

# SCH'NOL'S THEOREM FOR STRONGLY LOCAL FORMS

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*Dedicated to Shmuel Agmon on the occasion of his 85th birthday*

ABSTRACT

We prove a variant of Sch'nol's theorem in a general setting: for generators of strongly local Dirichlet forms perturbed by measures.

As an application, we discuss quantum graphs with  $\delta$ - or Kirchhoff boundary conditions.

## Introduction

The behavior of solutions to elliptic partial differential equations and its interplay with spectral properties of the associated partial differential operators is a topic of fundamental interest. Our knowledge today is in many aspects based on groundbreaking work by Shmuel Agmon (cf. [1, 2, 3, 4, 5, 6]) to whom this article is dedicated with great admiration and gratitude. Here we explore the well-known classical fact that the spectral values of Schrödinger operators  $H$  can be characterized in terms of the existence of appropriate “generalized eigenfunctions” or “eigensolutions.” One part of this characterization is Sch'nol's

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theorem stating that existence of an eigensolution of  $Hu = \lambda u$  “with enough decay” guarantees  $\lambda \in \sigma(H)$ . We refer to the original result [32] by Sch’*n*ol from 1957 that was rediscovered by Simon [34], as well as the discussion in [15].

Clearly, if  $u \in D(H)$  then  $\lambda$  is an eigenvalue. But much less restrictive growth conditions suffice to construct a Weyl sequence from  $u$  by a cut-off procedure. One of the main objectives of the present paper is to provide a proof along these lines for a great variety of operators. In our framework, the principal part  $H_0$  of  $H$  is the selfadjoint operator associated with a strongly local regular Dirichlet form  $\mathcal{E}$  and  $H = H_0 + \mu$  with a measure perturbation. This includes Schrödinger operators on manifolds and open subsets of Euclidean space, but much more singular coefficients are included. In our general Sch’*n*ol’s theorem potentials in  $L^1_{\text{loc}}$  with form small negative and arbitrary positive part are included, thereby generalizing results that require some Kato class condition. The appropriate “decay assumption” on  $u$  that is necessary can roughly be called subexponential growth and is phrased in terms of conditions like

$$\frac{\|u\chi_{B(x_0, r_n + \delta)}\|}{\|u\chi_{B(x_0, r_n)}\|} \rightarrow 1 \quad \text{for some } r_n \rightarrow \infty$$

and some fixed  $\delta > 0$ . Here,  $\chi_M$  is the characteristic function of  $M$  and  $B(p, s)$  denotes the closed ball in the intrinsic metric around  $p$  with radius  $s$ . A precise definition of the intrinsic metric is given below.

For uniformly bounded and strictly elliptic divergence form operators, one recovers the usual Euclidean balls.

It is interesting to note that we use a form analogue to Weyl sequences that enables us to treat partial differential operators with singular coefficients. Of course, the usual calculations of  $H(\eta u)$  for a smooth cut-off function  $\eta$  fail in the present general context. This holds for operators in divergence form with non-differentiable coefficients; and, to our knowledge, there does not exist Sch’*n*ol’s Theorem for this context in the literature. The calculations for  $\eta u$  referred to above have to be replaced by calculations with the corresponding forms. The crucial object in that respect is the **energy measure** of a strongly local Dirichlet form that supplies one with a calculus reminiscent of gradients. This leads to our version of Sch’*n*ol’s theorem, Theorem 4.4 below (one of the main results of the present paper). Apart from its generality it is also pretty simple conceptually. One step in this concept is a new version of the Caccioppoli inequality, Theorem 3.1 below. For the unperturbed operator  $H_0$  such an inequality can

be found in [12]. Our variant here, including measure perturbations, might be of interest in its own right.

**1. Assumptions and basic properties**

DIRICHLET FORMS. Throughout, we will work with a locally compact, separable metric space  $X$  endowed with a positive Radon measure  $m$  with  $\text{supp } m = X$ . Our exposition here follows closely along the same lines as those in [12, 40]. We refer to [18] as the classical standard reference as well as [13, 19, 30, 16] for literature on Dirichlet forms. Let us emphasize that in contrast to most of the work done on Dirichlet forms, we treat real and complex function spaces at the same time and write  $\mathbb{K}$  to denote either  $\mathbb{R}$  or  $\mathbb{C}$ .

The central object of our studies is a regular Dirichlet form  $\mathcal{E}$  with domain  $\mathcal{D}$  in  $L^2(X)$  and the selfadjoint operator  $H_0$  associated with  $\mathcal{E}$ . This means that  $\mathcal{D} \subset L^2(X, m)$  is a dense subspace,  $\mathcal{E} : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{K}$  is sesquilinear and  $\mathcal{D}$  is closed with respect to the energy norm  $\| \cdot \|_{\mathcal{E}}$ , given by

$$\|u\|_{\mathcal{E}}^2 = \mathcal{E}(u, u) + \|u\|_{L^2(X, m)}^2,$$

in which case one speaks of a **closed form** in  $L^2(X, m)$ . In the sequel we write

$$\mathcal{E}(u) := \mathcal{E}(u, u).$$

The unique selfadjoint operator  $H_0$  associated with  $\mathcal{E}$  is then characterized by

$$D(H_0) \subset \mathcal{D} \quad \text{and} \quad \mathcal{E}(f, v) = (H_0 f \mid v) \quad (f \in D(H_0), v \in \mathcal{D}).$$

Such a closed form is said to be a **Dirichlet form** if  $\mathcal{D}$  is stable under certain pointwise operations; more precisely,  $T : \mathbb{K} \rightarrow \mathbb{K}$  is called a **normal contraction** if  $T(0) = 0$  and  $|T(\xi) - T(\zeta)| \leq |\xi - \zeta|$  for any  $\xi, \zeta \in \mathbb{K}$  and we require that for any  $u \in \mathcal{D}$  also

$$T \circ u \in \mathcal{D} \quad \text{and} \quad \mathcal{E}(T \circ u) \leq \mathcal{E}(u).$$

Here we used the original condition from [11] that applies in the real and the complex case at the same time. Nowadays, particularly in the real case, it is mostly expressed in an equivalent but formally weaker statement involving  $u \vee 0$  and  $u \wedge 1$ , see [18, Thm. 1.4.1] and [30, Section I.4].

A Dirichlet form is called **regular** if  $\mathcal{D} \cap C_c(X)$  is dense both in  $(\mathcal{D}, \| \cdot \|_{\mathcal{E}})$  and  $(C_c(X), \| \cdot \|_{\infty})$ , where  $C_c(X)$  denotes the space of continuous functions with compact support.

STRONG LOCALITY AND THE ENERGY MEASURE.  $\mathcal{E}$  is called **strongly local** if

$$\mathcal{E}(u, v) = 0$$

whenever  $u$  is constant a.s. on the support of  $v$ .

The typical example one should keep in mind is the Laplacian

$$H_0 = -\Delta \text{ on } L^2(\Omega), \quad \Omega \subset \mathbb{R}^d \text{ open,}$$

in which case

$$\mathcal{D} = W_0^{1,2}(\Omega) \quad \text{and} \quad \mathcal{E}(u, v) = \int_{\Omega} (\nabla u | \nabla v) dx.$$

Now we turn to an important notion generalizing the measure  $(\nabla u | \nabla v) dx$  appearing above.

In fact, every strongly local, regular Dirichlet form  $\mathcal{E}$  can be represented in the form

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v),$$

where  $\Gamma$  is a nonnegative sesquilinear mapping from  $\mathcal{D} \times \mathcal{D}$  to the set of  $\mathbb{K}$ -valued Radon measures on  $X$ . It is determined by

$$\int_X \phi d\Gamma(u, u) = \mathcal{E}(u, \phi u) - \frac{1}{2} \mathcal{E}(u^2, \phi)$$

and called **energy measure**; see [13]. The energy measure satisfies the Leibniz rule,

$$d\Gamma(u \cdot v, w) = u d\Gamma(v, w) + v d\Gamma(u, w),$$

as well as the chain rule

$$d\Gamma(\eta(u), w) = \eta'(u) d\Gamma(u, w).$$

One can even insert functions from  $\mathcal{D}_{\text{loc}}$  into  $d\Gamma$ , where

$$\mathcal{D}_{\text{loc}} := \{u \in L^2_{\text{loc}} \text{ such that } \phi u \in \mathcal{D} \text{ for all } \phi \in \mathcal{D} \cap C_c(X)\},$$

as is readily seen from the following important property of the energy measure, **strong locality**:

Let  $U$  be an open set in  $X$  on which the function  $\eta \in \mathcal{D}_{\text{loc}}$  is constant, then

$$(1) \quad \chi_U d\Gamma(\eta, u) = 0,$$

for any  $u \in \mathcal{D}$ . This, in turn, is a consequence of the strong locality of  $\mathcal{E}$  and in fact equivalent to the validity of the Leibniz rule.

We write  $d\Gamma(u) := d\Gamma(u, u)$  and note that the energy measure satisfies the **Cauchy-Schwarz inequality**:

$$\begin{aligned} \int_X |fg|d\Gamma(u, v) &\leq \left( \int_X |f|^2 d\Gamma(u) \right)^{1/2} \left( \int_X |g|^2 d\Gamma(v) \right)^{1/2} \\ &\leq \frac{1}{2} \int_X |f|^2 d\Gamma(u) + \frac{1}{2} \int_X |g|^2 d\Gamma(v). \end{aligned}$$

**THE INTRINSIC METRIC.** Using the energy measure one can define the **intrinsic metric**  $\rho$  by

$$\rho(x, y) = \sup\{|u(x) - u(y)| : u \in \mathcal{D}_{\text{loc}} \cap C(X) \text{ and } d\Gamma(u) \leq dm\}$$

where the latter condition signifies that  $\Gamma(u)$  is absolutely continuous with respect to  $m$  and the Radon-Nikodym derivative is bounded by 1 on  $X$ . In general,  $\rho$  need not be a metric. (See the Appendix for a discussion of the finiteness of the sup.) However, here we will mostly rely on the following

**ASSUMPTION 1.1:** *The intrinsic metric  $\rho$  induces the original topology on  $X$ .*

We denote the intrinsic balls by

$$B(x, r) := \{y \in X : \rho(x, y) \leq r\}.$$

An important consequence of the latter assumption is that the distance function  $\rho_x(\cdot) := \rho(x, \cdot)$  itself is a function in  $\mathcal{D}_{\text{loc}}$  with  $d\Gamma(\rho_x) \leq dm$ , see [40]. This easily extends to the fact that for every closed  $E \subset X$  the function  $\rho_E(x) := \inf\{\rho(x, y) : y \in E\}$  enjoys the same properties (see the Appendix). This has a very important consequence. Whenever  $\zeta : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable, and  $\eta := \zeta \circ \rho_E$ , then  $\eta$  belongs to  $\mathcal{D}_{\text{loc}}$  and satisfies

$$(2) \quad d\Gamma(\eta) = (\zeta' \circ \rho_E)^2 d\Gamma(\rho_E) \leq (\zeta' \circ \rho_E)^2 dm.$$

**MEASURE PERTURBATIONS.** We will be dealing with Schrödinger type operators, i.e., perturbations  $H = H_0 + V$  for suitable potentials  $V$ . In fact, we can even include measures as potentials. Here we follow the approach from [36, 37]. Measure perturbations have been regarded by a number of authors in different contexts; see, e.g., [8, 20, 38] and the references there. To set up the framework, we first recall that every regular Dirichlet form  $\mathcal{E}$  defines a set function, the **capacity**, in the following way:

$$\text{cap}(U) := \inf\{\mathcal{E}(\phi) + \|\phi\|^2 : \phi \in \mathcal{D} \cap C_c(X), \phi \geq \chi_U\}$$

for open  $U$  and

$$\text{cap}(B) := \inf\{\text{cap}(U) : B \subset U, U \text{ open}\}.$$

It is clear that the capacity of a set  $B$  is bounded below by its measure  $m(B)$ . In most cases of interest, the capacity is larger and allows a finer distinction of sets. For example, for the classical Dirichlet form in one dimension, even a single point has positive capacity. We say that a property holds quasi-everywhere, q.e. for short, if it holds outside a set of capacity zero. We call a function  $g$  **quasi-continuous** if, for every  $\varepsilon > 0$  there is an open set  $U \subset X$  of capacity at most  $\varepsilon$  such that  $g$  is continuous on the complement  $X \setminus U$ . Every element  $u \in \mathcal{D}$  admits a quasi-continuous representative  $\tilde{u}$ . Most of the times we will be sloppy in our notation and just identify  $u$  with a quasi-continuous representative.

We denote by  $\mathcal{M}_0$  the set of nonnegative measures  $\mu : \mathcal{B} \rightarrow [0, \infty]$  that do not charge sets of capacity 0, i.e., those measures with  $\mu(B) = 0$  for every set  $B$  with  $\text{cap}(B) = 0$ . Here  $\mathcal{B}$  denotes the Borel subsets of  $X$  and we stress the fact that we do not assume our measures to be locally finite. Besides examples of the form  $V dm$ , where  $V$  is nonnegative and measurable, we should also mention the measure  $\infty_B$ , for a given  $B \subset X$ , defined by  $\infty_B(M) = \infty \cdot \text{cap}(B \cap M)$  with the usual convention  $\infty \cdot 0 = 0$ . For such a measure  $\mu_+ \in \mathcal{M}_0$ ,

$$D(\mathcal{E} + \mu_+) := \{u \in \mathcal{D} : \tilde{u} \in L^2(X, \mu_+)\},$$

$$(\mathcal{E} + \mu_+)(u, v) := \mathcal{E}(u, v) + \int_X \tilde{u} \tilde{v} d\mu_+$$

defines a closed form (not necessarily densely defined). We will use the notation  $\mu_+(u, v)$  for the integral in the above formula. It is well-defined since quasi-continuous versions of the same element in  $\mathcal{D}$  agree q.e. and so give the same integrals as the measure does not charge sets of capacity zero. The selfadjoint operator on the closure (in  $L^2(X, dm)$ ) of  $D(\mathcal{E} + \mu_+)$  associated with the form  $\mathcal{E} + \mu_+$  is denoted by  $H_0 + \mu_+$ . A little more restriction is needed for negative perturbations. We call  $\mu_-$  **admissible**, if

$$D(\mathcal{E} - \mu_-) := \mathcal{D},$$

$$(\mathcal{E} - \mu_-)(u, v) := \mathcal{E}(u, v) - \mu_-(u, v)$$

defines a semibounded closed form. Note that by regularity this implies that  $\mu_-$  is a Radon measure in the sense that it is finite on relatively compact sets. For an admissible  $\mu_-$  and  $\mu_+ \in \mathcal{M}_0$  the definition of  $\mathcal{E} + \mu_+ - \mu_-$  and the associated operator  $H = H_0 + \mu = H_0 + \mu_+ - \mu_-$  is obvious. To get better

properties of these operators we sometimes have to rely upon more restrictive assumptions concerning the negative part  $\mu_-$  of our measure perturbation. We write  $\mathcal{M}_1$  for those measures  $\mu$  that are  $\mathcal{E}$ -bounded with bound less than one; i.e., measures for which there is a  $\kappa < 1$  and a  $c_\kappa$  such that

$$\mu(u, u) \leq \kappa \mathcal{E}(u) + c_\kappa \|u\|^2.$$

By the KLMN theorem (see [31, p. 167]) these measures are admissible. An important class with very nice properties of the associated operators is the **Kato class** and the extended Kato class. In the present framework it can be defined in the following way: For  $\mu \in \mathcal{M}_0$  and  $\alpha > 0$  we set

$$\begin{aligned} \Phi(\mu, \alpha) &: C_c(X)_+ \rightarrow [0, \infty], \\ \Phi(\mu, \alpha)\varphi &:= \int_X ((H_0 + \alpha)^{-1}\varphi) \tilde{d}\mu. \end{aligned}$$

The extended Kato class is defined as

$$\hat{S}_K := \{\mu \in \mathcal{M}_0 \mid \exists \alpha > 0 : \Phi(\mu, \alpha) \in L^1(X, m)'\}$$

and, for  $\mu \in \hat{S}_K$  and  $\alpha > 0$ ,

$$c_\alpha(\mu) := \|\Phi(\mu, \alpha)\|_{L^\infty(X, m)} (= \|\Phi(\mu, \alpha)\|_{L^1(X, m)'}) , c_{\text{Kato}}(\mu) := \inf_{\alpha > 0} c_\alpha(\mu).$$

GENERALIZED EIGENFUNCTIONS. As usual an element  $u \in \mathcal{D}_{\text{loc}}$  is called a generalized eigenfunction for  $H$  or weak solution to the eigenvalue  $\lambda$  if

$$\mathcal{E}(u, v) + \mu(u, v) = \lambda(u, v)$$

for all  $v \in \mathcal{D}$  with compact support.

## 2. A Weyl type criterion

We include the following criterion for completeness; see [35, Lemma 1.4.4].

PROPOSITION 2.1: *Let  $h$  be a closed, semibounded form and  $H$  the associated selfadjoint operator. Then the following assertions are equivalent:*

- (i)  $\lambda \in \sigma(H)$ .
- (ii) *There exists a sequence  $(u_n)$  in  $\mathcal{D}(h)$  with  $\|u_n\| \rightarrow 1$  and*

$$\sup_{v \in \mathcal{D}(h), \|v\|_h \leq 1} |(h - \lambda)[u_n, v]| \rightarrow 0,$$

for  $n \rightarrow \infty$ .

*Proof.* (i) $\implies$ (ii): Choose a Weyl type sequence  $(u_n)$  if  $\lambda \in \sigma_{\text{ess}}(H)$  and  $u_n = u$  if there is a normalized eigenvector  $u \in D(H)$ .

(ii) $\implies$ (i): This is proven by contradiction. Assume  $\lambda \in \rho(H)$ . Then,

$$\sup_{n \in \mathbb{N}} \|(H - \lambda)^{-1}u_n\|_h =: C < \infty.$$

Therefore,

$$\|u_n\|^2 = |(h - \lambda)[u_n, (H - \lambda)^{-1}u_n]| \leq C \sup_{v \in \mathcal{D}(h), \|v\|_h \leq 1} |(h - \lambda)[u_n, v]|$$

and the latter term tends to zero for  $n \rightarrow \infty$ , by assumption. ■

We will produce a suitable sequence  $(u_n)$  as above by a suitable cutoff of generalized eigenfunctions. Note that to this end it is very convenient that we do not have to construct elements of the operator domain  $D(H)$ , a task that seems almost hopeless in the generality of forms we are aiming at. In fact, already for divergence form operators with singular coefficients there is no explicit description of the operator domain and the above criterion is of use in this important special case.

### 3. A Caccioppoli type inequality

In this section, we prove a bound on the energy measure of a generalized eigenfunction on a set in terms of bounds on the eigenfunction on certain neighborhood of the set.

We need the following notation: For  $E \in X$  and  $b > 0$  we define the  $b$ -neighborhood of  $E$  as

$$B_b(E) := \{y \in X : \rho(y, E) \leq b\}.$$

**THEOREM 3.1:** *Let  $\mathcal{E}$  be a strongly local regular Dirichlet form satisfying Assumption 1.1. Let  $\mu_+ \in \mathcal{M}_0$  and  $\mu_- \in \mathcal{M}_1$  be given. Let  $\lambda_0 \in \mathbb{R}$  be given. Then, there exists  $C = C(\lambda_0, \mu_-)$  such that for any generalized eigenfunctions  $u$  corresponding to an eigenvalue  $\lambda \leq \lambda_0$  of  $H_0 + \mu$  the inequality*

$$\int_E d\Gamma(u) \leq \frac{C}{b^2} \int_{B_b(E)} |u|^2 dm$$

*holds for any closed  $E \subset X$  and any  $b$  with  $1 > b > 0$ .*

*Remark 3.2:* The Caccioppoli inequality replaces the familiar commutator estimates that are used for Schrödinger operators.



We give a proof of the theorem at the end of this section after two auxiliary propositions.

**PROPOSITION 3.3:** *Let  $H_0 + \mu$  be given as in the theorem and  $u$  a generalized eigenfunction to the eigenvalue  $\lambda$ . Let  $\eta \in \mathcal{D}$ ,  $\eta$  real valued, be arbitrary. Then,*

$$\int \eta^2 d\Gamma(u) = (\lambda - \mu)(|\eta u|^2) - 2 \int \eta u d\Gamma(\eta, u).$$

*Proof.* A direct calculation invoking Leibniz rule and the chain rule gives

$$\begin{aligned} \int \eta^2 d\Gamma(u) &= \int d\Gamma(u, \eta^2 u) - \int u d\Gamma(u, \eta^2) \\ &= \int d\Gamma(u, \eta^2 u) - 2 \int u \eta d\Gamma(u, \eta) \\ &= \mathcal{E}(u, \eta^2 u) - 2 \int u \eta d\Gamma(u, \eta) \\ &= (h - \lambda)(u, \eta^2 u) + (\lambda - \mu)(|\eta u|^2) - 2 \int u \eta d\Gamma(u, \eta). \end{aligned}$$

As  $u$  is a generalized eigenfunction, the statement follows. ■

**PROPOSITION 3.4:** *Let  $u, \eta \in \mathcal{D}$ ,  $\eta$  real valued, be given. Then,*

$$\mathcal{E}(\eta u) = \int \eta^2 d\Gamma(u) + \int |u|^2 d\Gamma(\eta) + 2 \int \eta u d\Gamma(u, \eta).$$

*Proof.* This is a direct calculation. ■

We can now give the

*Proof of Theorem 3.1.* Let  $\omega = \rho_E$  and  $\zeta : [0, \infty) \rightarrow [0, 1]$  be continuously differentiable with  $\zeta(0) = 1$ ,  $\zeta \equiv 0$  on  $[b, \infty]$  and  $|\zeta'(t)| \leq 2/b$  for every  $t \in [0, \infty)$ . Set  $\eta := \zeta \circ \omega$ . Of course,

$$\int_E d\Gamma(u) \leq \int \eta^2 d\Gamma(u).$$

The main idea is now to use the previous two propositions to estimate  $\int \eta^2 d\Gamma(u)$  by terms of the form  $\int |u|^2 d\Gamma(\eta)$  and then to appeal to (2).

By assumption on  $\mu_-$ , there exists  $q < 1$  and  $C_q \geq 0$  with

$$\int \varphi^2 d\mu_- \leq q\mathcal{E}(\varphi) + C_q \|\varphi\|^2$$

for all  $\varphi \in \mathcal{D}$ . As  $\lambda \leq \lambda_0$ , this yields

$$\begin{aligned} (\lambda - \mu)(\|\eta u\|^2) &\leq \lambda \int \eta^2 |u|^2 dm + \int \eta^2 |u|^2 d\mu_- \\ &\leq \lambda_0 \|\eta u\|^2 + q\mathcal{E}(\eta u) + C_q \|\eta u\|^2 \\ &\leq q\mathcal{E}(\eta u) + (\lambda_0 + C_q) \|\eta u\|^2. \end{aligned}$$

Combining this with Proposition 3.3 we obtain

$$\int \eta^2 d\Gamma(u) \leq q\mathcal{E}(\eta u) + (\lambda_0 + C_q) \|\eta u\|^2 - 2 \int \eta u d\Gamma(u, \eta).$$

Invoking Proposition 3.4, we obtain

$$\begin{aligned} \int \eta^2 d\Gamma(u) &\leq q \int \eta^2 d\Gamma(u) + q \int |u|^2 d\Gamma(\eta) + (\lambda_0 + C_q) \|\eta u\|^2 \\ &\quad + 2(q - 1) \int \eta u d\Gamma(u, \eta). \end{aligned}$$

Application of Cauchy-Schwarz inequality to the last term yields

$$\begin{aligned} \int \eta^2 d\Gamma(u) &\leq q \int \eta^2 d\Gamma(u) + q \int |u|^2 d\Gamma(\eta) + (\lambda_0 + C_q) \|\eta u\|^2 \\ &\quad + \frac{1 - q}{S^2} \int |u|^2 d\Gamma(\eta) + S^2(1 - q) \int \eta^2 d\Gamma(u) \end{aligned}$$

for any  $S > 0$ . Hence

$$(1 - q - S^2(1 - q)) \int \eta^2 d\Gamma(u) \leq \left( q + \frac{1 - q}{S^2} \right) \int |u|^2 d\Gamma(\eta) + (\lambda_0 + C_q) \|\eta u\|^2$$

for any  $S > 0$ . As  $q < 1$  and  $S > 0$  is arbitrary, the statement follows with the help of (2). This finishes the proof.  $\blacksquare$

#### 4. A Sch'nol type result

In this section, we first prove an abstract Sch'nol type result. We need the following notation. For  $E \in X$  and  $b > 0$  we define the  $b$ -collar of  $E$  as

$$A_b(E) := \{y \in X : \rho(y, E) \leq b \text{ and } \rho(y, E^c) \leq b\}.$$

PROPOSITION 4.1: *Let  $\mathcal{E}$  be a strongly local regular Dirichlet form satisfying Assumption 1.1. Let  $\mu_+ \in \mathcal{M}_0$  and  $\mu_- \in \mathcal{M}_1$  be given. Let  $\lambda \in \mathbb{R}$  with*

generalized eigenfunction  $u$  be given. If there exists  $b > 0$  and a sequence  $(E_n)$  of closed subsets of  $X$  with

$$\frac{\|u\chi_{A_{3b}(E_n)}\|}{\|u\chi_{E_n}\|} \rightarrow 0, \quad n \rightarrow \infty,$$

then  $\lambda$  belongs to  $\sigma(H)$ .

*Proof.* Let  $\zeta : [0, \infty) \rightarrow [0, 1]$  be continuously differentiable, with  $\zeta(0) = 1$ ,  $\zeta \equiv 0$  on  $[b, \infty)$  and  $|\zeta'| \leq 2/b$ . Let  $\omega_n := \rho_{E_n}$  and  $\eta_n := \zeta \circ \omega_n$ . Let  $u_n := \eta_n^2 u / \|\eta_n^2 u\|$ . We show that  $(u_n)$  satisfies the assumption of Proposition 2.1: Let  $v \in \mathcal{E}$  be arbitrary. A direct calculation involving Leibniz rule gives

$$\int d\Gamma(\eta u, v) = \int d\Gamma(u, \eta v) + \int u d\Gamma(\eta, v) - \int \bar{v} d\Gamma(u, \eta)$$

for all  $\eta \in \mathcal{D}$ , which are real valued. This yields

$$\begin{aligned} (h - \lambda)[u_n, v] &= \frac{1}{\|\eta_n^2 u\|} \left( \int d\Gamma(\eta_n^2 u, v) + (\mu - \lambda)(\eta_n^2 u \bar{v}) \right) \\ &= \frac{1}{\|\eta_n^2 u\|} \left( \int d\Gamma(u, \eta_n^2 v) + \int u d\Gamma(\eta_n^2, v) \right. \\ &\quad \left. - \int \bar{v} d\Gamma(u, \eta_n^2) + (\mu - \lambda)(\eta_n^2 u \bar{v}) \right) \\ &= \frac{1}{\|\eta_n^2 u\|} \left( \int u d\Gamma(\eta_n^2, v) - \int \bar{v} d\Gamma(u, \eta_n^2) \right) \\ &= \frac{2}{\|\eta_n^2 u\|} \left( \int u \eta_n d\Gamma(\eta_n, v) - \int \bar{v} \eta_n d\Gamma(u, \eta_n) \right) \end{aligned}$$

where we used in the previous to the last step that  $u$  is a generalized eigenfunction. Cauchy-Schwarz now gives

$$\begin{aligned} &|(h - \lambda)[u_n, v]| \\ &\leq \frac{2}{\|\eta_n^2 u\|} \left( \left( \int |u|^2 d\Gamma(\eta_n) \right)^{1/2} \left( \int \eta_n^2 d\Gamma(v) \right)^{1/2} + \left| \int \eta_n \bar{v} d\Gamma(u, \eta_n) \right| \right). \end{aligned}$$

We will estimate the three terms on the right hand side.

As  $\eta_n$  is constant outside of  $A_b(E_n)$  we obtain from locality and (2)

$$\int |u|^2 d\Gamma(\eta_n) = \int_{A_{2b}(E_n)} |u|^2 d\Gamma(\eta_n) \leq \frac{4}{b^2} \|\chi_{A_{2b}(E_n)} u\|^2.$$

As for the second term, due to  $0 \leq \eta_n \leq 1$ , we easily find

$$\int \eta_n^2 d\Gamma(v) \leq \int d\Gamma(v) = \mathcal{E}(v) = \text{const.}$$

We now come to the last term. As  $\eta_n$  is constant outside of  $A_b(E_n)$ , locality again gives

$$\left| \int \eta_n v d\Gamma(u, \eta_n) \right| = \left| \int_{A_{2b}(E_n)} \eta_n v d\Gamma(u, \eta_n) \right|.$$

By Cauchy-Schwarz this can be estimated by

$$\left( \int_{A_{2b}(E_n)} \eta_n^2 d\Gamma(u) \right)^{1/2} \left( \int_{A_{2b}(E_n)} v^2 d\Gamma(\eta_n) \right)^{1/2}.$$

By (2) we can estimate  $\int_{A_{2b}(E_n)} v^2 d\Gamma(\eta_n)$  by  $4/b^2 \|v\|^2$ . By Theorem 3.1 and for  $0 \leq \eta_n \leq 1$ , we can estimate

$$\left( \int_{A_{2b}(E_n)} \eta_n^2 d\Gamma(u) \right)^{1/2} \leq \left( \int_{A_{2b}(E_n)} d\Gamma(u) \right)^{1/2} \leq \left( \frac{C}{b^2} \int_{A_{3b}(E_n)} |u|^2 dm \right)^{1/2}.$$

Putting these estimates together shows that there exists  $c > 0$  such that

$$|(h - \lambda)[u_n, v]| \leq c \frac{\|u\chi_{A_{3b}(E_n)}\|}{\|\chi_{E_n} u\|}$$

for all  $n \in \mathbb{N}$ . As the right hand side tends to zero by our assumption, so does the left hand side and Proposition 2.1 gives the desired result. ■

We will now specialize our considerations to subexponentially bounded eigenfunctions. We start with a piece of notation and two auxiliary lemmas.

A function  $J : [0, \infty) \rightarrow [0, \infty)$  is said to be subexponentially bounded if for any  $\alpha > 0$  there exists a  $C_\alpha \geq 0$  with  $J(r) \leq C_\alpha \exp(\alpha r)$  for all  $r > 0$ . A function  $f$  on a pseudo metric space  $(X, \rho)$  with measures  $m$  is said to be subexponentially bounded if for some  $x_0 \in X$  and  $\omega(x) = \rho(x_0, x)$  the function  $e^{-\alpha\omega} u$  belongs to  $L^2(X, m)$  for any  $\alpha > 0$ .

LEMMA 4.2: Let  $J : [0, \infty) \rightarrow [0, \infty)$  be subexponentially bounded. Let  $b > 0$  be arbitrary. Then, for any  $\delta > 0$  there exists an arbitrarily large  $r > 0$  such that  $J(r + b) \leq e^\delta J(r)$ .

*Proof.* Assume not. Then, there exists an  $R_0 \geq 0$  with

$$J(r + b) > e^\delta J(r)$$

for all  $r \geq R_0$ . Induction then shows

$$J(R_0 + nb) > e^{n\delta} J(R_0)$$

for any  $n \in \mathbb{N}$ . This gives a contradiction to the bounds on  $J$  for  $\alpha > 0$  with  $\alpha b < \delta$  and large  $n$ . ■

LEMMA 4.3: Let  $(X, \rho)$  be a (pseudo)metric space,  $m$  a measure on  $X$ ,  $x_0 \in X$  arbitrary and  $\omega(x) = \rho(x_0, x)$ ,  $B_r := B_r(x_0)$ . Let  $u : X \rightarrow \mathbb{C}$  be subexponentially bounded. Define

$$J : [0, \infty) \rightarrow [0, \infty), \quad J(r) := \int_{B_r} |u|^2 dm.$$

Then,  $J$  is subexponentially bounded.

*Proof.* For all  $\alpha > 0$ , we find

$$\begin{aligned} J(r) &= \int_{B_r} |u|^2 dm = \int_{B_r} |e^{\alpha\omega} e^{-\alpha\omega} u|^2 dm \\ &= \int_{B_r} e^{2\alpha\omega} |e^{-\alpha\omega} u|^2 dm \\ &\leq e^{2\alpha r} \int_{B_r} |e^{-\alpha\omega} u|^2 dm \\ &\leq \|e^{-\alpha\omega} u\|^2 e^{2\alpha r}. \end{aligned}$$

This proves the lemma. ■

THEOREM 4.4: Let  $\mathcal{E}$  be a strongly local regular Dirichlet form satisfying Assumption 1.1.,  $x_0 \in X$  arbitrary and  $\omega(x) = \rho(x_0, x)$ . Let  $\mu_+ \in \mathcal{M}_0$  and  $\mu_- \in \mathcal{M}_1$  be given. Let  $u$  be a generalized eigenfunction which is subexponentially bounded, i.e.,  $e^{-\alpha\omega} u \in L^2(X, m)$  for any  $\alpha > 0$ . Then,  $\lambda$  belongs to  $\sigma(H)$ .

*Proof.* As  $u$  is subexponentially growing, the function

$$J(r) := \int_{B_r} |u|^2 dm$$

is subexponentially bounded by the previous lemma. By Lemma 4.2, we are able to then choose  $b > 0$  and find a sequence  $(r_n)$  with  $r_n \rightarrow \infty$  and  $J(r_n + 3b)/J(r_n - 3b) \rightarrow 1$ ,  $n \rightarrow \infty$ . As  $J$  is monotonously increasing this

easily gives

$$\frac{J(r_n + 3b) - J(r_n - 3b)}{J(r_n)} \longrightarrow 0, \quad n \longrightarrow \infty.$$

Thus,  $u$  satisfies the assumption of Proposition 4.1 with  $E_n = B_{r_n}$  and the statement follows. ■

Let us now briefly discuss a converse of Sch'nol's Theorem as found in [14]. We need the following additional properties of the intrinsic geometry:

ASSUMPTION 4.5: *For each  $t > 0$  the semigroup  $e^{-tH_0}$  gives a map from  $L^2(X)$  to  $L^\infty(X)$  and all intrinsic balls have finite volume with subexponential growth:*

$$e^{-\alpha \cdot R} m(B(x, R)) \rightarrow 0 \text{ as } R \rightarrow \infty \text{ for all } x \in X, \alpha > 0.$$

With this assumption, [14, Corollary 3.1] gives:

THEOREM 4.6: *Let  $\mathcal{E}$  be a strongly local regular Dirichlet form satisfying Assumptions 1.1 and 4.5. Let  $\mu = \mu_+ - \mu_-$  with  $\mu_+ \in \mathcal{M}_0$  and  $\mu_- \in \hat{\mathcal{S}}_K$  with  $c_{\text{Kato}}(\mu) < 1$ . Define  $H := H_0 + \mu$ . Then for spectrally a.e.  $\lambda \in \sigma(H)$  there is a subexponentially bounded generalized eigenfunction  $u \neq 0$  with  $Hu = \lambda u$ .*

Thus, together with Theorem 4.4, we get the following characterization of the spectrum:

COROLLARY 4.7: *Let  $\mathcal{E}$  be a strongly local regular Dirichlet form satisfying Assumptions 1.1 and 4.5. Let  $\mu = \mu_+ - \mu_-$  with  $\mu_+ \in \mathcal{M}_0$  and  $\mu_- \in \hat{\mathcal{S}}_K$  with  $c_{\text{Kato}}(\mu) < 1$ . Define  $H := H_0 + \mu$ . Then the spectral measures of  $H$  are supported on*

$$\{\lambda \in \mathbb{R} : \exists \text{ subexponentially bounded } u \text{ with } Hu = \lambda u\}.$$

Remark 4.8: (1) Expansion in generalized eigenfunctions is a classical issue. We refer the reader to [9, 10] and the literature quoted in [14] for a different strategy.

(2) There is a number of results stating that the  $L^2$ -spectrum of Schrödinger type operators coincides with the  $L^p$ -spectra for all  $p$ . See, e.g. [21, 33, 39]. Of course, such a statement readily implies that every bounded generalized eigenfunction contributes to the spectrum.

**5. Application: Metric and Quantum Graphs**

We now introduce a class of examples that has attracted considerable interest in the physics as well as in the mathematical literature. We refer the reader to [7, 22, 26, 27, 28, 17, 23, 24, 25] and the references therein. Although different levels of generality and very different ways of notation can be found in the literature, the basic idea is the same: a **metric graph** consists of line segments – edges – that are glued together at vertices. In contrast to combinatorial graphs, these line segments are regarded as differentiable manifolds and in fact one is interested in the Laplacian on the union of the line segments. To get a self adjoint operator one has to specify boundary conditions at the vertices. More precisely, we work with the following

*Definition 5.1:* A **metric graph** is  $\Gamma = (E, V, i, j)$  where

- $E$  (edges) is a countable family of open intervals  $(0, l(e))$  and  $V$  (vertices) is a countable set.
- $i : E \rightarrow V$  defines the initial point of an edge and

$$j : \{e \in E : l(e) < \infty\} \rightarrow V$$

the end point for edges of finite length.

We let  $X_e := \{e\} \times e$ ,  $X = X_\Gamma = V \cup \bigcup_{e \in E} X_e$  and  $\overline{X_e} := X_e \cup \{i(e), j(e)\}$ .

Note that  $X_e$  is basically just the interval  $(0, l(e))$ , the first component is added to force the  $X_e$ 's to be mutually disjoint. The topology on  $X$  will be such that the mapping  $\pi_e : X_e \rightarrow (0, l(e)), (e, t) \mapsto t$  extends to a homeomorphism again denoted by  $\pi_e : \overline{X_e} \rightarrow \overline{(0, l(e))}$  that satisfies  $\pi_e(i(e)) = 0$  and  $\pi_e(j(e)) = l(e)$  (the latter is satisfied in case that  $l(e) < \infty$ ). To define a metric structure on  $X$  we proceed as follows: we say that  $p \in X^N$  is a **good polygon** if, for every  $k \in \{1, \dots, N - 1\}$  there is a unique edge  $e \in E$  such that  $\{x_k, x_{k+1}\} \subset \overline{X_e}$ . Using the usual distance in  $[0, l(e)]$  we get a distance  $d$  on  $\overline{X_e}$  and define

$$L(p) = \sum_{k=1}^{N-1} d(x_k, x_{k+1}).$$

Since multiple edges are, obviously, allowed, we had to restrict our attention to good polygons to exclude the case that  $\{x_k, x_{k+1}\}$  are joined by edges of different length. Provided the graph is connected and that the degree of every

vertex  $v \in V$  is

$$d_v := |\{e \in E : v \in \{i(e), j(e)\}\}| < \infty,$$

a metric on  $X$  is given by

$$d(x, y) := \inf\{L(p) : p \text{ a good polygon with } x_0 = x \text{ and } x_N = y\}.$$

In fact, symmetry and triangle inequality are evident and the separation of points follows from the finiteness. Clearly, with the topology induced by that metric,  $X$  is a locally compact, separable metric space. Note that in our setting we do allow loops, multiple edges and there is no on upper or lower bounds for the length of edges. In that respect, we allow more general graphs than those considered in the literature so far. To be able to use the framework of regular Dirichlet forms, we restrict our attention to certain boundary conditions, known as **Kirchhoff** and  $\delta$ -b.c. The operator with Kirchhoff b.c. is defined as the operator corresponding to the form

$$\mathcal{D} = \mathcal{D}(\mathcal{E}) := W_0^{1,2}(X), \quad \mathcal{E}(u, v) := \sum_e (u'_e | v'_e),$$

where  $u_e := u \circ \pi_e^{-1}$  defined on  $(0, l(e))$ ,

$$W^{1,2}(X) = \left\{ u \in C(X) : \sum_{e \in E} \|u_e\|_{W^{1,2}}^2 =: \|u\|_{W^{1,2}}^2 < \infty \right\},$$

$$W_0^{1,2}(X) := W^{1,2}(X) \cap C_0(X).$$

Clearly,  $\mathcal{E}$  is a regular Dirichlet form in  $L^2(X, m)$ , where  $m$  is the measure induced by the image of the Lebesgue measure on each  $X_e$ , so that  $L^2(X, m) \ni u \mapsto (u_e)_{e \in E} \in \bigoplus_{e \in E} L^2((0, l(e)), dt)$  is unitary.

This form is strongly local with energy measure

$$d\Gamma(u, v) = \sum_{e \in E} u'_e(\pi_e(x)) \overline{v'_e(\pi_e(x))} dm(x).$$

We denote by  $H_0$  the operator associated with  $\mathcal{E}$ . Note that every point  $x \in X$  has positive capacity by the Sobolev embedding theorem so that every measure  $\mu : \mathcal{B} \rightarrow [0, \infty]$  belongs to  $\mathcal{M}_0$ .

**COROLLARY 5.2:** *For  $\mathcal{E}$  as above, let  $\mu_+ : \mathcal{B} \rightarrow [0, \infty]$  and  $\mu_- \in \mathcal{M}_1$  be given. Let  $u \neq 0$  be a generalized eigenfunction for  $H := H_0 + \mu$  that is subexponentially bounded. Then,  $\lambda$  belongs to  $\sigma(H)$ .*



*Remark 5.3:* As mentioned above,  $\mu_+$  may include arbitrary sums of  $\delta$ -measures, in particular  $\delta$ -measures at points of  $V$  for which one gets a quantum graph with  $\delta$ -boundary conditions with positive coefficients.

While Sch' nol's theorem had already been known for quantum graphs [28], the way to interpret them as Dirichlet forms opens a powerful arsenal of analytic and probabilistic techniques. Quite a number of results in operator and perturbation theory have been established in the Dirichlet form setting and can readily be applied to quantum graphs.

For certain tree graphs an expansion in generalized eigenfunctions has been given in [22]. In a forthcoming work [29] we will prove that generalized eigenfunction expansions exist for much more general graphs than treated above.

**Appendix A. Properties of absolutely continuous elements, the distance function  $\rho_E$  and all that**

Let  $\mathcal{E}$  be a regular strongly local Dirichlet form with associated energy measure  $\Gamma$ . In this appendix, we discuss some properties of

$$\mathcal{A} := \{u \in \mathcal{D}_{\text{loc}} : u \text{ real valued with } d\Gamma(u) \leq dm\}.$$

We apply this to show that  $\rho_E$  belongs to  $\mathcal{A}$  for any closed  $E \subset X$  (and in fact for any  $E \subset X$ ) if Assumption 1.1 is satisfied. For  $E$  consisting of a single point this was first shown in [40]. For closed  $E$  this seems to be known. It is stated for example in [41], where a proof is attributed to [40]. As we did not find the proof there, we could not resist to produce one here. Along the way we will also reprove the case of a single point. Moreover, we will discuss connectedness of the space  $X$  in terms of the intrinsic metric.

We start by collecting basic properties of  $\mathcal{A}$ .

- PROPOSITION A.1: (a)  $\mathcal{A}$  is balanced, i.e. convex and closed under multiplication by  $-1$ .  
 (b)  $\mathcal{A}$  is closed under taking minima and maxima.  
 (c)  $\mathcal{A}$  is closed under adding constants.  
 (d)  $\mathcal{A}$  is closed under pointwise convergence of functions, which are uniformly bounded on compact sets.

*Proof.* (a) Obviously,  $\mathcal{A}$  is closed under multiplication by  $-1$ . Let  $u, v \in \mathcal{A}$  and  $\lambda, \mu \geq 0$  with  $\mu + \lambda = 1$  be given. Set  $w = \lambda u + \mu v$ . Then, for every  $\varphi \in C_c(X)$

we have

$$\begin{aligned} \int \varphi^2 d\Gamma(w) &= \lambda^2 \int \varphi^2 d\Gamma(u) + 2\lambda\mu \int \varphi^2 d\Gamma(u, v) + \mu \int \varphi^2 d\Gamma(v) \\ &\leq \lambda^2 \int \varphi^2 d\Gamma(u) + 2\lambda\mu \left( \int \varphi^2 d\Gamma(u) \int \varphi^2 d\Gamma(v) \right)^{1/2} + \mu \int \varphi^2 d\Gamma(v) \\ &\leq \int \varphi^2 dm. \end{aligned}$$

As  $\varphi$  was arbitrary the statement follows.

(b) As  $\mathcal{A}$  is closed under multiplication by  $-1$ , it suffices to consider minima. A direct consequence of locality is the truncation property

$$d\Gamma(u \wedge v, w) = \chi_{\{u < v\}} d\Gamma(u, w) + \chi_{\{u \geq v\}} d\Gamma(v, w)$$

for all  $u, v, w \in \mathcal{D}_{loc}$ . If  $w = u \wedge v$  we obtain

$$d\Gamma(u \wedge v, u \wedge v) = \chi_{\{u < v\}} d\Gamma(u, u) + \chi_{\{u \geq v\}} d\Gamma(v, v).$$

This shows that  $\mathcal{A}$  is closed under  $\wedge$ .

(c) This is obvious.

(d) Let  $(u_n)$  be a sequence in  $\mathcal{A}$  which converges pointwise to  $u$  and is uniformly bounded on each compact set. We first show that  $u$  belongs to  $\mathcal{D}_{loc}$ . Let  $\psi \in C_c(X) \cap \mathcal{D}$  be arbitrary. Leibniz rule, Cauchy-Schwarz inequality and locality of  $d\Gamma$  give

$$\begin{aligned} \mathcal{E}(\psi u_n) &= \int d\Gamma(\psi u_n) \\ &= \int \psi^2 d\Gamma(u_n) + 2 \int \psi u_n d\Gamma(u_n, \psi) + \int u_n^2 d\Gamma(\psi) \\ &\leq \int \psi^2 d\Gamma(u_n) + \int u_n^2 d\Gamma(\psi) + \int \psi^2 d\Gamma(u_n) + \int u_n^2 d\Gamma(\psi) \\ &\leq 2 \int \psi^2 dm + 2 \int u_n^2 \chi_{\text{supp } \psi} d\Gamma(\psi). \end{aligned}$$

The assumptions on  $(u_n)$  show that  $(\mathcal{E}(\psi u_n))$  remains bounded. By semicontinuity of  $\mathcal{E}$  we infer that  $\psi u$  belongs to  $\mathcal{D}$ . As  $\psi \in \mathcal{D} \cap C_c(X)$  is arbitrary, we obtain  $u \in \mathcal{D}_{loc}$ .

Let now an arbitrary  $\varphi \geq 0$  continuous with compact support be given. Choose  $\psi \in \mathcal{D} \cap C_c(X)$  with  $\psi \equiv 1$  on the support of  $\varphi$ . This is possible as  $\mathcal{E}$  is a Dirichlet form. Then, by Banach/Saks theorem, boundedness of  $(\mathcal{E}(\psi u_n))$  implies convergence of convex combinations  $(w_k)$  of the  $(\psi u_n)$  with respect to

the energy norm. By convexity of  $\mathcal{A}$ , these convex combinations have the form  $w_k = \psi v_k$  with  $v_k \in \mathcal{A}$ . As  $\psi u_n$  converges to  $\psi u$  in  $L^2$  we infer that the energy norm limit of the  $(w_k)$  is also  $\psi u$ . Locality and convergence of  $w_k = \psi v_k$  to  $u\psi$  with respect to the energy norm yield

$$\int \varphi d\Gamma(u) = \int \varphi d\Gamma(\psi u) = \lim \int \varphi d\Gamma(\psi v_n) = \lim \int \varphi d\Gamma(v_n) \leq \int \varphi dm.$$

As  $\varphi \geq 0$  with compact support is arbitrary, the statement follows. ■

The previous proposition implies that  $\mathcal{A}$  is also closed under taking suitable suprema and infima. This is discussed next.

LEMMA A.2: Let  $\mathcal{F} \subset \mathcal{A} \cap C(X)$  be stable under taking maxima (minima). If  $u := \sup\{v : v \in \mathcal{F}\}$  ( $u := \inf\{v : v \in \mathcal{F}\}$ ) is continuous, then  $u$  belongs to  $\mathcal{A}$ .

*Proof.* By our assumptions on  $X$ , there exist compact  $K_n \subset X$ ,  $n \in \mathbb{N}$ , with  $X = \bigcup_{n \in \mathbb{N}} K_n$  and  $K_n \subset K_{n+1}^\circ$ . By (d) of the previous proposition it suffices to construct  $u_n \in \mathcal{F}$  with  $|u_n - u| \leq 1/n$  on  $K_n$  for each  $n \in \mathbb{N}$ . This will be done next: For  $n \in \mathbb{N}$  and  $x \in K_n$ , we can find  $v_{x,n} \in \mathcal{F}$  with  $u(x) - 1/(2n) \leq v_{x,n}(x)$ . By continuity of  $v_{x,n}$  and  $u$ , there exists then an open neighbourhood  $U_{x,n}$  of  $x$  with

$$u(y) - 1/n \leq v_{x,n}(y)$$

for all  $y \in U_{x,n}$ . As  $K_n$  is compact, there exist  $x_1, \dots, x_l$  with  $K_n \subset \bigcup_{j=1}^l U_{x_j,n}$ . As  $\mathcal{F}$  is closed under taking maxima, the function

$$u_n := \max\{v_{x_j,n} : j = 1, \dots, l\}$$

belongs to  $\mathcal{F}$ . By construction

$$u(x) - 1/n \leq u_n \quad \text{on } K_n.$$

As the inequality  $u_n \leq u$  is clear, the proof is finished. ■

We now turn to the distance function  $\rho$ . By definition we have

$$\rho(x, y) := \sup\{u(x) - u(y) : u \in \mathcal{A} \cap C(X)\}.$$

Direct arguments show that  $\rho(x, y)$  is nonnegative, symmetric and satisfies the triangle inequality. As  $\mathcal{A} \cap C(X)$  is closed under adding constants, for each  $x \in X$ , the distance function  $\rho_x(y) := \rho(x, y)$  is then given by

$$\rho_x(y) := \sup\{u(y) : u \in \mathcal{F}_x\}$$

with  $\mathcal{F}_x = \{u \in \mathcal{A} \cap C(X) : u(x) = 0\}$ . The following proposition is essentially contained in [40, p. 191 and p. 194].

PROPOSITION A.3: *Assume 1.1. Let  $x \in X$  be arbitrary. Then,*

$$\{y : \rho_x(y) < \infty\}$$

*is exactly the connected component of  $x$ .*

*Proof.* Set  $C_x := \{y : \rho_x(y) < \infty\}$ . Of course, all functions which are constant on each component of  $X$  belong to  $\mathcal{A} \cap C(X)$ . Thus,  $\rho_x(y) = \infty$  whenever  $x$  and  $y$  belong to different components. Thus,  $C_x$  is contained in the connected component of  $x$ . We now show the reverse inclusion. To do so it suffices to show that  $C_x$  is both open and closed. By Assumption 1.1 the set  $C_x$  is open. Moreover, if  $y$  belongs to  $X \setminus C_x$ , then by

$$\infty = \rho(x, y) \leq \rho(y, z) + \rho(z, x)$$

we obtain that any  $z \in X$  with  $\rho(z, y) < 1$  belongs to  $X \setminus C_x$  as well. By Assumption 1.1 again the set of such  $z$  is open, and the complement  $X \setminus C_x$  is shown to be open as well. ■

PROPOSITION A.4: *Assume 1.1. Let  $x \in X$  be arbitrary and  $C_x$  be the connected component of  $x$ . Then,  $\chi_{C_x} \rho_x$  belongs to  $\mathcal{A} \cap C(X)$ .*

*Proof.* It suffices to consider the case that  $X$  is connected. By Assumption 1.1 and the previous lemma,  $\rho_x$  is then continuous. As

$$\mathcal{F}_x = \{u \in \mathcal{A} \cap C(X) : u(x) = 0\}$$

is closed under taking maxima and  $\rho_x(y) = \sup\{u : u \in \mathcal{F}_x\}$ , the statement now follows from Lemma A.2. ■

We now turn to distances from arbitrary sets. For  $E \subset X$  we define

$$\rho_E(z) := \inf\{\rho_x(z) : x \in E\}.$$

THEOREM A.5: *Assume 1.1. Let  $E \subset X$  be arbitrary and let  $C$  be the union of the connected components of the points of  $E$ . Then, the function  $\chi_C \rho_E$  belongs to  $\mathcal{A} \cap C(X)$ .*

*Proof.* As  $C$  is open and closed it suffices to consider the case  $C = X$ . By Assumption 1.1 and triangle inequality, the function  $\rho_E$  is continuous. Moreover,

as discussed above  $\rho_x$  belongs to  $\mathcal{A} \cap C(X)$  for any  $x \in X$ . The statement now follows from Lemma A.2. ■

We note a consequence of the previous theorem.

COROLLARY A.6: *Assume 1.1. For  $E \subset X$ , the equality*

$$\rho_E(z) = \sup\{u(z) : u \in \mathcal{F}_E\}$$

*holds, where  $\mathcal{F}_E := \{v \in \mathcal{A} \cap C(X) : v \equiv 0 \text{ on } E\}$ .*

*Proof.* Denote the supremum in the statement by  $\rho_E^*$ . As  $\rho_x(z) \geq u(z)$  for any  $u \in \mathcal{F}_E$  and  $x \in E$ , we have  $\rho_E \geq \rho_E^*$ . For the converse direction, we note that  $\rho_E$  belongs to  $\mathcal{F}_E$  by the previous theorem. ■

We finish this section by noting a strong closedness property of  $\mathcal{A}$ .

PROPOSITION A.7:  *$\mathcal{A}$  is closed under convergence in  $L_{loc}^2$ .*

*Proof.* Let  $K$  be an arbitrary compact subset of  $X$ . As  $\rho_x$  belongs to  $\mathcal{A}$  for any  $x \in X$ , we can find  $\psi \in C_c(X) \cap \mathcal{A}$  with  $\psi \equiv 1$  on  $K$  (take, e.g.,  $\psi := \max\{0, \frac{1}{R} \min\{R, 2R - \rho_x\}\}$  for  $x \in K$  and  $R$  large). The proof follows by mimicking the argument in the proof of Proposition A.1 (d) and using that  $d\Gamma(\psi) \leq dm$ . ■

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