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SCATTERING OF THE RAYLEIGH WAVE BY A

SEMI-CIRCULAR ROUGH SURFACE

Takao MOMOI

Earthquake Research Institute, University of Tokyo, Tokyo, Japan

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A theoretical study is made on the scattering of a Rayleigh wave which is in-
cident on a rough semi-circular surface. Partition of energy fluxes of the waves
scattered by the semi-circular rough surface is obtained. It is then revealed
that the resonances of P and S waves occurring in the semi-circular body con-
siderably suppress the transmission of the incident Rayleigh wave through the
semi-circular rough surface, while the Rayleigh wave resonance appearing along
the curved free surface of the semi-circular body strengthens the transmission
of the incident Rayleigh wave. Directivity of the energy flux of the waves
scattered into the semi-infinite elastic body is also discussed; partition of the
energy fluxes of the waves scattered into two quarter spaces which are on the in-
coming and leaving sides of the incident wave greatly depends on two factors,
the relative size of the semi-circular body to the wavelength of the incident Ray-
leigh wave and the appearance of the resonance of the Rayleigh wave in the semi-
circular body. In the semi-infinite elastic body, the zone of high energy density of
the scattered S waves runs at an angle of 45° to the plane free surface; this zone
is named breathing zone, since the zone is produced as the results of the inter-
change of energy between the generated Rayleigh waves and the scattered S
waves. As for the phase lag of the transmitted Rayleigh wave, it is retarded
in the range of kb〓1,0 (k: wave number of the incident wave, b: radius of the
semi-circular rough surface) owing to the generation of retarded waves in the
semi-circular body, while it is advanced in the range of kb〓1.0 due to the appear-
ance of the P and S waves which are transmitted through the medium below the
semi-circular body directly from its front root to the rear root.

1. Introduction

The propagation of a Rayleigh wave through a rough surface lying on a semi-infinite elastic body is a difficult as well as an interesting problem. The difficulty is primarily a mathematical one, as problems with other-type irregular boundaries encounter (SATO, 1949; SATO, 1967). Hence there exist no available theoretical studies on the problem with an irregular boundary of finite dimensions except such works which are done under the assumption that the slope and magnitude of the roughness are small as compared with the wave length of the incident wave (HOMMA, 1941, 1942a and b; SATO, 1955; GILBERT and KNOPOFF, 1960: and some other

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authors). The present work made by the author is a theoretical study on the propa-gation of a Rayleigh wave through a rough surface with a semi-circular configura-tion; our study then cover the behaviors of the waves in a model with a relatively large- as well as small-scale roughness. In order to explore the above kind of a problem, a new device is introduced. That is to say, the boundary is divided into a large number of smaller boundaries; the boundary condition is represented by the quantities averaged over each divided boundary; a set of the averaged boundary conditions obtained in such a way are used as an infinite system of simultaneous equations which determine the behavior of the waves in the field in question. This procedure is found to be very useful in the present problem. In section 2, a theory is developed by use of the newly introduced device. In section 3, the numerical computations and the discussions of the results are carried out.

2. Theory

2.1 Model and equations
The model used in this work is illustrated in Fig. 1. A semi-circular cylindrical
mountain range (with radius b) is located along the y-axis (normal to the x-axis) on
a semi-infinite elastic body which extends in the range z≧0. The elsatic medium is
assumed homogeneous and isotropic. It is assumed that the displacements of the in-cident wave lie in a plane perpendicular to the mountain range; the problem is,
therefore, two-dimensional. Let (u,w) be the displacement components of the
medium in the (x,z) directions. The equations of motion of the medium in the case
of periodic motion are expressed by

(1a)

(1b)

(1c)

Fig. 1. Model used and some nomenclatures. WR and LR denote the roots of the mountain, respectively, on the incoming (windward) and leaving (leeward) sides of the
incident waves.

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where ρ: density of the elastic medium, σ: angular frequency in the time factor eiσt (t: variable of time) and λ,μ: Lame's constants. In the later development of the
theory, Cartesian coordinates in (1a) are appropriately transformed to polar coor-
dinates (r,θ) with x=r cos θ and z=r sin θ.

2.2 Stress conditions and incident wave
On the surface (z=0), conditions of normal and shear stresses (Zz,Xz) are given by

(2a)

where

(2b)

On the cylindrical surface (r=b), radial and azimuthal stress components are expressed by

(2c)

where up and wp are the azimuthal and radial displacement components, respectively, (the subscript p denotes polar).
The incident Rayleigh wave which satisfies Eq. (1a) and conditions Zz=Xz=0
at z=0 for all x, instead of x restricted in (2a), is expressed by

(3a)

(3b)

where (uR,wR):(u,w) displacement components of the incident Rayleigh wave, and
C: constant. In the above, the time factor eiσt has been omitted. This convention
is used hereafter.

2.3 Formal expressions for waves
Let the entire portion of the elastic medium consist of two domains, i.e., Do:
domain in the range (z≧0,-∞<x<+∞) and DM: domain in the range (r≦b,

-π ≦ θ≦0), where the subscripts O and M signify open body and mountain, respec-

tively.

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Using the Fourier transform technique, the potential φo and ψo in Do satisfying
Eq. (1a) are expressed by

(4a)

where and A, B are unknown coefficients. The branch

cut of α (or β) runs from ξ=±h (or ±k), respectively, to 〓i∞ on the complex
ξ-plane.
The potentials φM and ψM in the mountain (domain DM) are expressed by use of
the Fourier expansion as

(4b)

where A(n)j, B(n)j (j=c,s) are unknown coefficients, Jn(z) the Bassel functions with

argument z, and the abbreviation for .

Substituting (4a) into (1b) and using the boundary conditions (2a, b), we have the displacements (uo,wo) in Do:

(5a)

(5b)

F(ξ)=(2ξ ξ2-kk2)2-4ξξ2α β, (5c)

(5d)

Further reduction of (5d) is made in the Appendix using the expression (4b) in

domain DM. The paths run on the upper and the lower sides of the real axis for
ξ 〓0, respectively.

In the derivation of the azimuthal and radial displacement components (up,wp)
in domain DM, the transformation of Cartesian coordinates into polar ones is re-quired. These displacements are expressed by

(6a)

where φM, ψM are given by (4b).
For the convenience of a later reduction, the displacement components (uM
,wM)i

n the x- and z-directions in domain DM at z=0 are given here by use of the azi-

muthal and radial dispalacement components (u
p,wp) in polar coordinates:

(6b)

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2.4 Infinite system of simultaneous equations
In the derivation of the theory, developed here, we introduced some device in
which continuously running boundaries are divided into a sequence of numerous
small boundaries where the mean values in each division are taken as the values on
real boundaries. Following this procedure, an infinite system of simultaneous equa-
tions is introduced.
The boundaries B_ and B+(-b≦x≦0 and 0≦x≦b at z=0) are divided respec-
tively into M small parts (with width 2Δx) with a reference point at the midpoint of
each subdivision. The reference points are then expressed as xp=(2p〓1)Δx(p=

±1,±2,…,±M) in B± with Δx=b/(2M). The sign ± or 〓 is to be taken in the
same order. The indices of the reference points in B_ are designated by negative p.

In order to obtain the solutions, the displacements at z=0 must be continuous.
Taking the mean value of these conditions in each division {xp-Δx<x<xp+Δx},
we have the set of conditions:

(7a)

Taking the mean value of (5a) in each subdivision and after some reductions, the left-hand side of (7a) is reduced to

(7b)

(7c)

(7d)

with G=(2ξξ2-kk2)-2α β. In (7b), {Xq,Zq} are given by (A.1 and 2) in the Appendix
with the substitution of xq for x. In the actual computations, the integrals (7c, d)
are carried out by use of Simpson's formula.

Taking the mean value of (6b) in each subdivision, the right-hand side of (7a)
becomes

(7e)

In the reduction of the above expression, the Bessel functions in the subdivisions

{xp-Δx<x<xp+Δx} are approximated by their values at the midpoint xp. If the
interval of the subdivision is small, this approximation is permissible.

The medium-free boundaries C_ and C+(π ≦ θ≦3π/2 and 3π/2≦ θ≦2π) in the
semi-circular mountain are divided, respectively, into N subdivisions with width
2Δ θ ・b(～2Δx) with a reference point at the midpoint of each subdivision. The

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reference points are then expressed by θp=π/2-(2p〓1)Δ θ with p from ±1 to ±N

in C± respectively. Taking the mean value of the condition (2c) in each division,

we have the set of conditions:

(8a)

Substituting (6a) with (4b) into RR and ΘR in (2c), we have

(8b)
with

and

(8c)
with

where p covers the integers from -N to -1 and from 1 to N,

and Jhn or Jkn=Jn(hb) or Jn(kb).

We have now arrived at the infinite system of simultaneous equations (7a) (4M eqs.) with (7b, c, d, e) and (8a) (4N eqs.) with (8b, c); these equations determine the 4(Nup+1) unknown coefficients {A(n)c,B(n)c(n=0 to Nup)} and {A(n)s,B(n)s(n=1 to

N)=4(Nup+1); the

Nup+1)}. The number of the uppermost term Nup is given by the relation 4(M+
tions given by

left-hand side of this relation denotes the number of the equa-

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(7a) and (8a). These computations were carried out by use of an
with an accuracy of double precisions or 14 decimal digits.

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2.5 Expressions for generated Rayleigh waves
If our consideration is limited to the behavior of the waves propagated a long distance away from the semi-circular body along the free surface, the waves gener-ated by the irregular surface are given by the pole contributions of the integrals (5a). These waves are the generated Rayleigh waves which are transmitted away from the wave source. After some reductions, the expressions of these waves are described by

(9a)

(9b)

where {αR,βR} and {Xz/μ,Zz/μ} are given by (3b) and by (A.3) in the Appendix, re-
spectively. In the above, {u±GR,w±GR} denote the x and z displacement components of
the "generated" Rayleigh waves which are transmitted away along the free surface
toward x=± ∞. Let {u±R,w±R} be the x and z displacement components of the

"resultant" Rayleigh waves leaving from the semi -circular body to x=± ∞. These

waves are given by

(9c)

where {uR,wR} are the displacement components of the incident Rayleigh waves given by (3a).

2.6 Expressions for the waves scattered into the semi-infinite body
Changing the variable in the integrals (5a) from ξ to φ by the expression ξ=
{h or k} sin φ for {P or S} waves, respectively, and using polar coordinates {x=
r cosθ,z=r sinθ}, the integrals (5a) are reduced to the forms

with

where the path of the integration is from -π/2-i∞ to +π/2+i∞ and {Gju(φ),
Gjw(φ)} are the integrands after the change of the variables. Applying the saddle
point method for large r, we can obtain the expressions of the displacements {uo,wo}
at r=∞ in Cartesian coordinates. Since the expressions obtained in such a way
are described in Cartesian coordinates, the conversion of the coordinates to polar
ones is required by use of the relations {up=-uo sinθ+wo cosθ, wp=uo cosθ+
wosinθ}. After these reductions, we have finally the azimuthal (up) and radial (wp)
displacement components at r=∞ in the semi-infinite elastic body:

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(10a)

(10b)

where {Xz/μ,Zz/μ} are given by (A. 3) in Appendix .

2.7 Expressions for energy fluxes

The energy E transmitted through the surface, with unit width along the y-axis
over one cycle in time is expressed as

(11)

where c: path depending on only (x,z) or (r,θ), ds: infinitesimal length along c, T: period, Xc and Zc: normal and shear stresses on path c, uc and wc: normal and shear displacements on path c (the axis of Zc or wc is positive in the clockwise direction
from that of Xc or uc).

First of all, the energy flux of the incident Rayleigh wave is considered. The
path is then z from 0 to ∞ and {Xc,Zc} are given by {(λ+2μ)∂uc/∂x+λ ∂wc/∂z,
μ(∂wc/∂x+∂uc/∂z)} with {uc, wc}=Re{uReiσt,wReiσt}. Using the expressions

{uR,wR} in (3a) and (11), the energy flux of the incident Rayleigh wave, ER, be- comes

ER=CC2kRπKHR, (12a)
with

(12b)
where the expression in the first or
derived from the expression related
second square brackets namely [ ]n or [ ]s,, is with normal (Xc) or shear (Zc) stress in the
energy integral (11), respectively.
Following the procedure mentioned above for the case of the incident Ray-leigh wave, the energy fluxes (over one cycle in time) of the Rayleigh waves (9c)

transmitted
are expressed as

through and reflected from the semi-circular obstacle, EtrR and EreR,
(then{uc,wc}=Re{u±Reiσt,w±Reiσt})

(12c)

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Consider next the energy flux (over one cycle in time) of the waves scattered
into the semi-infinite elastic body. The path c is then θ from 0 to π at r=∞ and
{Xc,Zc} are given by {RR,ΘR} in (2c), where {uc,wc} are Re{wpeiσt,upeiσt} given in
(10a). Using (11) with the help of the above stresses and displacements, the energy
scattered away into the semi-infinite elastic body, Esc, becomes

Esc=Epsc+Essc, (12d)

with

(12e)

where Epsc or Essc denotes the energy (over one cycle in time) scattered away into the elastic half space by P or S waves.
For the convenience of physical interpretation, the above-obtained energy fluxes are normalized here. The energy flux ER (over one cycle in time) of the incident Rayleigh wave (12a) is used as the normalization factor. Let EtrR, EreR, Esc and Ejsc (j=p,s) be, respectively, the normalized energies of EtrR, EreR, Esc and Ejsc ex-pressed in (12e, d, e). The normalized energies are expressed as

and (12f)

where j is p or s. The conservation of energy is then expressed as

Etotal=EtrR+EreR+Esc=1, (12g)

where Etotal is the total normalized energy.

3. Numerical Computation and Discussions

3.1 Accuracy
A numerical computation following the procedure described hitherto was car-
ried out. First of all, the accuracy of the computation is examined. If the number
of divisions M at the boundaries B±(|x|≦b,z=0) is 6(then Δx/b～0.08) and 8(then
Δx/b～0.06) in the ranges kb, respectively, from 0 to 2.0 and from 2.0 to 3.5 (k: wave
number of S, b: half a width of the semi-circular obstacle), the error of the total
energy |Etotal-1| in (12g) is suppressed to below 0.002; according to the numerical
computations, the accuracy of each energy flux EtrR (EreR,Epsc or Essc) is of the same order as or heiger than that of the total energy Etotal.

3.2 Energy fluxes
Variations of the energy fluxes (over one cycle in time) of the transmitted and
reflected Rayleigh waves (EtrR,EreR) and further the energy fluxes of the scattered

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Fig. 2. Variations ofenergy fluxes of the generated waves versus kb. First (second) figure: energy flux EtrR (EreR) ofthe transmitted (reflected) Rayleigh wave. Third figure: energy flux Esc of the resultant (sum of P and S) scattered waves. Fourth figure: nergy fluxes of scattered P and S waves (Epsc and Essc).

waves (Epsc,Essc,Esc) are given in Fig. 2 versus kb for λ/μ=0.1, 1.0 and 5.0. The
most conspicuous feature in this figure is a sudden decrease in the energy flux EtrR of
the transmitted Rayleigh wave (designated as "S-wave res." in the figure) which oc-
curs at kb〓1.0 or ks〓 π (s: circumferential distance along the semi-circular rough

surface); near this value of kb, resonance of S waves appears in the semi-circular
elastic body (see the following discussion concerning the energy density on the
curved free surface of the semi-circular body). The energy flux EreR (the second
figure in Fig. 2) of the reflected Rayleigh wave takes a maximum near kb=1.0 and
also a marked increase in the energy flux Esc of the scattered waves appears near

this value of kb (see the third figure in Fig. 2).

In order to examine the extent of the trapping of energy in the semi-circular
body, the variations of the energy densities along the circular free surface are con-
sidered here. Let δEiAM and δEiRM (i=P,S;A and R stand for "azimuthal" and
"radial") be the energy densities
, respectively, of ui-and wi-waves in the semi-circular
body, where (ui,wi) are the dilatational (i=P) and distorsional (i=S) components

of the displacements (up,wp), respectively, i.e., up=up+us and wp=wp+ws (up,wp:
azimuthal and radial displacement components in the semi-circular body). Using

the energy density δER of the incident Rayleigh wave on the plane free surface as the

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normalization factor, the normalized energy densities δEijM are expressed as

δEijM=δEijM/δER (i=P,S and j=A,R), (13a)
with

(13b)
and

(13c)

where {up=up+us,wp=wp+ws} and {uR,wR} are given by (6a) and (3a), respec-
tively (only the real parts of these displacements are considered here). Variations
of the energy densities δEijM (i=P,S and j=A,R) are given in Fig. 3 for a specified
value of λ/μ=1.0. The first figure in Fig. 3 shows the variations of these energy
densities at the top of the semi-circular body. The most significant feature in this
figure is the appearance of a maximum (designated as "in resonacce") of δESAM,
namely the energy density of the us-component of the displacement (u) in the azi-
muthal direction; the location (with respect to kb) of the appearance of the above
maximum i.e., kb〓0.8 accords with that of the sudden depression of the energy
flux EtrR of the transmitted Rayleigh wave in Fig. 2. This accordance indicates
that the sudden depression of the transmissivity of the Rayleigh wave near kb=0.8
is due to the occurrence of the resonance of S waves in the semi-circular body. In

Fig. 3. Variations of energy densities δEijM (i=P,S; j=A,R) at the top of the semi-
circular body versus kb (upper figure). Variations of the energy density (δESAM)
and phase (arg us) of the us-wave along the curved free surface for kb=0.8 (lower
figure). Both figures are given for a specified value λ/μ=1.0.

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order to inspect the behavior of us which is in resonance, the variations of δESAM
and arg us along the circular surface for kb=0.8 and λ/μ=1.0 are given in the second
figure of Fig. 3. The value of arg us is then nearly constant throughout the whole
semi-circular surface from the mountain roots WR to LR (for the notations WR and
LR, Fig. 1 should be referred to); this constancy of phase suggests the appearance
of standing waves along the semi-circular surface. The curve of δESAM in this figure
shows that the motion (in the circumferential direction) is stronger near the summit

(SM in the figure) of the circular body and becomes small near both roots (WR and
LR in the figure). This behavior indicates that the resonant standing us-wave (us:
the displacement component of S waves in the circumferential direction) has a loop
and two nodes, respectively, near the summit and the roots of the circular body. The
reason why the resonance along the circular surface occurs at kb〓0.8 instead of
kb=1.0 (then ks=π with s the distance along the free surface in the semi-circular
body) is due to an end-point correction, which corresponds to the mouth correction
in the case of harbor resonance; these end-point corrections, in general, have an effect
to extend the wave-length of standing waves or to decrease the value of the wave
number of resonance such as, in the present case, from kb=1.0 to kb=0.8. The
variation of EtrR near kb=1.0 in the first figure of Fig. 2 shows that the effect of the
end-point correction diminishes with increase of the value λ/μ, since the location

(with respect to kb) of the sudden depression of EtrR approaches the value of kb=1.0
with incresae of λ/μ.

The curve of energy flux of the transmitted Rayleigh wave (EtrR) in Fig. 2 runs
horizontally with a relatively small magnitude in the range of kb from 1.5 to 2.0;
this range is designated as "out-of-phase" in Fig. 2. For this behavior, some physi-
cal interpretation is given here. In and below the semi-circular body, two kinds of
Rayleigh waves are expected; one is the Rayleigh wave travelling along the free sur-
face of the semi-circular body and the other the Rayleigh wave transmitted directly
from the root WR to the root LR (for these notations, see Fig. 1) below the semi-
circular body. Right under the semi-circular body, the latter Rayleigh wave might
have a modulated form other than that of the incident Rayleigh wave owing to the

presence of the rough circular surface. These two kinds of Rayleigh waves arriving
at the root LR interfere with each other; the effect of this interference is evaluated
hereunder in the case of λ/μ=1.

SEZAWA (1927) developed a theory on the Rayleigh wave travelling on a cy-
lindrical surface in the circumferential direction. OLIVER (1955) modified Sezawa's
theory on comparing the theory with the experiment for the case of λ/μ=1, In
their works, the variation of the velocity of the Rayleigh wave in a form normalized
by the velocity of S waves is given versus L/b (L: wave length of the Rayleigh wave
upon a cylinder in the circumferential direction, b: radius of the cylinder). Accord-

ing to their results, the velocity of the Rayleigh wave on the cylindrical surface is

greater than that of S waves in the range of L/b above 0.4; this behavior differs from
that obtained in the case of the Rayleigh wave on the plane free surface in the semi-

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Fig. 4. Illustration of suppressed interference due to reverse displacements of two kinds
of the Rayleigh waves at the root LR of the semi-circular body in the case of kb=
1.5 and λ/μ=1.0.

infinite elastic body in such a way that the latter has a velocity less than that of S waves. The velocity of the Rayleigh wave on the cylinder can be evaluated from the figures given by SEZAWA (1927) and OLIVER (1955). In Fig. 2, the value of kb in the range of out-of-phase is about 1.5; this value is converted to the wave length (L/b) of the Rayleigh wave on the cylinder by use of the relation kcRb=kb/(7cR/Vs) (kcR and VcR: wave number and velocity of the Rayleigh wave on the cylinder surface, Vs: velocity of S waves), i.e.,

L/b=2π(VcR/Vs)/kb. (14)

By extrapolating SEZAWA'S (1927) and OLIVER'S (1955) curves, VcR/Vs〓1.5 is ob-
tained for L/b=2π. If one substitutes these values into the afore-mentioned relation
(14), this relation is found to be nearly satisfied for kb〓1.5. This result shows
that at kb〓1.5 the Rayleigh wave on the semi-circular body, though the evaluation
is made by use of the model of a cylinder, might have a phase difference of about π
(it is found from L/b=2π obtained above) between the two ends WR and LR of the
curved free surface; the above value of kb (〓1.5) corresponds to that in the out-of-
phase range (result (i)). On the other hand, the incident Rayleigh wave (being
subjected to modulation by the presence of the mountain) transmitted directly from
the root WR to the root LR also has a phase difference of kR(2b)〓 π between WR
and LR, since kR〓1.09・k (kb=1.5) for λ/μ=1 (result (ii)). From the above-men-
tioned two results (i) and (ii), it is found that both Rayleigh waves (one of which
runs along the circular surface and the other of which runs below the semi-circular
mountain) are in inverse displacement near the root LR as illustrated in Fig. 4. This
behabior, i.e., suppressed interference of the two kinds of Rayleigh waves due to
the reverse displacements accounts for the smallness of the energy flux EtrR of the
transmitted Rayleigh wave in the out-of phase range of kb in Fig. 2.
In the range of kb from 1.0 to 2.5, most of the energy, as found from the last
figure on Ejsc (j=p,s) in Fig. 2, is transmitted away by the scattered waves, primarily
by S waves.
In the range of kb over 3.0, rapid variations of energy curves occur in Fig. 2.
In order to examine the behaviors, the case of λ/μ=1 is considered here. The EtrR-

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Fig. 5. Variations of the energy density (right figure) and phase (left one) of the waves
along the free surface of the semi-circular body in the case of kb=3.0 and λ/μ=1.0.

curve (associated with the transmitted Rayleigh wave) for λ/μ=1 is elevated near

kb=3.0 (designated as "Rayl. W. res." in Fig. 2) and is suppressed down near kb=
3.5 (designated as "P-wave res." in the figure).
Consider first the elevation of the EtrR-curve near kb=3.0. Let δEM be the en-

ergy density, over one cycle in time, in the semi-circular body defined as

(15)

where δER (normalization factor) is that given by (13c) and {up,wp} are the azimuthal

and radial displacement components in the semi-circular body which are described
in (6a) only the real parts are considered here). Variation of the energy density δEM
along the semi-circular surface for kb=3.0, given on the right-hand side of Fig. 5,

shows the generation of standing waves with a wave length L〓2.8b. This wave
length corresponds to that of the Rayleigh wave (in the circumferential direction) on

the cylinder surface as evaluated below. The evaluation is made following the same

procedure as used in the estimation of the Rayleigh, wave length in the out-of-phase
range of kb; the relation (14) is then employed. When kb〓3.0 (in the present case),

the above relation is reduced to L/b〓2(VcR/Vs). By use of this equation and with
the help of SEZAWA'S (1927) and OLIVER'S (1955) curves, L/b=2.8 is obtained for

VeR/Vs〓1.4. That is to say, the wave length (L/b) of the Rayleigh wave for kb=3.0
on the semi-circular surface, though the evaluation is made by use of the model of the

cylinder, is considered to be about 2.8 (feature (i)). As already mentioned, the en-
ergy density δEM along the semi-circular cylinder surface (see Fig. 5) also has a
variation with wave length L/b〓2.8 (feature (ii)). The coincidence of the values of

the two wave lengths in the above features (i) and (ii) suggests the generation of a
standing Rayleigh wave on the semi-circular cylinder surface (in the circumferential

direction) near kb=3.0. The elevation of the EtrR-curve near kb=3 .0 is, therefore,

considered to be due to the appearance of a standing Rayleigh wave along the curved

free surface in the semi-circular body.
In order to inspect the behavior of the phase of the above standing wave
, the
curve of arg up and wp (for kb=3.0) along the semi-circular surface are given on the

left-hand side of Fig. 5. The curve of arg up in this figure shows a slowly advancing

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behavior of the wave along the curved surface from WR to LR, while that of arg wp
is nearly constant which suggest a standing wave. That is to say, the standing Ray-

leigh wave near kb=3.0 (in the case of λ/μ=1) is of a gradually advancing nature .

Consider next the behavior of the sudden depression of the EtrR-curve near kb=
3.5 in the case of λ/μ=1 which is designated as "P-wave res." in Fig. 2. The cause
of this depression is attributed to the P-wave resonance (the radially compressional
motion) occurring in the semi-circular body as described below. In order to ascertain
the presence of the P-wave resonance at kb〓3.5 (in the case of λ/μ=1), attention is
fixed on the expressions of the coefficients for A(0)c (the zeroth mode of P waves)
in the stress conditions (8b) and (8c) which are the conditions at the curved free
surface of the semi-circular body. The coefficient for A(0)c in the condition (8c) is
always zero owing to the presence of n(=0). If one, therefore, searches for the
value of kb where the expression, C(0)pR(0)Ahh2, of the coefficient for A(0)c in the other
condition (8b) vanishes, the resonance of P waves is expected at and near that value
of kb. According to the result of numerical computation, the function expressed by
C(0)pR(0)Ahh2 (the coefficient for A(0)c in (8b)) has a zero point at kb〓3.55 in the case
of λ/μ=1; that is to say, the zeroth mode of P-wave resonance is expected at kb〓
3.5 (result (i)). On the other hand, substitution of the expression φ ～A(0)cJ0(hr)
(which shows the above resonance) into (6a) yields wp～A(0)cdJ0(hr)/dr; the last ex-
pression denotes the radially compressional motion in the semi-circular body (result
(ii)). From the above two results (i) and (ii), it is considered that the sudden depres-
sion of the EtrR-curve (the curve of the energy flux of the transmitted Rayleigh
wave) at kb〓3.5 in the case of λ/μ=1 is due to the radially compressional P-wave
resonance occurring in the semi-circular body.
Now it is concluded that the P-wave or S-wave resonance occurring in the semi-
circular body weakens the transmissivity of the incident Rayleigh wave through
this semi-circular obstacle, while the Rayleigh wave resonance occurring along the
curved free surface in the semi-circular body strengthens the transmissivity of the
incident Rayleigh wave.
In the first figure (energy flux EtrR) in Fig. 2, the transmissivity of the incident
Rayleigh wave increases with increase of λ/μ. This behavior is due to that increase
of the transmissivity of P waves which is caused by the relative decrease of hb with
increase of λ/μ (for constant kb).
In Fig. 2, some peculiar feature is found in the second figure associated with the
energy flux EreR of the reflected Rayleigh wave; attention is then given only to the
case of λ/μ=1. In this figure, elevations of the EreR-curve appear near kb a little less

than 1.0 and near kb=3.5 which are designated as "UP" in the figure; the former and the latter of these kb-values coincide with those where, respectively, the S-wave and the P-wave resonances occur in the semi-circular body. That is to say, appear-ances of the S-wave and the P-wave resonances in the semi-circular body strengthen the intensity of the Rayleigh wave reflected from the semi-circular obstacle. On the other hand, the Rayleigh wave resonance in the semi-circular body does not

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have any influence on the behavior of the reflected Rayleigh wave (see the variation

of the EreR-curve near kb=3.0 where the Rayleigh wave resonance is expected in the

semi-circular body in the case of λ/μ=1).

3.3 Directivity of scattered waves

Consider next the directivity of the energy flux of the waves scattered away into

the elastic half space. Let Ejw and Ejl (j=p,s) be the energy fluxes (over one cycle

in time) of the j-waves (j=p,s) scattered into the elastic quarter spaces, on the

windward (for θ from π/2 to π) and leeward (for θ from 0 to π/2) sides of the elastic

half space, respectively, (terms windward and leeward are used here from an analogy

between the propagation of the incident Rayleigh wave and the flow of wind in air);

these energy fluxes are related to the scattered energies Ejsc (j=p,s) given in (12f)

such that Ejsc=Ejw+Ejl. Variations of Ejw or 1 (j=p,s) are shown in Fig. 6 for a

specified value of λ/μ=1. In the ranges of kb less than about 2.0 for P waves (the

first figure in Fig. 6) and less than 0.6 for S waves (the second figure in Fig. 6), the

energy fluxes Ejw (j=p,s) into the quarter space on the windward side (abbre-

viated as W.Q., i.e., windward quarter space, in what follows) are larger than the

energy fluxes Ejl into the quarter space on the leeward side (also abbreviated as

L.Q., in what follows). The reason for this behavior is that the waves which are

diffracted from the corner WR into the semi-circular body readily arrive at fhe circu-

lar surface on the leeward side (owing to small kb or hb) and from here the waves

are reflected primarily toward the W.Q,; this behavior is illustrated on the left-

hand side of Fig. 6. Reversely, in the ranges of medium kb from 2.0 to 3.0 for P waves

and kb from 0.6 to 3.0 for S waves, the energy fluxes Ejw (j=p,s) becomes smaller

than the fluxes Ejl (see Fig. 6). The reason is that, as kb increase, the waves diffract-

ed from the corner WR begin to arrive at the circular surface primarily on the wind-

Fig. 6. Variations of energy fluxes Ejw and Ejl versus kb in the case of λ/μ=1.0. Ejw

and Ejl: the energy fluxes of the j waves (j=p,s) scattered, respectively , into the
windward and leeward elastic quarter spaces.

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ward side (owing to largeness of kb) and these waves are directed mainly toward the
L.Q. as illustrated in Fig. 6. In Fig. 6, a noticeable variation of the curves appears

near kb=3.0. Some physical interpretation for this behavior is given here. In

Fig. 5, we have already discussed the appearance of a standing Rayleigh wave in the

semi-circular body at kb near 3.0 under the specified value of λ/μ=1. If one assumes

the generation of such a standing Rayleigh wave in the semi-circular body, the par-
ticular variation near kb=3.0 in Fig. 6 is explained in a physically acceptable form.
That is, as kb approaches 3.0, the energies Ejl (j=p,s) which are generated primarily

near the corner WR (and scattered toward the L.Q.) begin to be trapped along the

circular surface as standing waves instead of being reflected to the L.Q. As found in

Fig. 5, these standing waves are of advancing nature; this advancing nature causes
a high accumulation of energy on the circular surface near the root LR as the result
of the reflection. The high accumulated energy is directed toward the quarter space
W.Q. and, as a result, near kb=3.0 the energy Esw in the W.Q. becomes larger
than the energy Esl in the L.Q. as found in Fig. 6.
In Fig. 7, variations of the energy densities δEjsc (j=p,s) of P and S waves scat-
tered into the elastic half space are shown, where δEjsc (j=p,s) are defined as Ejsc=

with Ejsc (the energy fluxes of the scattered j waves (j=p,s)) given in (12f);

the variations in Fig. 7 are shown for specified values of kb=0.5 and 1.5 in the case
of λ/μ=1. As illustrated in Fig. 7 (in the uppermost figure), the abscissa runs
from the root WR to the root LR through V in the counter-clockwise direction.
The first and second figures in Fig. 7 reveal a very interesting feature, namely, that
the density of energy flux of S waves is very strong along the zone at an angle of
45° to the plane free surface, while P waves do not have such a behavior. Though the
strong directivity of S waves appears along the zone at a slightly larger angle than
45° in Fig. 7, the name of 45° zone is employed here for convenience. In the numer-
ical experiment done by SATO (1972) concerning the study on wave propagation in
an elastic quarter space, he also mentions a feature similar to that found by the au-
thor. However, he unfortunately did not notice its significance. According to
Fig. 1 of SATO'S work (1972), a strong wave packet of S waves is then propagated
along the 45° zone; one end of the wave front of the generated Rayleigh wave
then merges into this strong wave packet, while the other end is at the plane free
surface. For the strong directivity of the scattered S waves along the 45° zone,
some physical interpretation is given here. It is considered that the waves which
are scattered away from an irregular configuration (like the semi-circular body) on
the plane free surface produce a Rayleigh wave with a modulated form in the vicin-
ity of the irregular surface (this wave is named I.R.W. or immature Rayleigh wave),
because the distance from the irregular surface to the I.R.W. is short and therefore
the influence of the surface conditions to produce the Rayleigh wave does not extend
deeply into the elastic body; some of this behavior is found in the figures given by
SATO (1972). As the above I.R.W. leaves from the irrregular surface or wave
source, the influence of the surface conditions begins to reach more deeply into the

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Fig. 7. Variations of the densities δEjsc (j=p,s) of the energy fluxes of the scattered j
waves versus azimuth θ in the cases of kb=0.5 (first figure) and 1.5 (second one),
and λ/μ=1.0 (for both figures). Variation of the position (with respect to θ) of the
two maxima of the energy density δEssc versus kb (third figure); absence of open
circle in braces A and B in the figure is the result of indistinct appearance of 45° zone
due to weak scattered waves.

elastic body; the modification of the I.R.W. to the unmodulated Rayleigh wave

(i.e., the Rayleigh wave on the plane free surface without any irregularity) then pro-
ceeds in the deep inside of the elastic body interchanging energy between the
I.R.W. and the scattered S waves. This interchange of energy takes place along
the 45° zone in the semi-infinite elastic body. The above behavior results in a
concentration of energy of the scattered S waves along the 45° zone as found in
Fig. 7. Taking into account the generation process of the high energy density of
S waves along the 45° zone, this zone may be named "breathing zone" (breathing
stands for the interchanging process of the energy between the I.R.W. and the
scattered S waves); the terminology "breathing zone" is therefore employed in

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what follows. As seen in Fig. 7 (the first and second figures), the scattered P
waves do not have any breathing zone such as the scattered S waves have along the
45° zone. The reason for this behavior is attributed to the rapidness of the velocity
of P waves as compared with that of S waves (in the model of the semi-infinite
elastic body, the velocity of the Rayleigh waves is then the slowest of the three);

the interchange between the energy of P waves and that of the I.R.W. does not

appear as a result of interception by S waves.
As one of the possible causes for the energy concentration along the breathing
zone of S waves, interference between two kinds of waves which are scattered from
the two roots (namely, WR and LR) of the semi-circular body might be suggested;
this interference might produce the interference band shown in Fig. 7. If the exist-
ence of such an interference were assumed, the direction (azimuth θ) where the high-
est energy density of the scattered S waves appears would slide gradually depending
on the value of kb. In order to examine the above dependence on kb, the variation
of the location (with respect to the azimuth θ) of the highest energy density of the
scattered S versus kb is given in the last figure of Fig. 7 for λ/μ=1; the abscissa and
the ordinate denote, respectively, the azimuth θ (direction of the highest energy
density) and the variable kb. Figure 7 shows that the location (with respect to θ) of
the highest energy density is nearly independent of the value of kb; the possibility
of the interference band is therefore excluded.

3.4 Phase lag of the transmitted Rayleigh wave
We shall now consider the phase lag of the transmitted Rayleigh wave due to the presence of the semi-circular body. In order to make the physical interpreta-tion of the phase lag simple, the transmitted Rayleigh wave {u+R,w+R} of (9c) is normalized by the incident Rayleigh wave {uR,wR} in (3a), thus,

u+R/uR=w+R/wR=1+C+/C. (16)

The variation of arg (1+C+/C) is then shown in Fig. 8. This figure shows an inter-
esting feature that, after passing the semi-circular body, the transmitted Rayleigh
wave is retarded in the range of kb〓1.0 while it is accelerated in the range of kb〓
1.0; the retardation of the wave in the range of kb〓1.0 is most conspicuous when the
S-wave resonance appears in the semi-circular body. For these behaviors, some

physical interpretation s given here using the illustration nserted in Fig. 8. In the range of kb less than 1.0, some of the energy of S waves, which appear along the curved surface of the semi-circular body in the form of standing waves (see Fig. 3), are supplied to the transmitted waves; the above S waves have retarted phase (owing to the trapping in the semi-circular body) and hence cause a retardation of the transmitted wave (see the illustration on the left-hand side of Fig. 8). As kb increases beyond 1.0, the strength of the standing S waves in the semi-circular body weakens; on the other hand, the P and S waves transmitted directly from the root WR to the root LR become dominant. This domination of the P and S waves leads to the advancement of the phase of the transmitted Rayleigh wave, since P and S

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Fig. 8. Variations of the phase lag, arg (1+C+/C), of the transmitted Rayleigh wave
versus kb for specified values of λ/μ=0.5, 1.0 and 5.0.

waves have faster velocities than the Rayleigh wave on the plane free surface (see
the illustration on the right-hand side of Fig. 8).
Some mention is made here of the dependence of the phase lag in the trans-
mitted Rayleigh wave on the ratio λ/μ. As shown in Fig. 8, the retardation of phase
in the range of kb〓1.0 and its advancement in the range of kb〓1.0 decrease with
increase of λ/μ; the curve showing the phase lag approaches the abscissa which denotes
the absence of phase retardation or advancement in the transmitted Rayleigh wave,
i.e., arg (1+C+/C)=0. In other words, the influence of the presence of the semi-
circular body on the transmission of the Rayleigh wave becomes less with increase
of λ/μ. This behavior agrees with that of the transmitted Rayleigh wave near kb=
1.0 which has been discussed in relation to the EtrR-curve in Fig. 2. That is to say,
the transmissivity of the Rayleigh wave is then found to be strengthened with increase
of λ/μ (in other words, the influence of the semi-circular body becomes weaker).

4. Conclusion

The scattering of a Rayleigh wave by a rough semi-circular surface is considered in this work; the semi-circular surface is located on the free surface of a semi-infinite elastic body.
As for the energy fluxes of the generated Rayleigh waves, the following facts have been found. The most important feature is the appearance of a sudden de-crease in the energy flux of the transmitted Rayleigh wave near kb=1.0 (k: wave number of S, 2b: width of the semi-circular body). The sudden decrease is pro-duced by the appearance of the S-wave resonance in the semi-circular body (the

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motion of the S-wave resonance is then in the circumferential direction on the
curved free surface); the appearance of the above resonance scatters the energy of

the incident wave primarily into the elastic half space as scattered waves . In the

range of kb from 1.5 to 2.0, the energy of the transmitted Rayleigh wave is relatively
small owing to the interference of two kinds of Rayleigh waves; one of these waves
runs along the curved free surface of the semi-circular body and the other travels
under the semi-circular obstacle directly from the root in front of the semi-circular
body to the root behind it. Near kb=3.0, standing Rayleigh waves of gradually
advancing nature appear along the curved free surface of the semi-circular body;
the appearance of these standing waves leads to better transmissivity of the incident
Rayleigh wave. At kb a little more than 3.0, the P-wave resonance occurs in the
semicircular body (the motion of the resonance is then radially compressional);
this resonance leads to a sudden decrease in the energy of the transmitted Rayleigh
wave. As for the transmissivity of the Rayleigh wave through the semi-circular
body in response to the ratio λ/μ (λ,μ: Lame's constants), it strengthens with increase
of λ/μ, i.e., the decrease of hb (h: wave number of P waves). As for the Rayleigh
wave reflected from the semi-circular body, its energy becomes a maximum at the
resonance of S and P waves-not, however, at the resonance of the Rayleigh wave-
in the semicircular body.
As for the energy scattered away into the semi-infinite elastic body, most of it
is transmitted away by S waves. The directivity of the waves scattered away into
the semi-infinite elastic body is such that, in the range of small kb (0 to 2.0 for P and
0 to 0.6 for S in the case of λ/μ=1), the energy flux of the scattered waves into the
windward-side elastic quarter space is larger than that into the leeward-side one

(terms windward and leeward are used here from an analogy between the flow of
wind and the propagation of the incident Rayleigh wave), while, in the medium range
of kb (2.0 to 3.0 for P and 0.6 to 3.0 for S in the case of λ/μ=1), this behavior is the
reverse. The former behavior is the result of the dominating effect of the leeward-side
surface of the semi-circular body owing to small kb, while the latter is due to the effect
of the windward-side surface of the semi-circular body just behind the windward-
side corner owing to a relatively large kb. Near kb=3.0, no significant directivity
of the energy flux (toward the windward- and leeward-side quarter spaces) of the
scattered waves appears owing to the generation of standing Rayleigh waves in
the semi-circular body. As for the directivity of energy flux of the waves scattered
into the semi-infinite elastic body, one other conspicuous feature is found. That is,
a strong concentration of energy occurs along the zone named breathing zone; this
zone extends from the semi-circular body to infinity at an angle of 45° with the free
surface. The breathing zone is produced as the result of the interchange of energy
between the generated Rayleigh waves and the scattered S waves.
As for the phase lag of the transmitted Rayleigh wave, it is retarded in the range
of kb〓1.0 owing to the generation of the retarded waves from the semi-circular
body, while it is advanced in the range of kb〓1.0 due to the appearance of the P and
S waves which are transmitted through the medium just below the semi-circular body

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directly from the windward-side root to the leeward-side one. As the ratio λ/μ
increases (i.e., hb decreases), the phase retardation or acceleration of the transmitted
Rayleigh wave due to the presence of the semi-circular body becomes small.

APPENDIX

Further Reductions of Xz and Zz in (5d)
The expressions for the displacements (uM,wM) in the semi-circular body in Cartesian coordinates are obtained from (up,wp) of (6a) in polar coordinates by the conversion of the coordinates. Substituting these transformed displacements (uM,wM) into the stress conditions (2b) with the help of the expressions (4b), the stresses (Xz,Zz) at z=0 can be described in terms of unknown coefficients A(n)c, A(n)s,

B(n)c and B(n)s with the help of the Bessel functions J(ξ) (ξ: variable):

(A.1)

and

(A.2)

On the other hand, the integral (5d) is expressed by the sum of the integrals
in 2M small divisions (xq-Δx<x<xq+Δx: q from -M to -1 and from 1 to M).
If the Bessel functions included in {Xz,Zz} in (5d) namely (A.1 and A.2) in this Ap-
pendix are approximately expressed by the values at the reference points xq of the
small divisions, the integral (5d) is reduced to

(A.3)

where with a prime denotes the summation of q from -M to M excluding q=0 and {X,Z}q are the values of {X,Z} at x=xq.

REFERENCES

GILBERT, F. and L. KNOPOFF, Seismic scattering from topographic irregularities, J. Geophys. Res., 65, 3437-3444, 1960.
HOMMA, S., Effects of topographic irregularities upon the vibrations of the free surface (I), Kenshin-ziho (Q.J. Seismol), 11, 349-364, 1941 (in Japanese).

HOMMA, S., Effects of topographic irregularities upon the vibrations of the free surface (II), Ken-
shinjiho (Q.J. Seismol.), 12, 17-23, 1942a (in Japanese).

HOMMA, S., Effects of topographic irregularities upon the vibrations of the free surface (III), Ken-

Scattering of the Rayleigh Wave by a Semi-Circular Rough Sufrace 519

shinjiho (Q.J. Seismol), 12, 24-36, 1942b (in Japanese).
OLIVER, J., Rayleigh waves on a cylindrical curved surface, Earthquake Notes, 26, 24-25, 1955. SATO, R., On Rayleigh waves generated at rough surfaces (I) (Two dimensional case), Zisin (J. Seismol. Soc. Japan) [II], 8, 121-137, 1955 (in Japanese).
SATO, R., Some theoretical problems, Zisin (J. Seismol. Soc. Japan) [II], 20 (Special Vol.), 71-74, 1967 (in Japanese).
SATO, Y., Surface waves propagated on an elastic body, Kagaku, 19, 311-319, 1949 (in Japanese). SATO, Y., A numerical experiment on wave propagation in an elastic quater space, J. Phys. Earth, 20, 287-299, 1972.
SEZAWA, K., Dispersion of elastic waves propagated on the surface of the stratified bodies and on curved surfaces, Bull. Earthq. Res. Inst., 3, 1-19, 1927.