Scattering Theory for the Acoustic Equation in an Even Number of Space Dimensions

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Introduction. Early in the sixties the authors developed a theory of scattering for the acoustic equation in an odd number of space dimensions, an account of which can be found in [4]. This theory is based on the concept of incoming and outgoing subspaces, D_-^ρ and D_+^ρ respectively, where D_-^ρ [D_+^ρ] is defined as the set of initial data for solutions which vanish in the truncated backward [forward] cone: $|x| < \rho - t, t < 0$ [$|x| < \rho + t, t > 0$]; here ρ is chosen so that the scattering obstacle is contained in the ball $\{|x| < \rho\}$. Denoting the unitary group of solution operators acting on the Hilbert space H of initial data by $U(t)$, the basic properties of D_-^ρ and D_+^ρ are:

1) $U(t)D_-^\rho \subset D_-^\rho$ for $t < 0$, $U(t)D_+^\rho \subset D_+^\rho$ for $t > 0$;
2) $\wedge\ U(t)D_\pm^\rho = \{0\}$;
3) $\vee\ U(t)D_\pm^\rho = H$;
4) D_-^ρ is orthogonal to D_+^ρ.

From the existence of subspaces D_-^ρ and D_+^ρ having properties (1)–(3) it is easy (see Chapters II and V of [4]) to find the corresponding incoming and outgoing translation and spectral representations for U as well as the explicit form of the scattering matrix. In order to study the singularities of the scattering matrix we introduced the related semi-group of operators (see Chapters III and V of [4]):

$$Z(t) = P_+U(t)P_- , \qquad t \geqq 0, \tag{1}$$

acting on the subspace $K = H \ominus (D_-^\rho \oplus D_+^\rho)$; here P_- and P_+ are the orthogonal projections on the orthogonal complements of D_-^ρ and D_+^ρ, respectively. The semi-group property for Z results from the orthogonality of D_-^ρ and D_+^ρ. In our development most of the basic results were obtained by means of the theory of hyperbolic partial differential equations and throughout the Radon transform played a central role (see Chapter IV of [4]).

Somewhat later the corresponding scattering theory problem for the acoustic equation in an even number of space dimensions was treated, first by N. Shenk

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[10] and then by others using more traditional techniques which are essentially elliptic in nature. It was only after the appearance of N. Iwasaki's paper [2] on hyperbolic systems that it seemed feasible to us that our approach might work in the even dimensional case. Iwasaki used the Radon transform in the even dimensional case to define sets of data for which the solution vanishes in truncated forward and backward cones; it is not clear from his work that this defined the set of all such data. He also applied our technique to extend the Rellich uniqueness theorem to hyperbolic systems in an even number of space dimensions. These particular results are actually lemmas to Iwasaki's main theorem which concerns the local energy decay of solutions, but they served as our starting point.

This paper provides a self-contained adaption of our approach to the even dimensional problem. Section 1 contains a brief exposition of the basic properties of the Radon transform in an even number of space dimensions, tailored to fit our needs (cf. D. Ludwig [6]). This material is then applied in Section 2 to obtain a convenient expression of the solution to the initial value problem for the wave equation in free space; we follow Iwasaki in defining the subspaces D_-^0 and D_+^0 in terms of the Radon transform. Our basic result, which is proved in Section 3, is that D_- and D_+ are incoming and outgoing subspaces for the perturbed problem; that is we show that the properties (1)–(3) are satisfied. In order to prove this result we first show that U has no point spectrum and further that local energy decay is valid for the solutions of the exterior problem. Once we have established properties (1)–(3) for U it is an easy matter to prove the existence and completeness of the wave operators and from this the unitarity of the scattering operator.

In Section 4 we construct the incoming and outgoing translation representations for the unperturbed group U_0. The fact that these representations are distinct is peculiar to the even dimensional case; the two representations are related through the Hilbert transform. We also prove that D_-^0 and D_+^0, defined by means of the Radon transform of the initial data, contain all of the dato for which the solutions vanish in the respective forward and backward cones. In Section 5 we construct explicit incoming and outgoing spectral representations for U in terms of the scattered wave solutions of the reduced wave equation. The usual method of obtaining these scattered waves is to use surface potentials. We have devised an alternative construction which in effect reduces the exterior problem to an interior elliptic problem.

In Section 6 we obtain an explicit formula for the scattering matrix $\mathcal{S}$ and study its analytic extension. It is at this point that our previous approach to scattering theory breaks down, the reason for this is that property (4) is not valid in the even dimensional case and consequently Z, as defined in (1), is not a semi-group of operators. Thus in the even dimensional case it seems that we have lost our main tool for the study of the singularities of $\mathcal{S}$. However by factoring $\mathcal{S}$ into inner and outer factors, it can be shown that the so-obtained inner factor has precisely the same singularities as $\mathcal{S}$. Associated with this inner factor is a semi-group of operators which could serve the same purpose as Z; as yet we have not been able to make use of this fact.

It should be clear to the expert that the methods developed in the present paper apply to a much larger class of even dimensional problems than the acoustic equation. For instance, we can now treat first order symmetric hyperbolic systems of the kind considered in Chapter VI of [4] in the even dimensional case.

1. The Radon transform. We begin with a quick derivation of the basic properties of the Radon transform in an *even* number of space dimensions.

Definition. Let f be a function of class $\mathcal{S}$ defined in $\mathbf{R}^n$. Its *Radon transform*, denoted by $\tilde{f}$, is a function of $\{s, \omega\}$ in $\mathbf{R} \times S^{n-1}$ which is defined as follows:

$$\tilde{f}(s, \omega) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{x \cdot \omega = s} f(x)\, dS. \tag{1.1}$$

Two obvious properties are:

$$\tilde{f} \text{ is an even function of } \{s, \omega\}; \tag{1.2}$$

and

$$\widetilde{\partial_{x^j} f} = \omega_j\, \partial_s \tilde{f}. \tag{1.3}$$

Next we make use of the Fourier transform $\hat{f}$ of f to obtain further properties of $\tilde{f}$; here

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} f(x)\, dx. \tag{1.4}$$

Writing $\xi = \sigma\omega$, ω in S^{n-1}, we have

$$\hat{f}(\sigma\omega) = \frac{1}{(2\pi)^{n/2}} \int e^{i\sigma s} \int_{x \cdot \omega = s} f(x)\, dS\, ds;$$

note that $\hat{f}(\sigma\omega)$ is also an even function. Using the definition of $\tilde{f}$ this can be rewritten as

$$\hat{f}(\sigma\omega) = \frac{1}{(2\pi)^{1/2}} \int e^{i\sigma s} \tilde{f}(s, \omega)\, ds \equiv F\tilde{f}, \tag{1.5}$$

where we use the symbol F to denote Fourier transform with respect to s.

The Fourier inversion formula is

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{1.6}$$

which becomes in polar coordinates

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{n/2}} \iint_0^\infty e^{-i\sigma x\cdot\omega} \hat{f}(\sigma\omega) \sigma^{n-1}\, d\sigma\, d\omega \\ &= \int h_1(x\cdot\omega, \omega)\, d\omega, \end{aligned} \tag{1.7}$$

where

$$h_1(s, \omega) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-i\sigma s} \sigma^{n-1} \hat{f}(\sigma\omega) \, d\sigma. \tag{1.8}$$

The value of the integral on the right in (1.7) obviously depends only on the even part h of h_1 ; that is

$$f(x) = \int h(x \cdot \omega, \omega) \, d\omega \tag{1.9}$$

where

$$h(s, \omega) = \tfrac{1}{2}[h_1(s, \omega) + h_1(-s, -\omega)].$$

Using the definition (1.8) of h_1 we can write

$$h(s, \omega) = \frac{1}{2(2\pi)^{n/2}} \int_{-\infty}^\infty e^{-i\sigma s} \, |\sigma|^{n-1} \, \hat{f}(\sigma\omega) \, d\sigma = \frac{1}{2(2\pi)^{(n-1)/2}} F^{-1}(|\sigma|^{n-1} \, \tilde{f}). \tag{1.10}$$

Combining (1.5) and (1.10) we obtain

$$h = \frac{1}{2(2\pi)^{(n-1)/2}} F^{-1}(|\sigma|^{n-1} \, F\tilde{f});$$

and this can be written in the equivalent form

$$h = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K \, \partial_s^{n-1} \tilde{f}, \tag{1.11}$$

where K is the Hilbert transform in s, that is

$$K = F^{-1} \, (\operatorname{sgn} \sigma) F. \tag{1.12}$$

Note that $K\partial_s^{n-1}$ transforms even functions into even functions and odd into odd. Formulas (1.9) and (1.11) give the *inversion formula* for the Radon transform.

Denote the L_2 norm of f by $||f||$; then by the Parseval relation

$$||f||^2 = \frac{1}{2} \iint_{-\infty}^\infty |\hat{f}(\sigma\omega)|^2 \, |\sigma|^{n-1} \, d\sigma \, d\omega.$$

Using the Parseval relation again, but now with respect to σ, we get from (1.5) and (1.12) that

$$\int |\hat{f}(\sigma\omega)|^2 \, |\sigma|^{n-1} \, d\sigma = i^{n-1} \int \bar{\tilde{f}} K \, \partial_s^{n-1} \tilde{f} \, ds.$$

Finally combining these relations and denoting by [,] the L_2 scalar product with respect to $ds \, d\omega$, we get

$$||f||^2 = \frac{i^{n-1}}{2} [K \, \partial_s^{n-1} \tilde{f}, \, \tilde{f}]; \tag{1.13}$$

(1.13) is the *Parseval relation* for the Radon transform.

It is both suggestive and convenient to make use of the half-order Sobolev norm for functions $k(s, \omega)$:

$$||k||^2_{(n-1)/2} = \iint |\sigma|^{n-1} \, |Fk|^2 \, d\sigma \, d\omega, \tag{1.14}$$

in terms of which (1.13) can be rewritten as

$$||f|| = \frac{1}{2^{1/2}} ||\tilde{f}||_{(n-1)/2} \, . \tag{1.15}$$

The obvious inequality

$$|\sigma|^{n-1} \leqq \tfrac{1}{2} |\sigma|^{n-2} + \tfrac{1}{2} |\sigma|^n$$

implies

$$||k||^2_{(n-1)/2} \leqq \tfrac{1}{2} ||k||^2_{n/2-1} + \tfrac{1}{2} ||k||^2_{n/2} \, . \tag{1.16}$$

We shall make use of this inequality in Section 3. We note that the $(n - 1)/2$ scalar product can be written as:

$$[k, h]_{(n-1)/2} = i^{n-1}[K\partial_s^{n-1}k, h]. \tag{1.17}$$

Up to this point the Radon transform was defined only for functions f of class $\mathcal{S}$; using the Parseval relation (1.15) we can easily extend the transform of all f in L_2 ; so extended $\tilde{f}$ lies in $W^{\text{even}}_{(n-1)/2}$, the space of even functions with finite $(n - 1)/2$ norm. In order to show that the extended Radon tranfsorm maps L_2 *onto* $W^{\text{even}}_{(n-1)/2}$ it is enough to show that $\mathcal{S}$ maps onto a dense subset of $W^{\text{even}}_{(n-1)/2}$ and according to (1.5) and (1.14) this simply requires for the Fourier transforms $\hat{f}$ that the functions $|\sigma|^{(n-1)/2}\hat{f}(\sigma\omega)$ be dense in $L_2^{\text{even}}(\mathrm{R} \times S^{n-1})$. This is readily verified since the functions $g(\sigma, \omega)$ in $\mathcal{S}^{\text{even}}$ which vanish in a neighborhood of $\sigma = 0$ are obviously dense in $L_2^{\text{even}}(\mathrm{R} \times S^{n-1})$ and

$$\hat{f}(\sigma\omega) = g(\sigma, \omega)/|\sigma|^{(n-1)/2}$$

defines a function $\hat{f}$ in $\mathcal{S}$ whose inverse Fourier transform f lies in $\mathcal{S}$ and has as its Radon transform $\tilde{f} = F^{-1}\hat{f}$, as required.

The following result will be needed in Section 3:

Lemma 1.1. *If* $f(x) = 0$ *for* $|x| > \rho$, *then*

a) $\tilde{f}(s, \omega) = 0$ *for* $|s| > \rho$,
b) $|\tilde{f}|_{\max} \leqq \text{const.}\ \rho^{n-1} \, |f|_{\max}$,
c) $|\partial_s^j \tilde{f}|_{\max} \leqq \text{const.}\ \rho^{n-1} \sum_{|\alpha|=j} |\partial_x^\alpha f|_{\max}$.

Parts a) and b) are immediate consequences of the definition (1.1) of $\tilde{f}$; while c) follows from (1.1) and (1.3).

2. The wave equation in free space. The solutions to the wave equation

$$\partial_t^2 u - \triangle u = 0 \tag{2.1}$$

are uniquely determined by their initial data, $u(x, 0)$ and $\partial_t u(x, 0)$. The law of *conservation of energy* holds; that is

$$\int \{|\partial_x u|^2 + |\partial_t u|^2\}\, dx \tag{2.2}$$

is independent of t.

We denote by $d = \{f_1, f_2\}$ the initial data $\{u(0), \partial_t u(0)\}$ and by $\|d\|_E^2$ the *energy norm*, defined as

$$\|d\|_E^2 = \int \{|\partial_x f_1|^2 + |f_2|^2\}\, dx. \tag{2.3}$$

We denote by H_0 *the Hilbert space of all initial data with finite energy*, and by $U_0(t)$ *the operator mapping initial data at time* 0 *into data at time* t. In view of energy conservation, $U_0(t)$ is an isometric operator; since the backward initial value problem is properly posed, it follows that $U_0(t)$ is invertible; and since the equation in (2.1) is time independent, $U_0(t + s) = U_0(t)U_0(s)$. In summary, $U_0(t)$ *is a one-parameter family of unitary operators*. The infinitesimal generator of this group, denoted as A_0, is of the form:

$$A_0 = \begin{pmatrix} 0 & I \\ \triangle & 0 \end{pmatrix}. \tag{2.4}$$

The domain of A_0 consists of all pairs $\{f_1, f_2\}$ such that $\partial_x^\alpha f_1$ and $\partial_x^\beta f_2$ belong to L_2, for all $|\alpha| = 2$ and $|\beta| = 1$.

Next we describe the solutions of (2.1) in terms of the Radon transform. For this purpose we define the Radon transform Rd of the initial data $d = \{f_1, f_2\}$ as

$$Rd = \partial_s \tilde{f}_1 - \tilde{f}_2. \tag{2.5}$$

Note that

$$RA_0 d = R\{f_2, \triangle f_1\} = \partial_s \tilde{f}_2 - \partial_s^2 \tilde{f}_1$$

so that

$$RA_0 d = -\partial_s Rd; \tag{2.6}$$

that is, R *gives a translation representation of* A_0.

The energy norm of d is easily expressed in terms of Rd:

$$\|d\|_E^2 = \tfrac{1}{2} \|Rd\|_{(n-1)/2}^2. \tag{2.7}$$

To prove this we note that by (1.3), $\partial_x f_1 = \omega \partial_s \tilde{f}_1$. We apply Parseval's formula (1.15) to $\partial_x f_1$ and f_2 in (2.3) and get

$$\|d\|_E^2 = \|\partial_x f_1\|^2 + \|f_2\|^2 = \tfrac{1}{2} \|\partial_s \tilde{f}_1\|_{(n-1)/2}^2 + \tfrac{1}{2} \|\tilde{f}_2\|_{(n-1)/2}^2. \tag{2.8}$$

Now $\partial_s \tilde{f}_1$ is odd, $\tilde{f}_2$ is even, and it follows from the definition (1.14) that odd and even functions are orthogonal in the $(n-1)/2$ norm, hence using the definition (2.5) of Rd in (2.8) we get (2.7).

We saw at the end of Section 1 that the Radon transform maps L_2 onto $W^{\text{even}}_{(n-1)/2}$; a similar argument shows that R maps the set of all data with finite energy norm onto $W_{(n-1)/2}$. We note that K is unitary in $W_{(n-1)/2}$.

Next we take the Radon transform of the wave equation (2.1); using (1.3) we see that $\tilde{u}$ satisfies the one dimensional wave equation

$$\tilde{u}_{tt} - \tilde{u}_{ss} = 0.$$

This equation can be factored and written as

$$(\partial_t + \partial_s)(\partial_t - \partial_s)\tilde{u} = 0. \tag{2.9}$$

The quantity $(\partial_t - \partial_s)\tilde{u}$ was defined in (2.5) to be the Radon transform of the data $(\{u, u_t\})$. Abbreviating $R\{u, u_t\}$ as m, (2.9) can be written as

$$(\partial_t + \partial_s)m = 0. \tag{2.10}$$

It follows from (2.10) that m is a function of $s - t$:

$$m(s, \omega, t) = m_0(s - t, \omega), \tag{2.11}$$

where $m_0 = Rd$, $d = \{f_1, f_2\}$ being the initial data of u.

According to the inversion formula (1.9) and (1.11),

$$u(x, t) = \int h(x\cdot\omega, t)\, d\omega, \tag{2.12}$$

$$h = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K\, \partial_s^{n-1}\tilde{u}.$$

Using the definition (2.5) of $R\{u, u_t\} = m$ we see that h differs by const. $K\partial_s^{n-2}\tilde{u}_t$ from

$$\frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K\, \partial_s^{n-2}m. \tag{2.13}$$

This difference is an odd function of $\{s, \omega\}$; since adding an odd function to h does not alter the right side of (2.12) we may in (2.12) replace h by the function (2.13). We now use relation (2.11) to express m as translate of $m_0 = Rd$. Since translation in s commutes with ∂_s and K, we can state the result as follows:

Theorem 2.1. *The solution u of the wave equation* (2.1) *can be expressed in terms of its initial data d as follows:*

$$u(x, t) = \int k(x\cdot\omega - t, \omega)\, d\omega, \tag{2.14}$$

where

$$k = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K\, \partial_s^{n-2}Rd, \tag{2.15}$$

Rd being defined by (2.5).

Remark. It is easy to verify directly that (2.14) solves the wave equation and has initial data d. In case d is not smooth but merely has finite energy, k is a distribution of class $W_{(1-n)/2}$ and u defined by (2.14) is a distribution.

We now define the subspaces D_+ and D_- of H_0 to be the set of those data d for which the distribution $k(s, \omega)$ defined by (2.15) has its support in $s \geqq 0$, respectively $s \leqq 0$. Data in D_+, or D_-, are called *outgoing*, respectively *incoming*. It follows from formula (2.14) for solutions that if $k(s, \omega) = 0$ for $s < 0$, respectively $s > 0$, then $u(x, t) = 0$ in the *forward*, respectively *backward cone* $|x| < t$ or $|x| < -t$. Thus we have shown that *solutions of the wave equation with outgoing initial data vanish in the forward cone and those with incoming initial data vanish in the backward cone*; hence the terminology.

We denote by D_+^ρ and D_-^ρ, $\rho \geqq 0$, the set of those data d for which the distribution $k(s, \omega)$ defined by (2.15) have their supportive $s \geqq \rho$, respectively $s \leqq -\rho$. As above we conclude

Lemma 2.2. *Solutions whose initial data lie in D_+^ρ, or D_-^ρ, vanish in the cone $|x| \leqq \rho + t$, respectively $|x| < \rho - t$.*

The following observation will be needed in Section 3:

Lemma 2.3. *Let g be data in H_0 which is orthogonal to D_+^ρ in the energy norm; then for $s > \rho$, Rg is a polynomial of degree $< n/2 - 1$.*

Proof. According to (2.7),

$$(d, g)_E = \tfrac{1}{2}[Rd, Rg]_{(n-1)/2} ;$$

using (1.17) we get

$$(d, g)_E = \frac{i^{n-1}}{2} [K\partial_s^{n-1}Rd, Rg];$$

and by the definition (2.15) of k this can be rewritten as

$$(d, g)_E = (2\pi)^{(n-1)/2}[\partial_s k, Rg]. \tag{2.16}$$

For d in D_+^ρ, $k = 0$ for $s < \rho$; on the other hand, it follows from the fact that R maps H_0 onto $W_{(n-1)/2}$ that k is fairly arbitrary for $s > \rho$. In fact for any C_0^∞ function φ on $\mathbf{R} \times S^{n-1}$, the function $(\operatorname{sgn} \sigma)^{n/2-1} |\sigma|^{1/2} F\varphi$ is square integrable and since RH_0 fills out $W_{(n-1)/2}$ there exists a d in H_0 such that

$$|\sigma|^{(n-1)/2} FKRd = (\operatorname{sgn} \sigma)^{n/2-1} |\sigma|^{1/2} F\varphi.$$

Dividing through by $(\operatorname{sgn} \sigma)^{n/2-1} |\sigma|^{1/2}$ and taking the inverse Fourier transform we see that

$$\partial_s^{n/2-1} KRd = \varphi$$

and hence that

$$\partial_s k = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} \partial_s^{n/2} \varphi.$$

In particular for d in D_+^ρ, $\partial_s k$ can be any function of the form $\partial_s^{n/2}\varphi$ with support of φ in (ρ, ∞). Since g is orthogonal to D_+^ρ, it follows from (2.16) that Rg is a polynomial of degree $<n/2$ on (ρ, ∞).

A typical polynomial term is of the form

$$\ell = \psi(s)a(\omega)s^\gamma$$

where ψ in C^∞ is identically one for $s > \rho$ and vanishes for s sufficiently small. The Fourier transform of ℓ, in the sense of distributions, will be well behaved at infinity and singular near the origin; that is

$$F\ell \sim a(\omega)/\sigma^{\gamma+1}.$$

Hence ℓ belongs to $W_{(n-1)/2}$ if and only if $\gamma < n/2 - 1$ as asserted in the lemma.

3. The wave equation in the exterior of an obstacle. We shall study solutions of the wave equation

$$\partial_t^2 u - \Delta u = 0 \tag{3.1}$$

in the exterior G of an obstacle; we denote by ∂G the surface of the obstacle and impose on it a boundary condition

(3.2) i) $u = 0$ on ∂G,

or

ii) $\partial_n u + \kappa u = 0$ in ∂G, $\kappa \geqq 0$.

Under either boundary condition the law of *conservation of energy* holds:

$$\int_G [|\partial_x u|^2 + |\partial_t u|^2]\, dx + \int_{\partial G} \kappa\, |u|^2\, dS \equiv E(u) \tag{3.3}$$

is independent of t. It follows that solutions are uniquely determined by their initial data $\{u, u_t\}$.

We denote by H the space of all initial data d in the exterior G which have finite energy (3.3), and for which the first component vanishes at those parts of the boundary where Dirichlet boundary conditions are imposed. As so-defined H is a Hilbert space under the energy norm $E^{1/2}$ which we denote as $\|d\|_E$.

For simplicity we consider in the rest of this paper only Dirichlet boundary conditions. In this case the space H can be imbedded isometrically in the space H_0 by simply continuing the data as zero inside the obstacle.

The operator mapping initial data into data at time t is denoted by $U(t)$; clearly $U(t)$ forms a one-parameter group of unitary operators over H. The infinitesimal generator A of this group is of the form

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \tag{3.4}$$

and its domain consists of all pairs $\{f_1, f_2\}$ such that $\partial_x^\alpha f_1$ and $\partial_x^\beta f_2$ are in $L_2(G)$ for $|\alpha| = 1, 2$ and $|\beta| = 0, 1$, and f_1 and f_2 are zero on ∂G.

Part a) of the next theorem is Lichtenstein's inequality, and part b) the well-known interior estimate for the Laplace operator:

Theorem 3.1. a) *Suppose that the boundary of G is C^2; then*

$$\int (|\partial_x^2 f_1|^2 + |\partial_x f_2|^2)\, dx \leqq \text{const. } ||Ad||_E^2$$

b) *Let G' be any subdomain of G; denote by $E(d, G')$ the local energy*

$$E(d, G') = \int_{G'} [|\partial_x f_1|^2 + |f_2|^2]\, dx. \tag{3.5}$$

Let C be any compact subset of G', then

$$\int_C |\partial_x^{N+1} f_1|^2 + |\partial_x^N f_2|^2\, dx \leqq E(d, G') + E(A^N d, G') \tag{3.6}$$

where the constant depends on N, C and G'.

The following well-known result of Rellich will be needed:

Lemma 3.2. *A has no point spectrum.*

For $\lambda = 0$ this follows from the well known uniqueness theorem for harmonic functions which are zero on the boundary. For $\lambda \neq 0$ the proof is based on the observation that a solution of the reduced wave equation

$$\Delta u + \lambda^2 u = 0$$

which is defined and L_2 near infinity is zero. For a proof of this and more general propositions see [2] or [4].

The main result of this section is

Theorem 3.3. *Take ρ so large that the obstacle is contained in the ball of radius ρ around the origin. Since data in D_+^ρ are zero in $|x| \leqq \rho$, it follows that D_+^ρ is a subspace of H.*

Assertion. *The set of all data in*

$$[U(p)D_+^\rho\,; \qquad p \text{ in } \mathbf{R}]$$

is dense in H.

Before giving the proof of this theorem we give some of its consequences. As observed in Lemma 2.2 for any data f in D_+^ρ, $U_0(t)f$ vanishes in the cone: $|x| \leqq \rho + t$. Since the obstacle is contained in the ball $|x| \leqq \rho$, it follows that $U_0(t)f$ is, for $t \geqq 0$, a solution of the boundary value problem as well. In operator language,

$$U_0(t) = U(t)f \quad \text{for} \quad f \text{ in } D_+^\rho, \qquad t \geqq 0. \tag{3.7}$$

Let p be any number, and let g be data of the form

$$g = U(p)f, \quad f \text{ in } D_+^\rho . \tag{3.8}$$

Using (3.7) and the group property we get from (3.8) that for $t + p > 0$

$$\begin{aligned} U(t)g &= U(t+p)U(-p)g = U(t+p)f \\ &= U_0(t+p)f = U_0(t)U_0(p)f \end{aligned}$$

so

$$U_0(-t)U(t)g = U_0(p)f \quad \text{for} \quad t > -p.$$

In particular the limit

$$\lim_{t\to\infty} U_0(-t)U(t)g \tag{3.9}$$

exists for all g of form (3.8). If, as asserted by Theorem 3.3 these elements are dense in H, it follows from the isometric character of U_0 and U that the limit (3.9) exists for all g in H and is an isometry.

The analogue of Theorem 3.3 for the group U_0, *i.e.*, the density in H_0 of elements of form $U_0(p)D_+^\rho$, is trivially true since C_0^∞ functions of the form KRd are dense in $W_{(n-1)/2}$. As a consequence, the limit

$$\lim_{t\to\infty} U(-t)PU_0(t)g = W_+g \tag{3.10}$$

where P is the orthogonal projection of H_0 onto H, exists and is an isometric mapping of H_0 into H. It follows in the same way that the operator defined by (3.9) is inverse to the operator W_+ defined by (3.10).

The operator W_+ is called the *forward wave operator*; what we have shown can be expressed as:

Corollary 3.4. *The forward wave operator is a unitary mapping of H_0 onto H.*

Of course an analogous result holds for the backward wave operator.

Another important consequence of Theorem 3.3 is energy decay. Let G' be a subset of the exterior domain G. Suppose G' is contained in the ball of radius R, and suppose that g is of form (3.8). It follows from Lemma 2.2 that for $t > R - \rho - p$, $U(t)g$ is zero for $|x| < R$; therefore for such t

$$E(U(t)g, G') = 0.$$

Hence for all g of form (3.8), and all bounded subdomains G'

$$\lim_{t\to\infty} E(U(t)g, G') = 0. \tag{3.11}$$

If, as asserted in Theorem 3.3, these elements g are dense in H, then it follows that (3.11) holds for all g, that is,

Corollary 3.5. *For every solution of the mixed problem with finite energy all energy is eventually propagated out to infinity.*

In order to prove Theorem 3.3 we shall need

Lemma 3.6. For all g in H and every bounded subdomain G′

$$\liminf_{t\to\infty} E(U(t)g, G') = 0. \tag{3.12}$$

The proof of this for n even is the same as for n odd, described on pages 145–147 of [4]; for this reason we merely sketch the salient points of the proof.

It suffices to prove (3.12) for a dense subset of H; so we take g to belong to the domain of A. For such a g

$$||AU(t)g||_E = ||U(t)Ag||_E = ||Ag||_E .$$

According to Theorem 3.1, part a) we can estimate the square integral of the second order space derivatives of the first components of $U(t)g$, as well as the first order space derivatives of the second components, in terms of $||g||^2$ and $||Ag||^2$. So we deduce by the Rellich compactness theorem that $U(t)g$ lies in a compact set in the local energy norm. Therefore in order to prove (3.12) it suffices to construct a subsequence $\{t_N\} \to \infty$ such that

$$U(t_N)g \xrightarrow{weakly} 0,$$

i.e., such that

$$(U(t_N)g, d)_E \to 0 \tag{3.13}$$

for all d in H. Again it suffices to prove (3.13) for a dense subset of d.

By Stone's theorem

$$(U(t)g, d)_E = \int e^{it\lambda}\, d(g, P_\lambda d) = \int e^{it\lambda}\, dm. \tag{3.14}$$

According to Lemma 3.2, A has no point spectrum; consequently the spectral resolution P_λ and the measure $m = (g, P_\lambda d)$, have no point mass. Invoking a classical theorem of Wiener, we see that the Fourier transform m of a measure m with finite total variation and no point mass tends to zero in the mean:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\hat{m}(t)|^2\, dt = 0.$$

From this we can produce a sequence $t_N \to \infty$ such that $\hat{m}(t_N) \to 0$, for any denumerable collection of such measures m. Since according to (3.14)

$$\hat{m}(t) = (U(t)g, d),$$

we have succeeded in constructing a sequence $\{t_N\}$ such that (3.13) holds for a denumerable set of d. This completes the proof of Lemma 3.6.

The proof sketched above also shows the validity of

Corollary 3.7. For any finite collection $g_1, \cdots, g_k$ of data in H, and any bounded G′

$$\liminf_{t\to\infty} \sum E(U(t)g_i, G') = 0.$$

We now return to the proof of Theorem 3.3; our proof is indirect, that is we assume to the contrary that there exists a nonzero g' in H which is orthogonal to all elements of the form $U(p)f$, f in D_+^ρ ; since U is unitary, it follows that $U(-p)g'$ is orthogonal to D_+^ρ for all p. Any linear combination of the form

$$g = \int \varphi(t)U(t)g'\,dt$$

is also orthogonal to $U(p)D_+^\rho$ for any weight function φ. If φ is chosen to peak sharply enough near $t = 0$, g is near g' and therefore $\neq 0$; and if φ is a C_0^∞ function, *g belongs to the domain of A^j for all $j = 1, 2, \cdots$*.

Denote by G_R the set of those points of G for which $|x| < R$.

We now apply Corollary 3.7 to the domain $G' = G_{5\rho}$ and the data $g, Ag, \cdots, A^N g$; we conclude that, given any $\epsilon > 0$, there exist arbitrarily large values of T such that

$$E(U(T)A^j g, G_{5\rho}) < \epsilon, \qquad j = 0, 1, \cdots, N. \tag{3.15}$$

Next we make use of another fact about the wave equation: the domain of influence propagates with speed $\leqq 1$. In particular energy propagates with speed $\leqq 1$, from which we conclude that

$$E(U(t)d, G_{3\rho}) \leqq E(d, G_{5\rho})$$

for all data d and for $0 \leqq t \leqq 2\rho$. Applying this to $d = U(T)A^j g$ we deduce from (3.15) that for $j = 0, 1, \cdots, N$,

$$E(U(t)A^j g, G_{3\rho}) < \epsilon \quad \text{for} \quad T \leqq t \leqq T + 2\rho.$$

Since A commutes with U, this can be written as

$$E(A^j U(t)g, G_{3\rho}) < \epsilon \quad \text{for} \quad T \leqq t \leqq T + 2\rho, \quad j = 0, 1, \cdots, N. \tag{3.16}$$

We apply now inequality (3.6) of Theorem 3.1 to

$$d = U(t)g,\ G' = G_{3\rho},\ C = \{x \mid \rho \leqq |x| \leqq 2\rho\};$$

denoting the first component of $U(t)g$ by $w = w(x, t)$ we conclude from (3.16) that

$$\int_{\rho \leqq |x < 2\rho} |\partial_x^\alpha w|^2\,dx < O(\epsilon) \quad \text{for} \quad T < t < T + 2\rho$$

and for $|\alpha| \leqq N + 1$. Using Sobolev's inequality we deduce that

$$|\partial_x^\beta w(x, t)| < O(\epsilon) \tag{3.17}$$

for

$$\rho < |x| < 2\rho, \qquad T < t < T + 2\rho, \qquad |\beta| < N - \frac{n}{2} + 1. \tag{3.18}$$

Let ξ be a C^∞ function of x, such that

(3.19) $$\xi(x) = \begin{cases} 0 & \text{for} \quad |x| < \rho \\ 1 & \text{for} \quad 2\rho < |x|, \end{cases}$$

where again ρ is so chosen that the obstacle is contained in $\{|x| < \rho\}$. Define the function v as

$$v = \xi w.$$

Although w is defined only for x in G, v can be regarded as being defined for all x. Since w satisfies the wave equation for x in G, v satisfies the inhomogeneous wave equation

(3.20) $$v_{tt} - \Delta v = q$$

where

(3.21) $$q = -w\Delta\xi - 2w_x \cdot \xi_x .$$

Note that on account of (3.19),

(3.22) $$q(x, t) = 0 \quad \text{for} \quad |x| < \rho \quad \text{and for} \quad |x| > 2\rho.$$

From inequality (3.17), (3.18) for w and its derivatives we deduce the following estimate for q as expressed by (3.21):

(3.23) $$|\partial_x^\beta q(x, t)| < O(\epsilon)$$

for $T < t < T + 2\rho$, $|\beta| < N - n/2$; and because of (3.22) these inequalities are valid for all x.

Since $q = 0$ for $|x| > 2\rho$ we can apply Lemma 1.1 to estimate the Radon transform of q and its derivatives in terms of the partial derivatives of q. Using the estimate (3.23) for the latter we deduce that

(3.24) $$|\partial_s^j \tilde{q}| < O(\epsilon) \quad \text{for} \quad T < t < T + 2\rho, \quad j < N - n/2$$

and from part a) of Lemma 1.1 we deduce that

(3.25) $$\tilde{q} = 0 \quad \text{for} \quad |s| > 2\rho.$$

We introduce the abbreviation

(3.26) $$m = R(\xi U(t)g),$$

where R denotes the Radon transform as defined in (2.5).

By construction and assumption, $U(t)g$ is orthogonal to $D_+^{2\rho} \subset D_+^\rho$, for every value of t. Members of $D_+^{2\rho}$ are zero for $|x| < 2\rho$; since ξ was chosen to be $=1$ for $x > 2\rho$, it follows that also $\xi U(t)g$ is orthogonal to $D_+^{2\rho}$. The Radon transform of such functions is characterized in Lemma 2.3; using the abbreviation (3.26) we conclude:

$$m(s, \omega, t) \quad \textit{is a polynomial in } s$$

of degree $<n/2 - 1$ *for* $s > 2\rho$.

If we abbreviate $\partial_s^j m$ by m_j the above assertion is equivalent with

$$m_j(s, \omega, t) = 0 \quad \text{for} \quad s \geqq 2\rho \quad \text{and} \quad j \geqq n/2 - 1. \tag{3.27}$$

Now take the Radon transform of equation (3.20); we get

$$\tilde{v}_{tt} - \tilde{v}_{ss} = \tilde{q}. \tag{3.28}$$

Recalling that v was defined as the first component of $\xi U(t)g$ we can rewrite equation (3.28), as we have done in Section 2, in terms of $m = R\xi U(t)g$ as follows:

$$(\partial_t + \partial_s)m = \tilde{q}. \tag{3.29}$$

Lemma 3.8. Let T be one of the times chosen so that (3.15) *holds; then for T large enough*

$$||m(T)||_j < O(\epsilon)$$

for all integers j in $n/2 \leqq j \leqq N - n/2$.

Proof. Differentiate (3.29) j times with respect to s; abbreviating $\partial_s^j m$ by m_j as before, and $\partial_s^j \tilde{q}$ by $\tilde{q}_j$, we can write the resulting equation as

$$(\partial_t + \partial_s)m_j = \tilde{q}_j \, . \tag{3.30}$$

We write

$$||m||_j^2 = \int_{-\infty}^{\infty} |m_j|^2 \, ds = \int_{-\infty}^{-2\rho} + \int_{-2\rho}^{2\rho} + \int_{2\rho}^{\infty} = I_1 + I_2 + I_3 \, .$$

According to (3.27), $m_j = 0$ for $s > 2\rho$ and for $j \geqq n/2 - 1$; so $I_3 = 0$ for $j \geqq n/2 - 1$.

According to (3.25), $\tilde{q} = 0$ for $|s| > 2\rho$; therefore also $q_j = 0$ there, and we conclude from the differential equation (3.30) that for $s < -2\rho$, m_j is a constant along the characteristic of (3.30):

$$m_j(s, T) = m_j(s - T, 0).$$

Thus

$$I_1 = \int_{-\infty}^{-2\rho} |m_j(s, t)|^2 \, ds = \int_{-\infty}^{-2\rho - T} |m_j(s, 0)|^2 \, ds.$$

Now m_j is square integrable for $j \geqq n/2$; so it follows from the above expression for I_1 that $I_1 < \epsilon$ for T large enough.

To estimate I_2 we integrate the differential equation (3.30) along the characteristic between (s, T) and $(2\rho, T + 2\rho - s)$. Using the fact that according to (3.27), m_j is zero at the upper endpoint for $j \geqq n/2 - 1$ we get the following integral expression for m_j:

$$m_j(s, T) = -\int_0^{2\rho - s} \tilde{q}_j(s + t, T + t) \, dt.$$

For s in $[-2\rho, 2\rho]$ the interval of integration is at most 4ρ; using the estimate (3.24) for $\tilde{q}_j$ we conclude that

$$\tilde{m}_j(s, T) < O(\epsilon) \quad \text{for} \quad |s| < 2\rho;$$

this implies that also I_2 is $O(\epsilon)$ and the proof of Lemma 3.8 is complete.

Lemma 3.8 applies to integer values of j; using inequality (1.16) we can extend it to noninteger values. In particular setting $k = m_1$ we get from (1.16) that

$$||m_1(T)||^2_{(n-1)/2} \leqq \tfrac{1}{2} ||m_1(T)||^2_{n/2-1} + \tfrac{1}{2} ||m_1(T)||^2_{n/2} .$$

By definition $m_1 = \partial_s m$; so $||m_1||_j = ||m||_{j+1}$, and using Lemma 3.8 for $j = n/2$ and $n/2 + 1$ we deduce from the above inequality that

$$||m_1(T)||_{(n-1)/2} < O(\epsilon). \tag{3.31}$$

Combining (3.26) and (2.6) we see that

$$m_1 = -R(A_0 \xi U(t) g.$$

According to the Parseval relation (2.7)

$$||m_1(T)||_{(n-1)/2} = ||A_0 \xi U(T) g||_E ;$$

and using (3.31) we conclude that

$$||A \xi U(T) g||_E < O(\epsilon). \tag{3.32}$$

Note that we have replaced the operator A_0 by A, a move justified by the fact that $\xi(x) = 0$ for $|x| < \rho$.

Since the factor $\xi = 1$ for $|x| > 2\rho$, we see that

$$||AU(T)g||^2_E \leqq 2E(AU(T), G_{2\rho}) + 2 ||A\xi U(T)g||^2_E .$$

The terms on the right are estimated by inequality (3.16), respectively (3.32); we therefore conclude that there are values T where

$$||AU(T)g||_E < O(\epsilon).$$

Now $||AU(T)g||_E = ||Ag||_E$ is independent of T; since ϵ in the last inequality is arbitrary, it follows that $Ag = 0$. Hence by Lemma 3.2 also $g = 0$. This is a contradiction to our initial assumption, which proves Theorem 3.3.

4. **Incoming and outgoing free space translation representations.** The following theorem plays a basic role in our book [4]:

Let H be a Hilbert space, $U(t)$ a one-parameter group of unitary operators of $H \to H$, and D_+ a closed subspace of H with the following properties:

$$(4.1)_+ \qquad \begin{array}{ll} \text{i)} & U(t)D_+ \subset D_+ \text{ for } t > 0; \\ \text{ii)} & \wedge\, U(t)D_+ = \{0\}; \\ \text{iii)} & \vee\, U(t)D_+ = H. \end{array}$$

Conclusion: There is a mapping $\mathfrak{I}$ of H onto $L_2(\mathbf{R}, N)$, N an auxiliary Hilbert space, such that

(4.2)
a) $\mathfrak{I}$ is unitary;
b) $\mathfrak{I}$ maps D_+ onto $L_2(\mathbf{R}_+, N)$;
c) $\mathfrak{I}U(t) = T(t)\mathfrak{I}$, where $T(t)$ is right translation by t units.

$\mathfrak{I}$ is called the unitary *translation representation of U relative to D_+*.

An analogous result holds for a subspace D_- satisfying

$$\text{i)}_- \quad U(t)D_- \subset D_- \quad \text{for} \quad t < 0$$

along with (ii) and (iii) of $(4.1)_+$. In this case the corresponding mapping $\mathfrak{I}$ takes D_- onto $L_2(\mathbf{R}_-, N)$.

We recall that in Section 2, using the Radon transform:

$$d \rightarrow Rd,$$

we represented data in H_0 as functions of s and ω in $W_{(n-1)/2}$. Such a function can just as well be regarded as a vector-valued function of s with values in $N = L_2(S^{n-1})$. According to (2.11) the mapping R is a translation representation of the group U_0. Furthermore, the outgoing and incoming subspaces D_+ and D_- obviously satisfy the properties $(4.1)_+$ and $(4.1)_-$, respectively. By definition, data d belongs to D_+ if and only if $K\partial_s^{n-2}Rd$ vanishes for $s < 0$. However, as in the last part of the proof of Lemma 2.3, if we use the fact that Rd belongs to $W_{(n-1)/2}$ we see that $K\partial_s^{n-1}Rd$ vanishes for $s < 0$ if and only if $K\partial_s^{n-2}Rd$ vanishes for $s < 0$. Thus d belongs to D_+ if and only if

$$l = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K\partial_s^{n-1}Rd \tag{4.3}$$

vanishes for $s < 0$. It is clear from this and Lemma 2.3 that neither D_+ nor its complement consist of functions which vanish on a half line. Nevertheless R is very close to (4.2b) and motivates our development. Another difficulty arises from (2.7) according to which the energy norm of d is equal to the $(n - 1)/2$ norm instead of the L_2 norm of Rd.

In this section we show how to modify R to obtain two distinct translation representations, one for D_+ and the other for D_-, called outgoing and incoming translation representations, both of which are unitary. We note that for the case n odd, considered in our book, the outgoing and incoming unitary translation representations of U_0 were the same.

Our translation representations will be of the form

$$d \rightarrow JRd = \mathfrak{I}d, \tag{4.4}$$

J an operator to be determined. We recall the following fact, see (2.7) and (1.17):

$$||d||_E^2 = \tfrac{1}{2}\, [Rd, Rd]_{(n-1)/2}$$

where

$$[m, m]_{(n-1)/2} = [i^{n-1} K \partial_s^{n-1} m, m].$$

It follows that the isometry of (4.4) can be expressed as:

$$2\, \|d\|_E^2 = [i^{n-1} K \partial_s^{n-1} Rd, Rd]$$
$$= [JRd, JRd] = [J^* JRd, Rd].$$

Clearly, this is the case when

$$J^* J = i^{n-1} K \partial_s^{n-1}. \tag{4.5}$$

Using (4.5), and the definition (4.4) of $\mathfrak{I}$ as JR, relation (4.3) can be written as follows:

$$l = \text{const. } J^* JRd = \text{const. } J^* \mathfrak{I} d.$$

By definition, d belongs to D_+ if and only if l is supported on $\mathbf{R}_+$. Since by $(4.2)_b$ $\mathfrak{I}$ is supposed to map D_+ onto $L_2(\mathbf{R}_+, N)$, it follows that

β) J^* maps $L_2(\mathbf{R}_+, N)$ onto distributions supported in $\mathbf{R}_+$.

Since $d \to Rd$ is already a translation representation, condition $(4.2)_c$ means that

γ) J commutes with translation.

It is convenient to look at J in the Fourier representation:

$$\mathcal{J} = FJF^{-1}.$$

For J to commute with translation we have to take $\mathcal{J}$ in the form of multiplication by a scalar function $\mathcal{J}(\sigma)$. In this case equation (4.5) becomes

$$|\mathcal{J}(\sigma)|^2 = |\sigma|^{n-1} \tag{4.6}$$

and J^* corresponds to multiplication by $\overline{\mathcal{J}(\sigma)}$.

Property (β) requires that J^* take $L_2(\mathbf{R}_+, N)$ into functions vanishing on $\mathbf{R}_-$. In the Fourier representation the subspace of functions vanishing on $\mathbf{R}_-$ becomes a space of functions analytic in the upper half plane. Hence in order that J^* map $L_2(\mathbf{R}_+, N)$ into the distributions associated with D_+ we see that $\overline{\mathcal{J}(\sigma)}$ must have an *analytic extension into the upper half plane.* The only choice remaining at this point is

$$\mathcal{J}_+ = \begin{cases} \sigma^{1/2} \sigma^{n/2-1} & \text{for } \sigma > 0 \\ -i\, |\sigma|^{1/2}\, \sigma^{n/2-1} & \text{for } \sigma < 0; \end{cases} \tag{4.7$_+$}$$

and it follows from the Paley–Wiener theorem that $\overline{\mathcal{J}}_+$ does indeed take $L_2(\mathbf{R}_+, N)$ into the distributions associated with D_+. Similarly for the incoming representation, $\overline{\mathcal{J}}_-$ has to take functions analytic in the lower half plane into themselves and this is accomplished by choosing for $\mathcal{J}_-$:

$$\mathcal{J}_- = \begin{cases} \sigma^{1/2} \sigma^{n/2-1} & \text{for } \sigma > 0 \\ i\, |\sigma|^{1/2}\, \sigma^{n/2-1} & \text{for } \sigma < 0. \end{cases} \tag{4.7$_-$}$$

Here the square root of a positive number is taken to be positive. We note that $\mathcal{J}_+ = \overline{\mathcal{J}}_-$ so that $J_+ = J_-^*$.

In order to prove that $J_+^* L_2(\mathbf{R}_+, N)$ is onto the distributions associated with D_+ it suffices to show that data in $D_+^\perp$ correspond to functions in $L_2(\mathbf{R}_-, N)$; that is, that $J_+ R D_+^\perp$ maps into $L_2(\mathbf{R}_-, N)$. Now according to Lemma 2.3 the Radon transform of data in $D_+^\perp$ are polynomials of degree $<n/2 - 1$ for $s > 0$. The factor $\sigma^{n/2-1}$ in $\mathcal{J}_+$ differentiates away any such polynomial; that is, for d in $D_+^\perp$, $\partial_s^{n/2-1} Rd = 0$ for $s > 0$. By the Paley–Wiener theorem the remaining factor in $\mathcal{J}_+$, namely $\sigma^{1/2}$ for $\sigma > 0$ and $-i\,|\sigma|^{1/2}$ for $\sigma < 0$, takes Radon transforms with support in $\mathbf{R}_-$ into functions with support in $\mathbf{R}_-$ as desired. We summarize the above result in

Theorem 4.1. *Let J_+ and J_- denote operators whose symbols are given by* $(4.7)_+$ *and* $(4.7)_-$ *respectively. Then the representations*

$$d \to \frac{1}{2^{1/2}} J_\pm R d \tag{4.8}$$

are outgoing and incoming unitary translation representations of U_0 relative to D_+ and D_- respectively.

Another useful characterization of D_+ and D_- is given in the following corollary:

Corollary 4.2. *Data belong to D_+* [or D_-] *if and only if*

$$[U_0(t)d](x) = 0 \quad \textit{for all} \quad |x| < t \quad [\textit{or for all} \quad |x| < -t]. \tag{4.9}$$

Proof. The "only if" part of the assertion follows directly from the definition of D_+ and D_-. We now sketch the converse argument for D_+. Here we may suppose without loss of generality that d lies in $D(A^N)$ for all N. We represent the solution u with initial data d as in Theorem 2.1:

$$u(x, t) = \int k(x\cdot\omega - t, \omega)\, d\omega,$$

where

$$k = \frac{i^{n-1}}{2(2\pi)^{(n-1)/2}} K\, \partial_s^{n-2} Rd.$$

Assuming (4.9) it is clear that if φ lies in $C_0^\infty(0, \infty)$ then

$$\iint k(x\cdot\omega - t, \omega)\varphi'(t)\, d\omega\, dt = \int u(x, t)\varphi'(t)\, dt = 0$$

for all x in a sufficiently small neighborhood of 0. Applying ∂_x^α and setting $x = 0$ gives

$$0 = \iint \omega^\alpha\, \partial_s^{|\alpha|} k(s, \omega)\varphi'(-s)\, d\omega\, ds = \iint \omega^\alpha\, \partial_s^{|\alpha|+1} k(s, \omega)\varphi(-s)\, d\omega\, ds.$$

Combining (4.3) and (4.5) and replacing $\varphi(-s)$ by $\psi(s) \in C_0^\infty(-\infty, 0)$, this relation can be rewritten as

$$\iint \omega^\alpha \, \partial_s^{|\alpha|} J_+ Rd \overline{J_+ \psi} \, d\omega \, ds = 0.$$

Finally, setting

$$a_\alpha(s) = \int \omega^\alpha J_+ Rd \, d\omega \tag{4.10}$$

we get

$$\int \partial_s^{|\alpha|} a_\alpha(s) \overline{J_+ \psi} \, ds = 0 \tag{4.11}$$

for all multi-indices α. Note that since d belongs to $D(A^N)$ $\partial_s^{|\alpha|} a_\alpha$ is square integrable and that $J_+ C_0(\mathbf{R}_-)$ is dense in $L_2(\mathbf{R}_-)$; see the remark at the end of this section. It follows that $a_\alpha(s)$ is equal to a polynomial of degree $< |\alpha|$ for almost all $s < 0$ and since it is square integrable it is zero for almost all $s < 0$. Since the $\{\omega^\alpha\}$ are dense in $L_2(S^{n-1})$ we conclude that $J_+ Rd = 0$ for almost all $s < 0$ and by Theorem 4.1 this means that d belongs to D_+.

In the next section we construct explicit incoming and outgoing spectral representations for U and for this it will be convenient to have a well-behaved set of data dense in $D_\pm^\rho$. To this end we prove

Lemma 4.3. Let V_m denote the set of date $\{f_1, f_2\}$ for which

$$(1 + |x|^2)^{m/2} \partial_x f_1 \quad \text{and} \quad (1 + |x|^2)^{m/2} f_2 \tag{4.12}$$

are square integrable. Then $V_m \cap D_\pm^\rho \cap D(A)$ is dense in $D_\pm^\rho$.

Proof. We carry out the proof for the case D_+^ρ. In the outgoing translation representation described in Theorem 4.1, the subspace D_+^ρ corresponds to $L_2((\rho, \infty), N)$. We shall show that the set of all data d with representer of the form

$$k_+ = \frac{1}{2^{1/2}} J_+ Rd$$

with k_+ in $C_0^\infty(\rho, \infty)$, obviously in $D(A_0)$ and $D(A)$, and such that

$$\int s^j k_+(s) \, ds = 0 \quad \text{for} \quad j = 0, 1, \cdots, m + \frac{n}{2} - 1 \tag{4.13}$$

belongs to V_m. We note to begin with that the set of such data is clearly dense in D_+^ρ. It is clear that the Fourier transform Fk_+ is smooth, rapidly decreasing and by (4.13) has a zero of order $m + n/2$ at $\sigma = 0$. Consequently

$$FRd = 2^{1/2} \frac{Fk_+}{\mathcal{g}_+} \tag{4.14}$$

is rapidly decreasing and m times continuously differentiable.

If we denote the components of d by $\{f_1, f_2\}$ and their Fourier transforms by $\hat{f}_i(\xi)$, then as in (1.5)

(4.15) $$\hat{f}_i(\sigma\omega) = F\tilde{f}_i$$

and by (2.5)

(4.16) $$\partial_s \tilde{f}_1 = \text{odd part of } Rd,$$
$$\tilde{f}_2 = \text{even part of } Rd.$$

Combining (4.14), (4.15) and (4.16) we see both $\sigma\hat{f}_1$ and $\hat{f}_2$ are rapidly decreasing and m times continuously differentiable. It now follows by a familiar application of the Parseval relation that d belongs to V_m.

Remark. $J_+C_0(-\infty, 0)$ is dense in $L_2(\mathbf{R}_-)$: By construction, $J_+C_0^\infty(\mathbf{R}_-) \subset L_2(\mathbf{R}_-)$. If $J_+C_0^\infty(\mathbf{R}_-)$ were not dense in $L_2(\mathbf{R}_-)$, there would exist an $h \neq 0$ in $L_2(\mathbf{R}_-)$ orthogonal to $J_+C_0^\infty(\mathbf{R}_-)$. It follows that the distribution $J_+^* h = k$ is orthogonal to $C_0^\infty(\mathbf{R}_-)$, which means that the support of k lies on $\mathbf{R}_+$. By definition of J_+, the Fourier transform $\tilde{k} = \widetilde{J_+^* h}$ equals $\bar{\mathcal{g}}_+(\sigma)\tilde{h}(\sigma)$; moreover, according to $(4.7)_+$, $|\mathcal{g}_+(\sigma)| = |\sigma|^{n-1/2}$, so that $\tilde{k}/(i+\sigma)^n = (\bar{\mathcal{g}}_+/(i+\sigma)^n)\tilde{h}$ belongs to L_2.

Since the support of k and $F^{-1}(i+\sigma)^{-n}$ both lie in $\mathbf{R}_+$, the same is true of their convolution. It therefore follows from the above that

$$\frac{\tilde{k}(\sigma)}{(1 - i\epsilon\sigma)^n}$$

belongs to the Hardy class and from $(4.7)_+$ that

$$\tilde{h}(\sigma) = \frac{\tilde{k}(\sigma)}{\bar{\mathcal{g}}_+(\sigma)}$$

is also analytic in the upper half plane. We claim that $\tilde{h}$ actually belongs to the Hardy class, *i.e.*, that the L_2 norm of $\tilde{h}$ along lines $\operatorname{Im}\sigma = \text{const} > 0$ are bounded by the L_2 norm of $\tilde{h}$ along the real axis. To see this, form the functions

$$\tilde{h}_\epsilon(z) = \tilde{h}(z)\left(\frac{z}{z+i\epsilon}\right)^n\left(\frac{1}{1-i\epsilon z}\right)^n$$
$$= \frac{\tilde{k}(z)}{(1-i\epsilon z)^n}\cdot\frac{z^n}{\bar{\mathcal{g}}_+(z)}\cdot\frac{1}{(z+i\epsilon)^n}.$$

According to our previous observation about $\tilde{k}$, for every $\epsilon > 0$, $\tilde{h}_\epsilon$ is of Hardy class. As $\epsilon \to 0$, $\tilde{h}_\epsilon \to \tilde{h}$ in L_2; since the Hardy class is closed, it follows that $\tilde{h}$ belongs to it. Hence by the Paley–Wiener theorem, h is supported on $\mathbf{R}_+$. But this contradicts the fact that h is supported on $\mathbf{R}_-$.

5. The incoming and outgoing spectral representations. The outgoing and incoming spectral representations can be obtained directly from the respective translation representations by Fourier transform. In particular the U_0 spectral representations can be read off Theorem 4.1 and are given by

$$\mathfrak{F}^0_\pm : d \to \frac{1}{2^{1/2}} FJ_\pm Rd. \tag{5.1}$$

In this section we shall find a more explicit form of (5.1) and use this to obtain the incoming and outgoing spectral representations for the exterior problem. As a part of this development we will construct the scattered wave solutions for the reduced wave equation by reducing this to an interior problem.

Theorem 5.1. Let

$$\varphi_\pm(x, \omega, \sigma) = \begin{cases} -i \dfrac{\sigma^{n/2-2}}{(2\pi)^{n/2}} (\sigma/2)^{1/2} \{e^{-i\sigma x\cdot\omega}, i\sigma e^{-i\sigma x\cdot\omega}\} & \text{for} \quad \sigma > 0 \\ \mp \dfrac{\sigma^{n/2-2}}{(2\pi)^{n/2}} (|\sigma|/2)^{1/2} \{e^{-i\sigma x\cdot\omega}, i\sigma e^{-i\sigma x\cdot\omega}\} & \text{for} \quad \sigma < 0. \end{cases} \tag{5.2}$$

$$\mathfrak{F}^0_\pm : d \to (d, \varphi_\mp(\cdot, \omega, \sigma))_E ; \tag{5.3}$$

note the switch in signs in formula (5.3). Under the mapping $\mathfrak{F}^0_\pm$, H_0 is transformed unitarily onto $L_2(\mathbf{R}, N)$, the action of $U_0(t)$ goes over into multiplication by $e^{i\sigma t}$, and $D_+[D_-]$ corresponds in the outgoing [incoming] representation to the Hardy class $A_+(N)$ [the conjugate Hardy class $A_-(N)$] of vector-valued functions analytic in the upper [lower] half plane.

Proof. Aside from the formula (5.3) all of the assertions of the theorem are direct consequences of (5.1) and the properties of the outgoing and incoming translation representations as given in Theorem 4.1. To obtain (5.3) we rewrite (5.1) as

$$\mathfrak{F}^0_+ d = \frac{1}{2^{1/2}} \mathcal{J}_\pm(\sigma) FRd. \tag{5.4}$$

Now according to (1.5) and (2.5)

$$Rd = \partial_s \tilde{f}_1 - \tilde{f}_2 \quad \text{and} \quad F\tilde{f}_i = \frac{1}{(2\pi)^{1/2}} \hat{f}_i(\sigma\omega). \tag{5.5}$$

Combining (5.4) and (5.5) we obtain

$$\mathfrak{F}^0_\pm d = \frac{-1}{2^{1/2}} \mathcal{J}_\pm(\sigma)[i\sigma \hat{f}_1(\sigma\omega) + \hat{f}_2(\sigma\omega)],$$

which is readily computed to be the same as formula (5.3).

Remark. The incoming and outgoing representations of U_0 are Hilbert transforms of each other: that is

$$\mathfrak{F}^0_+ d = \mathcal{K}\mathfrak{F}^0_- d, \tag{5.6}$$

where

$$\mathcal{K}(\sigma) = \begin{cases} 1 & \text{for } \sigma > 0 \\ -1 & \text{for } \sigma < 0. \end{cases}$$

To construct the analogous incoming and outgoing spectral representations for the exterior problem we will need the scattered wave solutions of the reduced wave equation:

$$(5.7) \qquad \begin{aligned} \Delta v + \sigma^2 v &= 0 && \text{in } G, \\ v(x; \omega, \sigma) &= -e^{-i\sigma x\cdot\omega} && \text{on } \partial G, \end{aligned}$$

v_+ outgoing and v_- incoming in the sense that for large $r = |x|$, v_+ [or v_-] is a superposition of the outgoing [incoming] fundamental solution γ_+ [γ_-]:

$$(5.8)_+ \qquad \gamma_+(r, \sigma) = -\frac{i}{4}\left(\frac{\sigma}{2\pi r}\right)^{n/2-1} H^{(2)}(\sigma r)$$

$$(5.8)_- \qquad \gamma_-(r, \sigma) = \frac{i}{4}\left(\frac{\sigma}{2\pi r}\right)^{n/2-1} H^{(1)}(\sigma r),$$

written here in terms of Hankel functions. It should be noticed that $\gamma(r, \zeta)$ has a logarithmic branch point at $\zeta = 0$ when n is even. For large r, $\gamma_\pm$ behaves like $\exp(\mp i\sigma r)/r^{(n-1)/2}$ and the scattered wave v behaves like

$$(5.9) \qquad v_\pm(r\theta, \omega, \sigma) \sim \frac{e^{\mp i\sigma r}}{r^{(n-1)/2}}\, s_\pm(\theta, \omega; \sigma).$$

Theorem 5.2. There exist unique incoming and outgoing solutions $v_\mp$ of the reduced wave equation (5.7). These are continuous on $S^{n-1} \times (\mathbf{R}\backslash 0)$ in the local energy norm and are uniformly of order $r^{(1-n)/2}$ for σ in any compact subset of $\mathbf{R}\backslash 0$. Any $d_\pm$ in $V_m \cap D_\pm^\rho$, $m > n/2 + 1$, is orthogonal to $v_\mp$; that is

$$(d_\pm, v_\mp)_E = 0.$$

We postpone the proof of Theorem 5.2 until after we have shown how to use it in the construction of the incoming and outgoing spectral representations for U. For this we need the functions

$$(5.11) \qquad \psi_\pm(x, \omega, \sigma) = \begin{cases} -i\dfrac{\sigma^{n/2-2}}{(2\pi)^{n/2}}(\sigma/2)^{1/2}\{e^{-i\sigma x\cdot\omega} + v_\pm,\ i\sigma(e^{-i\sigma x\cdot\omega} + v_\pm)\} & \text{for } \sigma > 0 \\ \mp i\dfrac{\sigma^{n/2-2}}{(2\pi)^{n/2}}(|\sigma|/2)^{1/2}\{e^{-i\sigma x\cdot\omega} + v_\pm,\ i\sigma(e^{-i\sigma x\cdot\omega} + v_\pm)\} & \text{for } \sigma < 0. \end{cases}$$

Notice that $\psi_\pm$ belongs to $D(A)$ locally and that $A\psi_\pm = i\sigma\psi_\pm$.

Theorem 5.3. Under the mapping

$$(5.12) \qquad \mathfrak{F}_\pm : d \to (d, \psi_\mp)_E$$

H is carried unitarily onto $L_2(\mathbf{R}, N)$, the action of $U(t)$ goes into multiplication by $e^{i\sigma t}$ and in the outgoing [incoming] spectral representation D_+^ρ [D_-^ρ] corresponds to $e^{i\sigma\rho}A_+(N)$ [$e^{-i\sigma\rho}A_-(N)$].

Proof. Here we follow the proof of Theorem 5.3 in Chapter V of our book [4]. We shall construct the outgoing spectral representation. For d in V_m, $m > n + 1$, the expression $(d, \psi_-)_E$ is well-defined, linear in d and continuous in $\{\sigma, \omega\}$ on account of the continuity and boundedness properties of $v(\sigma, \omega)$ asserted in Theorem 5.2. The orthogonality assertion of Theorem 5.2 implies that

$$\mathfrak{F}_+d = \mathfrak{F}_+^0 d, \qquad d \text{ in } V_m \cap D_+^\rho \cap D(A), \tag{5.13}$$

and hence by Theorem 5.1, $\mathfrak{F}_+$ is an isometry on this subspace and it can be extended by continuity to map D_+^ρ onto $e^{i\sigma\rho}A_+(N)$.

Next set $d(t) = U(t)d$, again for d in $V_m \cap D_+^\rho \cap D(A)$, and let $\check{d}_+(t)$ denote its representer, $\mathfrak{F}_+d(t)$. Then

$$\frac{d}{dt}\check{d}_+(t) = (Ad(t), \psi_-)_E = -(d(t), A\psi_-) = i\sigma\check{d}_+(t).$$

Consequently

$$\check{d}_+(t) = e^{i\sigma t}\,\check{d}_+(0)$$

and it follows from this that the mapping is an isometry for all data e of the form

$$e = U(-T)d, \qquad T \text{ in } \mathbf{R},$$

and as above d is in $V_m \cap D_+^\rho \cap D(A)$. According to Theorem 3.1 and Lemma 4.3 the set of all such e is dense in H and therefore $\mathfrak{F}_+$ can be extended by continuity to be an isometry on all of H.

It remains only to show that the mapping is onto. But this follows from the fact that D_+^ρ maps onto $e^{i\sigma\rho}A_+(N)$. For then $U(T)D_+$ maps onto $e^{i\sigma(\rho+T)}A_+(N)$ and since these sets are dense in $L_2(\mathbf{R}, N)$ we conclude that $\mathfrak{F}_+$ is necessarily onto. This concludes the proof of Theorem 5.3.

We return now to Theorem 5.2, a fairly classical result first proved in the fifties by Kupradse [3], Müller [7] and Weyl [15] and later simplified by P. Werner [14]. We shall present a different proof which in effect reduces the exterior problem to the solution of an interior problem. In this respect our approach is similar to that of R. Leis [5]; however, perhaps because of its crudeness, our method is the more flexible of the two (see [8]).

Instead of (5.7) we consider the inhomogeneous problem:

$$\begin{aligned} \Delta u + \zeta^2 u &= f \quad \text{in } G,\\ u &= 0 \quad \text{in } \partial G, \end{aligned} \tag{5.14}$$

f of compact support and u outgoing for large r. We suppose that $\zeta \in C$ and

$f \in L_2(G_\rho)$. We shall treat this as a perturbation of the free space problem:

$$\Delta v + \zeta^2 v = g \quad \text{in} \quad \mathbf{R}^n,\ g \quad \text{in} \quad L_2(G_\rho), \tag{5.15}$$

whose solution can be given in terms of γ_+ as

$$v = R_\zeta^0 g = \int \gamma_+(x - y, \zeta) g(y)\, dy. \tag{5.16}$$

Accordingly we set

$$u = v - \beta h$$

where

$$\begin{aligned} \Delta h + \lambda^2 h &= 0 \quad \text{in} \quad G_\rho\,, \\ h &= v \quad \text{on} \quad \partial G, \\ h &= 0 \quad \text{on} \quad \{|x| = \rho\}. \end{aligned} \tag{5.17}$$

and $\beta \in C_0^\infty$ is chosen to be identically one in a neighborhood of the obstacle 0 and identically zero in a neighborhood of $\{|x| = \rho\}$. We note that (5.17) is always solvable if Im $\lambda^2 \neq 0$; λ will be so chosen and fixed throughout the discussion. Then v is completely determined by g and h is completely determined by v.

Our problem is to find g for which u satisfies (5.14), that is

$$\begin{aligned} f = \Delta u + \zeta^2 u &= \Delta v + \zeta^2 v - \Delta(\beta h) - \zeta^2 \beta h \\ &\equiv g - T_\zeta g \end{aligned} \tag{5.18}$$

where

$$T_\zeta g = 2\nabla\beta\cdot\nabla h + (\Delta\beta)h + (\zeta^2 - \lambda^2)\beta h. \tag{5.19}$$

Notice that $T_\zeta g$ contains no higher derivatives of h than the first.

The pertinant properties of T_ζ are listed in the following lemma:

Lemma 5.4. a) *T_ζ is a compact operator on $L_2(G_\rho)$ for each ζ in $C\backslash 0$;*

b) *$I - T_\zeta$ is one-to-one for* Im $\zeta \leqq 0$, $\zeta \neq 0$;

c) *T_ζ is a holomorphic function of ζ in the* log ζ *Riemann surface.*

Proof of a). Let $H_k(G_\rho)$ denote the Sobolev space of functions on G_ρ which are differentiable of order k, having as norm

$$\|h\|_k' = \left\{ \sum_{|\alpha|\leq k} \int_{G_\rho} |\partial^\alpha h|^2\, dx \right\}^{1/2}.$$

From (5.16) and the theory of singular integrals we see that

$$\|\beta v\|_2' \leqq C_\zeta \|g\|_0', \tag{5.20}$$

where C_ζ is uniformly bounded on compact subsets of the log Riemann surface.

Now $w = h - \beta v$ satisfies the equation

$$\triangle w + \lambda^2 w = -\triangle(\beta v) - \lambda^2 \beta v \quad \text{in} \quad G_\rho \,.$$
$$w = 0 \quad \text{on} \quad \partial G_\rho \,.$$

It therefore follows by elliptic theory that

(5.21) $$\|w\|_2' \leqq C \|\beta v\|_2' \,.$$

Combining this with (5.20) we obtain

$$\|h\|_2' \leqq CC_\zeta \|g\|_0' \,.$$

Since $T_\zeta g$ contains only first derivatives of h,

$$\|T_\zeta g\|_1' \leqq C \|h\|_2' \leqq C_\zeta' \|g\|_0' \,,$$

and hence by the Rellich compactness theorem we conclude that T_ζ is a compact linear operator on $L_2(G_\rho)$.

Proof of b). Suppose that $T_\zeta g = g$ for a nonzero g. Define v by (5.16); according to (5.18) this implies

$$\triangle v + \zeta^2 v = \triangle(\beta h) + \zeta^2 \beta h;$$

setting $u = v - \beta h$ we see that u is an outgoing solution of

(5.22) $$\triangle u + \zeta^2 u = 0 \quad \text{in} \quad G,$$
$$u = 0 \quad \text{on} \quad \partial G.$$

If Im $\zeta = 0$, $\zeta \neq 0$, then it follows by the Rellich uniqueness theorem that $u \equiv 0$ in G. For Im $\zeta < 0$, outgoing implies (see (5.9)) that v is in $L_2(G)$; since the spectrum of $\triangle$ lies in the negative real axis, $-\zeta^2$ is not in the spectrum of $\triangle$ and so (5.22) implies that $u \equiv 0$ in G. Since $u \equiv 0$ we see in particular that $h \equiv v$ near ∂G and hence h is a smooth continuation in G of v restricted to 0. Hence, making use of the fact that g is zero in 0 we see that

$$z = \begin{cases} v & \text{in} \quad 0 \\ h & \text{on} \quad \{|x| = \rho\}, \end{cases}$$

is a solution of

$$\Delta z + pz = 0 \quad \text{for} \quad |x| < \rho,$$
$$z = 0 \quad \text{on} \quad \{|x| = \rho\},$$

where $p = \zeta^2$ in 0 and λ^2 in G_ρ. Multiplying through by $\bar{z}$ and integrating by parts we obtain

$$\int [|\nabla z|^2 - p\, |z|^2]\, dx = 0$$

and since Im $\lambda^2 \neq 0$ we get

$$\int_{G_\rho} |z|^2 \, dx = 0$$

so that h, and with h also g, vanishes in G_ρ contrary to our assumption on g.

Proof of c). Considering R_ζ^0 as a transformation on $L_2(G_\rho)$ to the Sobolev space $H_2(G_\rho)$, it is clear from (5.16) that R_ζ^0 is holomorphic on the log Riemann surface. It then follows from the relation $h = w + \beta v$ and (5.21) that T_ζ is a holomorphic operator-valued function. This concludes the proof of Lemma 5.4.

We now define the operator Q by Q: $v \rightarrow \beta h$ and the transformation R_ζ by

$$u = R_\zeta f \equiv (I - Q)R_\zeta^0(I - T_\zeta)^{-1}f \tag{5.23}$$

on $L_2(G_\rho)$ to $H_2(G_\rho)$. Again by the relation $h = w + \beta v$ and (5.21) we see that Q is a bounded operator on $H_2(G_\rho)$. The assertions a)–c) of Lemma 5.4 imply (see for instance [12]) that $(I - T_\zeta)^{-1}$ is meromorphic on the domain of holomorphicity of T_ζ. Since u is determined by $R_\zeta^0(I - T_\zeta)^{-1}f$ outside of the ball $\{|x| < \rho\}$ we can just as well take the range space of R_ζ to be $H_2(G_R)$ for any $R > \rho$. Summarizing we have

Theorem 5.5. *Considered as a transformation-valued function (from* $L_2(G_\rho)$ *to* $H_2(G_R)$), $R > \rho$), R_ζ *is meromorphic on the log Riemann surface, holomorphic on one sheet covering* Im $\zeta \leqq 0$.

Proof of Theorem 5.2. In order to solve (5.7) we set

$$u = v - \beta e^{-i\sigma x \cdot \omega};$$

then u satisfies (5.12) with $\zeta = \sigma$ and

$$f(x, \omega, \sigma) = -2\nabla\beta \cdot \nabla e^{-i\sigma x \cdot \omega} - (\Delta\beta)e^{-i\sigma x \cdot \omega}.$$

It is clear that $f(\cdot\, , \omega, \sigma)$ is a continuous function of $\{\omega, \sigma\}$ in the $L_2(G_\rho)$ topology. It follows from Theorem 5.5 that $u(\cdot\, , \omega, \sigma) = R_\sigma f(\cdot\, , \omega, \sigma)$ is continuous in the $H_2(G_R)$ topology on $S^{n-1} \times (\mathbf{R}\backslash 0)$; this of course gives continuity in the local energy norm. Again making use of the fact that u, and hence v, is equal to $R_\sigma^0(I - T_\sigma)^{-1}f$ for $|x| > \rho$, it is easy to verify the uniform asymptotic behavior of v from (5.16) and the asymptotic behavior of $\gamma_\pm$. Uniqueness is a consequence of the Rellich uniqueness theorem.

To prove the orthogonality relation (5.10) we argue as follows: We construct v_+ as above; then for $|x| > \rho$ v_+ is of the form (5.16):

$$v_+ = R_\sigma^0 g, \qquad g \text{ in } L_2(G_\rho),$$

and so

$$(\{v_+\, , i\sigma v_+\}, d_-)_E = (\{R_\sigma^0 g, i\sigma R_\sigma^0 g\}, d_-)_E\, . \tag{5.24}$$

Since d_- in D_-^ρ vanishes for $|x| < \rho$,

$$(e, d_-)_E = 0 \quad \text{for} \quad e = \{0, -g\} \quad \text{and in} \quad g \quad \text{in} \quad L_2(G_\rho);$$

and since $U_0(-t)D_-^\rho \subset D_-^\rho$ for $t > 0$ we see that

$$(U_0(t)e, d_-)_E = (e, U_0(-t)d_-)_E = 0 \quad \text{for} \quad t > 0.$$

If we now take the Laplace transform of this expression we get

$$(5.25) \qquad (R_z(A_0)e, d_-)_E = \int_0^\infty e^{-zt}(U_0(t)e, d_-)_E\, dt = 0$$

for all z with Re $z > 0$. Setting $\{v_1, v_2\} = R_z(A_0)e$ and making use of the fact that $R_z(A_0)$ is the resolvent of A_0, we then obtain

$$zv_1 - v_2 = 0 \quad \text{and} \quad zv_2 - \Delta v_1 = g$$

so that

$$(5.26) \qquad \Delta v_1 - z^2 v_1 = g \quad \text{in} \quad \mathbf{R}^n.$$

For Re $z > 0$, $R_z(A_0)e$ lies in H_0 and in particular v_1 is square integrable. This uniquely determines v_1 as a solution of (5.26). It follows that

$$v_1 = R^0_{-iz}g.$$

If we replace z by $i\sigma + \delta$ and let δ tend to zero from above, then it is readily seen from (5.16) that $R_z(A_0)e$ tends to $\{R^0_\sigma g, i\sigma R^0_\sigma g\}$ in the topology dual to V_m. Hence passing to the limit in (5.25) and making use of (5.24), we obtain (5.10) as desired.

6. The scattering matrix. The scattering operator is defined in terms of the wave operators W_+ and W_- by

$$(6.1) \qquad S = W_+^{-1}W_- .$$

According to Corollary 3.4, W_+ and W_- are unitary transformations on H_0 to H, and therefore S is unitary on H_0 to itself. Moreover, it follows directly from the definition that the wave operators are intertwining operators; that is,

$$(6.2) \qquad W_\pm U_0(t) = U(t)W_\pm ,$$

and combining this with (6.1), we see that S commutes with $U_0(t)$:

$$(6.3) \qquad SU_0(t) = U_0(t)S.$$

Now as is well-known (see Corollary 4.2 of Chapter II in [4]), a unitary operator which commutes with the group of operators U_0 is representable as a multiplicative unitary operator on the spectral representation of U_0; that is, if $\mathcal{S} = \mathcal{F}_-^0 S \mathcal{F}_-^{0-1}$, then

$$(6.4) \qquad \mathcal{S}[\mathcal{F}_-^0 d] = \mathcal{S}(\sigma)[\mathcal{F}_-^0 d](\sigma),$$

and for each σ, $\mathcal{S}(\sigma)$ is a unitary operator on $N = L_2(S^{n-1})$. $\mathcal{S}$ is called the *scattering matrix.*

In this section, we derive an explicit formula for $\mathcal{S}(\sigma)$ and show that it can be continued analytically in σ onto the log Riemann surface. We begin by proving

Lemma 6.1. *Let $\mathcal{K}$ denote the spectral representer of the Hilbert transform as in* (5.6). *Then*

$$\mathcal{S} = \mathcal{K}\mathcal{F}_+\mathcal{F}_-^{-1}. \tag{6.5}$$

Proof. For d in D_+^ρ, it follows from (3.6) that $W_+d = d$, and from (5.13) that $\mathcal{F}_+d = \mathcal{F}_+^0 d$; so that on D_+^ρ the forward wave operator is an injection in the respective outgoing spectral representations. Combining this with (6.2), we see that for d in D_+^ρ,

$$\mathcal{F}_+W_+\mathcal{F}_+^{0-1} : \mathcal{F}_+^0 U_0(t)d = e^{i\sigma t}\mathcal{F}_+^0 d \rightarrow e^{i\sigma t}\mathcal{F}_+W_+d = e^{i\sigma t}\mathcal{F}_+d,$$

and since the $U_0(t)$ translates of D_+^ρ are dense in H_0, we conclude that

$$W_+ = \mathcal{F}_+^{-1}\mathcal{F}_+^0. \tag{6.6$_+$}$$

Likewise,

$$W_- = \mathcal{F}_-^{-1}\mathcal{F}_-^0. \tag{6.6$_-$}$$

Hence

$$S = \mathcal{F}_+^{0-1}\mathcal{F}_+\mathcal{F}_-^{-1}\mathcal{F}_-^0,$$

and therefore

$$\mathcal{S} = \mathcal{F}_-^0 S \mathcal{F}_-^{0-1} = \mathcal{F}_-^0\mathcal{F}_+^{0-1}\mathcal{F}_+\mathcal{F}_-^{-1},$$

which by (5.6) can be rewritten as (6.5).

It is customary and convenient to work with a modified form of the scattering matrix; namely,

$$\mathcal{S}_0 = \mathcal{F}_+\mathcal{F}_-^{-1} : \check{d}_- = \mathcal{F}_-d \rightarrow \check{d}_+ = \mathcal{F}_+d, \tag{6.7}$$

which differs from $\mathcal{S}$ itself by the factor $\mathcal{K}$. The explicit form of $\mathcal{S}_0$ which we are seeking is in terms of the asymptotic behavior of the scattered waves which was given in (5.9):

$$v_-(r\theta, \omega, \sigma) \sim \frac{e^{i\sigma r}}{r^{(n-1)/2}} s_-(\theta, \omega, \sigma). \tag{6.8}$$

Recall that v_- is an incoming solution of the reduced wave equation in G. Hence

$$u = \alpha v_-,$$

where $\alpha \in C^\infty$ vanishes on 0 and is identically one for $|x| > \rho$, is an incoming solution of

$$\Delta u + \sigma^2 u = g(x, \omega, \sigma) \quad \text{in} \quad \mathbf{R}^n,$$

and is given by $u = R_\sigma^0 g$. Note that g has its support in the ball $\{|x| < \rho\}$. For

$|x| > \rho$ we have

$$v_-(x, \omega, \sigma) = u(x, \omega, \sigma) = \int \gamma_-(|x - y|, \sigma) g(y, \omega, \sigma)\, dy. \tag{6.9}$$

Combining (6.8) and (6.9) we obtain

$$\begin{aligned} s_-(\theta, \omega, \sigma) &= \lim_{r\to\infty} r^{(n-1)/2} e^{-i\sigma r} v_-(r\theta, \omega, \sigma) \\ &= \frac{1}{4\pi}\left(\frac{-i\sigma}{2\pi}\right)^{(n-3)/2} \int e^{-i\sigma\theta\cdot y} g(y, \omega, \sigma)\, dy. \end{aligned} \tag{6.10}$$

Theorem 6.2. *The modified scattering matrix* $\mathcal{S}_0$ *is of the form: identity plus an integral operator with kernel const.* $\overline{s_-(-\theta, \omega, \sigma)}$; *more precisely,*

$$\check{d}_+(\omega, \sigma) = \check{d}_-(\omega, \sigma) + \left(\frac{-i\sigma}{2\pi}\right)^{(n-1)/2} \int \overline{s_-(-\theta, \omega, \sigma)}\, \check{d}_-(\theta, \sigma)\, d\theta. \tag{6.11}$$

Proof. We follow the proof of Theorem 5.4 in Chapter V of [4], and begin with the assumption that $\mathcal{S}_0(\sigma)$ can be expressed as the identity plus an integral operator with kernel $K(\theta, \omega, \sigma)$. In this case, the mapping: $\mathfrak{F}_-d \to \mathfrak{F}_+d$ takes the form

$$(f, \psi_-(\cdot, \omega, \sigma))_E = (f, \psi_+(\cdot, \omega, \sigma))_E + \int (f, \psi_+(\cdot, \theta, \sigma)K(\theta, \omega, \sigma)\, d\theta.$$

Since this holds for all f in C_0^∞, it suffices to show that

$$\varphi(x, \omega, \sigma) \equiv \psi_-(x, \omega, \sigma) - \psi_+(x, \omega, \sigma) - \int \psi_+(x, \theta, \sigma)\overline{K(\theta, \omega, \sigma)}\, d\theta \tag{6.12}$$

vanishes. It is clear to begin with that φ satisfies the reduced wave equation and vanishes on ∂G. We shall show for the asserted choice of K that φ is outgoing. It will then follow by the Rellich uniqueness theorem that φ vanishes identically.

Aside from a common factor, the right side of (6.12) is

$$v_-(x, \omega, \sigma) - v_+(x, \omega, \sigma) - \int [e^{-i\sigma x\cdot\theta} + v_+(x, \theta, \sigma)]\overline{K(\theta, \omega, \sigma)}\, d\theta. \tag{6.13}$$

We now make use of an ingenious device due to G. Schmidt ([9]). Replacing $\overline{K(\theta, \omega, \sigma)}$ by $(i\sigma/2\pi)^{(n-1)/2} s_-(-\theta, \omega, \sigma)$ and writing s_- in the form (6.10), we have

$$\int e^{-i\sigma x\cdot\theta}\overline{K(\theta, \omega, \sigma)}\, d\theta = \frac{i}{4\pi}\left(\frac{\sigma}{2\pi}\right)^{n-2} \iint e^{i\sigma\theta\cdot(y-x)} g(y, \omega, \sigma)\, d\theta\, dy.$$

As noted by Schmidt (see Watson [13, p. 48]),

$$\gamma_-(r, \sigma) - \gamma_+(r, \sigma) = \frac{i}{2}\left(\frac{\sigma}{2\pi r}\right)^{n/2-1} J_{n/2-1}(r\sigma)$$

$$= \frac{i\sigma^{n-2}}{2(2\pi)^{n-1}}\,\omega_{n-1}\int_{-1}^{1} e^{in\sigma t}(1-t^2)^{(n-3)/2}\,dt$$

$$= \frac{i}{4\pi}\left(\frac{\sigma}{2\pi}\right)^{n-2}\int e^{ir\sigma\theta\cdot\omega}\,d\theta.$$

Consequently,

$$\int e^{-i\sigma x\cdot\theta}\overline{K(\theta,\omega,\sigma)}\,d\theta = \int [\gamma_-(|x-y|,\sigma)-\gamma_+(|x-y|,\sigma)]g(y,\omega,\sigma)\,dy,$$

and substituting this and (6.9) in (6.13), we get

$$-v_+(x,\omega,\sigma) - \int v_+(x,\theta,\sigma)\overline{K(\theta,\omega,\sigma)}\,d\theta + \int \gamma_+(|x-y|,\sigma)g(y,\omega,\sigma)\,dy,$$

in which all of the terms are outgoing solutions of the reduced wave equation. This proves that φ is outgoing, and completes the proof of Theorem 6.2.

It follows from Theorem 5.5 that $g(y,\omega,\zeta)$ is an analytic vector-valued function of ζ with values in $L_2(G_\rho)\times C(S^{n-1})$. Hence by (6.10), $s_-(\theta,\omega,\zeta)$ is an analytic vector-valued function of ζ with values in $C(S^{n-1}\times S^{n-1})$. Combining this with (6.11), we obtain

Corollary 6.3. *$\mathcal{S}_0(\zeta)$ can be continued analytically to be meromorphic on the log Riemann surface, holomorphic in the lower half-plane of the principal sheet.*

Starting from the positive real axis $\sigma > 0$, we can thus continue $\mathcal{S}_0(\zeta)$ holomorphically throughout the lower half-plane $\operatorname{Im}\zeta < 0$. However, when ζ reaches the negative real axis $\sigma < 0$, the so-continued $\mathcal{S}_0(\zeta)$ will not take on the value of $\mathcal{S}_0(\sigma)$ but rather $\mathcal{S}_0(\sigma) + 2I$, as we now show. This result was obtained previously by Shenk and Thoe ([11]) by a different method.

Theorem 6.4. *Define the operator-valued function*

$$\mathcal{S}_1(\sigma) = \mathcal{S}_0(\sigma) - \mathcal{K}(\sigma) + I \tag{6.14}$$

on $\mathbf{R}\backslash 0$; *for each* σ, *these are operators on* $N = L_2(S^{n-1})$. *Then*

a) *$\mathcal{S}_1(\sigma)$ is the boundary value of an operator-valued function $\mathcal{S}_1(\zeta)$ holomorphic for $\operatorname{Im}\zeta < 0$ which converges along the lines $\operatorname{Re}\zeta = \sigma$ to $\mathcal{S}_1(\sigma)$ for almost all σ;*
b) $|\mathcal{S}_1(\zeta)| \leqq 1 + 2\,e^{2\rho|\operatorname{Im}\zeta|}$ *for all ζ with* $\operatorname{Im}\zeta < 0$;
c) $\mathcal{S}_0(\sigma) = \mathcal{S}_1(\sigma) - 2I$ *for* $\sigma < 0$.

Proof. For d in D_-^ρ and e in D_+^ρ, the energy inner product $(d,e)_E$ is the same for H_0 and H since both d and e vanish in the ball $\{|x| < \rho\}$. It follows that the orthogonal projection of d on D_+^ρ will be the same in both H_0 and H. Further, in both the unperturbed and the perturbed outgoing spectral representations, D_+^ρ corresponds to $e^{i\sigma\rho}A_+(N)$ and $\mathfrak{F}_+e = \mathfrak{F}_+^0e$. Consequently, for all d in D_+^ρ,

$$\mathfrak{F}_+d - \mathfrak{F}_+^0d$$

is orthogonal to $e^{i\sigma\rho}A_+(N)$ in $L_2(\mathbf{R}, N)$, and hence lies in $e^{i\sigma\rho}A_-(N)$. We conclude that the mapping

$$\mathcal{S}_0 - \mathcal{K} : \mathcal{F}_-f \to \mathcal{F}_+f - \mathcal{F}_+^0 f$$

has the following properties:

i) $|\mathcal{S}_0(\sigma) - \mathcal{K}(\sigma)| \leqq 2$;
ii) $[\mathcal{S}_0 - \mathcal{K}]e^{-i\sigma\rho}A_-(N) \subset e^{i\sigma\rho}A_-(N)$.

Thus $\mathcal{S}_0 - \mathcal{K}$ is causal in the sense of Fourès and Segal ([1]), and is therefore the boundary value of a holomorphic function in the lower half-plane with a bound

$$|[\mathcal{S}_0 - \mathcal{K}](\zeta)| \leqq 2\, e^{2\rho|\operatorname{Im}\zeta|}.$$

We see by the theorem of F. and M. Riesz that $[\mathcal{S}_0 - \mathcal{K}](\zeta)$ is equal to the principal branch of $\mathcal{S}_0(\zeta) - I$ in the lower half-plane. This proves the assertions a) and b); c) follows trivially from the fact that $I - \mathcal{K}(\sigma) = 0$ for $\sigma > 0$ and $= 2I$ for $\sigma < 0$.

When the spatial dimension is odd, the situation is considerably simpler. In this case $\mathcal{S}_0(\sigma)$ is itself the boundary value of an operator-valued function holomorphic in the lower half-plane, and since it is unitary for all σ, it is an inner factor in the sense of Beurling. Since $\mathcal{S}_0$ is an inner factor, it is associated with a semi-group of operators, in this case the semi-group Z defined in (1). It can be shown (see Chapter III of [4]) that ζ is a pole of the inner factor $\mathcal{S}_0(\zeta)$ if and only if $i\zeta$ is a pole of the infinitesimal generator of the associated semi-group of operators. In the odd dimensional case, this semi-group has turned out to be a very effective tool in the study of the singularities of $\mathcal{S}_0$.

From our point of view, the main difference between the odd and even dimensional cases is the lack of a natural semi-group of operators in the latter case. Before concluding this paper, we wish to establish the existence of a semi-group of this kind for the even dimensional case; the trouble with this semi-group is that we have not been able to find a direct and useful way of characterizing it.

Theorem 6.5. *$\mathcal{S}_1$ can be factored into inner and outer operator-valued functions Θ and Φ, respectively:*

(6.15) $$\mathcal{S}_1 = \Theta\Phi$$

where

a) *Φ and Φ^{-1} take $A_-(N)$ onto $A_-(N)$ and are the boundary values of holomorphic functions in the lower half-plane. Φ can be extended to be holomorphic in the cut plane $\mathbf{R}\backslash(-\infty, 0]$.*
b) *Θ is an inner factor, unitary on the real axis. It has an analytic extension into the cut plane $\mathbf{R}\backslash(-\infty, 0]$ where it has the same singularities as $\mathcal{S}_0(\zeta)$.*
c) *Set $\check{K} = A_-(N) \ominus e^{-2i\sigma\rho}\Theta A_-(N)$, and let Q denote the orthogonal projection of H onto $\mathcal{F}_-^{-1}\check{K}$. Then $Z'(t) = QU(t)Q$, $t > 0$, is an associated semi-group of operators for $\mathcal{S}_0$.*

We shall assume that the reader is familiar with the properties of inner and outer factors.

Proof. Since $\mathcal{S}_1(\sigma)$ is unitary for each $\sigma > 0$ and is equal to a unitary operator plus twice the identity for each $\sigma < 0$, it is obviously invertible for all $\sigma \neq 0$ and $|\mathcal{S}_1^{-1}(\sigma)| \leqq 1$. According to the proof of Theorem 6.4, $\mathcal{S}_1$ takes $A_-(N)$ into a subspace D of $e^{2i\sigma\rho}A_-(N)$, and it follows from the uniform boundedness of $\mathcal{S}_1^{-1}$ that D is a closed subspace. Associated with D is an inner factor $\Theta(\sigma)$, taking $A_-(N)$ onto D, which is unitary on the real axis and the boundary value of an operator-valued function, holomorphic in the lower half-plane with bound

$$|\Theta(\zeta)| \leqq e^{2\rho|\operatorname{Im}\zeta|} \quad \text{for} \quad \operatorname{Im}\zeta < 0. \tag{6.16}$$

We now define the outer factor as

$$\Phi(\sigma) = \Theta^{-1}(\sigma)\mathcal{S}_1(\sigma) \quad \text{for real} \quad \sigma. \tag{6.17}$$

Then both Φ and Φ^{-1} take $A_-(N)$ onto $A_-(N)$, and hence each is the boundary value of an operator-valued function holomorphic and uniformly bounded in the lower half-plane; we denote these functions by $\Phi(\zeta)$ and $\Phi'(\zeta)$, respectively. Note that $\Phi(\sigma)$ is unitary for each $\sigma > 0$ so that $\Phi(\sigma) = \Phi'^*(\sigma)$. It follows from this and the Schwarz reflection principle that $\Phi(\zeta)$ can be continued analytically across the positive real axis; this continuation is given by

$$\Phi(\zeta) = \Phi'^*(\bar{\zeta}) \quad \text{for} \quad \operatorname{Im}\zeta > 0. \tag{6.18}$$

Since $\Phi(\sigma)\Phi'(\sigma) = I$ for $\sigma > 0$, we see by analytic continuation that

$$\Phi'(\zeta) = \Phi^{-1}(\zeta) \quad \text{for} \quad \operatorname{Im}\zeta < 0,$$

and hence by (6.18) that the continuation of Φ in the upper half-plane is invertible as well as holomorphic. This completes the proof of (a).

As a consequence of (a),

$$\Theta(\zeta) = \mathcal{S}_1(\zeta)\Phi^{-1}(\zeta)$$

can be extended to be analytic in the upper half-plane through the positive real axis, and the so-extended $\Theta(\zeta)$ and $\mathcal{S}_1(\zeta)$ will have the same poles in the upper half-plane. This proves (b), and (c) is proved as in Chapter III of [4].

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