

# Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $\mathcal{O}(G^4)$

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We complete the calculation of conservative two-body scattering dynamics at fourth post-Minkowskian order, i.e.,  $\mathcal{O}(G^4)$  and all orders in velocity, including radiative contributions corresponding to the tail effect in general relativity. As in previous calculations, we harness powerful tools from the modern scattering amplitudes program including generalized unitarity, the double copy, and advanced multiloop integration methods, in combination with effective field theory. The classical amplitude involves complete elliptic integrals, and polylogarithms with up to transcendental weight 2. Using the amplitude-action relation, we obtain the radial action directly from the amplitude, and match the known overlapping terms in the post-Newtonian expansion.

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*Introduction.*—Gravitational-wave science has opened up a new direction in theoretical high energy physics: leveraging advances in quantum field theory (QFT) to develop new tools for the state-of-the-art prediction of gravitational-wave signals. This era of ever-increasing precision holds the promise of new and unexpected discoveries in astronomy, cosmology, and particle physics [1,2], but hinges crucially on complementary advances in our theoretical modeling of binary sources. In recent years, a new program for understanding the nature of gravitational-wave sources based on tools from scattering amplitudes and effective field theory (EFT) has emerged.

The central rationale of the new QFT-based approach is to derive classical binary dynamics from scattering amplitudes in order to take full advantage of relativity, on-shell methods [3], double-copy relations between gauge and gravity theories [4,5], advanced multiloop integration [6–9], and EFT methods [10–12]. These are the engines that drive modern scattering amplitude calculations in particle theory, but had not been fully integrated together and applied for gravitational-wave signals. This program is rooted in the well-known connection of scattering amplitudes to general relativity corrections to Newton’s potential [11,13,14], and in

the pioneering application of EFT to gravitational-wave physics [15]. It also complements traditional approaches to binary dynamics such as effective one-body [16], numerical relativity [17], gravitational self-force [18,19], and perturbation theory in the post-Newtonian (PN) [20–25], post-Minkowskian (PM) [26,27], and nonrelativistic general relativity [15,28] frameworks.

Scattering amplitude methods for determining binary dynamics from potential gravitons have been used in Refs. [29–31] to derive state-of-the-art results, which have been confirmed in multiple studies [24,25,32–36]. Until now, radiative effects at all orders in velocity have only been considered at  $\mathcal{O}(G^3)$ , where the radiation reaction is purely dissipative and can be cleanly separated and computed [37–42]. Radiative effects have new features at  $\mathcal{O}(G^4)$ , such as conservative contributions arising from two radiation gravitons known as the tail effect [43,44], as well as nonlinear dissipative effects that mix with time-reversal invariant conservative effects [42]. In this Letter we extend amplitude methods to include all conservative contributions to two-body scattering dynamics.

Our complete result for the conservative contributions ultimately follows from the same condition used to determine the potential graviton contributions [29–31]. Thus, while extracting conservative dynamics in the presence of radiation involves additional diagrams and expanding integrals in a new region, it is conceptually no more difficult. Remarkably, at this order in the  $G$  expansion, the result obtained with the standard Feynman  $i\epsilon$  prescription is consistent with using Wheeler-Feynman time-symmetric

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graviton propagators, which is the definition of conservative dynamics advocated in Ref. [42]. (For a thorough discussion of propagator prescriptions see Ref. [45].)

The classical amplitude presented in Eq. (3) below includes both the radiative contributions found here and the potential contributions from our previous work [31]. It determines the complete conservative two-body scattering dynamics at  $\mathcal{O}(G^4)$  and all orders in velocity. Using the amplitude-action relation introduced in Ref. [31] (see also Refs. [46]), we determine the radial action in Eq. (5). As usual, overlapping results from different techniques help guide new calculations and provide explicit benchmarks. Our result for the  $\mathcal{O}(G^4)$  scattering angle agrees with the sixth PN order result of Ref. [42]. We also agree with the fifth PN order result of Ref. [47] up to a single term of higher order in the self-force expansion, whose origin requires further study.

*Conservative radiative effects.*—The four-point amplitude  $\mathcal{M}$  of gravitationally interacting minimally coupled massive scalars encodes conservative two-body scattering dynamics in the classical limit. To compute the amplitude, we follow textbook QFT rules, such as the use of the Feynman  $i\epsilon$  prescription for all propagators. Our task is then to impose conditions that project out quantum and dissipative parts, leaving only the contributions to the full classical conservative dynamics.

We take the classical limit of  $\mathcal{M}$  by expanding in large angular momentum  $J \gg \hbar$  [12,29,30]. This amounts to rescaling the momentum transfer  $q$  and all graviton loop momenta  $\ell$  as  $q, \ell \rightarrow \lambda q, \lambda \ell$  and then expanding in small  $\lambda$ . The classical expansion is thus equivalent to an expansion in the soft region, defined by the loop-momentum scaling  $\ell = (\omega, \boldsymbol{\ell}) \sim (\lambda, \lambda)$ .

To define conservative dynamics in the presence of radiative effects, we evaluate integrals by picking up positive-energy residues of matter propagators [12,29–31]. A key implication of this is that we only need to consider cuts that have at least one on-shell matter line per loop. This is the same condition that identifies contributions from potential gravitons, and vastly simplifies the analysis. The relevant cuts are shown in Fig. 1.

Notably, the setup described above assumes the standard Feynman  $i\epsilon$  prescription for graviton propagators, but is consistent with using a principal-value (PV) prescription for the definition of conservative dynamics [42], corresponding to a time-symmetric propagator [48,49],

$$G^{\text{PV}}(k) = \mathcal{P} \frac{1}{k^2}, \quad (1)$$

where  $\mathcal{P}$  denotes the principal value. That is, for the part of the three-loop amplitude determined by these cuts, the PV prescription gives the same result as using the standard Feynman  $i\epsilon$  prescription for graviton propagators and then taking the real part of the final classical amplitude. Note

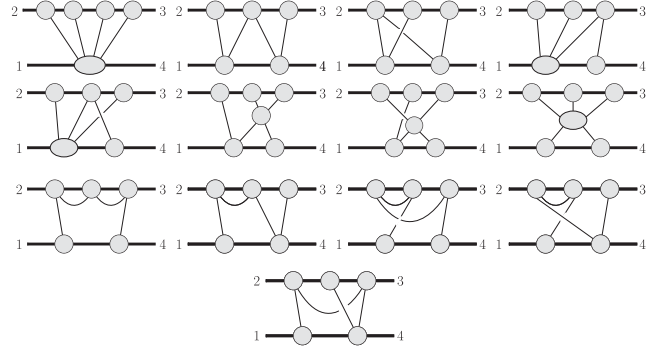


FIG. 1. Complete set of generalized unitarity cuts for conservative contributions at  $\mathcal{O}(G^4)$ . Exposed lines are on shell. Thick lines represent massive scalars and thin lines are gravitons. The first eight cuts contain both  $(ppp)$  and  $(prr)$  contributions, while the last five, which have gravitons starting and ending on the same matter line, have no  $(ppp)$  contributions.

that the prescription in Eq. (1) requires some care when applied to multiple propagators [50].

We use the method of regions [51], separating the soft region into the potential ( $p$ ) and radiation ( $r$ ) subregions defined by the following scalings of the loop momenta  $\ell$ :

$$(p): \ell \sim (v\lambda, \lambda), \quad (r): \ell \sim (v\lambda, v\lambda). \quad (2)$$

Here  $v$  denotes the typical velocity of the binary constituents, corresponding to the small velocity that defines the PN expansion. The classical contribution to any integral is obtained by expanding the three independent loop momenta  $\ell_1, \ell_2$ , and  $\ell_3$  in these regions:  $(\ell_1 \ell_2 \ell_3) \sim (ppp)$ ,  $(ppr)$ ,  $(prr)$ , and  $(rrr)$ . Note that  $\ell_1, \ell_2$ , and  $\ell_3$  are not the momenta of the three gravitons at the top of the middle diagram in Fig. 2. Of these combinations, only  $(ppp)$  and  $(prr)$  are relevant for conservative dynamics. The  $(rrr)$  contribution only has radiation gravitons and no potential gravitons, and thus leads to scaleless integrals. The  $(ppr)$  contribution yields odd-in- $v$  contributions, which capture dissipative effects and are thus removed by the PV prescription (1). Effects from the square of radiation reaction, which are dissipative but even in  $v$ , are also removed by the PV prescription. The contributions from the potential region  $(ppp)$  have been computed in our previous work [31] and confirmed in Refs. [25,36]. In this Letter we focus on the remaining conservative contribution, originating from the  $(prr)$  region.

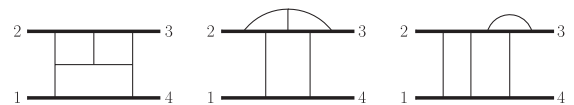


FIG. 2. Sample diagrams at  $\mathcal{O}(G^4)$ . From left to right: a diagram with both  $(ppp)$  and  $(prr)$  contributions, a diagram with only  $(prr)$  contributions, a diagram containing iteration terms that cancel in the total amplitude.

*Constructing the integrand.*—As in our previous work [29–31], we use generalized unitarity and the  $D$ -dimensional tree-level BCJ double copy to construct an integrand that captures the complete conservative dynamics. We sew together tree-level amplitudes in such a way that terms in the physical state projectors that depend on light-cone reference momenta drop out automatically [52]. The relevant radiative contributions come from generalized unitarity cuts containing one Compton tree amplitude, and two connected five-point tree amplitudes, each with one cut graviton line to separate the two matter particles. The subset of these cuts containing gravitons beginning and ending on the same matter line contribute only to  $(prr)$ , while the rest contribute to both  $(ppp)$  and  $(prr)$ . As before, graphs with graviton bubbles and matter contact interactions do not contribute. The relevant cuts including both radiative and potential contributions are shown in Fig. 1; others are obtained by relabeling the external momenta.

The resulting integrand is organized in terms of 93 distinct cubic-vertex Feynman-like diagrams, of which three are shown in Fig. 2. Upon expansion in the soft region, the propagators of certain diagrams, such as the second and third in Fig. 2, become linearly dependent. This is a feature observed in the method of regions [51], and described in Ref. [41] in the context of the soft expansion employed here. The products of independent propagators resulting from partial fraction decomposition can be assigned, upon multiplication and division by suitable propagators, to 51 distinct graphs with only cubic vertices.

In contrast to calculations at lower orders, the integrand at  $\mathcal{O}(G^4)$  depends on whether we use four- or  $D$ -dimensional states on cut lines. This is presumably related to subtleties associated with nonconventional forms of dimensional regularization (see, e.g., Ref. [53]). To avoid such issues we use conventional dimensional regularization where all states and momenta are uniformly continued to  $D = 4 - 2\epsilon$  dimensions.

*Evaluating the integrals.*—The resulting integrals are reduced to a basis of master integrals using integration-by-parts (IBP) identities [6] and graph symmetries. An important insight is that IBP relations are agnostic to the choice of graviton propagator prescriptions. This allows us to make use of automated tools such as FIRE6 [8] and KIRA [54], which are usually applied to cases with Feynman propagators, to reduce integrals with the prescription (1).

We evaluate the master integrals using the method of differential equations [7,9] with boundary conditions determined either in the  $(ppp)$  or  $(prr)$  region, and using the PV prescription (1). Since they are disjoint, they can be treated separately, and the complete system of linear differential equations splits into two partly overlapping sectors each with nonvanishing boundary conditions in one of the two regions. The overlap is given by integrals with nontrivial boundary conditions in both regions, which are those captured by the first eight cuts in Fig. 1.

The system of differential equations was solved in the  $(ppp)$  region in Ref. [31] in terms of classical polylogarithms up to weight three and complete elliptic integrals. The system for the integrals in the  $(prr)$  region can be brought to canonical form [55], which allows us to evaluate the integrals to arbitrary order in the dimensional regulator  $\epsilon = (4 - D)/2$ . The canonical form also implies that elliptic integrals are absent in the  $(prr)$  region. We have evaluated the integrals up to transcendental weight three. Note that due to the requirement of one on-shell matter line per loop, each of the  $L$  loops introduces a factor of  $\pi$ , and therefore the maximal weight is  $L$  instead of  $2L$  for a general  $L$ -loop amplitude [56]. We have also verified that our integration method is consistent with the nonrelativistic integration approach [12,29,30].

*Amplitude.*—The final result for the  $\mathcal{O}(G^4)$  conservative scattering amplitude in the classical limit including all contributions is

$$\begin{aligned}
 \mathcal{M}_4^{\text{cons}} &= G^4 M^7 \nu^2 |\mathbf{q}| \pi^2 \left\{ \mathcal{M}_4^p + \nu \left[ 4\mathcal{M}_4^t \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right] \right\} + \int_{\epsilon} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\epsilon} \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\epsilon} \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1} + \int_{\epsilon} \frac{\tilde{I}_{r,2}^2}{Z_1}, \\
 \mathcal{M}_4^p &= -\frac{35(1 - 18\sigma^2 + 33\sigma^4)}{8(\sigma^2 - 1)}, \\
 \mathcal{M}_4^t &= r_1 + r_2 \log\left(\frac{\sigma + 1}{2}\right) + r_3 \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}, \\
 \mathcal{M}_4^{\pi^2} &= r_4 \pi^2 + r_5 \mathbf{K}\left(\frac{\sigma - 1}{\sigma + 1}\right) \mathbf{E}\left(\frac{\sigma - 1}{\sigma + 1}\right) + r_6 \mathbf{K}^2\left(\frac{\sigma - 1}{\sigma + 1}\right) + r_7 \mathbf{E}^2\left(\frac{\sigma - 1}{\sigma + 1}\right), \\
 \mathcal{M}_4^{\text{rem}} &= r_8 + r_9 \log\left(\frac{\sigma + 1}{2}\right) + r_{10} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\frac{\sigma + 1}{2}\right) + r_{13} \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \log\left(\frac{\sigma + 1}{2}\right) + r_{14} \frac{\text{arccosh}^2(\sigma)}{\sigma^2 - 1} \\
 &\quad + r_{15} \text{Li}_2\left(\frac{1 - \sigma}{2}\right) + r_{16} \text{Li}_2\left(\frac{1 - \sigma}{1 + \sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2 - 1}} \left[ \text{Li}_2\left(-\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) - \text{Li}_2\left(\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) \right]. \tag{3}
 \end{aligned}$$

This result is for the scattering of two scalars with incoming four-momenta  $p_1$  and  $p_2$  and masses  $m_1$  and  $m_2$ . Here  $\mathbf{q}$  is the three-momentum transfer in the center of mass (c.m.) frame,  $M = m_1 + m_2$  is the total mass,  $\nu = m_1 m_2 / M^2$  is the symmetric mass ratio,  $\sigma = \mathbf{p}_1 \cdot \mathbf{p}_2 / m_1 m_2$  in the mostly minus signature, and  $p_\infty = \sqrt{\sigma^2 - 1}$ . For later convenience, we also introduce the three-momentum  $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$  in the c.m. frame, individual energies  $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$ , the total energy  $E = E_1 + E_2$ , and the symmetric energy ratio  $\xi = E_1 E_2 / E^2$ .

The contribution in the probe limit is given by  $\mathcal{M}_4^p$ , while the  $\mathcal{O}(\nu)$  contribution, i.e., first order in the self-force expansion, consists of the logarithmic tail coefficient  $\mathcal{M}_4^t$ , the potential  $\pi^2$  contribution  $\mathcal{M}_4^{\pi^2}$ , and the remainder  $\mathcal{M}_4^{\text{rem}}$ . There is no contribution of  $\mathcal{O}(\nu^2)$ , i.e., second order in the self-force expansion. The absence of such terms in our formalism is an immediate consequence of expanding in the soft region and dimensional analysis; for instance, the presence of  $\mathcal{O}(\nu^2)$  terms would require rational factors of  $1/(m_1 + m_2)$  which cannot appear. Note that this is in line with the so-called ‘‘mass polynomiality rule’’ [34,57], which imposes strong constraints on conservative dynamics [42].

Note that the basis of transcendental functions has been slightly rearranged with respect to Ref. [31], and that the rational coefficients, given in Table I, have simplified upon combining the  $(ppp)$  and  $(prr)$  contributions. This exposes

TABLE I. Functions specifying the amplitude in Eq. (3).

$r_1 = (1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6)/12(\sigma^2 - 1)$
$r_2 = 4g_1 - 2g_2 - \frac{1}{2}g_3$
$r_3 = (\sigma(-3 + 2\sigma^2)/4(\sigma^2 - 1))g_3$
$r_4 = -(16/3)g_1 + (4/3)g_2 - (1/3)g_4$
$r_5 = -(1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3)/2(\sigma^2 - 1)$
$r_6 = (834 + 2095\sigma + 1200\sigma^2)/2(\sigma^2 - 1)$
$r_7 = 7(169 + 380\sigma^2)/4(\sigma - 1)$
$r_8 = (1/72(\sigma^2 - 1)^2\sigma^7)(3600\sigma^{16} + 4320\sigma^{15} - 35360\sigma^{14} + 33249\sigma^{13} + 27952\sigma^{12} - 25145\sigma^{11} - 15056\sigma^{10} - 32177\sigma^9 + 64424\sigma^8 - 38135\sigma^7 + 13349\sigma^6 - 1471\sigma^4 + 207\sigma^2 - 45)$
$r_9 = -2(75\sigma^7 - 416\sigma^5 - 612\sigma^4 - 739\sigma^3 - 136\sigma^2 - 2520\sigma - 152)/3(\sigma^2 - 1)$
$r_{10} = -(\sigma(-3 + 2\sigma^2)/(\sigma^2 - 1))2r_1 + 28\sigma^2(-3 + 2\sigma^2)$
$r_{11} = (4\sigma/3(\sigma^2 - 1))(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6)$
$r_{12} = 6g_2 + g_3 - \frac{1}{2}g_4$
$r_{13} = -8(\sigma(-3 + 2\sigma^2)/(\sigma^2 - 1))g_1$
$r_{14} = -(\sigma^2(-3 + 2\sigma^2)^2/4(\sigma^2 - 1)^2)g_3$
$r_{15} = -16g_1 - 4g_2 - g_4$
$r_{16} = -2g_4$
$r_{17} = -(\sigma(-3 + 2\sigma^2)/(\sigma^2 - 1))(8g_1 - 4g_2)$
$g_1 = 2 + 15\sigma^2$
$g_2 = 19\sigma + 15\sigma^3$
$g_3 = 11 - 30\sigma^2 + 35\sigma^4$
$g_4 = -20 - 111\sigma^2 - 30\sigma^4 + 25\sigma^6$

additional structures; for example the coefficient of  $\text{arccosh}^2 \sigma$  is closely related to that of  $\text{arccosh} \sigma$  in  $\mathcal{M}_4^t$ , suggesting that this expression exponentiates. Furthermore, the coefficients of the weight two functions are expressible in terms of the polynomial functions  $g_1, g_2, g_3$ , and  $g_4$ . Note that the function  $g_3$  is related to the leading contribution of the electric Weyl squared tidal operator to two-body scattering [58].

The remaining integrals in Eq. (3) are contributions from iteration of lower-order amplitudes. At  $\mathcal{O}(G^4)$ , these arise only from the  $(ppp)$  region, and are thus the same as in our previous work [31]. Iteration terms in  $(prr)$ , such as from the last diagram in Fig. 2, cancel in the final amplitude. Similarly, terms of transcendental weight three appear in intermediate steps but cancel nontrivially in the final amplitude. Note also that the PV prescription eliminates the imaginary part of the amplitude, which is proportional to  $\mathcal{M}_4^t$ .

As for the  $\mathcal{O}(G^3)$  case, the result in Eq. (3) does not smoothly match onto the massless case. Taking the high energy limit  $|\mathbf{p}| \rightarrow \infty$ , we find a leading power discontinuity in the classical part of the amplitude

$$\mathcal{M}_4^{\text{cons}} = \frac{560\pi^2[2 - (2 \ln 2 + 1)^2](Gp^2)^4|q|}{M\nu} + \dots, \quad (4)$$

where the ellipses denote terms subleading in large  $|\mathbf{p}|$ . It would be interesting to study whether this mass singularity cancels with dissipative effect as it does at  $\mathcal{O}(G^3)$  [38].

*Radial action for hyperbolic orbits.*—Given the amplitude-action relation [31], it is straightforward to derive the radial action for hyperbolic orbits from Eq. (3) via an inverse two-dimensional Fourier transform. We find the radial action in the c.m. frame

$$I_{r,4}^{\text{hyp}} = -\frac{G^4 M^7 \nu^2 \pi p^2}{8EJ^3} \times \left\{ \mathcal{M}_4^p + \nu \left[ 4\mathcal{M}_4^t \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right] \right\}, \quad (5)$$

which inherits the simple mass dependence from the amplitude. The radial action (5) in *Mathematica* text format is given in the Supplemental Material [59]. As an explicit benchmark, let us consider the nonlogarithmic  $(prr)$  contributions to the radial action defined in Ref. [42] as  $\mathcal{M}_4^{\text{radgrav,f}} \equiv 14\mathcal{M}_4^t + \mathcal{M}_4^{\text{rem,prr}}$ , with  $\mathcal{M}_4^{\text{rem,prr}}$  given in the Supplemental Material [59]. Expanding this through the ninth PN order, we find

$$\begin{aligned} \mathcal{M}_4^{\text{radgrav,f}} = & \frac{12044}{75} p_\infty^2 + \frac{212077}{3675} p_\infty^4 + \frac{115917979}{793800} p_\infty^6 \\ & - \frac{9823091209}{76839840} p_\infty^8 + \frac{115240251793703}{1038874636800} p_\infty^{10} \\ & - \frac{411188753665637}{4155498547200} p_\infty^{12} + \dots, \end{aligned} \quad (6)$$

whose first three terms match the sixth PN order result in Eq. (6.20) of Ref. [42].

The scattering angle is given by  $\chi_4 = -\partial I_{r,4}^{\text{hyp}}/\partial J$ . After expanding in velocity, our result agrees with the sixth PN order result in Eq. (6.17) of Ref. [42]. In particular, we find that terms of second order in the self-force expansion are absent, implying the nontrivial cancellation among contributions from multipole moments in the world-line approach as discussed in Ref. [42]. This is expected since we adopt the same definition for conservative dynamics as in Ref. [42]. On the other hand, the result of Ref. [47] contains an  $\mathcal{O}(\nu^2)$  term [see Eq. (69) therein], which is also consistent with the computation of memory terms in Ref. [60]. Aside from the angle, another gauge-invariant observable is the time delay for hyperbolic orbits given by  $\partial I_{r,4}^{\text{hyp}}/\partial E$ .

*Local Hamiltonian for hyperbolic orbits.*—The presence of radiation leads to nonlocal-in-time contributions to classical dynamics [43]. Following previous analyses [12,30,31], a two-body Hamiltonian can be derived from the radial action in Eq. (5). This Hamiltonian is effectively local and captures the nonlocal-in-time Hamiltonian evaluated on the relevant orbits. For completeness, the effective local Hamiltonian valid for hyperbolic orbits is

$$H^{\text{hyp}} = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{r^n} c_n(\mathbf{p}^2), \quad (7)$$

where  $r$  is the distance between the bodies in the c.m. frame. We have checked that  $H^{\text{hyp}}$  reproduces the scattering angle. The lower-order coefficients  $c_1$ ,  $c_2$ , and  $c_3$  can be found in Eq. (10) of Ref. [29], while  $c_4 = c_4^{\text{hyp}}$  is

$$c_4^{\text{hyp}} = \frac{M^7 \nu^2}{4\xi E^2} \left\{ \mathcal{M}_4^p + \nu \left[ 4\mathcal{M}_4^l \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right] \right\} + \mathcal{D}^3 \left[ \frac{E^3 \xi^3}{3} c_1^4 \right] + \mathcal{D}^2 \left[ \left( \frac{E^3 \xi^3}{p^2} + \frac{E\xi(3\xi-1)}{2} \right) c_1^4 - 2E^2 \xi^2 c_1^2 c_2 \right] + \left( \mathcal{D} + \frac{1}{p^2} \right) \left[ E\xi(2c_1 c_3 + c_2^2) + \left( \frac{4\xi-1}{4E} + \frac{2E^3 \xi^3}{p^4} + \frac{E\xi(3\xi-1)}{p^2} \right) c_1^4 + \left( (1-3\xi) - \frac{4E^2 \xi^2}{p^2} \right) c_1^2 c_2 \right], \quad (8)$$

where  $\mathcal{D} = (d/d\mathbf{p}^2)$  denotes differentiation with respect to  $\mathbf{p}^2$ . The final explicit result for  $c_4^{\text{hyp}}$  is included in the ancillary file [59]. Again, the label “hyp” emphasizes that  $c_4^{\text{hyp}}$  applies only for hyperbolic orbits. Nonetheless, the potential contributions to  $\mathcal{M}_4^{\pi^2}$  and the coefficient of logarithm,  $\mathcal{M}_4^l$ , are local and should analytically continue between bound and unbound orbits [23,61].

*Conclusions.*—We extended amplitudes methods to include the full conservative contributions in classical two-body scattering at  $\mathcal{O}(G^4)$ . Compared to our previous work on pure potential contributions [31], this involves additional unitarity cuts and altered boundary conditions for the differential equations that determine the master integrals. An important next step will be to understand how to convert results for scattering in the presence of conservative radiative effects to ones for the bound-state problem [23,61]. It will also be interesting to study the impact of our results on waveform generation [62].

We expect that a setup similar to the one used at  $\mathcal{O}(G^3)$  [41] based on the formalism of Ref. [37] can be used to derive the dissipative radiation reaction contributions to the scattering angle at  $\mathcal{O}(G^4)$ . Perhaps the most exciting development is that we encountered no conceptual or technical obstructions to pushing PM scattering calculations to increasingly high orders and extracting from them the complete conservative scattering dynamics of two-body systems.

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