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Scattering of *SH* Waves by a Rough Half-Space of Arbitrary Slope

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Summary

The method of matched asymptotic expansions is used to look into the problem of the scattering of plane *SH* waves by topographic irregularities of a restricted range in an otherwise plane half-space when the characteristic length dimension of the irregularity is much smaller than the wavelength of the incident wave. In contrast to previous work the slope of the irregularity remains arbitrary. Expressions for the near and far scattered fields are obtained. Comparison between this theory and the regular perturbation technique (which also assumes that the irregularity has a small slope) show that both agree when the slope is small but differ in the general case. Results are given for irregularities in the shape of triangles, trapezia and semicircles.

1. Introduction

The San Fernando earthquake of 1971 February 9, recently stimulated interest in the effect of topography on the propagation of seismic waves. The contribution of the present work will be to analyse asymptotically in the limit of the long waves, the effect of an irregularity of finite extent on the otherwise plane surface of a half space. The problem of the scattering of elastic waves by such an irregularity is a complex one, and solutions for arbitrary wavelengths can only realistically be sought numerically (Boore 1972, 1973; Bouchon 1973). One exception to date is for the scattering of *SH* waves by a groove of semi-circular cross-section (Trifunac 1973); this solution is rather special but does provide at least one check on the validity of other approaches, numerical or analytical.

Analytical methods have so far been restricted to regular perturbation techniques, appropriate to small amplitude irregularities of gentle slope (see for instance Gilbert & Knopoff; McIvor 1969). These have the virtue of providing simple expressions for the scattered field far from the irregularity, so long as they are applicable, but do not give the motion on the scatterer itself. Hudson *et al.* (1973) have confirmed that the theory and experiment agree for *P* to Rayleigh wave scattering at surface irregularities with slopes of 25° or less from the median plane, when the irregularity is small.

In the present work, we employ the method of matched asymptotic expansions (Van Dyke 1964; Cole 1968; Eckhaus 1973; Nayfeh 1973). This removes the restriction on the slope of the irregularity, so that (finite) irregularities of any shape can be considered, but retains the restriction to long waves. Like the regular perturbation technique, it yields simple expressions for the far scattered field, but also yields expressions for the motion in the vicinity of the irregularity. Here, the method is applied to the scattering of *SH* waves; results concerning the scattering of Rayleigh waves will be reported separately.

2. Statement of the problem

We shall consider a medium of homogeneous, isotropic, elastic material of density ρ and elastic constants λ and μ in a state of anti-plane strain. Thus the displacement of the point given by (x', y', z') referred to Cartesian axes and at time t is given by $(0, 0, w'(x', y', t))$.

Let us consider the problem of the scattering of plane *SH* waves by an uneven surface as follows: when a plane *SH* wave w_i' is incident upon the free surface of a perfectly plane half-space, i.e. a half space with no surface irregularities, it produces a reflected plane *SH* wave w_r' such that a displacement $w' = w_i' + w_r'$ results. Now, if an irregularity is present it produces an additional wave w_s' which we shall refer to as the scattered *SH* wave produced by the irregular topography, so that a wave

$$w' = w_i' + w_r' + w_s' \tag{2.1}$$

arises.

We assume the wave $w'(x', y', t)$ to be a simple harmonic function of time with circular frequency ω. Hence the equation satisfied by the amplitude $w'(x', y')$ is the reduced wave equation, i.e.

$$\nabla^2_{x'y'} w' + k^2 w' = 0, \quad \text{in } d', \tag{2.2}$$

where

$$\nabla^2_{x'y'} \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$$

is the Laplacian operator in two dimensions, d' is the irregular half space given by $y' > f'(x')$, $k = \omega/\beta$ is the wavenumber and $\beta = (\mu/\rho)^{\frac{1}{2}}$ is the velocity of propagation of equivoluminal waves.

The free surface condition satisfied by the wave can be written as

$$\frac{\partial w'}{\partial n'} = 0, \quad \text{on } \partial d', \tag{2.3}$$

where $\mathbf{n}'$ is the inward unit normal to the boundary $\partial d'$ of d'.

The incident plane wave w_i', of unit amplitude, and reflected plane wave w_r', assumed known, satisfy (2.2) in d' and the free surface condition (2.3) on $y' = 0$. The scattered wave w_s' must then satisfy

$$\nabla^2_{x'y'} w_s' + k^2 w_s' = 0, \quad \text{in } d', \tag{2.4}$$

$$\frac{\partial w_s'}{\partial n'} = -\left(\frac{\partial w_i'}{\partial n'} + \frac{\partial w_r'}{\partial n'}\right), \quad \text{on } \partial d'. \tag{2.5}$$

To render the solution of (2.4) and (2.5) unique, we require w_s' to satisfy Sommerfeld's condition for outward radiation, namely

$$\lim_{r' \to \infty} r'^{\frac{1}{2}} \left(\frac{\partial w_s'}{\partial r'} + ikw_s'\right) = 0, \tag{2.6}$$

where $r' = (x'^2 + y'^2)^{\frac{1}{2}}$ (Sommerfeld 1949, p. 193).

The complete solution of the Neumann problem (2.4) to (2.6) cannot be found in closed form, so we shall restrict our attention to finding an asymptotic solution when the wavelength of the incident wave is much larger than the characteristic linear dimension l' of the irregularity. Or in other words, if we introduce the dimensionless parameter $\varepsilon = kl'$, we seek an asymptotic solution w_s' as ε tends to zero by the method of matched asymptotic expansions (Van Dyke 1964). We shall only deal with a finite surface irregularity on an otherwise flat half-space, with no restriction imposed on its slope.

3. The outer problem and expansion

In order to obtain the outer problem and the outer expansion we have to solve the exact differential equation to get solutions satisfying the radiation condition but not necessarily with the exact boundary condition imposed on the irregularity. We therefore choose the wavenumber k as a scaling factor so that independent space variables become dimensionless. Thus we define outer independent variables $\mathbf{x} = (x, y)$ and corresponding polar co-ordinates (r, θ) such that

$$\mathbf{x} = k\mathbf{x}' \quad \text{and} \quad r = kr', \quad \theta = \theta', \tag{3.1}$$

where $\mathbf{x}' = (x', y')$ and (r', θ') are the corresponding polar co-ordinates.

In the same way the characteristic linear dimension l' of the irregularity is normalized to $kl'(=\varepsilon)$. The surface irregularity is finite and so is contained in a neighbourhood of the origin in $\mathbf{x}$-space of order ε. In the limit as ε tends to zero with fixed outer co-ordinates x and y, this neighbourhood shrinks to a point, the origin. Then the outer domain d becomes the half-space $y > 0$ and it is to be expected that terms with singularities at the origin must be admitted in the outer expansion of w_s', whose amplitudes must be fixed by matching.

Also dimensionless dependent variables are given by

$$w_s = w_s', \quad w_i = w_i', \quad w_r = w_r' \tag{3.2}$$

since w_i' is of unit amplitude. In these new variables, equations (2.4), (2.5) and (2.6) become

$$\nabla^2_{xy} w_s + w_s = 0 \quad \text{in} \quad y > 0, \tag{3.3}$$

$$\frac{\partial w_s}{\partial y} = 0 \quad \text{on} \quad y = 0 \quad (x \neq 0), \tag{3.4}$$

$$\lim_{r \to \infty} r^{\frac{1}{2}} \left(\frac{\partial w_s}{\partial r} + i w_s \right) = 0. \tag{3.5}$$

The general solution of the outer problem (3.3) to (3.5) can readily be found by separation of variables. We shall assume the asymptotic expansion as $\varepsilon \to 0$

$$w_s(r, \theta) \sim \sum_{n=0}^{\infty} \varepsilon^n a_n(\varepsilon) H_n^{(2)}(r) \cos n\theta, \tag{3.6}$$

where $H_n^{(2)}(r)$ is the Hankel function of the second kind of order n. The coefficients $a_n (n \geqslant 0)$ are undetermined constants which are allowed to depend on ε in case the subsequent matching demands it. They are assumed to be such that $a_n = 0(1)$ or smaller for $n > 0$ and $a_0 = 0(\varepsilon)$ or smaller; in this situation terms like

$$\varepsilon^n a_n H_n^{(2)}(r) \cos n\theta, \quad \text{for} \quad n \geqslant 0,$$

remain bounded for each $(X, Y) \neq (0, 0)$ (see (4.1) below) when they are written in inner variables and expanded asymptotically for small ε. Moreover it proves convenient to write

$$a_n = a_n^{(0)} + \varepsilon a_n^{(1)} + \varepsilon^2 a_n^{(2)} + \ldots, \quad (n \geqslant 0), \tag{3.7}$$

where we take $a_0^{(0)} = 0$.

The solution (3.6) is obviously not uniformly valid throughout the outer domain d because each of the Hankel functions is singular at the origin. To study the solution in the vicinity of the origin, we shall solve, with the exact boundary condition on the irregularity an approximate differential equation without the radiation condition. Again unspecified constants will appear in this solution. They will have to be determined together with the a_n's by the matching which provides the extra boundary condition.

4. The inner problem and expansion

We stretch our variables by introducing inner independent variables **X** and **Y** with corresponding polar co-ordinates R and Θ such that

$$\mathbf{X} = \frac{\mathbf{x}}{\varepsilon} = \frac{\mathbf{x}'}{l'} \quad \text{and} \quad R = \frac{r}{\varepsilon} = \frac{r'}{l'}, \quad \Theta = \theta. \tag{4.1}$$

In terms of X and Y the equation for the free surface becomes $Y = F(X)$ where $F(X) = f'(l'X)/l'$.

We define the inner dependent variables by the equations

$$W_s = w_s, \quad W_i = w_i, \quad W_r = w_r. \tag{4.2}$$

After substituting (4.1) and (4.2) into (2.4) and (2.5) we obtain the equations of the inner problem, i.e.

$$\nabla^2_{XY} W_s + \varepsilon^2 W_s = 0, \quad \text{in } D, \tag{4.3}$$

$$\frac{\partial W_s}{\partial N} = -\frac{\partial W_i}{\partial N} - \frac{\partial W_r}{\partial N}, \quad \text{on } \partial D, \tag{4.4}$$

where D is the region given by $Y > F(X)$ and **N** is the inward normal to the boundary ∂D of D.

Instead of the radiation condition, we shall require $W_s(\mathbf{X})$ to be bounded in any finite domain of D containing the topographic irregularity.

Before solving the inner equations, we need to know the right-hand side of (4.4). The incident w_i' and reflected w_r' SH waves in a perfectly plane half-space satisfying a free surface condition, are, in terms of the inner variables,

$$\left.\begin{aligned} W_i &= \exp\left[i\varepsilon(X\cos\psi + Y\sin\psi)\right], \\ W_r &= \exp\left[i\varepsilon(X\cos\psi - Y\sin\psi)\right], \end{aligned}\right\} \tag{4.5}$$

where ψ is the angle of incidence.

The contribution of these waves to the right-hand side of (4.4) can now be written explicitly as

$$-\frac{\partial W_i}{\partial N} - \frac{\partial W_r}{\partial N} = \varepsilon M^{(1)}(\mathbf{X}) + \varepsilon^2 M^{(2)}(\mathbf{X}) + 0(\varepsilon^3), \quad \text{on } Y = F(X) \tag{4.6}$$

after expanding terms in powers of ε, where

$$\left.\begin{aligned} M^{(1)}(\mathbf{X}) &= -2i\cos\psi\sin\alpha_0\pi, \\ M^{(2)}(\mathbf{X}) &= 2(X\cos^2\psi\sin\alpha_0\pi - Y\sin^2\psi\cos\alpha_0\pi). \end{aligned}\right\} \tag{4.7}$$

Here $\alpha_0 \pi$ is the angle formed by the local tangent to the surface $Y = F(X)$ and the $X-$ axis.

Let us now construct an asymptotic expansion for W_s. We note that only positive integer powers of ε appear in the inner equations, so we may assume that as $\varepsilon \to 0$

$$W_s(\mathbf{X}) \sim \varepsilon W_s^{(1)}(\mathbf{X}) + \varepsilon^2 W_s^{(2)}(\mathbf{X}) + \dots \tag{4.8}$$

where, for $i = 1, 2$,

$$\nabla^2_{X\,Y}\, W_s^{(i)} = 0, \quad \text{in } D, \tag{4.9}$$

$$\frac{\partial W_s^{(i)}}{\partial N} = M^{(i)}(\mathbf{X}), \quad \text{on } \partial D, \tag{4.10}$$

and $W_s^{(i)}$ is bounded in any finite domain in D which contains the irregularity.

To solve the equations (4.9) and (4.10), we employ a conformal transformation of the $Z(=X+iY = R\exp(i\Theta))$ plane into the $\zeta(=\xi+i\eta = \rho\exp(i\chi))$ plane which transforms $Y \geqslant F(X)$ into the half-plane $\eta \geqslant 0$. Let this transformation be $Z = g(\zeta)$, chosen in such a way that the finite interval $[\lambda_1, \lambda_p]$ of the real axis in the ζ-plane is mapped into the topographic irregularity. Under this mapping the equations (4.9) and (4.10) transform, for $i = 1, 2$, to

$$\nabla^2_{\xi\eta}\, W_s^{(i)} = 0, \quad \text{in } \eta > 0, \tag{4.11}$$

$$\frac{\partial W_s^{(i)}}{\partial \eta} = -M^{(i)}(\zeta)\left|\frac{dZ}{d\zeta}\right|, \quad \text{on } \eta = 0 \tag{4.12}$$

and $W_s^{(i)}(\zeta)$ is bounded in any finite domain in $\eta > 0$ which contains the interval $[\lambda_1, \lambda_p]$. It is clear that outside the interval $[\lambda_1, \lambda_p]$ the Neumann condition (4.12) has zero right side.

It is reasonable to assume, further, that the conformal mapping has, for large ζ, the asymptotic expansion

$$Z = b\zeta + \sum_{n=1}^{\infty} b_n \zeta^{-n}, \tag{4.13}$$

where b is real and positive, since the irregularity of the boundary $Y = F(X)$ is of finite extent; this property is displayed by all of the mappings used in the examples given in Section 6. The function inverse to (4.13) has an expansion of the same form:

$$\zeta = b^{-1}Z + \sum_{n=1}^{\infty} c_n Z^{-n}. \tag{4.14}$$

Also the functions $\ln\zeta$ and ζ^m, m an integer, have similar expansions, i.e.

$$\ln\zeta = \ln(b^{-1}Z) + \sum_{n=1}^{\infty} d_n Z^{-n-1}, \tag{4.15}$$

$$\zeta^m = b^{-m}Z^m + \sum_{n=1}^{\infty} e_{m,n} Z^{-n+m-1}, \tag{4.16}$$

The coefficients appearing in (4.14) to (4.16) are related to those appearing in (4.13).

The solution of the equations (4.11) and (4.12) can readily be found. A particular solution $W_s^{(i)}$ is given by

$$\pi^{-1}\int_{\lambda_1}^{\lambda_p} M^{(i)}(\xi', 0)\left|\frac{dZ}{d\zeta'}\right|(\xi', 0)\ln|\xi' - \zeta|\,d\xi' \quad (i = 1, 2) \tag{4.17}$$

(*cf.*, e.g. Kantorovich & Krylov 1958, p. 579). The general solution of (4.11) and

(4.12) is formed by adding to (4.17) the general solution of the homogeneous problem associated with (4.11) and (4.12); i.e. Laplace's equation together with a homogeneous Neumann condition on $\eta = 0$ and boundedness in any finite domain in $\eta > 0$ containing $[\lambda_1, \lambda_p]$. This is given by

$$\sum_{n=0}^{\infty} A_n^{(i)}(\varepsilon)\rho^n \cos n\chi \quad (i = 1, 2) \tag{4.18}$$

where $A_n^{(i)}(\varepsilon)$, $n > 0$, are undetermined complex constants whose possible dependence upon ε is admitted at this stage in case the subsequent matching demands it. Now terms like $\varepsilon^i A_n^{(i)} Re[\zeta^n]$ for $n = i, i+1, \ldots$ and $i = 1, 2$ give rise to unbounded terms, when they are expanded asymptotically for small ε, after being written successively in inner and outer variables. For example

$$\varepsilon^i A_n^{(i)} Re[\zeta^n] = \varepsilon^i A_n^{(i)} Re\left[b^{-n} Z^n + \sum_{m=1}^{\infty} e_{n,m} Z^{-m+n-1}\right]$$

$$= \varepsilon^i A_n^{(i)} Re\left[\varepsilon^{-n} b^{-n} z^n + \sum_{m=1}^{\infty} \varepsilon^{m-n+1} e_{n,m} z^{-m+n-1}\right] \tag{4.19}$$

is bounded for each fixed $(x, y) \neq (0, 0)$ only when $A_n^{(i)} = 0(\varepsilon^{n-i})$ for $n = i, i+1, \ldots$ and $i = 1, 2$, or smaller. But terms like this, in any case, appear later in the expansion (4.8) and so may be neglected at this stage. Hence the admissible general solution of the homogeneous equations for $i = 1, 2$ is simply given by

$$\sum_{n=0}^{i} A_n^{(i)}(\varepsilon)\rho^n \cos n\chi. \tag{4.20}$$

Finally, the admissible general solution of (4.11) and (4.12) for $i = 1, 2$ is the sum of (4.17) and (4.20), i.e.

$$W_s^{(i)}(\zeta) = \sum_{n=0}^{i} A_n^{(i)}(\varepsilon)\rho^n \cos n\chi - \pi^{-1} \int_{\lambda_1}^{\lambda_p} M^{(i)}(\xi', 0)\left|\frac{dZ}{d\zeta'}\right|(\xi', 0)\ln|\xi' - \zeta|\, d\xi'. \tag{4.21}$$

If we expand the integrand in (4.21), we get

$$W_s^{(i)}(\zeta) = \sum_{n=0}^{i} A_n^{(i)}\rho^n \cos n\chi + \pi^{-1}\left[B_0^{(i)}\ln\rho - \sum_{n=1}^{\infty} B_n^{(i)} n^{-1} \rho^{-n} \cos n\chi\right] \tag{4.22}$$

valid for $|\zeta| > \max(|\lambda_1|, |\lambda_p|)$, where $B_n^{(i)}$, $n = 0, 1, 2, \ldots$ and $i = 1, 2$ are constants defined by

$$B_n^{(i)} = -\int_{\lambda_1}^{\lambda_p} \xi'^n M^{(i)}(\xi', 0)\left|\frac{dZ}{d\zeta'}\right|(\xi', 0)\, d\xi'. \tag{4.23}$$

The above constants for $i = 1$ are imaginary whereas those for $i = 2$ are real.

In particular, we have that

$$B_0^{(1)} = 0, \tag{4.24}$$

$$B_0^{(2)} = 2S, \tag{4.25}$$

where S is the area under the curve $Y = F(X)$.

5. The matching of inner and outer expansions

We shall now proceed to relate the inner and outer expansions of the previous sections in order to fix the unknown constants which arose in each of them. This may simply be done using Van Dyke's asymptotic matching principle (Van Dyke 1964, p. 64) which amounts to the following:

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outer expansion of (the p-term inner expansion). (5.1)

Here p and q may be taken as any two integers, equal or unequal. By definition the p-term inner expansion of the q-term outer expansion is found by rewriting it in inner variables, expanding asymptotically for small ε, and truncating the result to p terms; and conversely for the right-hand side (RHS) of (5.1).

When the matching is attempted, it can be seen that terms such as ε and $\varepsilon\ln\varepsilon$ appear, so let us assume that the first terms in the asymptotic sequence for both inner and outer expansions are $1, \varepsilon\ln\varepsilon, \varepsilon, \varepsilon^2\ln\varepsilon$ and ε^2. Initially, we apply (5.1) with $p = q = 3$, that is, up to terms of order ε in each expansion. Firstly, we form the RHS of (5.1) from the first three terms of the inner expansion, that is, from $W_s^{(1)}(\zeta)$ which, written in inner variables, becomes

$$\varepsilon\left\{A_0^{(1)}+A_1^{(1)}Re\left[b^{-1}Z+\sum_{n=1}^{\infty}c_nZ^{-n}\right]\right.$$
$$\left.-\sum_{n=1}^{\infty}(\pi n)^{-1}B_n^{(1)}Re\left[b^nZ^{-n}+\sum_{m=1}^{\infty}e_{-n,m}Z^{-m-n-1}\right]\right\} \quad (5.2)$$

valid for $|\zeta| > \max(|\lambda_1|, |\lambda_p|)$ where we have made use of (4.14), (4.16) and (4.24).

We rewrite (5.2) in outer variables and expand it asymptotically for small ε. Retaining 3 terms we have for the RHS of (5.1)

$$A_1^{(1)}b^{-1}r\cos\theta+\varepsilon A_0^{(1)}. \quad (5.3)$$

Next we construct the left-hand side (LHS) of (5.1) from the 3 terms of the outer expansion

$$\varepsilon[a_0^{(1)}H_0^{(2)}(r)+a_1^{(0)}H_1^{(2)}(r)\cos\theta], \quad (5.4)$$

which written in inner variables becomes

$$\varepsilon[a_0^{(1)}H_0^{(2)}(\varepsilon R)+a_1^{(0)}H_1^{(2)}(\varepsilon R)\cos\Theta]. \quad (5.5)$$

In the asymptotic expansion of (5.5) for small ε, we use the expansion of the Hankel function for small values of its argument given by

$$H_n^{(2)}(r)=\begin{cases}1-\dfrac{2i}{\pi}\left(\ln\dfrac{r}{2}+\gamma\right)+0(r^2\ln r), & n=0\\[2ex] i\dfrac{(n-1)!}{\pi}\left(\dfrac{2}{r}\right)^n+0(r^{-n-2}), & n\neq 0\end{cases} \quad (5.6)$$

where γ is the Euler constant ($\gamma = 0{\cdot}5772$). By retaining 3 terms we get the LHS of (5.1), i.e.

$$a_1^{(0)}\frac{2i}{\pi}\frac{\cos\Theta}{R}-\varepsilon\ln\varepsilon\frac{2i}{\pi}a_0^{(1)}+\varepsilon a_0^{(1)}\left[1-\frac{2i}{\pi}\left(\ln\frac{R}{2}+\gamma\right)\right]. \quad (5.7)$$

To match we write (5.7) in outer variables and compare similar terms with (5.3).

Comparison of the coefficients of 1, $\ln r$, $r\cos\theta$ and $r^{-1}\cos\theta$ yields four equations for the four constants $a_0^{(1)}$, $a_1^{(0)}$, $A_0^{(1)}$ and $A_1^{(1)}$. This produces

$$\left.\begin{aligned} a_0^{(1)} &= a_1^{(0)} = 0,\\ A_0^{(1)} &= A_1^{(1)} \sim 0. \end{aligned}\right\} \tag{5.8}$$

Hence, there is no term of order $\varepsilon\ln\varepsilon$ in the inner expansion and no term of order ε in the outer expansion.

The next step is to use the matching principle with $p = q = 5$, that is, up to terms of order ε^2 in each expansion. Then the RHS of (5.1) becomes

$$A_2^{(0)}b^{-2}r^2\cos\theta+\varepsilon A_1^{(2)}b^{-1}r\cos\theta-\varepsilon^2\ln\varepsilon B_0^{(2)}\pi^{-1}$$
$$+\varepsilon^2\{-B_1^{(1)}b\pi^{-1}r^{-1}\cos\theta+A_0^{(2)}+A_2^{(2)}Re[e_{2,1}]+B_0^{(2)}\pi^{-1}\ln r-B_0^{(2)}\pi^{-1}\ln|b|\} \tag{5.9}$$

and the LHS of (5.1) is given by

$$a_2^{(0)}\frac{i}{\pi}\left[\frac{2}{R}\right]^2\cos 2\Theta+\varepsilon a_1^{(1)}\frac{2i}{\pi}\frac{\cos\Theta}{R}-\varepsilon^2\ln\varepsilon\frac{2i}{\pi}a_0^{(2)}$$
$$+\varepsilon^2 a_0^{(2)}\left[1-\frac{2i}{\pi}\left(\ln\frac{R}{2}+\gamma\right)\right]. \tag{5.10}$$

After writing (5.10) in outer variables and comparing with (5.9) the coefficients of $r^2\cos 2\theta$, $r\cos\theta$, 1, $\ln r$, $r^{-1}\cos\theta$ and $r^{-2}\cos 2\theta$ we have six equations to fix the six unknowns: $a_0^{(2)}$, $a_1^{(1)}$, $a_2^{(0)}$, $A_0^{(2)}$, $A_1^{(2)}$ and $A_2^{(2)}$. The results are

$$\left.\begin{aligned} a_0^{(2)} &= \frac{i}{2}B_0^{(2)},\\ a_1^{(1)} &= \frac{ib}{2}B_1^{(1)},\\ a_2^{(0)} &= 0,\\ A_0^{(2)} &\sim \frac{iB_0^{(2)}}{2}\left[1-\frac{2i}{\pi}\left(\ln\frac{|b|}{2}+\gamma\right)\right]+\ln\varepsilon\frac{B_0^{(2)}}{\pi},\\ A_1^{(2)} &\sim 0,\\ A_2^{(2)} &\sim 0. \end{aligned}\right\} \tag{5.11}$$

Hence the inner and outer expansions become as $\varepsilon \to 0$

$$W_s(\zeta) \sim \frac{\varepsilon}{\pi}\int_{\lambda_1}^{\lambda_p} M^{(1)}(\xi',0)\left|\frac{dZ}{d\zeta'}\right|(\xi',0)\ln|\xi'-\zeta|d\xi'$$
$$+\varepsilon^2\ln\varepsilon\frac{B_0^{(2)}}{\pi}+\varepsilon^2\frac{iB_0^{(2)}}{2}\left[1-\frac{2i}{\pi}\left(\ln\frac{|b|}{2}+\gamma\right)\right]$$
$$-\frac{\varepsilon^2}{\pi}\int_{\lambda_1}^{\lambda_p} M^{(2)}(\xi',0)\left|\frac{dZ}{d\zeta'}\right|(\xi',0)\ln|\xi'-\zeta|d\xi', \tag{5.12}$$

and

$$w_s(\mathbf{x}) \sim \varepsilon^2\left[\frac{iB_0^{(2)}}{2}H_0^{(2)}(r)+\frac{ib}{2}B_1^{(1)}H_1^{(2)}(r)\cos\theta\right], \tag{5.13}$$

respectively. The solution given by (5.12) and (5.13) is valid asymptotically as ε tends to zero, and so may be expected to give a satisfactory approximation to the solution when ε is small. It contains the smallest number of terms that could conceivably model the solution adequately, since the far field (5.13) is of order ε^2. In practice, equations (5.12) and (5.13) should provide a good approximation to the solution when $\varepsilon < 0{\cdot}1$, although this cannot be justified rigorously. In the following section, it will be demonstrated that (5.13) predicts the far field within 10 per cent when ε is as large as 0.3, in the one case for which an exact solution is available for comparison.

If we introduce the real coefficients a_M and a_D, which depend exclusively on the shape of the topography, defined by

$$a_M = S, \tag{5.14}$$

$$a_D = \frac{ib}{2}\,\frac{B_1^{(1)}}{\cos\psi}, \tag{5.15}$$

the expression (5.13) becomes as $\varepsilon \to 0$

$$w_s(\mathbf{x}) \sim \varepsilon^2[ia_M H_0^{(2)}(r) + a_D H_1^{(2)}(r)\cos\theta\cos\psi] \tag{5.16}$$

and a_M, a_D have the significance of monopole and dipole source amplitudes. An expression similar to (5.13) or (5.16) is obtained in Appendix A under the assumption that the slope of the irregularity is small. If we write (A.5) in outer variables, we have

$$w_s(\mathbf{x}) = \varepsilon^2\, S[iH_0^{(2)}(r) + H_1^{(2)}(r)\cos\theta\cos\psi] + 0(\varepsilon^3). \tag{5.17}$$

Comparing (5.16) with (5.17) we see that the coefficient of $H_0^{(2)}(r)$ is the same in both cases in spite of the fact that the latter is derived under a stronger restriction than that of the former, namely, it assumes a very small slope. On the other hand, the coefficient of $H_1^{(2)}(r)$ differs in general.

6. Examples

Although the major part of this section will relate to irregularities of polygonal shape, a simple solution exists for a surface irregularity, consisting of a semi-circular groove. To fix ideas, therefore, this will be examined first. The function

$$\zeta = z + 1/z \tag{6.1}$$

maps the grooved half-plane into the upper half of the ζ-plane (Morse & Feshbach 1953, p. 1227). First, it is readily obtained from (5.14) and (5.15) that

$$\left.\begin{aligned} a_M &= \pi/2, \\ a_D &= \pi, \end{aligned}\right\} \tag{6.2}$$

having evaluated $B_1^{(1)}$ from (4.23). The displacement of the groove itself is also of interest, and this can be obtained directly from the inner solution. Equation (5.12) gives, when $z = \exp(i\theta_0)$,

$$\begin{aligned} W_s \sim{}& \frac{2i\varepsilon}{\pi}\cos\psi\int_0^\pi \cos\theta\ln|\cos\theta-\cos\theta_0|d\theta + \varepsilon^2\ln\varepsilon + \frac{i\varepsilon^2\pi}{2} \\ &-\varepsilon^2(\ln 2-\gamma) + \frac{2\varepsilon^2}{\pi}\left[\cos 2\psi\int_0^\pi \cos^2\theta\ln|\cos\theta-\cos\theta_0|d\theta\right. \\ &\left.+\sin^2\psi\int_0^\pi \ln|\cos\theta-\cos\theta_0|d\theta\right]. \end{aligned} \tag{6.3}$$

The total displacement W is the sum of W_i, W_r and W_s, where the former two are given by equations (4.5). Expanding W_i+W_r to order ε^2 gives, when $z = \exp(i\theta_0)$,

$$W_i+W_r \sim 2[1+i\varepsilon\cos\psi\cos\theta_0-(\varepsilon^2/2)(\cos^2\psi\cos^2\theta_0+\sin^2\psi\sin^2\theta_0)]. \tag{6.4}$$

Hence, from (6.3) and (6.4), the change in amplitude induced by W_s is of order ε^2. The change in phase is of order ε, except that, at the bottom of the groove, $\theta_0 = \pi/2$ and this term vanishes. When $\psi = \pi/2$, so that the incident wave is normal to the plane free surface, the total displacement at the bottom of the groove is given by

$$W_i+W_r+W_s \sim 2+\varepsilon^2\ln\varepsilon+\frac{i\varepsilon^2\pi}{2}-\varepsilon^2(\ln 2-\gamma)-\frac{2\varepsilon^2}{\pi}\int_0^\pi \sin^2\theta\ln|\cos\theta|d\theta, \tag{6.5}$$

in which the integral has the value $-\pi(2\ln 2+1)/4$, precisely.

The problem of scattering by a semi-circular groove has been solved exactly by Trifunac (1973), by separation of variables. The solution was in the form of an infinite series like (3.6) whose coefficients were determined directly from the boundary conditions on surface $r = 1$ of the groove. When r is large, the scattered field is given asymptotically by the first two terms in the series, and so has the form (5.16). In our notation, Trifunac's results become

$$a_M = \frac{2iJ_1(\varepsilon)}{\varepsilon^2 H_1^{(2)}(\varepsilon)}, \qquad a_D = \frac{-4i[\varepsilon J_0(\varepsilon)-J_1(\varepsilon)]}{\varepsilon^2[\varepsilon H_0^{(2)}(\varepsilon)-H_1^{(2)}(\varepsilon)]},$$

which agree precisely with (6.2) as $\varepsilon \to 0$, and which are within 10 per cent of (6.2) when $\varepsilon < 0.3$. Unfortunately no such simple check can be made on (6.5), because each term of Trifunac's series is of order ε^2, so that the integral in (6.5) would need to be compared with the sum of the series. Also, the smallest value of ε considered by Trifunac was $\pi/4$, so that (6.5) cannot be compared directly against his computed results.

We shall now consider examples for which the shape of the surface $Y = F(X)$ is piecewise linear. The conformal transformation $Z = g(\zeta)$ of the upper half of the ζ-plane into D can be effected by means of the Schwartz–Christoffel transformation (see for instance Kantorovich & Krylov 1958, p. 521), i.e.

$$g(\zeta) = g_1\int_{\lambda_0}^{\zeta}\prod_{i=1}^{p}(\zeta-\lambda_i)^{\alpha_i-1}d\zeta+g_2, \tag{6.6}$$

where g_1, g_2 and λ_0 are complex constants, λ_i $(i = 1, 2, \ldots, p)$ are points on the real axis of the ζ-plane which correspond to the vertices Λ_i $(i = 1, 2, \ldots, p)$ of the polygon in the Z-plane and $\alpha_i\pi$ $(i = 1, 2, \ldots, p)$ are the magnitudes of the interior angles of the polygon, as shown in Fig. 1. They satisfy the equality

$$\sum_{i=1}^{p}\alpha_i = p. \tag{6.7}$$

The points λ_i are chosen so that they lie in a finite interval. Also we select the origin of co-ordinates so that $-\lambda_1 = \lambda_p$. The points outside the interval $[\lambda_1, \lambda_p]$ on the real axis are mapped into the points on $Y = 0$ exterior to the irregularity.

We shall now show that (6.6) has an expansion in the neighbourhood of the point at infinity of the form (4.13).

Rewrite the integrand of (6.6) in the form

$$\prod_{i=1}^{p}\left(1-\frac{\lambda_i}{\zeta}\right)^{\alpha_i-1}. \tag{6.8}$$

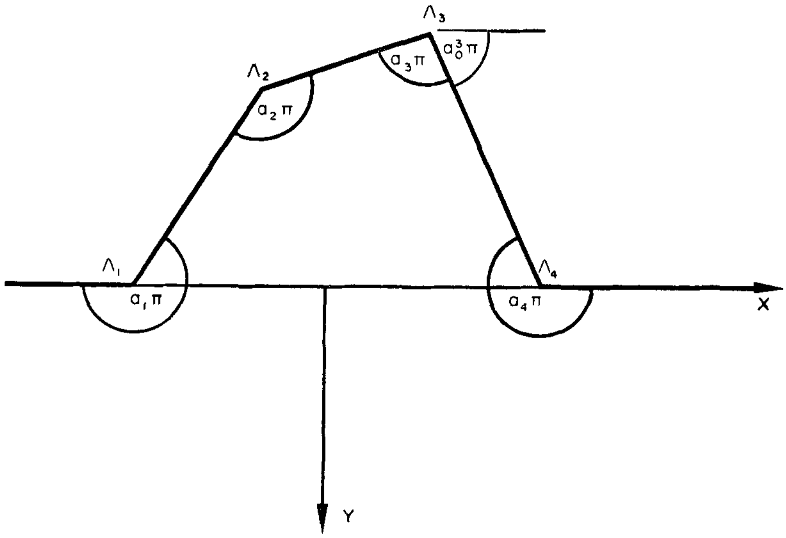

FIG. 1. Illustration of a polygonal irregularity, with $p = 4$. $\alpha_0{}^i\pi$ is the angle made by the side $\Lambda_i\ \Lambda_{i+1}$ with the X-axis and is shown for $i = 3$.

Here we used the relationship (6.7). Applying the binomial theorem we obtain the following expansion for (6.8) in the neighbourhood of the point at infinity

$$1+\sum_{n=1}^{\infty} \bar{b}_n \zeta^{-n}, \tag{6.9}$$

valid for $|\zeta| > |\lambda_1|$. After integration the right-hand side of (6.6) follows as

$$b\zeta+b^* \ln \zeta+b_0+\sum_{n=1}^{\infty} b_n \zeta^{-n}. \tag{6.10}$$

Clearly

$$b = g_1 \tag{6.11}$$

and

$$b^* = g_1 \sum_{i=1}^{p} (1-\alpha_i)\lambda_i. \tag{6.12}$$

The expansion (6.10) is of the same type as (4.13) provided $b^* = 0$. Since three of the parameters λ_i ($i = 1, 2, \ldots, p$) are at our disposal (Kantorovich & Krylov 1958, p. 524), this can always be achieved, and our formalism is applicable.

Now let (L_i, H_i) be the co-ordinates of the ith vertex of the polygon, so that $H_1 = H_p = 0$ and let (L_0, H_0) correspond to the point $\zeta = \lambda_0$. If the normalizing factor l' is chosen as $|L_2-L_1|$, it follows that

$$g_1 = [(L_2-L_1)^2+H_2{}^2]^{\frac{1}{2}}/J_0(\lambda_1, \lambda_2), \tag{6.13}$$

$$g_2 = L_0+iH_0 \tag{6.14}$$

and

$$B_1{}^{(1)} = 2|g_1| \cos\psi \sum_{i=1}^{p-1} \sin \alpha_0{}^i \pi J_1(\lambda_i, \lambda_{i+1}), \tag{6.15}$$

where

$$J_j(\lambda_i, \lambda_{i+1}) = \int_{\lambda i}^{\lambda i+1} \xi^j \prod_{k=1}^{i} (\xi-\lambda_k)^{\alpha k-1} \prod_{k=i+1}^{p} (\lambda_l-\xi)^{\alpha_l-1}\, d\xi, \tag{6.16}$$

for $i = 1, 2, \ldots, p-1$, $j = 0, 1$. ${\alpha_0}^i \pi$ is the angle formed between the side $\Lambda_i \Lambda_{i+1}$ of the polygon and the X-axis. (See Fig. 1.)

Now we shall examine some particular examples and compare the coefficient a_D derived by the regular perturbation technique (RPT) for the case where the slope of the topography is assumed to be small, and that derived by the method of matched asymptotic expansions (MAE) where the magnitude of the slope is arbitrary.

As a first example we consider a surface irregularity in the form of an isosceles triangle with vertices at the points $\Lambda_1(-1, 0)$, $\Lambda_2(0, H)$, $\Lambda_3(1, 0)$ and interior angles $\alpha\pi$, $(3-2\alpha)\pi$, $\alpha\pi$ at each vertex respectively; $H \gtrless 0$ according to $\alpha \lesseqgtr 1$. We map the points $-1, 0$ and 1 on the real axis of the ζ-plane into the points Λ_1, Λ_2 and Λ_3 of the Z-plane. The condition $b^* = 0$ is thus satisfied. Choosing $\lambda_0 = 0$, the mapping function takes the form

$$Z = \frac{2[1+H^2]^{\frac{1}{2}}}{B(\frac{3}{2}-\alpha, \alpha)} \int_0^{\zeta} \left(\frac{\zeta^2-1}{\zeta^2}\right)^{\alpha-1} d\zeta + iH, \tag{6.17}$$

where

$$B(\beta_1, \beta_2) = \int_0^1 t^{\beta_1-1}(1-t)^{\beta_2-1}\, dt \quad (\beta_1, \beta_2 > 0) \tag{6.18}$$

is the Beta function.

Hence

$$a_D = 4H[1+H^2]^{\frac{1}{2}} B(2-\alpha, \alpha)[B(\tfrac{3}{2}-\alpha, \alpha)]^{-2}. \tag{6.19}$$

We plot a_D against H in Fig. 2. We recall that H is normalized with respect to one half of the triangle base, so, in other words, in Fig. 2 a_D is plotted against a measure of the slope of the irregularity (i.e. $(dF/dX) = H$). We distinguish two cases: (a) a grooved surface or $H > 0$ and (b) a ridged surface or $H < 0$. For the former case, a_D obtained by MAE is approximately linear in the range from $H = 0$ to about $H = -0{\cdot}1$ where both theories agree. Outside this region, RPT and MAE disagree; a_D derived from MAE appears to tend to the value $-0{\cdot}57$ well below the values predicted by RPT. For the latter case, a_D obtained by MAE is approximately linear from $H = 0$ to about $H = 0{\cdot}4$ where both theories agree. Outside this range, the coefficient a_D derived from MAE increases monotonically very rapidly with values above the ones predicted by RPT. We note that $|a_M| < |a_D|$ for MAE.

As a second example we shall consider an isosceles trapeziform topography with vertices at the points $\Lambda_1(-1, 0)$, $\Lambda_2(-L, H)$, $\Lambda_3(L, H)$, $\Lambda_4(1, 0)$ and interior angles $\alpha\pi$, $(2-\alpha)\pi$, $(2-\alpha)\pi$, $\alpha\pi$ at each vertex respectively. The value of L is taken to be positive while $H \gtrless 0$ according to $\alpha \gtrless 1$. We map the points $-1/K$, -1, 1 and $1/K$, where $0 < K < 1$, of the real axis of the ζ-plane, into the points Λ_1, Λ_2, Λ_3 and Λ_4 of the Z-plane. Again $b^* = 0$ is satisfied. The value of K remains to be determined by solving an equation which relates the known ratio of the lengths of two sides of the trapezium, i.e.

$$J_0(-1, 1) = PJ_0(1, 1/K) \tag{6.20}$$

where

$$P = \frac{\overline{\Lambda_2 \Lambda_3}}{\overline{\Lambda_3 \Lambda_4}} = 2L[(1-L)^2+H^2]^{-\frac{1}{2}} \tag{6.21}$$

and $J_0(-1, 1)$ and $J_0(1, 1/K)$ are defined in (6.16). This equation may be solved

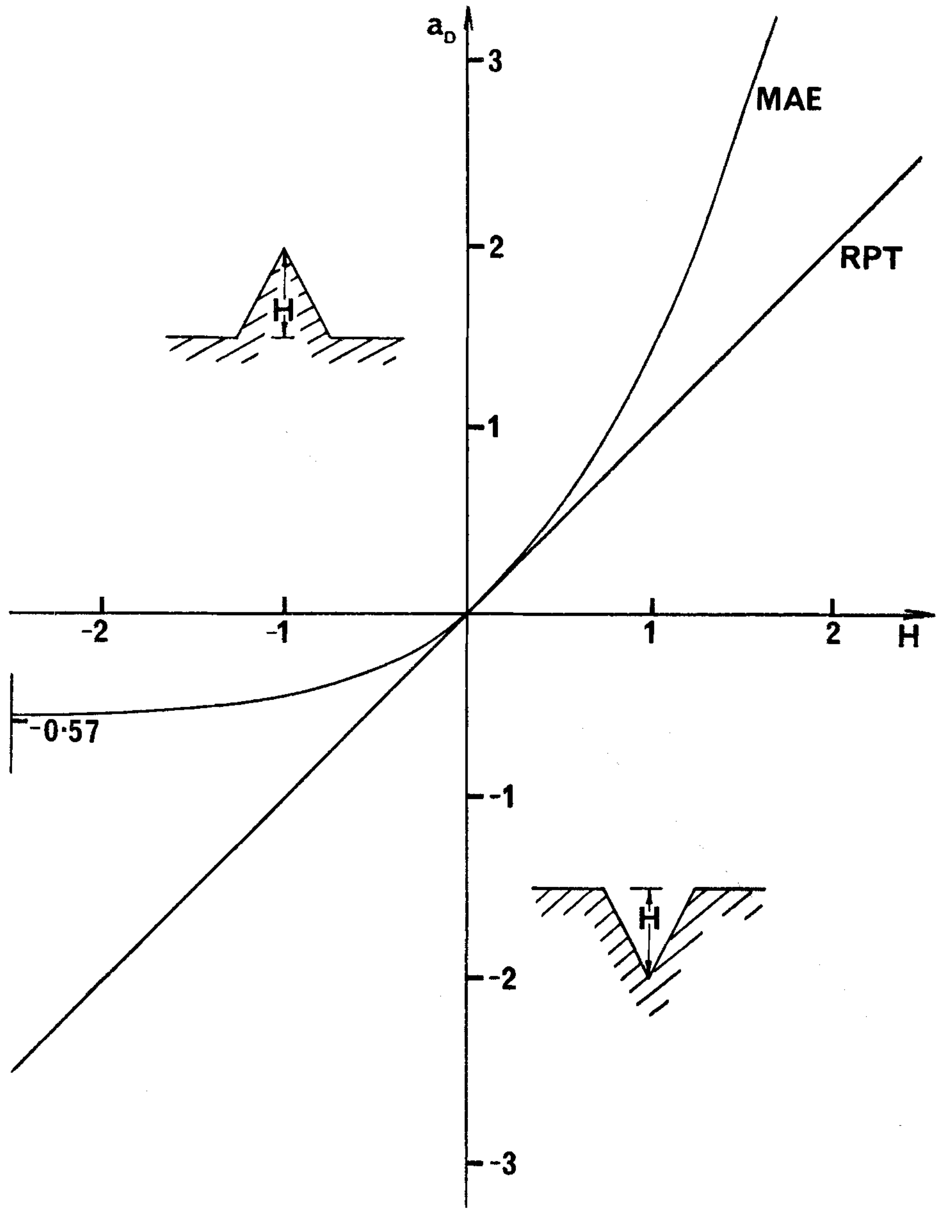

FIG. 2. The coefficient a_D plotted against normalized height H for a triangular irregularity. A groove corresponds to $H > 0$ whereas a ridge is for $H < 0$. The results from the regular perturbation technique (RPT) and the method of matched asymptotic expansions (MAE) are shown. Note the agreement of both theories only when H (or the slope) is small, as expected.

numerically for K by, for instance, an iterative procedure such as the Newton–Raphson method.

The mapping function becomes

$$Z = \frac{[(1-L)^2+H^2]^{\frac{1}{2}}}{J_0(1, 1/K)} \int_{\lambda 0}^{\zeta} \left(\frac{\zeta^2-1/K^2}{\zeta^2-1}\right)^{\alpha-1} d\zeta + L_0 + iH_0. \qquad (6.22)$$

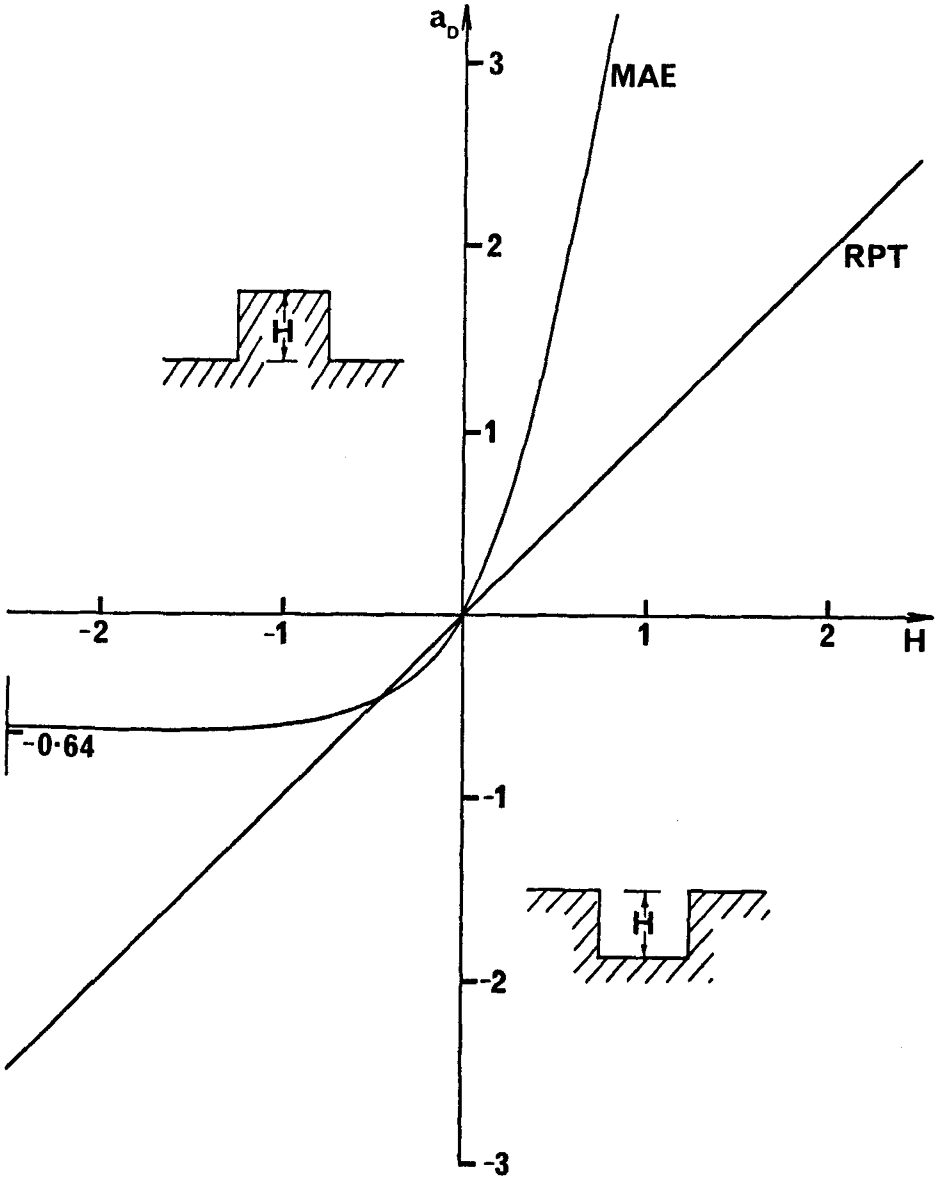

FIG. 3. As in Fig. 2 but for a rectangular irregularity. Observe that RPT is not valid here even for small H. MAE yields values which RPT cannot produce.

Hence

$$a_D = 2H[(1-L)^2+H^2]^{\frac{1}{2}} J_1(1, 1/K)[J_0(1, 1/K)]^2. \tag{6.23}$$

We now consider two trapezia: (a) a rectangle with $L = 1$ and (b) a trapezium somewhat in between a rectangle and a triangle, so to speak, with $L = 1/4$.

The RPT cannot strictly apply for the rectangular irregularity, since the slope can never be small. Fig. 3 shows plots of a_D against H for this case, obtained from both

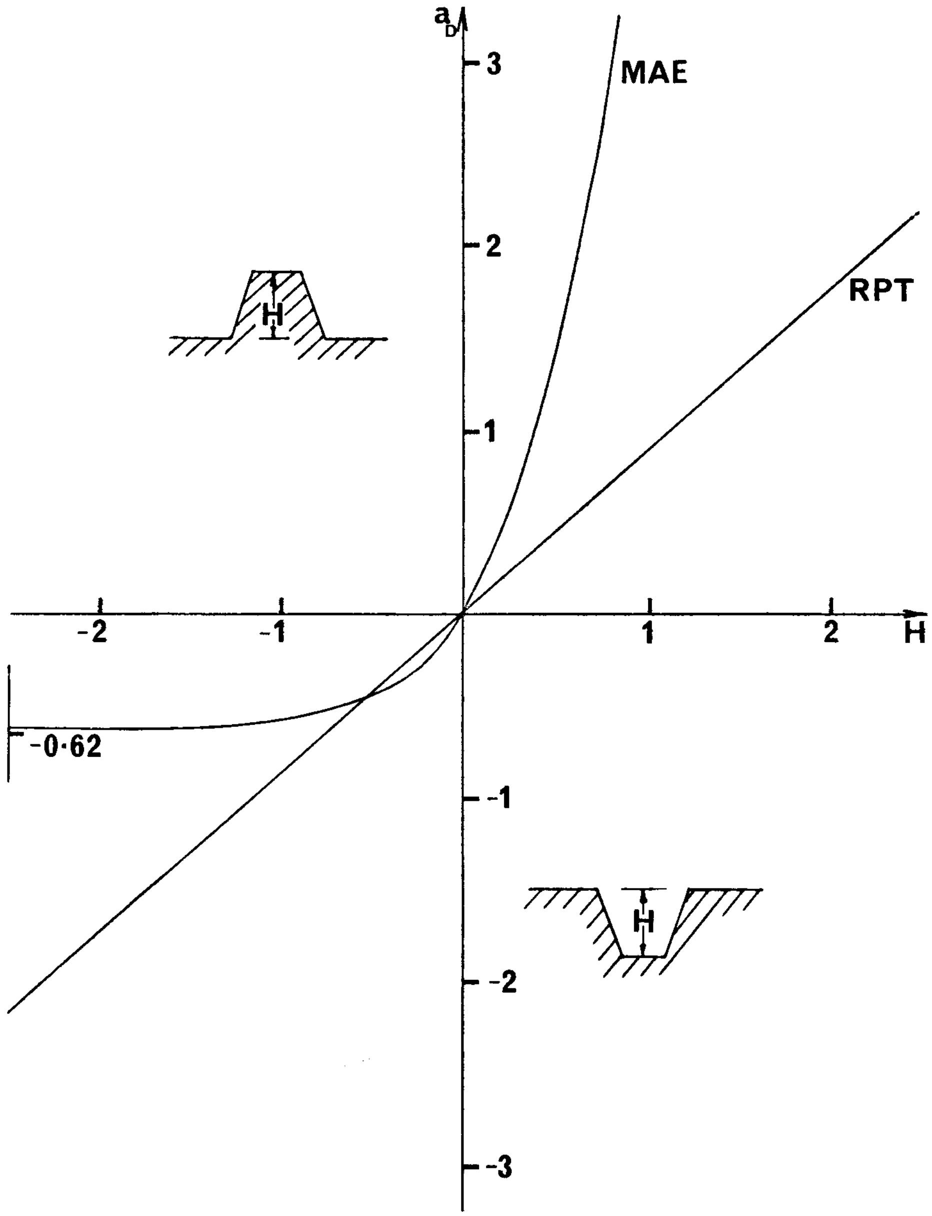

FIG. 4. As in Fig. 2 but for a trapeziform irregularity. The common region of coincidence of RPT and MAE is even smaller than that of the triangular irregularity.

MAE and from the formal application of RPT and shows a discrepancy, even when H is small. But we can compare the a_D's for the other trapezium, whose slope may be very small when its height is very small. The curve describing a_D against H, shown in Fig. 4, is very similar to that obtained for the triangular irregularity. The common region of validity of RPT and MAE is smaller than that for the triangular shape. For MAE the magnitude of a_D is never smaller than the one for a_M save in the range $[-0{\cdot}55, 0]$, more or less. This range becomes approximately $[-0{\cdot}5, 0]$ for the rectangular ridge.

The results show that $|a_D|$ is the greatest for the rectangular irregularity and the smallest for the triangular one.

Finally, the scattered field W_s has been evaluated on the surface of the irregularity when this has the form of an isosceles triangle. In this case, equation (5.12) reduces to the form

$$W_s(\zeta) \sim \frac{2H\varepsilon^2}{\pi}\ln\varepsilon + \frac{4H\varepsilon^2}{\pi^2}\left[\ln\left(\frac{b}{2}\right)+\gamma\right] + \frac{\varepsilon^2}{\pi}P(\zeta) + \frac{i\varepsilon}{\pi}Q(\zeta) + \frac{2Hi\varepsilon^2}{\pi}, \tag{6.24}$$

where

$$b = 2(1+H^2)^{\frac{1}{2}}/B(\tfrac{3}{2}-\alpha, \alpha) \tag{6.25}$$

and the functions $P(\zeta)$, $Q(\zeta)$ are given below, for the case $\zeta = \xi$, corresponding to points on the free surface.

$$P(\xi) = \frac{4H}{B(\frac{3}{2}-\alpha, \alpha)}\{R_1(\xi)\cos 2\psi + R_2(\xi)\sin^2\psi\}, \tag{6.26}$$

$$Q(\xi) = \frac{4H\cos\psi}{B(\frac{3}{2}-\alpha, \alpha)}Q_2(\xi), \tag{6.27}$$

where

$$R_1(\xi) = \frac{2}{B(\frac{3}{2}-\alpha, \alpha)}\int_0^1 dx\left(\frac{1-x^2}{x^2}\right)^{\alpha-1}[\ln|x-\xi|+\ln|x+\xi|]\int_0^x\left(\frac{1-y^2}{y^2}\right)^{\alpha-1}dy, \tag{6.28}$$

$$R_2(\xi) = \int_0^1\left(\frac{1-x^2}{x^2}\right)^{\alpha-1}[\ln|x-\xi|+\ln|x+\xi|]dx \tag{6.29}$$

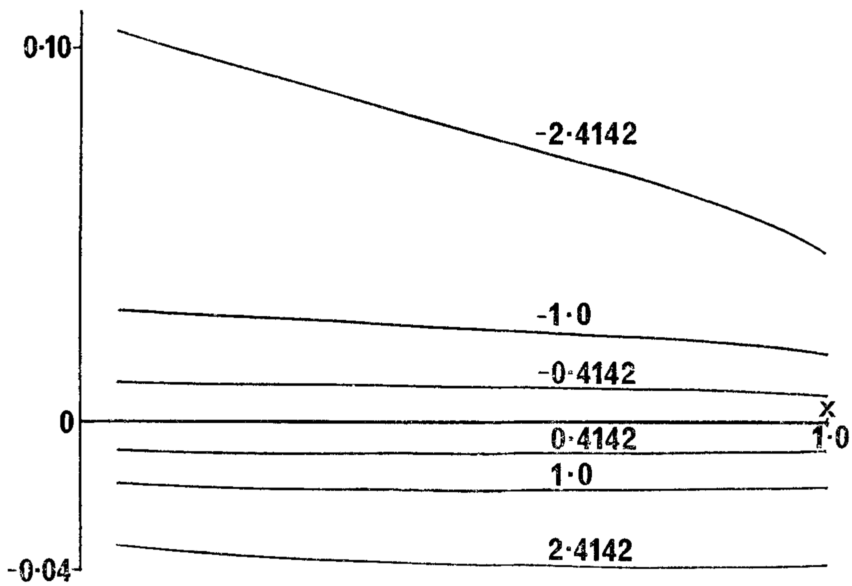

FIG. 5. Plots of the real part of W_s, as given by equation (6·24), for a range of heights H, against the corresponding co-ordinate X, when $\psi = \pi/4$ and $\varepsilon = 0{\cdot}1$.

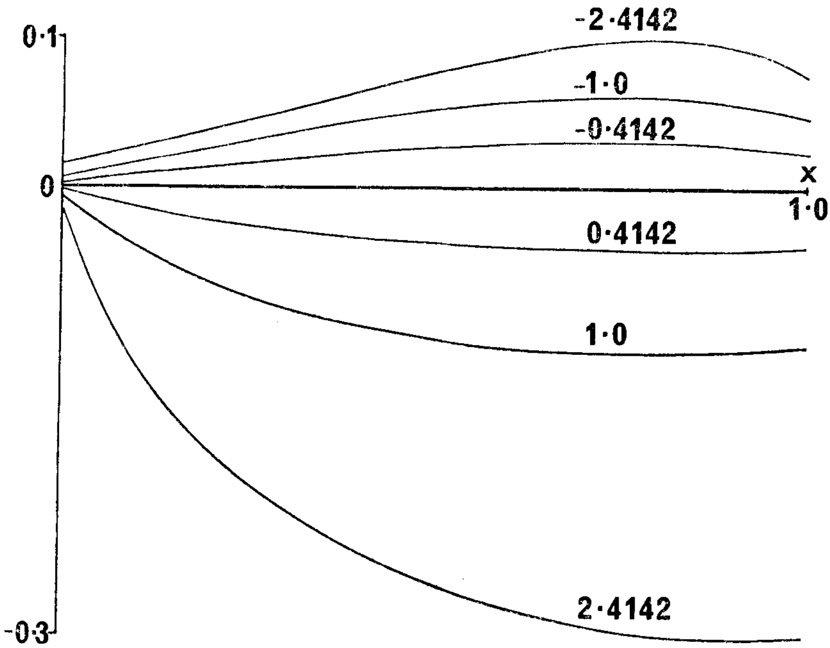

FIG. 6. Plots of the imaginary part of W_s, as given by equation (6·24), for a range of heights H, against the corresponding co-ordinate X, when $\psi = \pi/4$ and $\varepsilon = 0{\cdot}1$.

and

$$Q_2(\xi) = \int_0^1 \left(\frac{1-x^2}{x^2}\right)^{\alpha-1} [\ln|x-\xi| - \ln|x+\xi|]dx, \tag{6.30}$$

the variable x being taken just above the real axis. The co-ordinates (X, Y) corresponding to $\zeta = \xi$ can be obtained from (6.17); they are given as

$$\left.\begin{aligned} X &= \frac{2}{B(\frac{3}{2}-\alpha, \alpha)} \int_0^\xi \left(\frac{1-x^2}{x^2}\right)^{\alpha-1} dx, \\ Y &= H(1-|X|) \end{aligned}\right\} \tag{6.31}$$

when $|\xi| < 1$.

It may be noted that, as for the semi-circular groove, the amplitude change induced by W_s is of order ε^2, while the phase change is of order ε, except near the vertex $X = 0$ or when $\psi = \pi/2$. Values of $P(\xi)$, $Q(\xi)$ and the corresponding (X, Y) have been calculated for a range of values of α(or H) and angles of incidence ψ. Representative plots of the real and imaginary parts of the scattered field W_s are given in Figs 5 and 6 respectively, for $\psi = \pi/4$ with $\varepsilon = 0{\cdot}1$.

7. Conclusions

We have obtained explicit expressions for the far- and near-scattered fields when the slope of the isolated surface irregularity is arbitrary and its amplitude is small in comparison with the wavelength of the incident wave. This contrasts with earlier work, employing regular perturbation techniques, which also requires slopes to be small.

The far-scattered field agrees with that derived from regular perturbation theory when the irregularity has small slope but differs in the general case. The amplitude of the scattered wave, far from the irregularity, is of second order in the ratio characteristic dimension of irregularity: wavelength, as is the perturbation of the amplitude of the field in the vicinity of the irregularity, but the phase of the latter is of first order in this ratio.

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Appendix A

In this appendix we consider, by an alternative method, the two-dimensional problem of the scattering of plane harmonic *SH* waves incident upon the surface of a homogeneous, isotropic, elastic half-space with an approximately plane surface $y' = f'(x')$, where the magnitude of $f'(x')$ compared with the wavelength of the incident wave and the slope $df'(x')/dx'$ are both assumed to be small.

Following the regular perturbation technique (see, for instance, Gilbert & Knopoff 1960) we replace the topography by an equivalent stress distribution $\mu T'(x')$ applied to the plane $y' = 0$, where

$$T'(x') = \frac{df'}{dx}\left[\frac{\partial w_i'}{\partial x'} + \frac{\partial w_r'}{\partial x'}\right]_{y'=0} - f'\left[\frac{\partial^2 w_i'}{\partial y'^2} + \frac{\partial^2 w_r'}{\partial y'^2}\right]_{y'=0} \tag{A.1}$$

to the first order. The stress distribution $\mu T'(x')$ is chosen to correspond to the loading of the plane $y' = 0$ due to the *SH* wave incident and reflected upon the original irregular topography $y' = f'(x')$. We then solve

$$\left.\begin{aligned} &\nabla^2_{x',y'} w_s' + k^2 w_s' = 0, \quad \text{in } y' > 0, \\ &\frac{\partial w_s'}{\partial y'} = T'(x'), \quad \text{on } y' = 0, \\ &\lim_{r'\to\infty} r'^{\frac{1}{2}}\left(\frac{\partial w_s'}{\partial r'} + ikw_s'\right) = 0, \end{aligned}\right\} \tag{A.2}$$

by the Green function technique. Our problem has the formal solution

$$w_s'(x', y') = \frac{i}{2}\int_{l'_1}^{l'_p} H_0^{(2)}(kR)\,T'(x^*)\,dx^*, \tag{A.3}$$

where $R = [(x'-x^*)^2 + y^{*2}]^{\frac{1}{2}}$, since the irregularity is of finite extent.

We now find the far-field scattered displacement. Assuming that $|\mathbf{x}^*|/|\mathbf{x}'|$ is small, we may write

$$R = |x'| - \frac{\mathbf{x}'.\mathbf{x}^*}{|x'|} \tag{A.4}$$

to the first order, and hence

$$w_s'(x', y') = k^2 s'[iH_0^{(2)}(kr') + H_1^{(2)}(kr')\cos\theta'\cos\psi] + 0(k^3) \tag{A.5}$$

after expanding the resulting integrals in powers of k. Here s' is the area under the curve $y' = f'(x')$.