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**SCATTERING THEORY FOR
THE HARTREE EQUATION**

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SCATTERING THEORY FOR THE HARTREE EQUATION

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Abstract. We study the scattering problem for the Hartree equation

$$\begin{cases} i\partial_t u = -(1/2)\Delta u + f(|u|^2)u, & (t, x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \quad (*)$$

where

$$f(|u|^2) = V * |u|^2, \quad V(x) = \lambda|x|^{-1}, \lambda \in \mathbf{R}, \quad n \geq 2.$$

We prove that for any $u_0 \in H^{0,\gamma} \cap H^{\gamma,0}$, with $\frac{1}{2} < \gamma < \frac{n}{2}$, such that the value $\epsilon = \|u_0\|_{0,\gamma} + \|u_0\|_{\gamma,0}$ is sufficiently small, there exists unique $u_{\pm} \in H^{\sigma,0} \cap H^{0,\sigma}$ with $\frac{1}{2} < \sigma < \gamma$, such that for all $|t| \geq 1$

$$\|u(t) - \exp(\mp if(|\hat{u}_{\pm}|^2)(\frac{x}{t}) \log |t|)U(t)u_{\pm}\|_{L^2} \leq C\epsilon|t|^{-\mu+C\epsilon},$$

where $\mu = \min(1, \frac{\gamma}{2})$, $\hat{\phi}$ denotes the Fourier transform of ϕ defined by

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\xi} \phi(\xi) d\xi,$$

$U(t)$ is the free Schrödinger evolution group and $H^{m,s}$ is the weighted Sobolev space defined by

$$H^{m,s} = \{\phi \in \mathcal{S}' ; \|\phi\|_{m,s} = \|(1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} \phi\|_{L^2} < \infty\}.$$

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§1 Introduction. This paper is devoted to the study of the asymptotic behavior for large time of solutions to the Cauchy problem for the Hartree equation

$$\begin{cases} i\partial_t u = -(1/2)\Delta u + f(|u|^2)u, & (t, x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \quad (1.1)$$

where

$$f(|u|^2) = V * |u|^2 = \int V(x-y)|u|^2(y)dy, \quad V(x) = \lambda|x|^{-1}, \quad \lambda \in \mathbf{R} \quad \text{and} \quad n \geq 2.$$

There is a large amount of literature on the Cauchy problem for (1.1) and the asymptotic behavior in time of solutions for (1.1) with $\lambda > 0$ [2,3,6-13]. The scattering problem for (1.1) has only been studied in the framework of the nonexistence of scattering states [11,12] or of the existence of modified wave operators [5]. It seems that the decay rates of solutions to (1.1) obtained in [3,7-10] through the pseudo-conformal conservation law are not sufficient to obtain the existence and uniqueness of the modified scattering states (see [14] for a study in this direction). The only exception is [9], where the existence of solutions with the same decay rate as in the free case is proved and the asymptotic profile of the solutions is studied. In the present paper we propose a new setting for the study of large time behavior of solutions to (1.1) to make clear the connection with the theory of long range scattering for (1.1). Although the method of the paper follows [9], the argument here is different in some respects from the previous one and requires a number of sharp estimates.

We denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transformation of ϕ defined by

$$\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ix\xi} \phi(x) dx$$

and let $\mathcal{F}^{-1}\phi(x)$ be the inverse Fourier transformation of ϕ :

$$\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\xi} \phi(\xi) d\xi.$$

The free Schrödinger evolution group $U(t) = e^{\frac{it}{2}\Delta}$ is given by

$$U(t)\phi = \frac{1}{(2\pi it)^{\frac{n}{2}}} \int e^{\frac{i|x-y|^2}{2t}} \phi(y) dy = \mathcal{F}^{-1} e^{-\frac{it}{2}|\xi|^2} \mathcal{F}\phi.$$

We define the weighted Sobolev space $H^{m,s}$ by $H^{m,s} = \{\phi \in \mathcal{S}' ; \|\phi\|_{m,s} = \|(1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} \phi\| < \infty\}$, where $m, s \in \mathbf{R}$ and $\|\cdot\|$ denotes the usual L^2 norm.

We now state our results in this paper.

Theorem 1.1. We assume that $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$ and $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon$, where ϵ is sufficiently small and $\frac{1}{2} < \gamma < \frac{n}{2}$, $n \geq 2$.

Then there exists a unique global solution u of the Hartree equation (1.1) such that $u \in C(\mathbf{R}; H^{\gamma,0} \cap H^{0,\gamma})$ and

$$\sup_{\alpha \in [\frac{1}{2}, \sigma]} \sup_{t \in \mathbf{R}} (1 + |t|)^{-\alpha} \|u(t)\|_{p(\alpha)} \leq C\epsilon,$$

where $\frac{1}{2} < \sigma < \gamma$, $p(\alpha) = \frac{2n}{n-2\alpha}$.

Theorem 1.2. Let u be the solution of (1.1) obtained in Theorem 1.1.

Then for any u_0 satisfying the condition of Theorem 1.1, there exist unique functions $u_{\pm} \in H^{\sigma,0} \cap H^{0,\sigma}$, $\frac{1}{2} < \sigma < \gamma$, such that for all $|t| \geq 1$

$$\|u(t) - \exp(\mp if(|\hat{u}_{\pm}|^2)(\frac{x}{t}) \log |t|) U(t) u_{\pm}\| \leq C\epsilon |t|^{-\mu+C\epsilon}, \quad (1.2)$$

where $\mu = \min(1, \frac{\gamma}{2})$.

In the previous paper [9] the following results were shown.

Proposition 1.1. We assume that $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$ and $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon$, where ϵ is sufficiently small and $\gamma > n/2$.

Then there exists a unique global solution u of the Hartree equation (1.1) such that $u \in C(\mathbf{R}; H^{\gamma,0} \cap H^{0,\gamma})$ and

$$\|u(t)\|_{\infty} \leq C\epsilon(1 + |t|)^{-n/2}.$$

Proposition 1.2. Let u be the solution of (1.1) stated in Proposition 1.1.

Then for any $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$ there exist unique functions $\Phi \in L^{\infty}$ and $\hat{u}_+ \in L^{\infty} \cap L^2$ such that

$$\left\| \int_1^t f(|\hat{u}(\tau)|^2) \frac{d\tau}{\tau} - f(|\hat{u}_+|^2) \log t - \Phi \right\|_{\infty} \leq C\epsilon t^{-\xi\eta} \quad \text{for } t \geq 1,$$

and

$$\|\mathcal{F}(U(-t)u)(t) \exp(i \int_1^t f(|\hat{u}(\tau)|^2) \frac{d\tau}{\tau}) - \hat{u}_+ e^{i\Phi}\|_k \leq C\epsilon t^{-\xi} \quad \text{for } t \geq 1,$$

where $k = 2$ or ∞ , $0 < \eta < 2/n$, $n/2 + 2\xi < \gamma$, and $0 < \xi < 1$. Furthermore we have the asymptotic formula for large time t uniformly with respect to $x \in \mathbf{R}^n$

$$u(t, x) = \frac{1}{(2\pi it)^{\frac{n}{2}}} \hat{u}_+(\frac{x}{t}) \exp(i \frac{|x|^2}{2t} - if(|\hat{u}_+|^2)(\frac{x}{t}) \log t) + O(\epsilon t^{-\frac{n}{2}-\xi\eta}) \quad (1.3)$$

and the estimate

$$\|\mathcal{F}(U(-t)u)(t) - \hat{u}_+ \exp(-if(|\hat{u}_+|^2) \log t)\|_k \leq C\epsilon t^{-\xi\eta},$$

where $k = 2$ or ∞ .

If we write the formula (1.3) in the form

$$u(t, x) = \exp(-if(|\hat{u}_+|^2)\left(\frac{x}{t}\right) \log t)U(t)u_+ + O(\epsilon t^{-\frac{n}{2}-\xi\eta})$$

then we get the inequality

$$\|u(t) - \exp(-if(|\hat{u}_+|^2)\left(\frac{x}{t}\right) \log t)U(t)u_+\|_\infty \leq C\epsilon t^{-\frac{n}{2}-\xi\eta},$$

which is similar to (1.2) but gives the estimate in the uniform norm. The setting in Theorem 1.2 fits more closely to the theory of long-range scattering developed in [5], where the existence of modified wave operators for (1.1) has been proved. Roughly speaking the result of Theorem 1.2 implies the asymptotic completeness of the modified wave operators. Another advantage of the setting in Theorem 1.2 over the previous one in Proposition 1.2 is that for the L^2 theory of scattering one needs only the requirement $\gamma > 1/2$ for the index of the weighted Sobolev spaces as compared to the previous assumption, namely $\gamma > n/2$.

We organize our paper as follows. Below we explain the necessary notations and then in Section 2 we give some preliminary results. In Lemma 2.1 we formulate well-known embedding results for the Sobolev spaces. Lemma 2.2 gives the sharp time decay estimate of L^p norm of the function in term of the free evolution group $U(t)$. And in Lemma 2.3 we prove the estimates of the nonlinearity of equation (1.1) in the weighted Sobolev spaces. In Section 3 we prove Theorems 1.1-1.2 by using a priori estimates of the solutions obtained in Lemmas 3.1 - 3.3 in a space X_T . The function space X_T is the following

$$X_T = \{\phi \in C([-T, T]; \mathcal{S}'); \|\phi\|_{X_T} = \sup_{t \in [-T, T]} (1 + |t|)^{-C\epsilon} \|\phi(t)\|_{\gamma, 0} \\ + \sup_{t \in [-T, T]} (1 + |t|)^{-C\epsilon} \|U(-t)\phi(t)\|_{0, \gamma} + \sup_{\alpha \in [\frac{1}{2}, \sigma]} \sup_{t \in [-T, T]} (1 + |t|)^{-\alpha} \|\phi(t)\|_{p(\alpha)} < \infty\},$$

where $p(\alpha) = \frac{2n}{n-2\alpha}$, $\frac{1}{2} < \sigma < \gamma < \frac{n}{2}$.

We consider below the case $t > 0$ only since the opposite case is treated analogously.

Notation and function spaces. We let $\partial_j = \partial/\partial x_j$, $\partial^l = \partial_1^{l_1} \dots \partial_n^{l_n}$, $l \in (\mathbf{N} \cup \{0\})^n$, $M = M(t) = \exp(i|x|^2/2t)$, $J_j = J_j(t) = (x_j + it\partial_j) = U(t)x_jU(-t)$,

$J = (J_1, \dots, J_n) = U(t)xU(-t)$, and $|J|^\zeta = U(t)|x|^\zeta U(-t)$, $\zeta \in [0, \infty)$. We introduce some function spaces. $L^p = \{\phi \in \mathcal{S}' ; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = (\int |\phi(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\{|\phi(x)| ; x \in \mathbf{R}^n\}$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $H_p^{m,s}$ defined by $H_p^{m,s} = \{\phi \in \mathcal{S}' ; \|\phi\|_{m,s,p} = \|(1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} \phi\|_p < \infty\}$, $m, s \in \mathbf{R}$, $1 \leq p \leq \infty$; $H^{m,s} = H_2^{m,s}$, $\|\cdot\|_{m,s} = \|\cdot\|_{m,s,2}$. We let $(\psi, \phi) = \int \psi \cdot \bar{\phi} dx$. Denote by $\dot{B}_{p,q}^s$ the homogeneous Besov space with the semi-norm

$$\|\psi\|_{\dot{B}_{p,q}^s} = \left(\int_0^\infty y^{-1-\xi q} \sup_{|z| \leq y} \sum_{|k| \leq [s]} \|\partial^k (\psi(z) - \psi)\|_p^q dy \right)^{1/q},$$

where $s = [s] + \xi$, $0 < \xi < 1$, $\psi_{(z)}(x) = \psi(x + z)$ and $[s]$ is the largest integer less than s . We let $C(I; E)$ be the space of continuous functions from an interval I to a Banach space E . Different positive constants might be denoted by the same letter C . If necessary, by $C(*, \dots, *)$ we denote constants depending on the quantities appearing in parentheses.

We note that $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $D(t)$ is the dilation operator defined by $(D(t)\psi)(x) = \frac{1}{(it)^{\frac{n}{2}}} \psi\left(\frac{x}{t}\right)$ and

$$U(-t) = M(-t)\mathcal{F}^{-1}D(t)^{-1}M(-t) = M(-t)i^n \mathcal{F}^{-1}D\left(\frac{1}{t}\right)M(-t), \quad (1.4)$$

since $D(t)^{-1} = i^n D\left(\frac{1}{t}\right)$. By using the above identities we easily see that

$$\begin{aligned} J_j(t) &= U(t)x_j U(-t) = M(t)D(t)\mathcal{F}M(t)x_j M(-t)\mathcal{F}^{-1}i^n D\left(\frac{1}{t}\right)M(-t) \\ &= M(t)D(t)i^n (i\partial_j)D\left(\frac{1}{t}\right)M(-t) = M(t)D(t)i^n D\left(\frac{1}{t}\right)(it\partial_j)M(-t) \\ &= M(t)(it\partial_j)M(-t) \end{aligned}$$

and $|J|^\zeta(t) = M(t)(-t^2\Delta)^{\frac{\zeta}{2}}M(-t)$, $\zeta \in [0, \infty)$.

§2 Preliminaries.

Lemma 2.1. *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$, and let j, m be any real numbers satisfying $0 \leq j < m$. If $u \in H_r^{m,0} \cap L^q$, then*

$$\|(-\Delta)^{j/2}u\|_p \leq C \|(-\Delta)^{m/2}u\|_r^a \|u\|_q^{1-a}, \quad (2.1)$$

where $(1/p) = (j/n) + a((1/r) - (m/n)) + (1-a)/q$ for all a in the interval $(j/m) \leq a \leq 1$, where C is a constant depending only on n, m, j, q, r, a , with the following exception: if $m - j - (n/r)$ is a nonnegative integer, then (2.1) holds for any $(j/m) \leq a < 1$.

For Lemma 2.1 see, e.g., [4].

Lemma 2.2. *We let $u(t, x)$ be a smooth function. Then we have*

$$\|u(t)\|_{p(\alpha)} \leq C|t|^{-\alpha} \|\mathcal{F}U(-t)u(t)\|_{p(\alpha)} + C|t|^{-\alpha-\nu} \|U(-t)u(t)\|_{0,\gamma} \quad \text{for } |t| \geq 1,$$

where $p(\alpha) = \frac{2n}{n-2\alpha}$, $\alpha \in [\frac{1}{2}, \frac{n}{2})$, $n \geq 2$, $\nu \in [0, 1]$, $\gamma = \alpha + 2\nu$.

Proof. We have the identity with $v(t) = U(-t)u(t)$ and $w(t, x) = (e^{\frac{i|x|^2}{2t}} - 1)v(t, x)$

$$\begin{aligned} u(t) &= U(t)v(t) = M(t)D(t)\mathcal{F}v(t) + M(t)D(t)\mathcal{F}(M(t) - 1)v(t) \\ &= \frac{e^{\frac{i|x|^2}{2t}}}{(2\pi it)^{\frac{n}{2}}} \int e^{-iy\frac{x}{t}} v(t, y) \{1 + (e^{\frac{i|y|^2}{2t}} - 1)\} dy \\ &= \frac{e^{\frac{i|x|^2}{2t}}}{(it)^{\frac{n}{2}}} \left(\hat{v}(t, \frac{x}{t}) + \hat{w}(t, \frac{x}{t}) \right). \end{aligned} \quad (2.2)$$

We get the estimate

$$|e^{\frac{i|y|^2}{2t}} - 1| = 2 \left| \sin \frac{|y|^2}{4t} \right| \leq \min(2, \frac{|y|^2}{2|t|}) \leq 2^{1-2\nu} \frac{|y|^{2\nu}}{|t|^\nu} \quad (2.3)$$

for any ν satisfying $0 \leq \nu \leq 1$, and by a direct calculation we see that

$$\|f(\frac{\cdot}{t})\|_p = |t|^{n/p} \|f\|_p. \quad (2.4)$$

Applying equality (2.4) and estimate (2.3) to identity (2.2) and using Lemma 2.1 with $p = p(\alpha)$, $a = 1$, $r = 2$, $j = 0$, $m = \alpha$, we get

$$\begin{aligned} \|u(t)\|_p &\leq C|t|^{-\alpha} (\|\hat{v}(t)\|_p + \|\hat{w}(t)\|_p) \leq C|t|^{-\alpha} (\|\hat{v}(t)\|_p + \|\hat{w}(t)\|_{\alpha,0}) \\ &\leq C|t|^{-\alpha} (\|\hat{v}(t)\|_p + \|w(t)\|_{0,\alpha}) \leq C|t|^{-\alpha} (\|\hat{v}(t)\|_p + |t|^{-\nu} \|v(t)\|_{0,\gamma}). \end{aligned}$$

This implies the lemma. \square

Lemma 2.3. *We let $u(t, x)$ be a smooth function and $0 < \gamma < \frac{n}{2}$. Then the following estimates*

$$|\text{Im}(|x|^\gamma U(-t)f(|u|^2)u(t), |x|^\gamma U(-t)u(t))| \leq C \|u\|_{2n/(n-1)}^2 \|U(-t)u\|_{\dot{H}^{0,\gamma}}^2,$$

and

$$|\text{Im}((-\Delta)^{\gamma/2} f(|u|^2)u(t), (-\Delta)^{\gamma/2} U(-t)u(t))| \leq C \|u\|_{2n/(n-1)}^2 \|u\|_{\dot{H}^{\gamma,0}}^2,$$

are valid.

Proof. Let us consider only the case $0 < \gamma < 1$ since the other cases are treated analogously. By the relation $M(t)(-t^2\Delta)^{\gamma/2}M(-t) = U(t)|x|^\gamma U(-t)$ we have with $g = M(-t)u$ and $f = f(|u|^2) = f(|g|^2)$

$$\begin{aligned} & |\operatorname{Im}(|x|^\gamma U(-t)f(|u|^2)u, |x|^\gamma U(-t)u)| = |\operatorname{Im}((-t^2\Delta)^{\gamma/2}fg, (-t^2\Delta)^{\gamma/2}g)| \\ & = |\operatorname{Im}((-t^2\Delta)^{\gamma/2}fg - f(-t^2\Delta)^{\gamma/2}g, (-t^2\Delta)^{\gamma/2}g)| \\ & \leq C\|g\|_{2n/(n-2\gamma)}\|(-t^2\Delta)^{\gamma/2}f\|_{n/\gamma}\|(-t^2\Delta)^{\gamma/2}g\|, \end{aligned}$$

where we have used the fractional Leibniz rule which is proved in [15]. Since $f = V * |u|^2 = C(-\Delta)^{-(n-1)/2}|g|^2$ with a certain constant C (see [16]) we obtain by Lemma 2.1 with $j = 0, m = n - 1 - \gamma, p = \frac{n}{\gamma}, r = \frac{n}{n-1}, a = 1$

$$\begin{aligned} & |\operatorname{Im}(|x|^\gamma U(-t)(V * |u|^2)u, |x|^\gamma U(-t)u)| \\ & \leq C|t|^\gamma\|g\|_{2n/(n-2\gamma)}\|(-\Delta)^{-(n-1-\gamma)/2}|g|^2\|_{n/\gamma}\| |x|^\gamma U(-t)u \| \\ & \leq C|t|^\gamma\|g\|_{2n/(n-2\gamma)}\|u\|_{2n/(n-1)}^2\| |x|^\gamma U(-t)u \| \\ & \leq C\|u\|_{2n/(n-1)}^2\|U(-t)u\|_{\dot{H}^{0,\gamma}}^2. \end{aligned}$$

The second estimate of the lemma follows from the same argument as in the proof of the first one, and so we omit it. \square

§3 Proofs of Theorems 1.1-1.2.

To clarify the idea of the proof of the Theorems we only show a priori estimates of local solutions to (NLS). For that purpose we assume that the following local existence theorem holds.

Theorem 3.1. *We assume that $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon \leq \epsilon'$ and ϵ is sufficiently small, where $\frac{1}{2} < \gamma < \frac{n}{2}$. Then there exists a finite time interval $[-T, T]$ with $T > 1$ and a unique solution u of (1.1) such that*

$$\|u\|_{X_T} \leq C\epsilon'.$$

For the proof of Theorem 3.1, see, e.g. [1],[7],[13],[15].

Lemma 3.1. *Let u be the local solutions to (1.1) stated in Theorem 3.1. Then we have*

$$\sup_{t \in [-T, T]} (1 + |t|)^{-C\epsilon'} (\|u(t)\|_{\gamma, 0} + \|U(-t)u(t)\|_{0, \gamma}) \leq C\epsilon.$$

Proof. By using the commutation relation $[L, |J|^\gamma] = 0$, where $L = i\partial_t + \frac{1}{2}\Delta$ we get from the Hartree equation (1.1)

$$L|J|^\gamma u = |J|^\gamma f(|u|^2)u.$$

Multiplying both sides of this equation by $\overline{|J|^\gamma u}$ and using Lemma 2.3, we obtain

$$\| |J|^\gamma u(t) \|^2 \leq \| |x|^\gamma u_0 \|^2 + C \int_0^t \|u(s)\|_{2n/(n-1)}^2 \| |J|^\gamma u(s) \|^2 ds.$$

We use the estimate $\|u(t)\|_{p(\frac{1}{2})} \leq C\epsilon(1 + |t|)^{-\frac{1}{2}}$ of Theorem 3.1 to get

$$\| |J|^\gamma u(t) \|^2 \leq \| |x|^\gamma u_0 \|^2 + C\epsilon' \int_0^t (1 + s)^{-1} \| |J|^\gamma u(s) \|^2 ds,$$

whence via the Gronwall's inequality we find the estimate $\| |J|^\gamma u(t) \| \leq \| |x|^\gamma u_0 \| (1 + t)^{C\epsilon'}$ which implies

$$(1 + t)^{-C\epsilon'} \| |x|^\gamma U(-t)u(t) \| \leq \| |x|^\gamma u_0 \|. \quad (3.1)$$

In the same way as in the proof of (3.1) we have

$$(1 + t)^{-C\epsilon'} \|u(t)\|_{\gamma, 0} \leq C\|u_0\|_{\gamma, 0}. \quad (3.2)$$

The lemma follows from (3.1) and (3.2).

Lemma 3.2. *Let u be the local solutions to (1.1) stated in Theorem 3.1. Then we have*

$$\sup_{\alpha \in [\frac{1}{2}, \sigma]} \sup_{t \in [-T, T]} (1 + |t|)^\alpha \|u(t)\|_{p(\alpha)} \leq C\epsilon,$$

where $\frac{1}{2} < \sigma < \gamma$, $p(\alpha) = \frac{2n}{n-2\alpha}$.

Proof. By Lemma 3.1 and Theorem 3.1 we have

$$\sup_{\alpha \in [\frac{1}{2}, \sigma]} \sup_{t \in [-1, 1]} (1 + |t|)^\alpha \|u(t)\|_{p(\alpha)} \leq C\epsilon. \quad (3.3)$$

We assume that $t \geq 1$. From Lemma 2.2 and Lemma 3.1 it follows that

$$\|u(t)\|_{p(\alpha)} \leq C\epsilon t^{-\alpha-\beta+C\epsilon} + Ct^{-\alpha}\|\mathcal{F}U(-t)u(t)\|_{p(\alpha)} \quad (3.4)$$

for any $\alpha \in [\frac{1}{2}, \sigma]$, where $\beta = \min(1, \frac{\gamma-\sigma}{2})$. Multiplying both sides of (1.1) by $U(-t)$ we obtain

$$i(U(-t)u(t))_t + U(-t)f(|u|^2)u = 0,$$

whence in view of identity (1.4) we have

$$i\hat{v}_t - t^{-1}f(|\hat{v}|^2)\hat{v} = t^{-1}(I_1(t) + I_2(t)), \quad (3.5)$$

where

$$I_1(t) = \mathcal{F}(M(-t) - 1)\mathcal{F}^{-1}f(|\mathcal{F}M(t)v|^2)\mathcal{F}M(t)v,$$

$$I_2(t) = f(|\mathcal{F}M(t)v|^2)\mathcal{F}M(t)v - f(|\hat{v}|^2)\hat{v}.$$

Then by (3.5)

$$i\hat{w}_t = B(t)(t^{-1}\{I_1(t) + I_2(t)\}) \quad (3.6)$$

with $\hat{v}B(t) = \hat{w}$ and $B(t) = \exp(i \int_1^t f(|\hat{v}|^2) \frac{d\tau}{\tau})$. Integrating (3.6) with respect to t from 1 to t , we get

$$\hat{w}(t) = \hat{w}(1) + i \int_1^t B(\tau)(\{I_1(\tau) + I_2(\tau)\}) \frac{d\tau}{\tau}. \quad (3.7)$$

In the same way as in the proof of Lemma 2.3 we have with $\frac{1}{2} < \theta \leq \gamma$ (We let now $\gamma < 1$, since the case $1 \leq \gamma < \frac{n}{2}$ is treated analogously.)

$$\begin{aligned} \|f(h_1 h_2)h_3\|_{\theta,0} &\leq C(\|f\|_{\infty}\|h_3\| + \|(-\Delta)^{\theta/2}(fh_3)\|) \\ &\leq C(\|f\|_{\infty}\|h_3\| + \|(-\Delta)^{\theta/2}(fh_3) - f(-\Delta)^{\theta/2}h_3\| + \|f(-\Delta)^{\theta/2}h_3\|) \\ &\leq C(\|f\|_{\infty}\|h_3\|_{\theta,0} + \|h_3\|_{p(\theta)}\|(-\Delta)^{\theta/2}f\|_{n/\theta}) \\ &\leq C(\|f\|_{\infty}\|h_3\|_{\theta,0} + \|h_3\|_{p(\theta)}\|(-\Delta)^{-(n-1-\theta)/2}h_1 h_2\|_{n/\theta}) \\ &\leq C\|h_3\|_{\theta,0}(\|f\|_{\infty} + \|h_1\|_{p(\frac{1}{2})}\|h_2\|_{p(\frac{1}{2})}) \\ &\leq C\|h_3\|_{\theta,0}(\|h_1\|_q\|h_2\|_q + \|h_1\|_r\|h_2\|_r + \|h_1\|_{\frac{1}{2},0}\|h_2\|_{\frac{1}{2},0}) \leq C \prod_{j=1}^3 \|h_j\|_{\theta,0}, \end{aligned} \quad (3.8)$$

where $q < p(\frac{1}{2}) < r < p(\theta)$. We use (3.8) and (2.3) to obtain with $h = M(t)v$ and $0 \leq \theta \leq \sigma$

$$\begin{aligned} \|I_1(t)\|_{\theta,0} &\leq C\|\mathcal{F}(M(-t) - 1)\mathcal{F}^{-1}f(|\hat{h}|^2)\hat{h}\|_{\theta,0} \leq Ct^{-\omega}\|\mathcal{F}^{-1}f(|\hat{h}|^2)\hat{h}\|_{0,\gamma} \\ &\leq Ct^{-\omega}\|f(|\hat{h}|^2)\hat{h}\|_{\gamma,0} \leq Ct^{-\omega}\|\hat{h}\|_{\gamma,0}^3 \leq Ct^{-\omega}\|v\|_{0,\gamma}^3, \end{aligned} \quad (3.9)$$

where $\omega = \min(1, \frac{\gamma-\theta}{2})$. We easily see that

$$f(|\hat{h}|^2)\hat{h} - f(|\hat{v}|^2)\hat{v} = f(|\hat{h}|^2)(\hat{h} - \hat{v}) + f((\hat{h} - \hat{v})\bar{\hat{h}})\hat{v} + f((\bar{\hat{h}} - \bar{\hat{v}})\hat{v})\hat{v} \quad (3.10)$$

Applying (3.8) to (3.10), using (2.3) and Lemma 2.1, we obtain

$$\begin{aligned} \|I_2(t)\|_{\sigma,0} &= \|f(|\hat{h}|^2)\hat{h} - f(|\hat{v}|^2)\hat{v}\|_{\sigma,0} \leq C\|\hat{h} - \hat{v}\|_{\sigma,0}(\|\hat{h}\|_{\sigma,0}^2 + \|\hat{v}\|_{\sigma,0}^2) \\ &\leq C\|(M(t) - 1)v\|_{0,\sigma}\|v\|_{0,\sigma}^2 \leq Ct^{-\beta}\|v\|_{0,\gamma}^3, \end{aligned} \quad (3.11)$$

where $\frac{1}{2} < \sigma < \gamma < \frac{n}{2}$, $\beta = \min(1, \frac{\gamma-\sigma}{2})$. By (3.7), (3.9), (3.11) and Lemma 3.1 we have

$$\begin{aligned} \|\mathcal{F}U(-t)u\|_{p(\alpha)} &= \|\hat{v}\|_{p(\alpha)} = \|\hat{w}\|_{p(\alpha)} \leq C\epsilon + C \int_1^t (\|I_1(\tau)\|_{p(\alpha)} + \|I_2(\tau)\|_{p(\alpha)}) \frac{d\tau}{\tau} \\ &\leq C\epsilon + C \int_1^t (\|I_1(\tau)\|_{\alpha,0} + \|I_2(\tau)\|_{\alpha,0}) \frac{d\tau}{\tau} \leq C\epsilon \int_1^t \tau^{-1-\beta+C\epsilon} d\tau \leq C\epsilon. \end{aligned} \quad (3.12)$$

We apply (3.12) to (3.4) to get the lemma. \square

In the same way as in the proof of Lemma 3.1 we have by Theorem 3.1 and Lemma 3.2

Lemma 3.3. *Let u be the local solutions to (1.1) stated in Theorem 3.1. Then we have*

$$\sup_{t \in [-T, T]} (1 + |t|)^{-C\epsilon} (\|u(t)\|_{\gamma,0} + \|U(-t)u(t)\|_{0,\gamma}) \leq C\epsilon.$$

We are now in a position to prove Theorems 1.1-1.2.

Proof of Theorem 1.1. We have by Lemma 3.2 and Lemma 3.3

$$\|u\|_{X_T} \leq C(\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma}) = C\epsilon.$$

We take ϵ satisfying $C\epsilon \leq \epsilon'$. Then the standard continuation argument yields the result because our constant C does not depend on the existence time T . \square

Proof of Theorem 1.2. By (3.7), (3.9) and (3.11) and Theorem 1.1 we have

$$\|\hat{w}(t) - \hat{w}(s)\|_{\sigma,0} \leq C\epsilon s^{-\beta+C\epsilon} \quad (3.13)$$

for $t > s \geq 1$, where $\frac{1}{2} < \sigma < \gamma$, $\beta = \min(1, \frac{\gamma-\sigma}{2})$. Therefore we find that there exists a unique function $\hat{w}_+ \in H^{\sigma,0}$ such that

$$\|\hat{w}(t) - \hat{w}_+\|_{\sigma,0} \leq C\epsilon t^{-\beta+C\epsilon}. \quad (3.14)$$

In the same way as in the proof of (3.9) and (3.11) we have

$$\|I_1(t)\|_{0,\sigma} + \|I_2(t)\|_{0,\sigma} \leq C t^{-\beta+C\epsilon} (\|v\|_{0,\gamma} + \|v\|_{\gamma,0})^3 \leq C\epsilon t^{-\beta+C\epsilon}. \quad (3.15)$$

Hence from (3.7) we see that $\hat{w}_+ \in H^{0,\sigma}$ and

$$\|\hat{w}(t) - \hat{w}_+\|_{0,\sigma} \leq C\epsilon t^{-\beta+C\epsilon}. \quad (3.16)$$

By Lemma 2.1 with $p = 2n, j = 0, a = 1, m = n - 1, r = \frac{2n}{2n-1}$ we find that

$$\|f(h_1 h_2)\|_{2n} = C \|(-\Delta)^{-(n-1)/2} (h_1 h_2)\|_{2n} \leq C \|h_1\| \|h_2\|_{p(\frac{1}{2})} \leq C \|h_1\| \|h_2\|_{\frac{1}{2},0}. \quad (3.17)$$

Let us consider now the case $\frac{1}{2} < \sigma < 1$. The other cases can be treated analogously. We use estimate (3.17) to get the following estimate with $\frac{1}{2} < \sigma < \gamma$ (Let us consider now the case $\frac{1}{2} < \gamma < 1$, since the case $1 \leq \gamma < \frac{n}{2}$ is treated analogously.)

$$\begin{aligned} \|f(h_1 h_2)\|_{\dot{B}_{2n,2}^\sigma} &= \left(\int_0^\infty y^{-1-2\sigma} \sup_{|z| \leq y} \|(f(z) - f)\|_{2n}^2 dy \right)^{1/2} \\ &\leq C \|h_2\|_{\frac{1}{2},0} \left(\int_0^\infty y^{-1-2\sigma} \sup_{|z| \leq y} \|(h_1(z) - h_1)\|^2 dy \right)^{1/2} \\ &\quad + C \|h_1\|_{\frac{1}{2},0} \left(\int_0^\infty y^{-1-2\sigma} \sup_{|z| \leq y} \|(h_2(z) - h_2)\|^2 dy \right)^{1/2} \leq C \|h_1\|_{\sigma,0} \|h_2\|_{\sigma,0}, \end{aligned} \quad (3.18)$$

whence we obtain

$$\begin{aligned} \|f(|\hat{w}(t)|^2) - f(|\hat{w}(s)|^2)\|_{\dot{B}_{2n,2}^\sigma} &= \|f((\hat{w}(t) - \hat{w}(s))\overline{\hat{w}(t)}) + f(\overline{(\hat{w}(t) - \hat{w}(s))}\hat{w}(t))\|_{\dot{B}_{2n,2}^\sigma} \\ &\leq C \|\hat{w}(t) - \hat{w}(s)\|_{\sigma,0} (\|\hat{w}(t)\|_{\sigma,0} + \|\hat{w}(s)\|_{\sigma,0}) \leq C s^{-\beta+C\epsilon} (\|u_0\|_{0,\gamma} + \|u_0\|_{\gamma,0}), \end{aligned} \quad (3.19)$$

for all $1 < s < t$, where $\beta = \min(1, \frac{\gamma-\sigma}{2})$.

We now let $\Psi(t) = \int_1^t (f(|\hat{w}(\tau)|^2) - f(|\hat{w}(t)|^2)) \frac{d\tau}{\tau}$. Then

$$\Psi(t) - \Psi(s) = \int_s^t (f(|\hat{w}(\tau)|^2) - f(|\hat{w}(t)|^2)) \frac{d\tau}{\tau} - (f(|\hat{w}(t)|^2) - f(|\hat{w}(s)|^2)) \log s, \quad (3.20)$$

where $1 < s < \tau < t$. We apply (3.19) to (3.20) to get for all $1 < s < t$

$$\|\Psi(t) - \Psi(s)\|_{\dot{B}_{2n,2}^\sigma} \leq C\epsilon s^{-\beta+C\epsilon}. \quad (3.21)$$

This implies that there exists a unique real valued function $\Phi \in \dot{B}_{2n,2}^\sigma$ such that $\lim_{t \rightarrow \infty} \Psi(t) = \Phi$ in $\dot{B}_{2n,2}^\sigma$. We let $t \rightarrow \infty$ in (3.21). Then

$$\|\Phi - \Psi(t)\|_{\dot{B}_{2n,2}^\sigma} \leq C\epsilon t^{-\beta+C\epsilon}. \quad (3.22)$$

We denote $\hat{u}_+ = \hat{w}_+ \exp(-i\Phi)$. By the facts that $\Phi \in \dot{B}_{2n,2}^\sigma$, $\hat{w}_+ \in H^{\sigma,0} \cap H^{0,\sigma}$ and Lemma 2.1 we see that $\hat{u}_+ \in H^{\sigma,0} \cap H^{0,\sigma}$ and we also have by Theorem 1.1, (3.14), (3.21) and (3.22)

$$\begin{aligned} \|u_+\|_{0,\sigma} &= \|\hat{u}_+\|_{\sigma,0} = \|\exp(i\Phi)\hat{w}_+\|_{\sigma,0} \leq C(\|w_+\| + \|\hat{w}_+ \exp(i\Phi)\|_{\dot{B}_{2,2}^\sigma}) \\ &\leq C\epsilon + C \left(\int_0^\infty y^{-1-2\sigma} \sup_{|z| \leq y} \|(\hat{w}_{+(z)} - \hat{w}_+) \exp(i\Phi_{(z)})\|^2 dy \right)^{1/2} \\ &\quad + C\|\hat{w}_+\|_{p(\frac{1}{2})} \left(\int_0^\infty y^{-1-2\sigma} \sup_{|z| \leq y} \|\Phi_{(z)} - \Phi\|_{2n}^2 dy \right)^{1/2} \\ &\leq C\epsilon + C\|\hat{w}_+\|_{\sigma,0} + C\|\hat{w}_+\|_{\frac{1}{2},0} \|\Phi\|_{\dot{B}_{2n,2}^\sigma} \leq C\epsilon. \end{aligned} \quad (3.23)$$

We easily find that the following identity holds

$$\int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau} = f(|\hat{w}_+|^2) \log t + \Phi + (\Psi(t) - \Phi) + (f(|\hat{w}(t)|^2) - f(|\hat{w}_+|^2)) \log t. \quad (3.24)$$

In the same way as in the proof of (3.11), (3.17) we have

$$\begin{aligned} \|I_2(t)\| &= \|f(|\hat{h}|^2)\hat{h} - f(|\hat{v}|^2)\hat{v}\| \leq \|\hat{h} - \hat{v}\| \|f(|\hat{h}|^2)\|_\infty \\ &\quad + \|\hat{v}\|_{p(\frac{1}{2})} (\|f((\hat{h} - \hat{v})\bar{\hat{h}})\|_{2n} + \|f((\bar{\hat{h}} - \bar{\hat{v}})\hat{v})\|_{2n}) \leq C\|\hat{h} - \hat{v}\| (\|\hat{h}\|_{\sigma,0}^2 + \|\hat{v}\|_{\sigma,0}^2) \\ &\leq C\|(M(t) - 1)v\| \|v\|_{0,\sigma}^2 \leq Ct^{-\mu} \|v\|_{0,\gamma}^3 \leq Ct^{-\mu+C\epsilon} (\|v\|_{0,\gamma} + \|v\|_{\gamma,0})^3 \\ &\leq C\epsilon t^{-\mu+C\epsilon}, \end{aligned} \quad (3.25)$$

where $\mu = \min(1, \frac{\gamma}{2})$. Hence by virtue of (3.9) with $\theta = 0$ and (3.25) we see from (3.7) that

$$\|\hat{w}(t) - \hat{w}_+\| \leq C\epsilon t^{-\mu+C\epsilon}. \quad (3.26)$$

Using (3.17) and (3.26), analogously to (3.19) we get

$$\begin{aligned} \|f(|\hat{w}(t)|^2) - f(|\hat{w}(s)|^2)\|_{2n} &= \|f((\hat{w}(t) - \hat{w}(s))\overline{\hat{w}(t)}) + f((\overline{\hat{w}(t)} - \overline{\hat{w}(s)})\hat{w}(t))\|_{2n} \\ &\leq C\|\hat{w}(t) - \hat{w}(s)\| (\|\hat{w}(t)\|_{p(\frac{1}{2})} + \|\hat{w}(s)\|_{p(\frac{1}{2})}) \leq Cs^{-\mu+C\epsilon} (\|u_0\|_{0,\gamma} + \|u_0\|_{\gamma,0}), \end{aligned} \quad (3.27)$$

and therefore we obtain

$$\|\Phi - \Psi(t)\|_{2n} \leq C\epsilon t^{-\mu+C\epsilon}. \quad (3.28)$$

By (3.24), (3.27) and (3.28) we have

$$\left\| \int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau} - f(|\hat{w}_+|^2) \log t - \Phi \right\|_{2n} \leq C\epsilon t^{-\mu+C\epsilon}. \quad (3.29)$$

Since $\hat{w}(t) = B(t)\mathcal{F}U(-t)u(t) = \exp(i \int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau})\mathcal{F}U(-t)u(t)$, we have in view of (3.14), (3.26) and (3.29)

$$\begin{aligned} & \|\mathcal{F}U(-t)u(t) - \hat{u}_+ \exp(-if(|\hat{u}_+|^2) \log t)\| \\ &= \|\hat{w}(t) \exp(-i \int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau}) - \hat{u}_+ \exp(-if(|\hat{u}_+|^2) \log t)\| \\ &\leq \|\hat{w}(t) - \hat{w}_+\| + C\|\hat{w}_+\| \left\| \int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau} - f(|\hat{u}_+|^2) \log t - \Phi \right\| \\ &\leq C\epsilon t^{-\mu+C\epsilon} + C\|\hat{w}_+\|_{p(\frac{1}{2})} \left\| \int_1^t f(|\hat{w}(\tau)|^2) \frac{d\tau}{\tau} - f(|\hat{u}_+|^2) \log t - \Phi \right\|_{2n} \\ &\leq C\epsilon t^{-\mu+C\epsilon}. \end{aligned} \quad (3.30)$$

By (2.3) and Theorem 1.1

$$\begin{aligned} \|u(t) - M(t)D(t)\mathcal{F}U(-t)u(t)\| &= \|M(t)D(t)\mathcal{F}(M(t) - 1)U(-t)u(t)\| \\ &= \|(M(t) - 1)U(-t)u(t)\| \leq Ct^{-\mu}\|U(-t)u(t)\|_{0,\gamma} \leq C\epsilon t^{-\mu+C\epsilon}. \end{aligned} \quad (3.31)$$

Via (3.23), (3.30) and (3.31) it follows that

$$\begin{aligned} & \|u(t) - \exp(-i(f(|\hat{u}_+|^2)(\frac{x}{t}) \log t)U(t)u_+\| \\ &= \|u(t) - M(t)D(t) \exp(-if(|\hat{u}_+|^2) \log t)\mathcal{F}M(t)u_+\| \\ &\leq \|u(t) - M(t)D(t)\mathcal{F}U(-t)u(t)\| \\ &+ \|M(t)D(t)(\mathcal{F}U(-t)u(t) - \hat{u}_+ \exp(-if(|\hat{u}_+|^2) \log t))\| \\ &+ \|M(t)D(t) \exp(-if(|\hat{u}_+|^2) \log t)\mathcal{F}(M(t) - 1)u_+\| \\ &\leq C\epsilon t^{-\mu+C\epsilon} + Ct^{-\mu+C\epsilon}\|u_+\|_{0,\sigma} \leq C\epsilon t^{-\mu+C\epsilon}, \end{aligned} \quad (3.32)$$

where we chose the value $\sigma \in (\frac{1}{2}, \gamma)$ to be sufficiently close to γ . From (3.32) Theorem 1.2 follows.

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