $\tilde{\mathcal{P}}$. $\square$

In the following theorem, we show, given a pseudoregulus, how to construct a linear set defining this pseudoregulus.

Theorem 22. *Let $q>2$. Let $\mathcal{P}$ be a pseudoregulus in $\mathrm{PG}(2n-1,q^3)$, let P be a point of $\tilde{\mathcal{P}}$, on the line L of $\mathcal{P}$, not lying on one of the transversal spaces to $\mathcal{P}$. Let $T=\{L_1, L_2, \ldots\}$ be the set of $(q+1)$-secants through P to $\tilde{\mathcal{P}}$, let $P(T)$ be the set of points on the lines of T in $\tilde{\mathcal{P}}$. Let Π_i be the plane $\langle L, L_i\rangle$, and let D_i be the set of directions on L, determined by the intersection of Π_i with $\tilde{\mathcal{P}}$. Then $D_i = D_1$, for all i, and $P(T)$, together with the points of D_1 form a linear set $\mathcal{L}$ of rank $3n$ determining the pseudoregulus $\mathcal{P}$.*

Proof. By Lemma 20, there are $q^2(q^{3n-3}-1)/(q-1)$ lines in T, each defining a subline through P, that is contained in $\tilde{P}$. In the spread representation, this implies that there are $q^2(q^{3n-3}-1)/(q-1)$ lines ℓ_i through a point x of the spread element corresponding to P, such that $\mathcal{B}(\ell_i) \subset \tilde{\mathcal{P}}$. By Lemma 21, $\mathcal{B}(\langle \ell_i, \ell_j\rangle) \subset \tilde{\mathcal{P}}$, and since the number of $(q+1)$-secants through P is exactly $q^2(q^{3n-3}-1)/(q-1)$, this implies that $\nu := \langle \ell_1, \ell_2, \ldots\rangle$ is a subspace of dimension $3n-1$. Then $P(T) \subset \mathcal{B}(\nu)$, by construction.

Each plane $\langle L, L_i\rangle$ contains q^3 points of $\tilde{\mathcal{P}}$ and q^2 $(q+1)$-secants $\langle \mathcal{B}(\ell_{i_1})\rangle, \langle \mathcal{B}(\ell_{i_2})\rangle, \ldots,$ $\langle \mathcal{B}(\ell_{i_{q^2}})\rangle$ through P, and determines a set D_i of directions on L. The lines $\ell_{i_1}, \ldots, \ell_{i_{q^2}}$

span a subspace ν_i of ν and each direction of D_i is of the form $\mathcal{B}(y)$, for some $y \in \nu_i$, and hence each set of directions D_i on L determined by the points of $P(T)$ is contained in $\mathcal{B}(\nu) \cap L$. Since $\mathcal{B}(\nu)$ intersects L in a linear set, and each D_i contains at least q^2+q+1 points, by Lemma 19, $\mathcal{B}(\nu) \cap L$ is a linear set of rank at least 3. On the other hand, since $\mathcal{B}(\nu)$ contains the $(q^{3n}-q^3)/(q-1)$ points of $P(T)$ and ν has dimension $3n-1$, it follows that $\mathcal{B}(\nu)$ is a scattered linear set $\mathcal{L}$ of rank $3n$ and $\mathcal{L} \cap L = D_i$. The scattered linear set $\mathcal{L}$ of rank $3n$ defines a pseudoregulus $\mathcal{P}(\mathcal{L})$, so we need to show that $\mathcal{P} = \mathcal{P}(\mathcal{L})$. The $(q^{3n}-1)/(q-1)$ points of $\mathcal{L}$ all lie on one of the lines of $\mathcal{P}$, hence, a line of $\mathcal{P}$ has on average q^2+q+1 points of $\mathcal{L}$, and by Lemma 5(i), it is not possible that one of the lines of $\mathcal{P}$ contains more than q^2+q+1 points of $\mathcal{L}$. This implies that $\mathcal{P} = \mathcal{P}(\mathcal{L})$. $\square$

Corollary 23. *Let $q > 2$. If $\mathcal{P}$ is a pseudoregulus, then there are $q-1$ scattered linear sets having $\mathcal{P}$ as associated pseudoregulus.*

Proof. Counting the number of couples $(P, \mathcal{L})$, where P is a point of the pseudoregulus, not on one of the transversal spaces and $\mathcal{L}$ is a scattered linear set through P having $\mathcal{P}$ as pseudoregulus yields that the number of scattered linear sets having $\mathcal{P}$ as pseudoregulus is equal to $\frac{q^{3n}-1}{q^3-1}(q^3-1)\frac{q-1}{q^{3n}-1}$. $\square$

5 A characterisation of reguli and pseudoreguli in $\mathrm{PG}(3, q^3)$

Theorem 24. *Let $q > 2$. Let $\tilde{\mathcal{S}}$ be the point set of a set $\mathcal{S}$ of q^3+1 mutually disjoint lines in $\mathrm{PG}(3, q^3)$ such that the subline defined by three collinear points of $\tilde{\mathcal{S}}$ is contained in $\tilde{\mathcal{S}}$, then $\mathcal{S}$ is a regulus or pseudoregulus.*

Proof. By Lemma 16, a line meets $\tilde{\mathcal{S}}$ in $0, 1, 2, q+1$ or q^3+1 points.

Case 1: Suppose first that every line meets $\tilde{\mathcal{S}}$ in $0, 1, 2$ or q^3+1 points. Let L be a line of $\mathcal{S}$ and let Π be a plane through L. Since Π meets all lines of $\mathcal{S}$ and all lines of $\mathcal{S}$ are disjoint, there are exactly q^3 points of $\tilde{\mathcal{S}}$ in Π, not on L. Let P and Q be two points of $\tilde{\mathcal{S}} \setminus L$ in Π. Since the line PQ has to contain q^3 points of $\tilde{\mathcal{S}} \setminus L$, the q^3 points of $\tilde{\mathcal{S}}$ in Π are collinear. In this way, we find a line $\notin \mathcal{S}$ contained in $\tilde{\mathcal{S}}$, in every of the q^3+1 planes through L. If two of those lines meet, then the lines of $\mathcal{S}$ would not be disjoint, a contradiction. Hence, we find a set of q^3+1 mutually disjoint lines $\mathcal{S}'$, meeting the lines of $\mathcal{S}$. This shows that $\mathcal{S}$ is the opposite regulus to $\mathcal{S}'$ and vice versa.

Case 2: There is a line M meeting $\tilde{\mathcal{S}}$ in exactly $q+1$ points. Let P be a point of M, let L_0 be the line of $\mathcal{S}$ through P and let $L_1, \ldots, L_{q^3}$ be the other lines of $\mathcal{S}$. A plane $\langle L_i, P\rangle$, $i = 1, \ldots, q^3$, meets q^3 points of $\tilde{\mathcal{S}}$ that do not lie on L_i. Suppose that in one of the planes, these q^3 points are collinear, say on N, then the plane $\langle M, N\rangle$ meets q lines of $\mathcal{S}$ in 2 different points, a contradiction since the lines of $\mathcal{S}$ are mutually disjoint. By Lemma 18, this implies that in every plane $\langle P, L_i\rangle$, there are exactly q^2+q+1 $(q+1)$-secants through P. Let p be a point of the spread element corresponding to P. By Lemma 18, there is a 3-space ν_i such that $\mathcal{B}(\nu_i) \subset \langle P, L_i\rangle \cap \tilde{S}$; w.l.o.g. we may choose ν_i

through p. Let μ_i be the plane $\nu_i \cap \langle \mathcal{B}(L_i) \rangle$. The 3-space ν_i is the unique 3-space through p such that $\mathcal{B}(\nu_i) \subset \langle P, L_i \rangle \cap \tilde{S}$ since pr_j, with $r_j \in \mu_i$, is the unique transversal line to the regulus $\langle P, \mathcal{B}(r_j) \rangle \cap \tilde{S}$.

The q^3 planes $\mu_1, \dots, \mu_{q^3}$ are mutually disjoint and satisfy the condition that the line $\langle p, x \rangle$, where x is a point on one of the planes μ_i, corresponds to a subline contained in $\tilde{\mathcal{S}}$. We get that the 3-space $\langle p, \mu_i \rangle$ intersects the plane μ_j for all j non-trivially, and hence, since the planes μ_i are mutually disjoint, $\langle p, \mu_i \rangle$ and μ_j meet in a point if $i \neq j$. This implies that $\langle p, \mu_1, \mu_2 \rangle$ is 5-dimensional.

We will prove that $\langle p, \mu_1, \dots, \mu_{q^3} \rangle$ is 5-dimensional. W.l.o.g. suppose that μ_3 does not go through the line $\langle p, \mu_1 \rangle \cap \langle p, \mu_2 \rangle$.

It is clear that the space $\rho := \langle p, \mu_1, \mu_2, \mu_3 \rangle$ is at most 6-dimensional, so assume that ρ is 6-dimensional. Since every plane μ_i has to meet the spaces $\langle p, \mu_1 \rangle$, $\langle p, \mu_2 \rangle$, and $\langle p, \mu_3 \rangle$, it is clear that if μ_i is not going through one of the 3 lines $\ell_1 := \langle p, \mu_1 \rangle \cap \langle p, \mu_2 \rangle$, $\ell_2 := \langle p, \mu_1 \rangle \cap \langle p, \mu_3 \rangle$ or $\ell_3 := \langle p, \mu_2 \rangle \cap \langle p, \mu_3 \rangle$, then μ_i is contained in $\langle p, \mu_1, \mu_2, \mu_3 \rangle$. This means that at least $q^3 - 3q$ planes μ_i are in ρ. Let μ_i be a plane, through one of the lines ℓ_j, $j = 1, 2, 3$. Repeating the same argument with 3 planes in ρ such that μ_i is not on the intersection lines of the cones defined by p and these 3 planes shows that all planes μ_i are contained in ρ.

Now let m be a line through p, such that $\langle \mathcal{B}(m) \rangle$ is not the line L_0. Suppose that m does not meet any of the planes μ_i. There are $q^4 + q^2 + q + 1$ planes through m in ρ and there are $q^3(q^2 + q + 1)$ points in ρ contained in one of the planes μ_i. This implies that there is a plane ν through m containing at least 3 points lying on one of the planes μ_i. Since m does not meet any of the planes μ_i, the 3 points belong to different planes, say μ_1, μ_2 and μ_3. Hence, in the plane ν, there are 3 lines n_1, n_2, n_3 through p such that $\mathcal{B}(n_i)$ is contained in $\tilde{S}$. Let n_4 be a line meeting n_1, n_2, n_3 in different points. As $\mathcal{B}(n_4)$ is a subline containing 3 points of $\tilde{S}$, $\mathcal{B}(\pi_4)$ is contained in $\tilde{S}$. This implies that the intersection point $p' := n_4 \cap m$ has necessarily $\mathcal{B}(p')$ contained in a line, say L_1 of $\mathcal{S}$. Since we have assumed that p' is not on one of the planes μ_i, p' does not lie on μ_1 and the 3-space $\langle p', \mu \rangle$ is contained in $\langle \mathcal{B}(L_1) \rangle \cap \rho$, which means that L_1 is entirely contained in $\mathcal{B}(\rho)$. Repeating the same argument for a line meeting n_1, n_2, n_3 in three distinct points and meeting n_4 in a point p'', different from p' shows that, if p'' is not on μ_i, there is a second line of $\mathcal{S}$, say L_2 contained in $\mathcal{B}(\rho)$. But then $L_1 \cap \mathcal{B}(\rho) = \sigma_1$ and $L_2 \cap \mathcal{B}(\rho) = \sigma_2$ with σ_1 and σ_2 three-spaces in the 6-space ρ. Since σ_1 and σ_2 necessarily meet in a point, the lines L_1 and L_2 meet in a point, a contradiction. This implies that every line through p in ρ such that $\langle \mathcal{B}(m) \rangle$ is not the line L_0, meets one of the planes μ_i. There are at least $q^5 + q^4 + q^3$ such lines, but as there are only q^3 planes and every line through a point of p and a point of a plane μ_i contains q points, lying on a plane μ_i, the number of these lines is exactly $q^2(q^2 + q + 1)$, a contradiction. Hence, ρ is 5-dimensional.

Let r be a point of the 5-space ρ, not on one of the q^3 planes μ_i, then there is a line through r meeting at least 3 different planes of $\{\mu_i | i = 1, \dots, q^3\}$. This gives rise to a subline meeting 3 points of $\tilde{S}$, hence, contained in $\tilde{S}$, which implies that $\mathcal{B}(r)$ is on the line L_0. We conclude that ρ meets the space $\langle \mathcal{B}(L_0) \rangle$ in a plane.

Now ρ is scattered: suppose that there is a spread element $\mathcal{B}(\pi)$ meeting ρ in a subspace

π of dimension at least one, then every line through $\mathcal{B}(\pi)$ would contain q^2+1 points of $\tilde{S}$, a contradiction. As seen in Lemma 5, the scattered linear set ρ of rank 6 defines a pseudoregulus in $\mathrm{PG}(3,q^3)$ and the lines of $\mathcal{S}$ are the (q^2+q+1)-secants to $\mathcal{B}(\rho)$, hence, $\mathcal{S}$ is the associated pseudoregulus. □

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