Scattering of light from quasi-homogeneous sources by quasi-homogeneous media

Taco D. Visser*

Department of Physics and Astronomy, University of Rochester, Rochester, New York, 14627

David G. Fischer

Microgravity Science Division, NASA Glenn Research Center, 21000 Brookpark Road, MS 110-3, Cleveland, Ohio 44135

Emil Wolf†

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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The field generated by scattering of light from a quasi-homogeneous source on a quasi-homogeneous, random medium is investigated. It is found that, within the accuracy of the first-order Born approximation, the far field satisfies two reciprocity relations (sometimes called uncertainty relations). One of them implies that the spectral density (or spectral intensity) is proportional to the convolution of the spectral density of the source and the spatial Fourier transform of the correlation coefficient of the scattering potential. The other implies that the spectral degree of coherence of the far field is proportional to the convolution of the correlation coefficient of the source and the spatial Fourier transform of the strength of the scattering potential. While the case we consider might seem restrictive, it is actually quite general. For instance, the quasi-homogeneous source model can be used to describe the generation of beams with different coherence properties and different angular spreads. In addition, the quasi-homogeneous scattering model adequately describes a wide class of turbulent media, including a stratified, turbulent atmosphere and confined plasmas. © 2006 Optical Society of America

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1. INTRODUCTION

A class of model random sources that play an important role in statistical optics are the so-called quasi-homogeneous sources (see, for example, Ref. 1, Sec. 5.2.2). They have the property that their spectral density (spectral intensity) $S_Q(\mathbf{r},\omega)$ at a particular frequency ω varies much more slowly with position than the correlation coefficient of the source $\mu_Q(\mathbf{r}_1,\mathbf{r}_2;\omega)\equiv\mu_Q(\mathbf{r}_2-\mathbf{r}_1;\omega)$ varies with $\mathbf{r}'=\mathbf{r}_2-\mathbf{r}_1$. Well-known members of this class are Lambertian sources (Ref. 1, Sec. 5.3.3). The far field generated by a quasi-homogeneous source satisfies two reciprocity relations. One implies that the angular distribution of the spectral density is proportional to the spatial Fourier transform of the spectral degree of coherence of the source. The other implies that the spectral degree of coherence of the far field is proportional to the spatial Fourier transform of the spectral density of the source.

The intimate relation that exists between radiation and scattering is well illustrated by so-called quasi-homogeneous, random scatterers. Such scatterers were introduced by Silverman (who called them locally homogeneous media).[2] Quasi-homogeneous media are characterized by the property that the strength of their scattering potential $\mathcal{S}_F(\mathbf{r},\omega)$ at a particular frequency ω varies much more slowly with position than the correlation coefficient $\eta_F(\mathbf{r}_1,\mathbf{r}_2;\omega)\equiv\eta_F(\mathbf{r}_2-\mathbf{r}_1;\omega)$ varies with the difference $\mathbf{r}'=\mathbf{r}_2-\mathbf{r}_1$. The troposphere, for example, is sometimes modeled as such a medium,[3] as are confined plasmas.[4] An analysis of the spectral changes produced by scattering from a quasi-homogeneous, anisotropic random medium was carried out in Ref. 5.

The far field generated by scattering of a plane monochromatic wave that is incident on a quasi-homogeneous, random medium is known to satisfy two reciprocity relations that are strictly analogous to those pertaining to the radiation from three-dimensional quasi-homogeneous sources. More specifically, it can be shown that, within the accuracy of the first-order Born approximation, the angular distribution of the spectral density of the far field is proportional to the spatial Fourier transform of the correlation coefficient of the scattering potential and that the spectral degree of coherence of the far field is proportional to the spatial Fourier transform of the strength of the scattering potential.[6] These two reciprocity relations were used to study certain inverse problems.[7] Because these reciprocity relations are less well known, and also to establish our notation, a short derivation of them is presented in Section 2. The significance of these results is illustrated by applying them, in Section 3, to the case of a plane monochromatic wave scattered by a Gaussian-correlated spherical medium.

The more general problem of scattering of light from a quasi-homogeneous source on a quasi-homogeneous medium does not appear to have been studied so far. In Section 4 we show that, within the accuracy of the first-order Born approximation, the scattered field satisfies two generalized reciprocity relations. One pertains to the spectral density of the far field; it connects the spectral density of the source and the correlation coefficient of the scattering potential. The other pertains to the spectral degree of coherence of the far field and relates the correlation coefficient of the source and the strength of the scattering potential.

2. SCATTERING ON A QUASI-HOMOGENEOUS, RANDOM POTENTIAL

Let us consider a monochromatic plane wave $V^{(i)}(\mathbf{r},t)$ of frequency ω and (possibly complex) amplitude a propagating in a direction specified by a unit vector $\mathbf{s}_0$, i.e.,

$$V^{(i)}(\mathbf{r},t) = U^{(i)}(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t), \tag{1}$$

where $\mathbf{r}$ is a position vector of a field point, t denotes the time, and

$$U^{(i)}(\mathbf{r},\omega) = a(\omega)\exp(\mathrm{i}k\mathbf{s}_0\cdot\mathbf{r}) \quad (\mathbf{s}_0^2=1), \tag{2}$$

with $k=\omega/c$, c being the speed of light in vacuum. Suppose that the wave is incident on a deterministic scatterer that occupies a finite domain D. The space-dependent part of the scattered field $U^{(s)}(\mathbf{r},\omega)$ is, within the accuracy of the first-order Born approximation, given by the expression (Ref. 8, Sec. 13.1.2)

$$U^{(s)}(\mathbf{r},\omega) = \int_D F(\mathbf{r}',\omega)U^{(i)}(\mathbf{r}',\omega)G(\mathbf{r},\mathbf{r}';\omega)\mathrm{d}^3r', \tag{3}$$

where

$$F(\mathbf{r},\omega) = \frac{k^2}{4\pi}[n^2(\mathbf{r},\omega)-1] \tag{4}$$

is the scattering potential, $n(\mathbf{r},\omega)$ being the refractive index of the medium, and

$$G(\mathbf{r},\mathbf{r}';\omega) = \frac{\exp(\mathrm{i}k|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{5}$$

is the outgoing free-space Green's function of the Helmholtz operator.

For a random scatterer the scattering potential is a random function of position. Let

$$C_F(\mathbf{r}_1',\mathbf{r}_2';\omega) = \langle F^*(\mathbf{r}_1',\omega)F(\mathbf{r}_2',\omega)\rangle \tag{6}$$

be its correlation function. The angle brackets denote the average, taken over an ensemble of realizations of the scattering potential. Because of the random nature of the scatterer, the scattered field will, of course, also be random. Its spatial coherence properties may be characterized by its cross-spectral density function (Ref. 1, Sec. 4.3.2)

$$W^{(s)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \langle U^{(s)*}(\mathbf{r}_1,\omega)U^{(s)}(\mathbf{r}_2,\omega)\rangle. \tag{7}$$

On substituting from Eqs. (3) and (6) into Eq. (7) and on interchanging the order of integration and ensemble averaging we obtain the formula

$$W^{(s)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \int_D\int_D W^{(i)}(\mathbf{r}_1',\mathbf{r}_2';\omega)C_F(\mathbf{r}_1',\mathbf{r}_2';\omega) \times G^*(\mathbf{r}_1,\mathbf{r}_1';\omega)G(\mathbf{r}_2,\mathbf{r}_2';\omega)\mathrm{d}^3r_1'\mathrm{d}^3r_2', \tag{8}$$

where $W^{(i)}$ denotes the cross-spectral density of the (monochromatic) incident plane wave, i.e.,

$$W^{(i)}(\mathbf{r}_1',\mathbf{r}_2';\omega) = U^{(i)*}(\mathbf{r}_1',\omega)U^{(i)}(\mathbf{r}_2',\omega) = I^{(i)}(\omega)\exp[\mathrm{i}k\mathbf{s}_0\cdot(\mathbf{r}_2',-\mathbf{r}_1')], \tag{9}$$

with $I^{(i)}(\omega)=|a(\omega)|^2$. We choose the origin O of a Cartesian coordinate system in the region containing the scatterer and consider the field at a point $\mathbf{r}$ in the far zone. Setting $\mathbf{r}=r\mathbf{u}$, with $\mathbf{u}^2=1$, we have for the Green's function the well-known asymptotic approximation

$$G(r\mathbf{u},\mathbf{r}';\omega) \sim \frac{\exp(\mathrm{i}kr)}{r}\exp[-\mathrm{i}k\mathbf{u}\cdot\mathbf{r}'] \quad \text{as } kr\to\infty, \tag{10}$$

$\mathbf{u}$ being kept fixed. Let

$$\eta_F(\mathbf{r}_1',\mathbf{r}_2';\omega) = \frac{C_F(\mathbf{r}_1',\mathbf{r}_2';\omega)}{\sqrt{\mathcal{S}_F(\mathbf{r}_1',\omega)\mathcal{S}_F(\mathbf{r}_2',\omega)}}, \tag{11}$$

where

$$\mathcal{S}_F(\mathbf{r},\omega) = C_F(\mathbf{r},\mathbf{r};\omega). \tag{12}$$

Since, according to Eq. (6), $\mathcal{S}_F(\mathbf{r},\omega)=\langle F^*(\mathbf{r},\omega)F(\mathbf{r},\omega)\rangle$, we call $\mathcal{S}_F$ the strength of the scattering potential. The function η_F is the normalized correlation coefficient of the scattering potential. We assume that the scatterer is homogeneous in the sense that $\eta_F(\mathbf{r}_1',\mathbf{r}_2';\omega)$ depends on $\mathbf{r}_1'$, and $\mathbf{r}_2'$ only through the difference $\mathbf{r}_2'-\mathbf{r}_1'$, i.e.,

$$\eta_F(\mathbf{r}_1',\mathbf{r}_2';\omega) \equiv \eta_F(\mathbf{r}_2'-\mathbf{r}_1';\omega). \tag{13}$$

We also assume that the strength of the scattering potential varies so slowly with position that over the effective width of $|\eta_F|$ the function $\mathcal{S}_F(\mathbf{r},\omega)$ is essentially constant. Such a situation may be described by saying that $\mathcal{S}_F(\mathbf{r},\omega)$ is a slow function of $\mathbf{r}$ and that $\eta_F(\mathbf{r}';\omega)$ is a fast function of $\mathbf{r}'$. Evidently, over regions of the scatterer for which $|\eta_F(\mathbf{r}_2'-\mathbf{r}_1';\omega)|$ is appreciable, we may make the approximations

$$\mathcal{S}_F(\mathbf{r}_1',\omega) \approx \mathcal{S}_F(\mathbf{r}_2',\omega) \approx \mathcal{S}_F[(\mathbf{r}_1'+\mathbf{r}_2')/2,\omega]. \tag{14}$$

We then see from Eq. (11) that

$$C_F(\mathbf{r}_1',\mathbf{r}_2';\omega) = \sqrt{\mathcal{S}_F(\mathbf{r}_1',\omega)\mathcal{S}_F(\mathbf{r}_2',\omega)}\,\eta_F(\mathbf{r}_1',\mathbf{r}_2';\omega), \tag{15}$$

$$\approx \mathcal{S}_F[(\mathbf{r}_1'+\mathbf{r}_2')/2,\omega]\,\eta_F(\mathbf{r}_2'-\mathbf{r}_1';\omega). \tag{16}$$

Scatterers whose correlation functions have this form, with $\mathcal{S}_F$ being a slow function of its spatial argument and η_F being a fast function of its spatial argument, may be said to be quasi-homogeneous.

It will be useful to make the change of variables:

$$\mathbf{R}_S^- = \mathbf{r}_2' - \mathbf{r}_1', \quad \mathbf{R}_S^+ = (\mathbf{r}_1' + \mathbf{r}_2')/2. \tag{17}$$

On substituting from expressions (9), (10), and (16) into Eq. (8), we obtain for the cross-spectral density of the scattered field in the far zone the expression

$$W^{(s)}(r_1\mathbf{u}_1, r_2\mathbf{u}_2;\omega) \approx \Lambda(r_1,r_2;\omega)\tilde{\mathcal{S}}_F[k(\mathbf{u}_1-\mathbf{u}_2),\omega] \times \tilde{\eta}_F[k(\mathbf{s}_0-(\mathbf{u}_1+\mathbf{u}_2)/2);\omega] \quad (\mathbf{u}_1^2=\mathbf{u}_2^2=1). \tag{18}$$

Here

$$\Lambda(r_1,r_2;\omega) = I^{(i)}(\omega)\frac{\exp[\mathrm{i}k(r_2-r_1)]}{r_1r_2}, \tag{19}$$

and

$$\tilde{\mathcal{S}}_F[k\mathbf{u},\omega] = \int \mathcal{S}_F(\mathbf{R}_S^+,\omega)\exp(\mathrm{i}k\mathbf{R}_S^+\cdot\mathbf{u})\mathrm{d}^3R_S^+, \tag{20}$$

$$\tilde{\eta}_F[k\mathbf{u},\omega] = \int \eta_F(\mathbf{R}_S^-,\omega)\exp(\mathrm{i}k\mathbf{R}_S^+\cdot\mathbf{u})\mathrm{d}^3R_S^- \tag{21}$$

are the three-dimensional spatial Fourier transforms of $\mathcal{S}_F$ and η_F, respectively, the integrals extending formally over the whole space. The spectral degree of coherence (Ref. 1, Sec. 4.3.2) of the scattered field is defined by the formula

$$\mu^{(s)}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \frac{W^{(s)}(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S^{(s)}(\mathbf{r}_1,\omega)S^{(s)}(\mathbf{r}_2,\omega)}}, \tag{22}$$

and the spectral density of the field is, evidently, given by the expression

$$S^{(s)}(\mathbf{r},\omega) = W^{(s)}(\mathbf{r},\mathbf{r};\omega). \tag{23}$$

On substituting from expression (18) into Eqs. (22) and (23), we find that in the far zone

$$S^{(s)}(r\mathbf{u},\omega) = \frac{I^{(i)}(\omega)\tilde{\mathcal{S}}_F(0,\omega)}{r^2}\tilde{\eta}_F[k(\mathbf{s}_0-\mathbf{u}),\omega]; \tag{24}$$

and

$$\mu^{(s)}(r_1\mathbf{u}_1,r_2\mathbf{u}_2;\omega) = \frac{\tilde{\eta}_F[k(\mathbf{s}_0-(\mathbf{u}_1+\mathbf{u}_2)/2),\omega]\tilde{\mathcal{S}}_F[k(\mathbf{u}_1-\mathbf{u}_2),\omega]}{\sqrt{\tilde{\eta}_F[k(\mathbf{s}_0-\mathbf{u}_1),\omega]\tilde{\eta}_F[k(\mathbf{s}_0-\mathbf{u}_2),\omega]}} \times \frac{\exp[\mathrm{i}k(r_2-r_1)]}{\tilde{\mathcal{S}}_F(0,\omega)}. \tag{25}$$

Because η_F is a fast function of its spatial argument, it follows from well-known properties of Fourier transforms that $\tilde{\eta}_F$ is a slow function of $k\mathbf{u}$. Hence

$$\tilde{\eta}_F[k(\mathbf{s}_0-\mathbf{u}_1),\omega] \approx \tilde{\eta}_F[k(\mathbf{s}_0-\mathbf{u}_2),\omega] \approx \tilde{\eta}_F[k(\mathbf{s}_0-(\mathbf{u}_1+\mathbf{u}_2)/2),\omega]. \tag{26}$$

On making use of these approximations in Eq. (25) we obtain for the spectral degree of coherence of the far field the formula

$$\mu^{(s)}(r_1\mathbf{u}_1,r_2\mathbf{u}_2;\omega) = \frac{\tilde{\mathcal{S}}_F[k(\mathbf{u}_1-\mathbf{u}_2),\omega]}{\tilde{\mathcal{S}}_F(0,\omega)}\exp[\mathrm{i}k(r_2-r_1)]. \tag{27}$$

Equations (24) and (27) bring into evidence the following two reciprocity relations, valid within the accuracy of the first-order Born approximation, for scattering of a plane monochromatic wave on a quasi-homogeneous, random scattering potential:

1. The angular distribution of the spectral density of the scattered field in the far zone is proportional to the three-dimensional spatial Fourier transform of the normalized correlation coefficient of the scattering potential.
2. The spectral degree of coherence of the scattered field in the far zone is, apart from a simple geometrical phase factor, equal to the normalized three-dimensional spatial Fourier transform of the strength of the scattering potential.

The second result will be recognized as an analogue to the van Cittert–Zernike theorem for the field in the far zone generated by a spatially incoherent source (Ref. 1, Sec. 4.4.4). It is known that this theorem also holds for radiation from quasi-homogeneous, random sources (Ref. 1, Sec. 5.2.2).

The above two results can be used to study inverse problems pertaining to quasi-homogeneous, random media (cf. Ref. 7).

3. SPHERICAL, GAUSSIAN-CORRELATED SCATTERER

We will now apply the two reciprocity relations that we just derived to the scattering of a plane monochromatic wave on a nonuniform, quasi-homogeneous, Gaussian-correlated spherical scatterer with radius α. A similar, but somewhat less general, analysis dealing with the angular distribution of the intensity of the far field generated by scattering on a homogeneous random medium was presented in Ref. 9.

One has, in this case,

$$\eta_F(\mathbf{R}_S^-;\omega) = \exp[-(\mathbf{R}_S^-)^2/2\sigma_\eta^2], \tag{28}$$

$$\mathcal{S}_F(\mathbf{R}_S^+,\omega) = A\exp[-(\mathbf{R}_S^+)^2/2\sigma_S^2], \tag{29}$$

with A, σ_η, and σ_S positive constants and $\alpha \gg \sigma_S \gg \sigma_\eta$. The Fourier transforms of these two expressions are

$$\tilde{\eta}_F(k\mathbf{u},\omega) \approx (\sigma_\eta\sqrt{2\pi})^3\exp[-(k\sigma_\eta\mathbf{u})^2/2], \tag{30}$$

$$\tilde{\mathcal{S}}_F(k\mathbf{u},\omega) \approx A(\sigma_S\sqrt{2\pi})^3\exp[-(k\sigma_S\mathbf{u})^2/2]. \tag{31}$$

On substituting from expressions (30) and (31) into Eq. (24), one finds that

$$S^{(s)}(r\mathbf{u},\omega) = \frac{(\sigma_\eta \sigma_S 2\pi)^3 A I^{(i)}(\omega)}{r^2} \exp[-2k^2\sigma_\eta^2 \sin^2(\theta/2)], \tag{32}$$

where θ denotes the angle between the direction of incidence $\mathbf{s}_0$ and the direction of scattering $\mathbf{u}$, i.e., $\mathbf{s}_0 \cdot \mathbf{u} = \cos\theta$ (see Fig. 1). A striking example of the correspondence between scattering and radiation is evident when one compares Eq. (32) with Eq. (5.2–45) of Ref. 1. The latter deals with the spectral density of the far field radiated by a three-dimensional, Gaussian-correlated, quasi-homogeneous source. It is seen that the spectral density in that case has the same functional dependence on the correlation length of the source (σ_g) as does the spectral density of the far field generated by scattering a plane monochromatic wave on a Gaussian-correlated, quasi-homogeneous sphere on the correlation length of the scattering potential (σ_η).

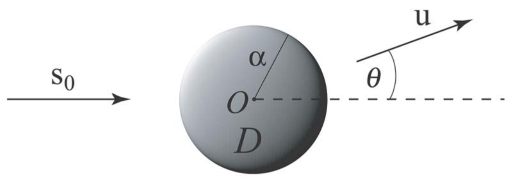

Fig. 1. Illustration of the notation.

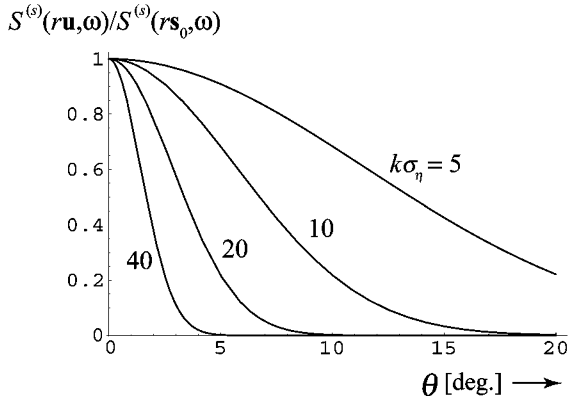

Fig. 2. Normalized spectral density $S^{(s)}(r\mathbf{u},\omega)/S^{(s)}(r\mathbf{s}_0,\omega)$ of the far field [Eq. (32)], as a function of the angle θ between the direction of incidence $\mathbf{s}_0$ and the direction of scattering $\mathbf{u}$, for selected values of the scaled correlation length $k\sigma_\eta$.

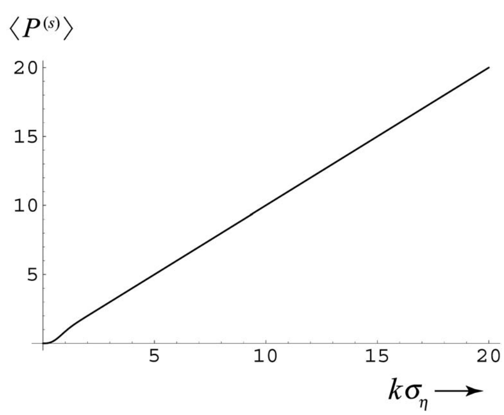

Fig. 3. (Color online) Normalized time-averaged total scattered power $\langle P^{(s)}\rangle k^3/AI^{(i)}(\omega)\sigma_S^3(2\pi)^4$, given by Eq. (34), as a function of the scaled correlation length $k\sigma_\eta$ of a Gaussian-correlated, random spherical scatterer.

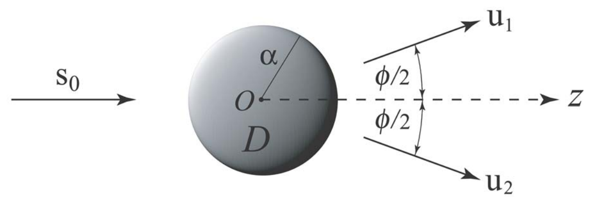

Fig. 4. Directions of observation $\mathbf{u}_1$ and $\mathbf{u}_2$, located symmetrically with respect to the direction of incidence $\mathbf{s}_0$.

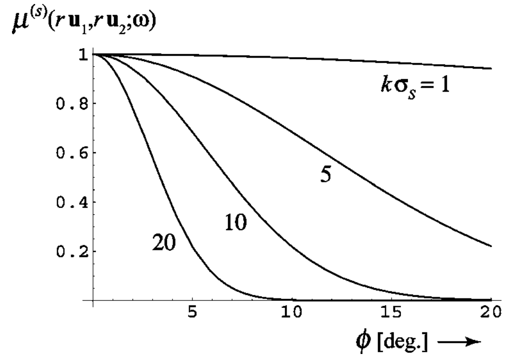

Fig. 5. (Color online) Spectral degree of coherence of the far field [Eq. (36)], for two symmetrically located directions of scattering, $\mathbf{u}_1$ and $\mathbf{u}_2$, as a function of the angle ϕ between the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, for selected values of the scaled effective radius $k\sigma_S$ of the sphere.

The behavior of the spectral density of the scattered field in the far zone, given by Eq. (32), is shown in Fig. 2 for selected values of the scaled correlation length $k\sigma_\eta$. It is seen that when the correlation length of the quasi-homogeneous scatterer increases the effective angular width of the spectral density of the far-zone field decreases, in agreement with the general considerations of Section 2.

The (time-averaged) total scattered power is given by the expression

$$\langle P^{(s)}\rangle = \int_0^\pi \int_0^{2\pi} S^{(s)}(r\mathbf{u},\omega) r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi, \tag{33}$$

$$= AI^{(i)}(\omega)\sigma_\eta^3\sigma_S^3(2\pi)^4 \left[\frac{1-\exp(-2k^2\sigma_\eta^2)}{k^2\sigma_\eta^2}\right]. \tag{34}$$

The behavior of this quantity, as calculated from Eq. (34), is shown in Fig. 3. It is seen that, within the validity of the first-order Born approximation, for a completely random, uncorrelated spherical scatterer ($k\sigma_\eta \to 0$), the averaged scattered power $\langle P^{(s)}\rangle \to 0$, i.e., no scattered power is generated. On increasing the correlation length, the total scattered power is seen to increase approximately linearly with $k\sigma_\eta$.

On substituting from expression (31) into Eq. (27), one obtains for the spectral degree of coherence of the far field the expression

$$\mu^{(s)}(r_1\mathbf{u}_1, r_2\mathbf{u}_2;\omega) = \exp[-k^2\sigma_S^2(\mathbf{u}_1-\mathbf{u}_2)^2/2]\exp[ik(r_2-r_1)]. \tag{35}$$

Without loss of generality we may take the z direction to be along the direction of incidence $\mathbf{s}_0$ and the x direction along the vector $\mathbf{u}_1-\mathbf{u}_2$. If we restrict our attention to pairs of observation points placed symmetrically along the direction of incidence, then $r\mathbf{u}_1=r(u_x,0,u_z)$, and $r\mathbf{u}_2=r(-u_x,0,u_z)$ (see Fig. 4). Formula (35) then takes on the simple form

$$\mu^{(s)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega) = \exp[-2k^2\sigma_S^2\sin^2(\phi/2)], \tag{36}$$

where ϕ is the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$. The behavior of $\mu^{(s)}(r\mathbf{u}_1, r\mathbf{u}_2;\omega)$ is shown in Fig. 5 for selected values of the scaled effective scatterer radius $k\sigma_S$. It is seen that when the effective radius of the scatterer increases the angular width of the spectral degree of coherence of the far-zone field decreases, in agreement with one of the general results of the previous section.

4. QUASI-HOMOGENEOUS SOURCE ILLUMINATING A QUASI-HOMOGENEOUS SCATTERER

The correlation properties of a three-dimensional random source Q may be characterized by its cross-spectral density function

$$W_Q(\mathbf{r}_1,\mathbf{r}_2;\omega) = \langle U_Q^*(\mathbf{r}_1,\omega)U_Q(\mathbf{r}_2,\omega)\rangle. \tag{37}$$

Here the brackets denote the average taken over an ensemble of realizations $U_Q(\mathbf{r},\omega)$ of the source distribution. The spectral density of the source is given by the diagonal element of its cross-spectral density, i.e.,

$$S_Q(\mathbf{r},\omega) = W_Q(\mathbf{r},\mathbf{r};\omega), \tag{38}$$

and its spectral degree of coherence is given by the expression

$$\mu_Q(\mathbf{r}_1,\mathbf{r}_2;\omega) = \frac{W_Q(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S_Q(\mathbf{r}_1,\omega)S_Q(\mathbf{r}_2,\omega)}}. \tag{39}$$

We will from now on not display the frequency dependence of the various quantities. As already mentioned, when a source is quasi-homogeneous its spectral degree of coherence depends on $\mathbf{r}_1$ and $\mathbf{r}_2$ only through the difference $\mathbf{r}_2-\mathbf{r}_1$, i.e.,

$$\mu_Q(\mathbf{r}_1,\mathbf{r}_2) \equiv \mu_Q(\mathbf{r}_2-\mathbf{r}_1); \tag{40}$$

and if, in addition, its spectral density varies so slowly with position that over the region where $|\mu_Q(\mathbf{r}_2-\mathbf{r}_1)|$ is appreciable $S_Q(\mathbf{r},\omega)$ is essentially constant. So one has, to a good approximation,

$$W_Q(\mathbf{r}_1,\mathbf{r}_2) \approx S_Q[(\mathbf{r}_1+\mathbf{r}_2)/2]\mu_Q(\mathbf{r}_2-\mathbf{r}_1). \tag{41}$$

If the field generated by a quasi-homogeneous source is incident on a scatterer that is located in the far zone of the source, the cross-spectral density of the incident field at the scatterer is given by the formula (Ref. 1, Sec. 4.4.5)

$$W^{(i)}(\mathbf{r}_1,\mathbf{r}_2) = \int\int W_Q(\mathbf{r}_1',\mathbf{r}_2')G^*(\mathbf{r}_1,\mathbf{r}_1')G(\mathbf{r}_2,\mathbf{r}_2')\mathrm{d}^3r_1'\mathrm{d}^3r_2', \tag{42}$$

with the integrals extending over the domain that is occupied by the source and the far-zone expression for the Green's function $G(\mathbf{r},\mathbf{r}')$ given by expression (10). On introducing new spatial variables

$$\mathbf{R}_Q^- = \mathbf{r}_2' - \mathbf{r}_1', \quad \mathbf{R}_Q^+ = (\mathbf{r}_1' + \mathbf{r}_2')/2 \tag{43}$$

and on substituting from expression (41) into Eq. (42), we obtain for the cross-spectral density function of the incident field the expression

$$\begin{aligned} W^{(i)}(\mathbf{r}_1,\mathbf{r}_2) \approx \frac{\exp[ik(r_2-r_1)]}{r_1r_2}\int\int & S_Q(\mathbf{R}_Q^+)\mu_Q(\mathbf{R}_Q^-) \\ &\times\exp[ik\mathbf{r}_1\cdot(\mathbf{R}_Q^+ - \mathbf{R}_Q^-/2)/r_1] \\ &\times\exp[-ik\mathbf{r}_2\cdot(\mathbf{R}_Q^+ + \mathbf{R}_Q^-/2)/r_2]\mathrm{d}^3R_Q^+\mathrm{d}^3R_Q^-, \end{aligned} \tag{44}$$

the integrals formally extending over the entire space. In the denominator of the factor in front of the integral we can make the approximation

$$r_1r_2 \approx R^2, \tag{45}$$

$R=|\mathbf{R}|$ being the distance from the origin (located in the source region) to the region of the scatterer (see Fig. 6). Further, each of the factors r_1 and r_2 appearing in the denominator of the two exponentials in the integrand of expression (44) may also be approximated by R. It is shown in Appendix A that this latter approximation introduces an error in the phase of the exponential functions, Δ_p, say, that is bounded in absolute value by

$$|\Delta_p| \leqslant \frac{kL_QL_S}{2R}, \tag{46}$$

with L_Q and L_S denoting the linear dimensions of the source and of the scatterer, respectively. It is seen from this inequality that $|\Delta_p|$ may be made arbitrarily small by taking the distance R between the source and the scattering medium large enough. Within the validity of the first-order Born approximation, the cross-spectral density of the scattered field in the far zone has the general form [see Eq. (8)]

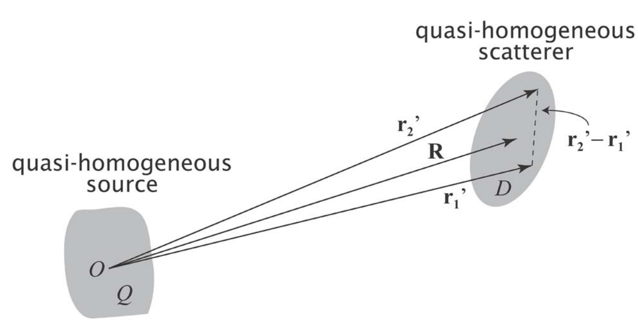

Fig. 6. Illustration of the notation.

$$W^{(\infty)}(\mathbf{r}_1,\mathbf{r}_2) = \int_D \int_D W^{(i)} \times(\mathbf{r}_1',\mathbf{r}_2') C_F(\mathbf{r}_1',\mathbf{r}_2') G^*(\mathbf{r}_1,\mathbf{r}_1') G(\mathbf{r}_2,\mathbf{r}_2') \mathrm{d}^3 r_1' \mathrm{d}^3 r_2', \tag{47}$$

with the integrations extending over the domain D occupied by the scatterer. On substituting from expression (44) into Eq. (47), it is advantageous to absorb the factor $\exp[\mathrm{i}k(r_2'-r_1')]$ into the function C_F. Accordingly, we define a modified correlation function:

$$\bar{C}_F(\mathbf{r}_1',\mathbf{r}_2') = C_F(\mathbf{r}_1',\mathbf{r}_2') \exp[\mathrm{i}k(r_2'-r_1')]. \tag{48}$$

It is seen from Eq. (12) that the strength of the scattering potential, $\mathcal{S}_F$, remains unchanged when this modification is made. It follows immediately from Eq. (11) that the modified correlation coefficient, $\bar{\eta}_F$, is given by the expression

$$\bar{\eta}_F(\mathbf{r}_1',\mathbf{r}_2') = \eta_F(\mathbf{r}_1',\mathbf{r}_2') \exp[\mathrm{i}k(r_2'-r_1')]. \tag{49}$$

The term $(r_2'-r_1')$ appearing in Eq. (49) may be approximated by [see Fig. 6 and Ref. 8, Sec. 8.8.1, Eq. (2)]

$$r_2' - r_1' \approx (\mathbf{r}_2' - \mathbf{r}_1') \cdot \mathbf{r}_2'/r_2' \approx (\mathbf{r}_2' - \mathbf{r}_1') \cdot \mathbf{R}/R. \tag{50}$$

The last step follows from the fact that the scatterer is located sufficiently far from the source. Since $|\bar{\eta}_F(\mathbf{r}_1',\mathbf{r}_2')| = |\eta_F(\mathbf{r}_1',\mathbf{r}_2')|$, it follows that $\bar{C}_F(\mathbf{r}_1',\mathbf{r}_2')$ can also be expressed as the product of a fast function and a slow function. Moreover, since $\eta = \eta(\mathbf{R}_S^-)$, it follows from expression (50) that $\bar{\eta} = \bar{\eta}(\mathbf{R}_S^-)$. Hence

$$\bar{C}_F(\mathbf{r}_1' - \mathbf{r}_2') = \bar{\eta}_F(\mathbf{R}_S^-) \mathcal{S}_F(\mathbf{R}_S^+), \tag{51}$$

with the variables $\mathbf{R}_S^+$ and $\mathbf{R}_S^-$ defined by Eqs. (17). We thus obtain for the cross-spectral density function of the far-zone scattered field the expression

$$W^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2) \approx \frac{\exp[\mathrm{i}k(r_2-r_1)]}{R^2 r_1 r_2} \int\int\int\int S_Q(\mathbf{R}_Q^+) \mu_Q(\mathbf{R}_Q^-) \mathcal{S}_F(\mathbf{R}_S^+) \times \bar{\eta}_F(\mathbf{R}_S^-) \exp(\mathrm{i}k\Phi) \mathrm{d}^3 R_Q^+ \mathrm{d}^3 R_Q^- \mathrm{d}^3 R_S^+ \mathrm{d}^3 R_S^-, \tag{52}$$

with the integrals extending over the entire space. Also, $\mathbf{s}_1^2 = \mathbf{s}_2^2 = 1$, and the phase Φ of the propagator term is given by the formula

$$\Phi = \mathbf{r}_1' \cdot (\mathbf{R}_Q^+ - \mathbf{R}_Q^-/2)/R - \mathbf{r}_2' \cdot (\mathbf{R}_Q^+ + \mathbf{R}_Q^-/2)/R - \mathbf{R}_S^+ \cdot (\mathbf{s}_2 - \mathbf{s}_1) - \mathbf{R}_S^- \cdot (\mathbf{s}_1 - \mathbf{s}_2)/2, \tag{53}$$

$$= - \mathbf{R}_S^+ \cdot [(\mathbf{s}_2 - \mathbf{s}_1) + \mathbf{R}_Q^-/R] - \mathbf{R}_S^- \cdot [(\mathbf{s}_1 + \mathbf{s}_2)/2 + \mathbf{R}_Q^+/R], \tag{54}$$

where we have used Eqs. (17). On carrying out the integrations over $\mathbf{R}_S^+$ and $\mathbf{R}_S^-$, we find that

$$W^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2) = \frac{\exp[\mathrm{i}k(r_2-r_1)]}{R^2 r_1 r_2} A(\mathbf{s}_2 - \mathbf{s}_1) B[(\mathbf{s}_1 + \mathbf{s}_2)/2], \tag{55}$$

with

$$A(\mathbf{s}_2 - \mathbf{s}_1) = \int \mu_Q(\mathbf{R}_Q^-) \tilde{S}_F\{-k[(\mathbf{s}_2 - \mathbf{s}_1) + \mathbf{R}_Q^-/R]\} \mathrm{d}^3 R_Q^-, \tag{56}$$

$$B[(\mathbf{s}_1 + \mathbf{s}_2)/2] = \int S_Q(\mathbf{R}_Q^+) \tilde{\eta}_F\{-k[(\mathbf{s}_1 + \mathbf{s}_2)/2 + \mathbf{R}_Q^+/R]\} \mathrm{d}^3 R_Q^+, \tag{57}$$

and

$$\tilde{S}_F(k\mathbf{u}) = \int S_F(\mathbf{R}_S^+) \exp(\mathrm{i}k\mathbf{R}_S^+ \cdot \mathbf{u}) \mathrm{d}^3 R_S^+, \tag{58}$$

$$\tilde{\eta}_F(k\mathbf{u}) = \int \bar{\eta}_F(\mathbf{R}_S^-) \exp(\mathrm{i}k\mathbf{R}_S^- \cdot \mathbf{u}) \mathrm{d}^3 R_S^-, \tag{59}$$

begin the three-dimensional spatial Fourier transforms of the strength of the scattering potential, S_F, and the modified correlation coefficient of the scattering potential, $\bar{\eta}_F$, respectively. We note that both functions A and B have the form of a (scaled) convolution. It follows from Eq. (55) that the spectral density of the far field is given by the formula

$$S^{(\infty)}(r\mathbf{s}) = W^{(\infty)}(r\mathbf{s}, r\mathbf{s}) = \frac{1}{R^2 r^2} A(0) B(\mathbf{s}). \tag{60}$$

On using Eqs. (22) and (55) we obtain for the spectral degree of coherence of the far field the expression

$$\mu^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2) = \frac{A(\mathbf{s}_2 - \mathbf{s}_1) B[(\mathbf{s}_1 + \mathbf{s}_2)/2]}{A(0)\sqrt{B(\mathbf{s}_1) B(\mathbf{s}_2)}} \exp[\mathrm{i}k(r_2 - r_1)]. \tag{61}$$

Since $\bar{\eta}_F$ is a fast function of $\mathbf{R}_S^-$, it follows that $\tilde{\eta}_F$ is a slow function of $k\mathbf{u}$. Hence

$$B(\mathbf{s}_1) \approx B(\mathbf{s}_2) \approx B[(\mathbf{s}_1 + \mathbf{s}_2)/2]. \tag{62}$$

On making use of these approximations in Eq. (61), we obtain for the spectral degree of coherence of the far field the expression

$$\mu^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2) = \frac{A(\mathbf{s}_2 - \mathbf{s}_1)}{A(0)} \exp[\mathrm{i}k(r_2 - r_1)]. \tag{63}$$

Equations (60) and (63) are generalized reciprocity relations that may be stated as follows:

1. The spectral density of the far field generated by radiation from a quasi-homogeneous source scattered on a quasi-homogeneous medium is proportional to the convolution of the spectral density of the source, S_Q, and the spatial Fourier transform of the modified correlation coefficient of the scatterer, $\tilde{\eta}_F$.

2. The spectral degree of coherence of the far field generated by radiation from a quasi-homogeneous source scattered on a quasi-homogeneous medium is, apart from a geometrical factor, given by the convolution of the correlation coefficient of the source, μ_Q, and the spatial Fourier transform of the strength of the scattering potential, $\tilde{S}_F$.

We mention in passing that the convolution of two Gaussian functions with widths w_1 and w_2 is again a Gaussian function with width $w_3=(w_1^2+w_2^2)^{1/2}$. Therefore, if, for example, both S_Q and $\tilde{\eta}_F$ are Gaussians, then the spectral density of the far-zone field will also be a Gaussian. This observation implies the existence of an equivalence theorem in which both S_Q and $\tilde{\eta}_F$ are altered, without affecting the spectral density of the far field. In other words, a suitably chosen trade-off between the width of the spectral density of the source and the width of the correlation length of the scattering potential will leave the spectral density of the far field unchanged.

5. CONCLUSIONS

We reviewed and extended various reciprocity relations relating to quasi-homogeneous sources and quasi-homogeneous scatterers. The usual reciprocity relations were applied to analyze the far-zone field generated by a monochromatic plane wave incident on a Gaussian-correlated, spherical scatterer. Further, we analyzed the more general case when light emitted by a quasi-homogeneous, random source is incident on a quasi-homogeneous, random medium. Two generalized reciprocity relations for the scattered field in the far zone were derived. These relations were found to have the form of convolutions of a function describing properties of the source and a function describing properties of the scatterer.

In recent years, there has been much interest in the scattering of partially coherent light by the turbulent atmosphere (see, for example, Refs. 10–12), and our results may find application in the analysis of problems of this kind.

APPENDIX A: DERIVATION OF AN APPROXIMATION RELATING TO EXPRESSION (44)

We will derive here the approximation that is applied in expression (44), viz.,

$$\exp[-\mathrm{i}k\mathbf{r}_2\cdot(\mathbf{R}_Q^+ + \mathbf{R}_Q^-/2)/r_2] = \exp[-\mathrm{i}k\mathbf{r}_2\cdot\mathbf{r}_2'/r_2] \approx \exp[-\mathrm{i}k\mathbf{r}_2\cdot\mathbf{r}_2'/R]. \tag{A1}$$

Making the approximation in expression (A1) introduces an error Δ_p in the phase of the exponential function, i.e.,

$$\Delta_p = -k\mathbf{r}_2\cdot\mathbf{r}_2'\left[\frac{1}{R}-\frac{1}{r_2}\right]. \tag{A2}$$

For the factors $\mathbf{r}_2$ and $\mathbf{r}_2'$ appearing in Eq. (A2), we have (see Fig. 6)

$$|\mathbf{r}_2| \approx R, \tag{A3}$$

$$|\mathbf{r}_2'| \leqslant L_Q, \tag{A4}$$

where L_Q denotes the linear dimension of the source. On making use of these two expressions in Eq. (A2), we find that

$$|\Delta_p| \leqslant kRL_Q\left|\frac{1}{R}-\frac{1}{r_2}\right|. \tag{A5}$$

Because the scatterer is in the far zone of the source, we have

$$R \gg L_S, \tag{A6}$$

where L_S denotes the linear dimension of the scatterer. Also (see Fig. 6),

$$R - L_S/2 \leqslant r_2 \leqslant R + L_S/2. \tag{A7}$$

Hence,

$$\left[\frac{1}{R}-\frac{1}{R\pm L_S/2}\right] = \frac{\pm L_S/2}{R^2 \pm RL_S/2} \approx \frac{\pm L_S}{2R^2}, \tag{A8}$$

and consequently

$$\left|\frac{1}{R}-\frac{1}{r_2}\right| \leqslant \frac{L_S}{2R^2}. \tag{A9}$$

On substituting from expression (A9) into expression (A5), we obtain the inequality

$$|\Delta_p| \leqslant \frac{kL_QL_S}{2R}, \tag{A10}$$

which is expression (46). One can derive the approximation of the other exponential in expression (44) in a strictly analogous manner.

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T. D. Visser's e-mail address is tvisser@nat.vu.nl.

*Permanent address. Department of Physics and Astronomy, Free University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands.

†Also with The Institute of Optics, University of Rochester, and College of Optics, CREOL, University of Central Florida, Orlando, Florida 32816.

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