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SCATTERING OF PULSAR RADIATION IN THE INTERSTELLAR MEDIUM

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SUMMARY

The scattering of pulsar radiation in the interstellar medium is investigated under the assumptions of both 'strong' and 'multiple' scattering, and a gaussian spatial autocorrelation function for the scattering irregularities. For a thin screen located midway between the pulsar and the observer it is shown that the frequency autocorrelation function for the resulting scintillations is the Fourier Transform of the pulse broadening function $\exp(-t/\Delta t)$ arising from multipath propagation. The width of the frequency auto-correlation function to half power Δf, is thus given by $2\pi\Delta f\Delta t = 1$, and this relationship is fulfilled approximately for all scattering screens. Relationships between Δf and the decorrelation frequencies used by different observers are established.

Measured values of Δf and Δt show significant deviations from the theoretical relationship $\Delta f \propto DM^{-2}$ where DM is the dispersion measure of the pulsar. These deviations can be explained by a combination of two effects-violations of the assumptions of strong and multiple scattering for nearby pulsars, and regions of anomalously strong scattering along the lines of sight to distant pulsars.

1. INTRODUCTION

Electron scattering in the interstellar medium is responsible for some of the observed properties of pulsar radiation (Scheuer 1968; Salpeter 1969; Rickett 1970; Lang 1971a). This paper is concerned with two of these properties, the pulse broadening Δt and the range of frequency over which intensity fluctuations are correlated Δf. Both Δf and Δt are strongly frequency dependent but the product $\Delta f\Delta t$ is approximately 1 if both quantities are measured at the same frequency. It is the purpose of this paper to compare the measurements of Δf made by different observers (allowing for their different definitions), to establish the relationship between Δf and Δt more precisely, and finally to use these results to investigate the relationship between Δf and the integrated electron content DM.

2. OBSERVATIONS OF PULSE BROADENING

Dramatic pulse broadening at low frequencies has been observed in a number of pulsars, particularly those with large dispersion measures. Pulses are extended on the trailing side and pulse widths vary as λ^4. The observed broadening is adequately described as convolution of the intrinsic pulse shape with a pulse broadening function which is a truncated exponential with width proportional to λ^4. In the case of PSR 0531+21 there is some disagreement concerning the shape of the pulse broadening function. Sutton, Staelin & Price (1971), from measurements of individual strong pulses, find it to be a truncated exponential. Rankin *et al.* (1970) using average pulse shapes and assuming that the intrinsic pulse shape is the same at all frequencies, find it to be of the form $x\,e^{-x}$. Recent measurements reveal marked spectral differences between the different pulse components (Manchester 1971; Rankin, Heiles & Comella 1971) and it is to be expected that these will modify the solution of Rankin *et al.* For the remainder of this paper it is assumed that the observed pulse broadening function is a truncated exponential.

The λ^4 broadening is not to be confused with the much weaker effects observed in PSR 0525+21 (Zeissig & Richards 1969), PSR 1133+16 (Craft & Comella 1968) and PSR 2045−16 (Komesaroff, Morris & Cooke 1970). In these cases the wavelength dependence is typically $\lambda^{0.2}$ and the changes in the separation of pulse components with wavelength show that the phenomenon is intrinsic to the pulsar.

Observed values of pulse broadening Δt, corresponding to a broadening function $\exp(-t/\Delta t)$ for $t>0$ and zero for $t<0$, are given in Table I. All tabulated

TABLE I

Pulse broadening at 318 MHz

Pulsar	Galactic latitude (degrees)	Dispersion measure (electrons pc cm^{-3})	Pulse broadening Δt at 318 MHz (m sec)	Reference
PSR 0531+21	−6	57	0·23	Staelin & Sutton (1970)
0823+26	32	19	0·005*	Craft (1970)
0833−45	−3	69	6	Ables *et al.* (1970)
1858+03	−1	402	187	Lang (1971b)
1919+21	4	12·4	0·006*	Bash *et al.* (1970)
1933+16	−2	159	1·0	Lang (1971b)
1946+35	5	129	32	Lang (1971b)
2003+31	0	225	10	Lang (1971b)

* Based on observations at 40 MHz. Scaling by λ^4 from 40 MHz to 318 MHz may introduce an appreciable error.

values are for a frequency of 318 MHz and, where necessary, were scaled to 318 MHz according to λ^4. Following the interpretation of Lang (1971b), Table I includes values of Δt for PSR 0823+26 and PSR 1919+21 derived from asymmetric pulse shapes measured by Craft (1970) and Bash, Boyzan & Torrence (1970) at 40 MHz. These values should be used with caution because of the large uncertainties involved in scaling from 40 MHz to 318 MHz, and because the 40 MHz pulse shapes could be due to a change in the intrinsic pulse shape at low frequencies.

3. OBSERVATIONS OF FINE FREQUENCY STRUCTURE IN PULSAR RADIATION

As described in the following section, the application of scattering theory to the interstellar medium predicts that the scintillation pattern observed at the Earth will contain fine frequency structure with a characteristic bandwidth which varies as $f^4 DM^{-2}$. There have been several sets of measurements designed to investigate this dependence on frequency f and dispersion measure DM. Attention will be restricted to measurements described by Rickett (1970), Lang (1971a), and Ewing *et al.* (1970). The earlier work of Huguenin, Taylor & Jura (1969) will not be included. Their quantitiative results are often in disagreement with those of the other observers, which is possibly due to inadequate calibration of their photographic technique.

Rickett observed with several bandpass filters of different widths, all centred on the same frequency, and measured the scintillation index (m) for the pulses received in each. m is defined as the RMS deviation from the mean pulse intensity, divided by the mean pulse intensity. He defines the half-visibility bandwidth B_h as the bandwidth at which m has fallen to half its value for zero bandwidth. B_h is determined by fitting a theoretical curve to the measured values of m, assuming a rectangular bandpass and gaussian autocorrelation function (a.c.f.) for the variations of intensity with frequency. The derived values of m for zero bandwidth are in the range 0·5 to 1·4, in reasonable agreement with the value of ~ 1 expected for strong scattering (Salpeter 1967). The measurements are primarily at 408 MHz, with some at 151 and 610 MHz.

Lang observed with identical narrow bandpasses at several closely spaced frequencies. He derives the cross-correlation coefficient between each pair of frequencies defined as

$$\Gamma_{12}(0) = \frac{\overline{\Delta I_1(t)\Delta I_2(t)}}{\text{RMS } \Delta I_1(t) \text{ RMS } \Delta I_2(t)}$$

where $\Delta I(t) = I(t) - \overline{I(t)}$ and the bar denotes a time average. $\Gamma_{11}(0)$ is always unity, regardless of the depth of modulation at zero bandwidth. This definition has the merit of giving the correct result even in cases where there is appreciable decorrelation across the bandpass, for all $\Delta I(t)$ are reduced by the same factor. In many cases $\Gamma_{12}(0)$ does not fall to zero for large frequency separations but reaches a constant value, p, which indicates correlated intensities between the different channels. Decorrelation frequency, f_ν, is defined as the frequency separation at which $\Gamma_{12}(0)$ has fallen to $0{\cdot}5\,(1+p)$. The measurements are at 318 and 111 MHz.

Ewing *et al.* (1970) observed with an autocorrelation receiver and several filter bank receivers, and display their data as contour diagrams of intensity as a function of frequency and time. The scintillation pattern is clearly recognizable and instantaneous widths in frequency between half-power intensity points B are measured for many features. Such a definition, which depends on the selection of features, is naturally more subjective than those of B_h and f_ν. B has been measured in this way at a number of frequencies between 112 MHz and 560 MHz.

Some indication of the relationship between B_h, f_ν and B can be obtained from the data for PSR 0329+54 at 408 MHz given by Rickett (1970). Using the set of bandpass filters he measured B_h = 800 KHz. He also presents data obtained with an autocorrelation receiver and displayed in a manner similar to that used by Ewing *et al.* The widths between half power points for strong features vary from 100 to 300 KHz, say typically 150 KHz. In a two-dimensional autocorrelation of this same data the correlation coefficient falls to 0·5 at a displacement of 100 KHz along the frequency axis. This is equivalent to Lang's f_ν. Assuming that the statistical properties of PSR 0329+54 were the same at the two times of observation, $f_\nu \approx B_h/8 \approx B/1{\cdot}5$.

There are several ways in which the measurements of Δf may be in error, apart from violation of the assumptions given at the beginning of Section 4.

(1) Faraday rotation in the interstellar medium can generate fine frequency structure in cases where the mean pulse shape is appreciably linearly polarized (Manchester 1970) and the observations are made with a linearly polarized antenna. Linear polarization was used by Ewing *et al.* for frequencies above 250 MHz and by Lang at 318 MHz. Rickett does not state the type of polarization used. For a typical rotation measure of 40 rad m^{-2} (Smith 1968; Staelin & Reifenstein 1969; Ekers *et al.* 1969), the features will simulate $B \sim 5$ MHz at 318 MHz. This is comparable with the larger values of B at 318 MHz, and it is possible that a few values of Δf will be affected.

(2) Variations of intensity which are intrinsic to the pulsar and relatively wideband will distort the measurements of B_h and f_ν. Such pulse to pulse variations have been observed (Lyne & Rickett 1968), but there are no data concerning variations on longer time scales. Rickett and Ewing *et al.* removed most of the pulse to pulse effects by using intensities averaged over a minute or so. Lang (private communication) did not average intensities prior to cross-correlation and, presumably, his non-zero value of Γ_{12} for large frequency separations is caused by intensity variations intrinsic to the pulsar. Their effect on Γ_{12} can be calculated by considering $I(t)$ as $x(t)y(t)$ where $x(t)$ is the intrinsic pulse intensity and $y(t)$ describes the scintillation superimposed by the interstellar medium. Writing $x(t) = \bar{x} + \epsilon x(t)$ and $y(t) = \bar{y} + \epsilon y(t)$, $\Delta I(t) = \bar{x}\epsilon y(t) + \epsilon x(t)\bar{y} + \epsilon x(t)\epsilon y(t)$: assuming that the intrinsic variations are wideband, i.e. $x_1(t) = x_2(t)$ and that the observing time is sufficiently long that $\bar{y}_1 = \bar{y}_2$, $\text{RMS}(\Delta I_1) = \text{RMS}(\Delta I_2)$ and crossproducts of uncorrelated terms can be ignored, then

$$\Gamma_{12} = \frac{\overline{\epsilon y_1 \epsilon y_2} + A}{\overline{(\epsilon y_1)^2} + A} \qquad \text{where} \qquad A = \frac{\overline{(\epsilon x)^2}(\bar{y})^2}{(\bar{x})^2 + \overline{(\epsilon x)^2}}.$$

In particular, for large frequency separations, $p = A/[\overline{(\epsilon y_1)^2} + A]$. The value of Γ_{12} which would occur in the absence of intrinsic variations is thus

$$\overline{\epsilon y_1 \epsilon y_2}/\overline{(\epsilon y_1)^2} = (\Gamma_{12} - p)/(1 - p).$$

This leads to Lang's empirical result that f_ν is the frequency separation at which the observed value of Γ_{12} has fallen to $0 \cdot 5\ (1 + p)$.

The value of p can be related to the modulation index for the variations intrinsic to the pulsar, $\alpha = \text{RMS}(\epsilon x)/\bar{x}$. If the modulation index associated with the interstellar medium alone is $\beta = \text{RMS}(\epsilon y)/\bar{y}$, the observed scintillation index m is given by

$$\begin{aligned} m^2 &= \overline{(\Delta I)^2}/(\bar{I})^2 \\ &= [(\bar{x})^2\overline{(\epsilon y)^2} + \overline{(\epsilon x)^2}(\bar{y})^2 + \overline{(\epsilon x)^2}\,\overline{(\epsilon y)^2}]/(\bar{x})^2(\bar{y})^2 \\ &= \alpha^2 + \beta^2 + \alpha^2\beta^2. \end{aligned}$$

Similarly, algebraic manipulation of the expressions for A and p leads to $p = \alpha^2/m^2$. Typical measured values of $m = 1 \cdot 0$ and $p = 0 \cdot 3$ correspond to modulation indices of $0 \cdot 5$ for the pulsar and $0 \cdot 7$ for the medium.

(3) Ewing *et al.* (1970) have found that scintillation features in PSR 1133+16 and PSR 1919+21 often drift systematically in frequency with time by several times their instantaneous width in frequency. They explain this in terms of a small number of beams dominating the scintillation in such a way that the interference

pattern at the distance of the Earth is similar to that of a two-element interferometer. The drifting features arise from the motion of the Earth across the frequency-dependent fringe pattern. In such circumstances, where the scintillation is determined by a small number of rays, the measured decorrelation frequencies cannot be adequately described by the scattering theory presented below.

Drifting features can also be explained in terms of the Earth's motion across a scintillation pattern which has been displaced perpendicular to the line of sight by frequency-dependent deviation of the entire wavefront, as by a prism. A linear variation of integrated electron content across the wavefront would constitute a suitable prism. In this latter case, the value of Δf corresponding to the undisturbed scintillation pattern is the overall width of the drifting feature, rather than the instantaneous width measured by B_h, f_ν or B.

4. SCATTERING THEORY

Theoretical expressions for Δt, B_h and $\Delta f(=f_\nu)$ will first be derived for strong and multiple scattering by a thin screen situated midway between the pulsar and the observer. It is then shown that the product $\Delta f \Delta t$ is the same for all cases of strong and multiple scattering regardless of the location or extent of the scattering screen.

Consider a scattering screen of thickness D, consisting of irregularities having a characteristic diameter a and r.m.s. fluctuation in electron density Δn_e. Following the treatment of Scheuer (1968), θ_{scat} the angular scattering by the screen, θ_0 the angular diameter of the received power, and ϕ_0 the r.m.s. fluctuation in phase across the wavefront imposed by the scattering screen, are related by

$$\theta_0 \approx \theta_{scat}/2 \approx \phi_0\lambda/4\pi a \approx \left(\frac{D}{a}\right)^{1/2} \Delta n_e r_0 \lambda^2/4\pi \tag{1}$$

where λ is the wavelength and r_0 is the classical radius of the electron, $2 \cdot 8 \times 10^{-13}$ cm.

By strong scattering it is meant that $\phi_0^2 \gg 1$, and by multiple scattering it is meant that the received radiation results from the interference of many independent beams. This requires that the scattered radiation is displaced transverse to the line of sight by many times the characteristic size of the scattering irregularities, i.e. $L\theta_0 \gg a$, where L is the distance from the pulsar to the observer. This condition is from Scheuer's (1968) treatment for a thin screen. For a thick screen, Rickett (1970) obtains a similar result based on the work of Uscinski (1968). An equivalent statement is that ϕ_0 is much less than the phase delay associated with the additional path length $L\theta_0^2/2$. The inequality $L\theta_0 \gg a$ is also satisfied if $\phi_0 \gg 1$ and $L\lambda/4\pi a^2 \gg 1$, which are the condition for strong scattering and the requirement that the observer is further from the screen than the Fresnel distance of one scattering irregularity.

Radiation arriving at the observer at an angle θ to the direct path is delayed by $L\theta^2/2c$. If it is assumed that the angular distribution of the received power is $\exp(-(\theta/\theta_0)^2)$, it then follows from incoherent addition of the radiation arriving from all directions that the power is distributed in time as $\exp(-t/\Delta t)$ where $\Delta t = L\theta_0^2/2c$ and $t > 0$ (Cronyn 1970). It is noted that if the spatial autocorrelation function of the scattering irregularities is gaussian, it can then be shown that the observed angular power distribution is also gaussian (see Ratcliffe 1956).

For theoretical purposes, Lang's frequency autocorrelation function $\Gamma_{12}(0)$ will be written in the form

$$\rho(f, B) = \frac{\int_{-\infty}^{\infty} \Delta I(f, \tau)\, \Delta I(f+B, \tau)\, d\tau}{\int_{-\infty}^{\infty} \{\Delta I(f, \tau)\}^2\, d\tau} \tag{2}$$

where $\Delta I(f, \tau)$ is the deviation from the mean intensity (power) at frequency f and time τ, and $\rho(f, B)$ is its a.c.f. between frequencies f and $f+B$. In what follows the denominator will be omitted, but it is understood that $\rho(f, B)$ is normalized so that $\rho(f, 0) = 1$.

$\Delta I(f, \tau)$ and $\Delta I(f+B, \tau)$ are simply simultaneous samples of the scintillation pattern at two frequencies separated by B. Integration over time produces an average across the scintillation pattern. Alternatively, integration over frequency will produce the same result if the decorrelation bandwidth of the scintillation pattern is the same at all frequencies, i.e. for all frequencies ν, $\rho(\nu, B) = \rho(f, B)$. As described in the previous section, the observations show that ρ is strongly dependent on frequency. However, for the present purpose of determining ρ at frequency f, which depends only on the properties of the scintillation pattern within a small range of frequency about f, it is convenient to replace the true scintillation pattern by one in which $\rho(\nu, B)$ does equal $\rho(f, B)$ at all frequencies. (This corresponds physically to a scattering medium in which the average refractive index, and hence θ_0, are independent of frequency.) The integral can then be written

$$\int_{-\infty}^{\infty} \Delta I(\nu) \Delta I(\nu+B)\, d\nu$$

where the ΔI refer to the artificial scintillation pattern and are all evaluated for the same time.

If $\Delta I(f)$ can be written in the form

$$\int_{-\infty}^{\infty} A(t) \exp(i2\pi f t)\, dt,$$

it then follows by application of the Wiener–Khintchine theorem (see Ratcliffe 1956) that

$$\rho(f, B) = \int_{-\infty}^{\infty} |A(t)|^2 \exp(i2\pi B t)\, dt;$$

i.e. if $\Delta I(f)$ is the Fourier Transform of $A(t)$, then $\rho(f, B)$ is the Fourier Transform of $|A(t)|^2$.

Adapting a result for interplanetary scintillations (Little 1968, equation (10)) to the present problem. $\Delta I(f)$ can be written

$$\Delta I(f) = \sum_{j \neq k}\sum F_j F_k \cos\left[\frac{\pi L}{\lambda}(\theta_j^2 - \theta_k^2) + \phi_j - \phi_k\right] \tag{3}$$

where F_j is the amplitude (voltage) of radiation received from direction j (which can also be specified by a radial angle θ_j and azimuthal angle α_j) and ϕ_j is the additional phase introduced by the scattering screen. Assuming that the angular power distribution is a circularly symmetric gaussian of the form $F_j^2 \propto \exp(-(\theta_j/\theta_0)^2)$, the expression can be written

$$\Delta I(f) \propto \mathscr{R} \sum_{j \neq k}\sum \exp(-(\theta_j^2 + \theta_k^2)/2\theta_0^2) \exp(i\pi L(\theta_j^2 - \theta_k^2)/\lambda) \exp(i(\phi_j - \phi_k)).$$

Substituting $t_j = L\theta_j^2/2c$ and $t_k = L\theta_k^2/2c$

$$\Delta I(f) \propto \mathscr{R} \sum_{t_j=0}^{\infty} \sum_{t_k=0}^{\infty} \exp(-c(t_j+t_k)/L\theta_0^2) \exp(i2\pi f(t_j-t_k)) \times \left\{ \sum_{\alpha_m=0}^{2\pi} \sum_{\alpha_n=0}^{2\pi} \exp(i[\phi(t_j, \alpha_m) - \phi(t_k, \alpha_n)]) \right\}$$

which can be considered as a double summation over all annular rings in the (t, α) plane, each ring of radius t and width dt resulting from a summation over α from 0 to 2π. It is easily shown that each ring of width dt arises from an equal area of the wavefront, independent of the value of t. Providing the r.m.s. value of ϕ is $\gg 1$ (the condition for strong scattering), equal areas of the wavefront will contain an equal number of scattering irregularities, each with an independent value of ϕ (modulo 2π). Consequently, for all rings of width dt, the summation over α_m and α_n is a two dimensional random walk containing the same number of steps, independent of the values of t_j or t_k. Its value is proportional to $\exp(-i\epsilon)$, on average, where $\epsilon(t_j, t_k)$ is randomly distributed between 0 and 2π.

Writing the expression for $\Delta I(f)$ in integral form, and substituting $x = t_j - t_k$ and $y = t_j + t_k$, it can then be shown that

$$\Delta I(f) \propto \int_{-\infty}^{\infty} A(x) \exp(i2\pi fx)\, dx$$

where

$$A(x) = \exp(i\eta(|x|)) \exp\left(-\frac{c|x|}{L\theta_0^2}\right) = \exp(i\eta(|x|)) \exp\left(-\frac{|x|}{2\Delta t}\right)$$

and $\eta(x)$ is randomly distributed between 0 and 2π. $\Delta I(f)$ is now in the desired form. Consequently, by application of the Wiener–Khintchine Theorem,

$$\rho(f, B) \propto \int_{-\infty}^{\infty} \exp\left(-\frac{|x|}{\Delta t}\right) \exp(i2\pi Bx)\, dx \tag{4}$$

and hence

$$\rho(f, B) = \frac{1}{1 + (2\pi B\Delta t)^2}. \tag{5}$$

In equation (4), x has the dimension of $L\theta^2/2c$ and can thus be associated with the time delay t in the pulse broadening function $\exp(-t/\Delta t)$. The frequency a.c.f. is thus the real part of the Fourier Transform of the pulse broadening function.

Defining Δf as the value of B at which $\rho(f, B)$ has fallen to 0·5, so that Δf is equivalent to Lang's f_ν,

$$2\pi \Delta f \Delta t = 1. \tag{6}$$

It follows from equation (1) and $\Delta t = L\theta_0^2/2c$ that

$$\Delta f \propto f^4 DM^{-2}. \tag{7}$$

The same results are also true for weak scattering ($\phi_0^2 \ll 1$) by a single screen with a gaussian spatial autocorrelation function. Indeed, in the case of weak scattering, the Fourier Transform relationship between the frequency a.c.f. and the pulse broadening distribution holds for any circulatory symmetrical angular power distribution.

The scintillation index m for observations with a finite bandwidth is related to the frequency a.c.f. by

$$\left(\frac{m}{m_0}\right)^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(f_1)G(f_2)\rho(f, f_1-f_2)\, df_1\, df_2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(f_1)G(f_2)\, df_1\, df_2} \tag{8}$$

where m_0 is the scintillation index for bandwidths approaching zero and $G(f)$ is the bandpass. This result follows directly from the definitions of m and ρ by considering the contributions to the total power from the different frequencies.

For a rectangular bandpass of total width W, this reduces to

$$\left(\frac{m}{m_0}\right)^2 = \frac{2}{W^2}\int_0^W \frac{W-x}{1+(2\pi x\Delta t)^2}\, dx$$

$$= \frac{2}{M}\tan^{-1} M - \frac{1}{M^2}\ln(1+M) \tag{9}$$

where $M = 2\pi W\Delta t$. m/m_0 falls to 0·5 at $M = 9{\cdot}9$ and it follows that B_h is then $9{\cdot}9/(2\pi\Delta t)$.

For a gaussian bandpass of shape $\exp(-(f_1-f)^2/w^2)$

$$\left(\frac{m}{m_0}\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(-(a^2+b^2)/w^2)}{1+[2\pi(a-b)\Delta t]^2}\, da\, db \bigg/ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-(a^2+b^2)/w^2)\, da\, db$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\exp(x^2/2)}{1+(Kx)^2}\, dx$$

where $K = 2\pi w\Delta t$. Multiplying the identity

$$\int_{-\infty}^{\infty} \exp(-kx^2/2)\, dx = \sqrt{\frac{2\pi}{k}} \quad \text{by } \exp(-k/2)$$

and integrating over k from $1/K^2$ to $+\infty$ this expression can be written

$$\left(\frac{m}{m_0}\right)^2 = \frac{1}{K}\exp\left(\frac{1}{2K^2}\right)\int_{1/K}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \tag{10}$$

which is equivalent to the result obtained by Little (1968) by a different method. m/m_0 falls to 0·5 at $K = 4{\cdot}2$, and it follows that B_h (the width of the bandpass between half-power points, which equals $1{\cdot}67w$) is then $7{\cdot}0/(2\pi\Delta t)$.

The results for ρ and m/m_0 can also be obtained by the method which Little (1968) used to study the effect of a finite bandwidth on interplanetary scintillations, by suitably modifying his bandpass function $\rho_B(\tau)$. However, the derivations given above better illustrate the Fourier Transform relationship between $\rho(f, B)$ and the pulse broadening function, and also show how m/m_0 can be derived from the more fundamental quantity $\rho(f, B)$. The expressions for both m/m_0 and $\rho(f, B)$ are the same for weak scattering as for strong and multiple scattering, and will be valid for scattering media other than the interstellar medium. For example, they can be applied to interplanetary scintillations by replacing L with Z, the distance from the Earth to the scattering screen.

The relationship $2\pi\Delta f\Delta t = 1$ has been established for a thin screen located midway between the pulsar and observer, under the conditions of strong and multiple scattering, and a gaussian distribution for the received power. It is now shown that this relationship holds approximately for any screen, thick or thin, and regardless of its location, providing these conditions are fulfilled. Consider the relative phases of radiation following paths P and Q from the pulsar to the observer though an arbitrary scattering screen, where P is the path of shortest time and Q is that followed by radiation delayed by time T. At the centre of a spectral feature, at frequency f, the difference in phase between paths P and Q is $2\pi fT$, while at frequency $f+F$ the difference is $2\pi(f+F)T$. Hence, between frequencies f and $f+F$, the difference in phase between the paths P and Q has changed by $2\pi FT$. Now the decorrelation in intensity between two frequencies is determined by some combination of the numerical values of $2\pi FT$ for all paths. Providing the forms of the pulse broadening function and the frequency a.c.f. are the same for all the screens it follows that the effects of F and T can be represented by those at specified parts of the broadening and frequency a.c.f. curves, such as Δf and Δt. Hence the numerical value of $2\pi\Delta f\Delta t$ will represent a particular degree of decorrelation and this value will be independent of the screen involved. In particular, as described above, it equals unity for a thin screen situated midway between the pulsar and the observer. Hence the relationship $2\pi\Delta f\Delta t = 1$ holds for all screens. Even if the forms of the pulse broadening and frequency a.c.f. are not exactly those for a thin screen, it is unlikely that they are sufficiently different for the relationship between Δf and Δt to change by more than a factor 2.

5. THE COMPOSITE Δf–DM DIAGRAM

In Table II the measured values of decorrelation frequency have been scaled to 318 MHz according to f^4 and converted to decorrelation frequencies using the relationships $\Delta f = f_\nu = B_h/9{\cdot}9 = B/1{\cdot}5$. The result for B is from Rickett's measurements, while the result for B_h is that given by equation (9), assuming

TABLE II

Decorrelation frequencies at 318 MHz

Pulsar	Galactic latitude (degrees)	Dispersion measure (electrons cm^{-3})	$B_h/9{\cdot}9$ (Rickett) (MHz)	f_ν (Lang) (MHz)	$B/1{\cdot}5$ (Ewing) (MHz)	Adopted Δf (MHz)
PSR 0329+54	−1	27	0·03[a]	—	0·015	0·03
0808+74	32	5·8	2·3[b]	—	—	2·3
0823+26	32	19	—	0·5	—	0·5
0834+06	26	12·8	0·20	0·5	0·29	0·3
0950+08	44	3·0	⩾0·7	9[c]	—	9
1133+16	69	4·9	⩾0·4	2·0	0·26	0·7
1237+25	87	8·5	—	0·7	—	0·7
1508+55	52	20	0·06	—	—	0·06
1749−28	−1	51	0·006	—	—	0·006
1919+21	4	12·6	0·11	0·8	0·40	0·3
1929+10	−4	3·2	—	0·4[d]	—	0·4
2016+28	−4	14·2	0·03	<0·5	—	0·03
2045−16	−33	11·4	⩾0·7	—	—	⩾0·7

[a] Autocorrelation receiver observations gave 0·037 MHz.
[b] Scaled from 151 MHz.
[c] Derived from Lang's Γ_{12} data at 111 MHz.
[d] Derived from Lang's Γ_{12} data at 318 MHz.

that Rickett observed with a rectangular bandpass. Except where noted in the table, the values of B_h and f_ν were derived from those tabulated by Rickett for 408 MHz and by Lang for 318 MHz. It is noted that Rickett's assumed form for the frequency a.c.f. was sufficiently similar to the expression given in equation (5) that no revisions to Rickett's curves of best fit, and hence values of B_h, were necessary. The values of B lie on curves of best fit, $B \propto f^4$, to the tabulated data of Ewing *et al.* (1970). The final column in Table II contains a subjective ' average ' of the three values of Δf.

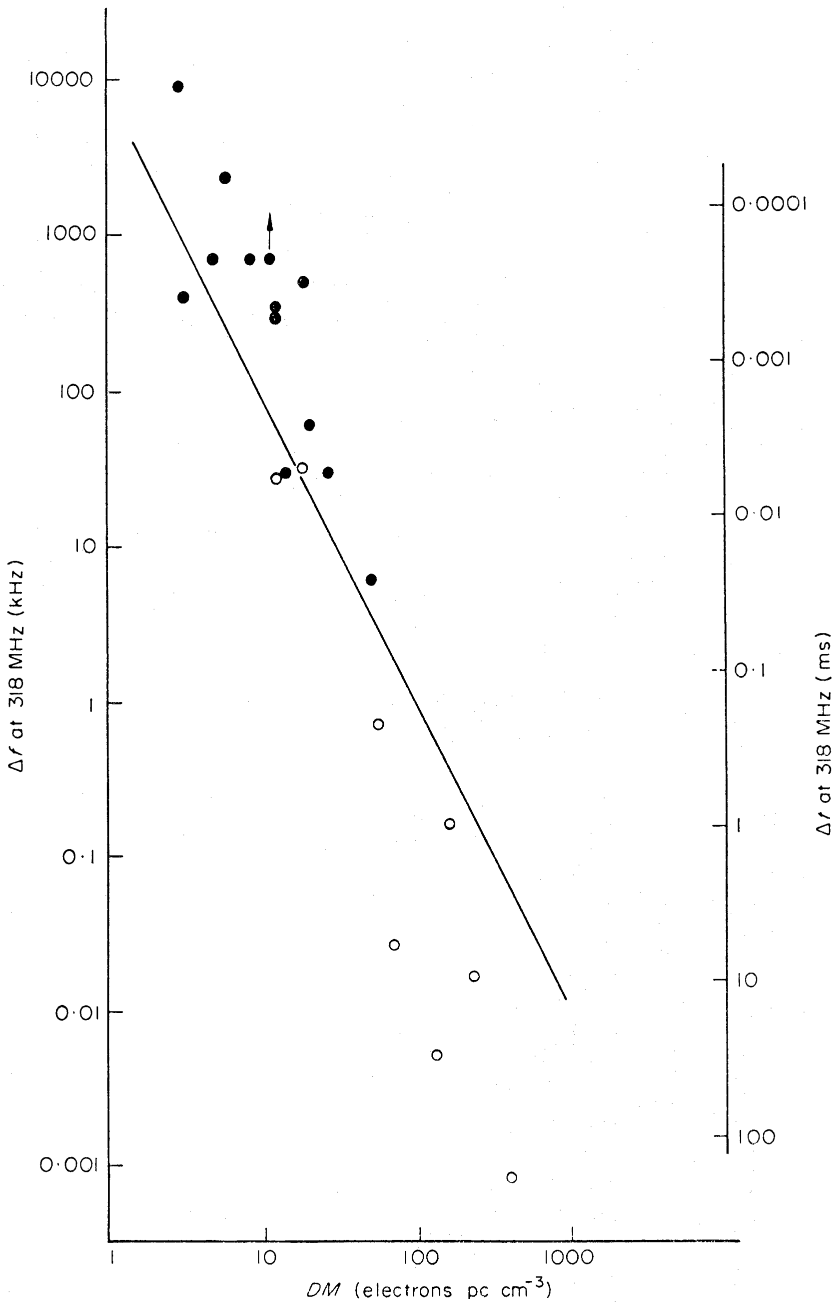

FIG. 1. *The observed relationship between decorrelation frequency* Δf *and pulse broadening* Δt (*both at* 318 *MHz*), *and dispersion measure* DM. *The scales of* Δf *and* Δt *are related by* $2\pi\Delta f\Delta t = 1$. *The line indicates the slope of the theoretical relationship,* $\Delta f \propto (\mathrm{DM})^{-2}$. *Filled circles represent direct measurements of decorrelation frequency while open circles represent those derived from pulse broadening.*

In compiling Tables I and II it has been assumed that $\Delta f \propto f^4$ and $\Delta t \propto \lambda^4$, as predicted by the theory. The multifrequency measurements of B by Ewing *et al.* (1970) and of pulse broadening (see references to Table I) all show that the exponent is in the range 3 to 5 with the single exception of PSR 1133+16 for which it is $2 \cdot 6 \pm 0 \cdot 6$. Since most measurements were at frequencies close to 318 MHz the errors in Δf due to scaling are, with very few exceptions, less than a factor 2.

The values of Δf, Δt and DM for each pulsar are plotted in Fig. 1, where the scales of Δf and Δt are related by $2\pi \Delta f \Delta t = 1$. A line representing the slope of the theoretical relationship $\Delta f \propto (DM)^{-2}$ is shown. The error for each point is typically a factor 3, based on the inaccuracies of measurements and the uncertainties associated with scaling to 318 MHz and converting from Δt, B_h and B to Δf. The Δt points for PSR 0823+26 and PSR 1919+21 have considerably larger errors. The data presented in Fig. 1 are clearly inconsistent with the scattering theory described above. There is an apparent relationship $\Delta f \propto DM^{-4}$, but it will be assumed that this is due to systematic deviations from the ideal relationship rather than a complete breakdown of the theory. However it is not clear which values lie on the mean curve and which are deviations. The results of Fig. 1 cannot be attributed to errors in the Δf–Δt relationship for it is established to within a factor 2, providing the assumptions are satisfied.

Lang (1971) has presented results similar to those in Fig. 1; but using $\Delta f \Delta t = 1$, which differs by a factor 2π from the value used here, and $\Delta f = f_\nu = B_h/4$. His treatment has the effect of reducing deviations from the theoretical relationship $\Delta f \propto DM^{-2}$.

6. DISCUSSION

The Δf–DM diagram is essentially a comparison of the scattering and dispersing properties of the medium between the pulsar and the observer. The observations reveal significant deviations from the theoretical relationship. Several explanations are considered.

(1) Violation of the conditions for which the Δf–Δt relationship was derived.

(2) Departures from isotropy in the interstellar medium, (a) systematic variations of scattering and dispersing properties with galactic latitude, and (b) H II regions along the line of sight.

(3) Two populations of scattering irregularities.

(1) The Δf–Δt relationship was derived with the following assumptions, some of which are interdependent; (a) there is strong scattering, $\phi_0^2 \gg 1$; (b) there is multiple scattering, $L\theta_0 \gg a$; (c) scattering irregularities are characterized by a diameter a and fluctuation in electron density Δn_e; (d) the forms of the frequency a.c.f. $\rho(f, B)$ and the pulse broadening function are approximately the same for all screens both thick and thin; (e) the angular distribution of the received power is gaussian or, more fundamentally, the spatial autocorrelation function for the scattering irregularities is gaussian.

Using values of $\Delta n_e \sim 4 \cdot 7 \times 10^{-5}$ electrons pc cm^{-3} and $a \sim 10^{11}$ cm, derived by Rickett (1970), ϕ_0 at 318 MHz for a pulsar at distance 100 pc is ~ 7 rad. This result, and Rickett's measured value of 0·5 for the modulation index of PSR 0950+08 at 408 MHz suggest that radiation from some of the nearest pulsars may not experience strong scattering at 318 MHz.

If the condition of multiple scattering is barely satisfied, so that $L\theta_0 \sim a$, the maximum width of the wavefront along the line of sight will then be $\sim a$, and the scintillation pattern will be determined by the interference of only a small number of beams. This will cause Δf to exceed its theoretical value and the instantaneous value of Δt to be less than its theoretical value. However, Δt derived from pulse shapes averaged over sufficient time (many scintillation features) will be unchanged, and consequently $2\pi\Delta f\Delta t$ will be greater than 1. Using Rickett's values of Δn_e and a, the condition for multiple scattering is barely satisfied at 318 MHz for a pulsar at a distance of 100 pc. For an average electron density of 0·04 cm^{-3} (Prentice & ter Haar 1969) this corresponds to $DM = 4$. Although Δn_e and a are not precisely determined, and will probably vary throughout the interstellar medium, the Δf–Δt relationship is almost certainly violated for nearby pulsars such as PSR 0950+08 and PSR 1133+16, and there may be violations for distances up to 500 pc. This view is supported by measurements of both Δf and Δt for PSR 0823+26 and PSR 1919+21, for which the values in Tables I and II correspond to $2\pi\Delta f\Delta t \sim 14$ (but with a large uncertainty due to the uncertainties in these values of Δt). Further evidence that the intensity fluctuation of nearby pulsars are determined by a small number of beams is provided by the observations and interpretation of scintillation features which drift is frequency with time (see Section 3 and Ewing *et al.* 1970).

Unfortunately it is difficult to check the Δf–Δt relationship directly by measurements of both Δf and Δt at the same frequency for the same pulsar because, even at the most favourable frequency values will be of the order of 1 KHz and 1 ms. If Δf is measured at a slightly higher frequency and Δt at a slightly lower frequency, larger and more easily measured quantities are involved, but the two frequencies should be sufficiently close together that uncertainties in scaling results to a common frequency are small. Furthermore, to avoid possible violations of the condition for multiple scattering, measurements should be made on pulsars of as high a dispersion measure as possible.

It has been assumed that the scattering irregularities in the interstellar medium do have a characteristic linear size a and electron density fluctuation Δn_e which are independent of frequency f and distance L. The irregularities will include large ranges of size a_i and density fluctuation Δn_i, and the effective values of a and Δn_e may well depend on f and L. Salpeter (1969) has considered ranges of size and density fluctuation such that Δn_i is proportional to $a_i^{0\cdot5+\gamma}$ (with the implied assumption that the number of irregularities of size a_i per unit volume is proportional to a_i^{-3}) and finds that Δf is proportional to $f^{4/(1-\gamma)}L^{-(2+\gamma)}$. Since the observed frequency dependence of Δf means that $|\gamma| < 0\cdot3$, such effects cannot account for an L dependence stronger than $L^{-2\cdot3}$ (or $DM^{-2\cdot3}$).

(2) It is expected that the scattering and dispersive properties of the interstellar medium vary systematically with height above the galactic plane. Inspection of Fig. 1 reveals a tendency for pulsars close to the galactic plane to lie lower relative to the theoretical line than those of higher galactic latitude, but the effect is insufficient to explain the observed gross systematic deviations.

Alternatively, the line of sight to the pulsar may pass through H II regions (Davidson & Terzian 1968; Prentice & ter Haar 1969; Grewing & Walmsley 1970) where the increased electron density and turbulence will increase DM and Δt respectively. As to whether a point will be displaced above or below the normal $\Delta f \propto DM^{-2}$ line depends on the diameter of the H II region, its position along the line of sight and the values of n_e, Δn_e and a within it relative to the average values for the interstellar medium. For example, for an H II region located approximately midway between the pulsar and the observer and contributing no more than 50 per cent to the total dispersion measure, Δf will be displaced below the normal $\Delta f \propto DM^{-2}$ line if $(\Delta n_e)^2/an_e$ within the H II region is more than several times the value in the interstellar medium. Similar conditions can also be derived for regions deficient in electrons, as may be the case in regions between the spiral arms.

The abnormally large values of DM and Δt for PSR 0833–45 are attributed to the Gum Nebula (Prentice & ter Haar 1969; Ables *et al.* 1970). The abnormally large values of Δt for PSR's 1858+03, 1946+35 and 2003+31 can probably be explained in a similar way by less spectacular H II regions or simply regions where there is increased scattering with relatively little change to DM. If this interpretation is accepted, the results indicate that the probability of a suitable region lying along the line of sight is reasonably high for $DM > 50$.

(3) Fig. 1 can also be explained in terms of two populations of scattering irregularities distributed uniformly throughout the interstellar medium. The pulse broadening is determined by one scattering population and the frequency structure by the other. Thus, although $2\pi\Delta f\Delta t = 1$ and $\Delta f \propto DM^{-2}$ for each population, the measured values of Δf and Δt are related by $2\pi\Delta f\Delta t = K$ where K is a constant not equal to unity. In this interpretation the angular power distribution, broadening function and frequency a.c.f. should each be the superposition of two distributions of equal area, having similar shapes but different widths and amplitudes.

The discrepancies between the 'Δf' and 'Δt' points in Fig. 1 show that $K \sim 20$, in which case the observed pulse broadening and frequency a.c.f. must both correspond to the broader distributions. Existing measurements of both pulse broadening and frequency a.c.f. show no indication of a second narrower distribution superimposed on the first. These failures to detect the second distributions are sufficient evidence to reject the interpretation of Fig. 1 in terms of two populations of scattering irregularities.

7. CONCLUSIONS

Theoretical relationships between the pulse broadening Δt, the decorrelation frequency Δf and the dispersion measure DM have been derived for conditions of scattering which are believed to approximate those in the interstellar medium. These include strong and multiple scattering and the assumption that the spatial auto-correlation function of the scattering irregularities is gaussian. The observations reveal significant deviations from the theoretical Δf–Δt–DM relationship. These deviations are probably due to a combination of two effects. First, for small dispersion measures, the condition for multiple scattering is not satisfied sufficiently strongly for the Δf–Δt relationship to be valid. Secondly, for some large dispersion measures there is abnormally strong scattering by turbulent regions such as H II regions. It is estimated that the undisturbed relationship is approximately the line shown in Fig. 1. All values of Δt should lie close to this line, providing they are determined from averages over a sufficient length of time and there is no abnormal scattering. The Δf–Δt relationship and deviations from it can be examined by further pairs of measurements of Δf and Δt, for as many pulsars as possible. Suspected cases of abnormally strong scattering can be further investigated by direct measurements of θ_0. It is expected that pulsars with anomalously large values of Δt or low values of Δf will also have anomalously large values of θ_0.

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NOTE ADDED IN PROOF

For PSR 1933+16 Guelin, Encrenaz and Bonazzola (*Astr. Astrophys.* in press) have recently measured $\Delta f = 100$ KHz at 1420 MHz and derived a distance of $\geqslant 6$ kpc from the H I absorption profile. This value of Δf, scaled to 318 MHz according to λ^4 and combined with the value of Δt from Table I, gives $2\pi \Delta f \Delta t = 1{\cdot}6$, close to the theoretical result of 1·0. The conditions of strong and multiple scattering are better satisfied than for PSR 0823+26 and PSR 1919+21, the only other pulsars with measurements of both Δf and Δt.