

## Scattering Amplitudes in Gauge Theory and Gravity

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Providing a comprehensive, pedagogical introduction to scattering amplitudes in gauge theory and gravity, this book is ideal for graduate students and researchers. It offers a smooth transition from basic knowledge of quantum field theory to the frontier of modern research.

Building on basic quantum field theory, the book starts with an introduction to the spinor helicity formalism in the context of Feynman rules for tree-level amplitudes. The material covered includes on-shell recursion relations, superamplitudes, symmetries of  $N = 4$  super Yang–Mills theory, twistors and momentum twistors, Grassmannians, and polytopes. The presentation also covers amplitudes in perturbative supergravity, 3d Chern–Simons matter theories, and color-kinematics duality and its connection to “gravity = (gauge theory)<sup>2</sup>.”

Basic knowledge of Feynman rules in scalar field theory and quantum electrodynamics is assumed, but all other tools are introduced as needed. Worked examples demonstrate the techniques discussed, and over 150 exercises help readers absorb and master the material.

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“In recent years, a series of surprising insights and new methods have transformed the understanding of gauge and gravitational scattering amplitudes. These advances are important both for practical calculations in particle physics, and for the fundamental structure of relativistic quantum theory. Elvang and Huang have written the first comprehensive text on this subject, and their clear and pedagogical approach will make these new ideas accessible to a wide range of students.”

Joseph Polchinski, *University of California, Santa Barbara*

“This book provides a much needed text covering modern techniques that have given radical new insights into the structure of quantum field theory. It gathers together a very large body of recent literature and presents it in a coherent style. The book should appeal to the wide body of researchers who wish to use quantum field theory as a tool for describing physical phenomena or who are intending to gain insight by studying its mathematical structure.”

Michael B. Green, *University of Cambridge*

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**For mom, dad, and Coco.**  
Yu-tin

**Thank you, Mormor.**  
Henriette

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## Preface

This book grew out of a need to have a set of easily accessible notes that introduced the basic techniques used in modern research on scattering amplitudes. In addition to the key tools, such a review should collect some of the small results and intuitions the authors had acquired from their work in the field and which had not previously been exposed in the literature. As the authors quickly realized, such an introduction would bring the reader only part of the way towards some of the most exciting topics in the field, so they decided to add “a little extra” material. While doing so – and this took quite a while – the authors remained in full and complete denial about writing a book. It was only at the end of the process that they faced their worst fears: the review was becoming a book. You now hold the result in your hands. Because the authors were not writing a book, they actually thoroughly enjoyed the work. Their hope is that you will enjoy it too and that you will find it useful.

It is a pleasure to thank our friends and collaborators who have worked with us and helped us learn the subject of scattering amplitudes: Ratin Akhoury, Nima Arkani-Hamed, Zvi Bern, Freddy Cachazo, John Joseph Carrasco, Simon Caron-Huot, Tim Cohen, Scott Davies, Tristan Dennen, Lance Dixon, Dan Freedman, David Kosower, Johannes Henn, Harald Ita, Henrik Johansson, Michael Kiermaier, Sangmin Lee, Arthur Lipstein, Thomas Lam, David McGady, Timothy Olson, Cheng Peng, Jan Plefka, Radu Roiban, Mark Srednicki, Warren Siegel, David Speyer, and Jaroslav Trnka. H.E. is grateful for the hospitality offered by Stanford/SLAC during her visit in February/March 2013 and KITP/UCSB during January–March 2014.

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Yu-tin Huang and Henriette Elvang