SCHUR–WEYL DUALITY FOR TWIN GROUPS

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Abstract. The twin group TW_n on n strands is the group generated by $t_1, \dots, t_{n-1}$ with defining relations $t_i^2 = 1$, $t_i t_j = t_j t_i$ if $|i-j| > 1$. We find a new instance of semisimple Schur–Weyl duality for tensor powers of a natural n-dimensional reflection representation of TW_n, depending on a parameter q. At $q = 1$ the representation coincides with the natural permutation representation of the symmetric group, so the new Schur–Weyl duality may be regarded as a q-analogue of the one motivating the definition of the partition algebra.

1. Introduction

Let $\mathbf{E} = \mathbb{C}^n$ with standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. The symmetric group on n letters, realized as the Weyl group W_n of permutation matrices in $\mathrm{GL}(\mathbf{E})$, acts as permutations on the basis. The transposition (i, j) acts as a reflection, sending $\mathbf{e}_i - \mathbf{e}_j$ to its negative and fixing pointwise the orthogonal complement (with respect to the standard bilinear form $\langle e_i, e_j\rangle = \delta_{ij}$). The line spanned by $\mathbf{e}_1 + \cdots + \mathbf{e}_n$ is fixed pointwise by W_n and its orthogonal complement $(\mathbf{e}_1 + \cdots + \mathbf{e}_n)^\perp$ is an irreducible $(n-1)$-dimensional reflection representation of W_n. Study of the centralizer algebra $\mathrm{End}_{W_n}(\mathbf{E}^{\otimes r})$ leads to the partition algebra of [Mar91, Mar94, Jon94, Mar96] and the corresponding Deligne category [Del07].

We are interested in q-analogues of the above picture. One such, previously studied in [DG21], is obtained by replacing the above representation $W_n \to \mathrm{GL}(\mathbf{E})$ with the Burau representation $B_n \to \mathrm{GL}(\mathbf{E})$ of Artin's braid group B_n. This is done by replacing the standard bilinear form with a q-analogue. In effect, we perturb the eigenvalues of the generating reflections $s_i = (i, i+1)$ from $(1, -1)$ to parameters (q_1, q_2); thus the representation $\mathbb{C}[B_n] \to \mathrm{GL}(\mathbf{E})$ factors through the quotient map $\mathbb{C}[B_n] \to H_n(q_1, q_2)$, where $H_n(q_1, q_2)$ is the two-parameter Iwahori–Hecke algebra of [BW89, Big06].

In this paper, we study a second q-analogue of the situation of paragraph one, related to that of the preceding paragraph through the algebra $H_n(q_1, q_2)$. Keeping $\mathbf{E}$ the same, we introduce certain operators S_i, preserving the bilinear form, which act on $\mathbf{E}$ as reflections fixing $(q\mathbf{e}_i - \mathbf{e}_{i+1})^\perp$ pointwise. This gives a reflection representation $\rho : TW_n \to \mathrm{O}(\mathbf{E})$ of the twin group TW_n defined in the Abstract. The twin group TW_n and the braid group B_n are both covering groups of W_n, respectively obtained by omitting the quadratic relation and cubic braid relation from the standard Coxeter presentation of W_n. As long as $[n]_q = 1 + q + \cdots + q^{n-1} \neq 0$, where $q = -q_2/q_1$ is the negative ratio of the eigenvalues, $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ decomposes as the direct sum of the line $\mathbf{L}$ spanned by $\mathbf{e}_1 + \cdots + \mathbf{e}_n$ and its orthogonal complement $\mathbf{F} = (\mathbf{e}_1 + \cdots + \mathbf{e}_n)^\perp$, and these subspaces are irreducible for both B_n and TW_n. In case $q = 1$ we recover the situation of the first paragraph.

The twin group TW_n has previously appeared in a variety of contexts. It serves as an analogue of Artin's braid group B_n in the study of doodles [Kho97], which are configurations of a finite number of closed curves on a surface without triple intersections, and it appeared in [Gia91, Gia92, GGS92] in relation to certain constructions of quantum groups.

When $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$, the rth tensor power $\mathbf{E}^{\otimes r}$ is a semisimple $\mathbb{C}[TW_n]$-module, with TW_n acting diagonally. Our main result is a combinatorial description of its centralizer $\mathrm{End}_{TW_n}(\mathbf{E}^{\otimes r})$ in case q avoids a certain well-defined set of algebraic numbers in the union of the positive real axis and the unit circle. The centralizer is isomorphic to a homomorphic image of the two-parameter partial Brauer algebra $\mathcal{P}\mathfrak{B}_r(n, \delta')$, where $\delta' \neq 0$, studied in [MM14, Hd14], and is isomorphic to that algebra if $n > r$. The parameter δ' can be any nonzero scalar (all the corresponding partial Brauer algebras are isomorphic). This leads to the new Schur–Weyl duality statement of Theorem 8.6, extending the Schur–Weyl duality of [MM14, Hd14].

The main technical fact underlying our results is Theorem 6.1, that (under certain restrictions on q) the image $\rho(TW_n)$ of the representation is Zariski-dense in $\mathrm{O}(\mathbf{L}) \oplus \mathrm{O}(\mathbf{F})$. This enables an easy proof of Theorem 8.6 and also gives another new instance (Theorem 8.9) of Schur–Weyl duality for the commuting actions of the group TW_n and the Brauer algebra $\mathfrak{B}_r(n-1)$ on $\mathbf{F}^{\otimes r}$; this is a new variant of Brauer's original result in [Bra37].

Sections 2–5 derive the main properties of the twin group and its reflection representation $\mathbf{E}$; an appendix also constructs an explicit orthonormal basis and applies it to obtain an alternative proof of the conclusion of Theorem 6.1, under stronger hypotheses. Section 6 is devoted to proving the density result of Theorem 6.1 under the weaker hypotheses needed for the main results. Section 7 defines the partial Brauer algebra and gives its presentation by generators and relations (due to [MM14]) and the main results are deduced in Section 8.

2. Preliminaries

The ground field is always $\mathbb{C}$ in this paper, unless stated otherwise. Begin with the two-parameter [BW89, Big06] Iwahori–Hecke algebra $H_n(q_1, q_2)$, defined by generators $T_1, \dots, T_{n-1}$ subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ if } |i - j| > 1 \tag{1}$$

$$(T_i - q_1)(T_i - q_2) = 0. \tag{2}$$

Although it is easy to eliminate one of the parameters, carrying both causes no trouble. We assume that $q_1q_2 \neq 0$, so that the generators T_i are invertible elements in $H_n(q_1, q_2)$, with

$$T_i^{-1} = (T_i - q_1 - q_2)/(q_1q_2).$$

Artin's braid group B_n may be defined by generators $\sigma_1, \dots, \sigma_{n-1}$ subject to defining relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1. \tag{3}$$

The algebra $H_n(q_1, q_2)$ is isomorphic to the quotient algebra of the group algebra $\mathbb{C}[B_n]$ via the quotient map determined by $\sigma_i \mapsto T_i$, with kernel the ideal generated by the $(\sigma_i - q_1)(\sigma_i - q_2)$ for $i = 1, \dots, n-1$.

Let $\mathbf{E}$ be an n-dimensional complex vector space with basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Consider the action of $H_n(q_1, q_2)$ defined on generators by

$$\begin{gathered} T_i \cdot \mathbf{e}_i = (q_1 + q_2)\mathbf{e}_i + q_1\mathbf{e}_{i+1}, \quad T_i \cdot \mathbf{e}_{i+1} = -q_2\mathbf{e}_i \\ T_i \cdot \mathbf{e}_j = q_1\mathbf{e}_j \;\text{ if }\; j \neq i, i+1. \end{gathered} \tag{4}$$

In other words, if we set $Q = \begin{bmatrix} q_1 + q_2 & -q_2 \\ q_1 & 0 \end{bmatrix}$ then T_i acts on $\mathbf{E}$ via the $n \times n$ block diagonal matrix

$$\overline{T}_i = \begin{bmatrix} q_1 I_{i-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & q_1 I_{n-i-1} \end{bmatrix}.$$

The $\overline{T}_i$ satisfy the defining relations of $H_n(q_1, q_2)$, so the map defined on generators by $T_i \mapsto \overline{T}_i$ is a representation $H_n(q_1, q_2) \to \mathrm{End}(\mathbf{E})$. By composing with the quotient map $\mathbb{C}[B_n] \to H_n(q_1, q_2)$ given above, we obtain a $\mathbb{C}[B_n]$-module structure on $\mathbf{E}$. This is essentially the Burau representation; it differs from the standard definition [Bur35, Jon87, BLM94] by a simple change of parameters (see [DG21]).

2.1. **Remark.** Set $q = -q_2/q_1$. There are well-known algebra isomorphisms

$$H_n(q_1, q_2) \cong H_n(-1, q), \quad H_n(q_1, q_2) \cong H_n(1, -q)$$

defined by sending $T_i \mapsto -q_1T_i$, $T_i \mapsto q_1T_i$ respectively. Moreover, the map $T_i \mapsto q^{-1/2}T_i$ defines an algebra isomorphism

$$H_n(-q^{-1/2}, q^{1/2}) \cong H_n(-1, q),$$

so $H_n(-q^{-1/2}, q^{1/2}) \cong H_n(q_1, q_2)$. The algebra $H_n(-q^{-1/2}, q^{1/2})$, the "balanced" form of the Iwahori–Hecke algebra, is often preferred in the theory of quantum groups. We choose to work with the generalized Burau representation because it makes sense in general, including in all the one-parameter versions of $H_n(q_1, q_2)$.

3. The decomposition $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$

By direct computation, we notice the following explicit eigenvectors for the $\overline{T}_i$ operators defined in the previous section.

3.1. **Lemma.** *Assume that $q_1 q_2 \neq 0$ and set $q = -q_2/q_1$. For any $i = 1, \dots, n-1$ the operator $\overline{T}_i$ has eigenvectors:*

(a) $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_i + \mathbf{e}_{i+1}, \mathbf{e}_{i+2}, \dots, \mathbf{e}_n$ *with eigenvalue q_1.*
(b) $q\mathbf{e}_i - \mathbf{e}_{i+1}$ *with eigenvalue q_2.*

In particular, $\overline{T}_i$ is diagonalizable if and only if $q_1 \neq q_2$. Moreover, the vector $\boldsymbol{\ell}_0 := \mathbf{e}_1 + \cdots + \mathbf{e}_n$ is a simultaneous eigenvector for all the $\overline{T}_i$.

Proof. Parts (a), (b) are easily checked. Observe that $\mathbf{e}_i + \mathbf{e}_{i+1}$ and $q\mathbf{e}_i - \mathbf{e}_{i+1}$ are linearly dependent if and only if $q = -1$, which proves the diagonalizability claim, as $q = -1 \iff q_1 = q_2$. □

3.2. **Remark.** If $q_1 = q_2$ then the $\overline{T}_i$ have only one eigenvalue and the corresponding eigenspace has dimension $n - 1$.

As in [DG21, §3] we consider the following $H_n(q_1, q_2)$-submodules of $\mathbf{E}$:

$$\mathbf{L} = \mathbb{C}\boldsymbol{\ell}_0 = \mathbb{C}(\mathbf{e}_1 + \cdots + \mathbf{e}_n), \quad \mathbf{F} = \textstyle\bigoplus_{i=1}^{n-1} \mathbb{C}(q\mathbf{e}_i - \mathbf{e}_{i+1}).$$

Since $q \neq 0$, the spanning vectors $q\mathbf{e}_i - \mathbf{e}_{i+1}$ are linearly independent, so $\dim \mathbf{F} = n - 1$.

We aim now to show that $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ under suitable hypotheses. The following result is standard; see e.g., [Mat99, Exercise 1.4].

3.3. **Proposition.** *Suppose that $q_1 q_2 \neq 0$. Set $q = -q_2/q_1$ and $[n]_q = 1 + q + \cdots + q^{n-1}$.*

(a) $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ *if and only if $[n]_q \neq 0$.*
(b) *If $n > 2$ then $\mathbf{F}$ is irreducible as an $H_n(q_1, q_2)$-module if and only if $[n]_q \neq 0$. (If $n = 2$ then $\mathbf{F}$ is irreducible for any q.)*

Proof. For (a), observe that the determinant of the matrix with the columns $q\mathbf{e}_1 - \mathbf{e}_2, \dots, q\mathbf{e}_{n-1} - \mathbf{e}_n, \boldsymbol{\ell}_0$ is equal to $[n]_q$. For (b), a direct argument can be found in [DG21]. □

3.4. **Remark.** As $[n]_1 = n$, the decomposition $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ holds at $q = 1$.

4. Orthogonal group

Recall that we always assume that $q_1 q_2 \neq 0$ (hence $q \neq 0$). We will need the nondegenerate symmetric bilinear form $\langle -, - \rangle$ on $\mathbf{E}$ defined by the rule

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} q^{j-1} \tag{5}$$

extended bilinearly, where $q = -q_2/q_1$. Let $J = \operatorname{diag}(1, q, \dots, q^{n-1})$ be the matrix of the form with respect to the $\{\mathbf{e}_i\}$-basis.

Observe that $\langle q\mathbf{e}_i - \mathbf{e}_{i+1}, \boldsymbol{\ell}_0\rangle = 0$ for all $i = 1, \dots, n-1$. Thus $\mathbf{F} \subset \mathbf{L}^\perp$ (the orthogonal complement with respect to the form). Since $\dim \mathbf{L}^\perp = n-1$ by the standard theory of bilinear forms, it follows by dimension comparison that $\mathbf{F} = \mathbf{L}^\perp$.

Now we consider certain orthogonal operators on $\mathbf{E}$. Assume from now on that $q_1 \neq q_2$. (This is equivalent to assuming that $q \neq -1$.) Then we may define elements $S_i \in H_n(q_1, q_2)$ by

$$S_i = \tfrac{1}{q_1 - q_2}\big(2T_i - (q_1 + q_2)\big) \tag{6}$$

for $i = 1, \dots, n-1$. A simple calculation with the defining quadratic relation in $H_n(q_1, q_2)$ shows that

$$S_i^2 = 1. \tag{7}$$

Let $\overline{S}_i \in \mathrm{End}(\mathbf{E})$ be the corresponding linear operator, defined by replacing T_i by its image $\overline{T}_i$. The $\overline{T}_i$-eigenvectors are also $\overline{S}_i$-eigenvectors, and the $\overline{T}_i$-eigenvalues q_1, q_2 have been "deformed" to $\overline{S}_i$-eigenvalues $1, -1$ respectively.

Recall [Bou02, Ch. 5, §2, Nos. 1–2] that a linear endomorphism s in $\mathrm{End}(\mathbf{E})$ is:

- a *pseudo-reflection* if $1 - s$ has rank 1.
- a *reflection* if, additionally, $s^2 = 1$.

Lemma 3.1 and the fact that $S_i^2 = 1$ implies that $\overline{S}_i$ is a reflection, so the group generated by the $\overline{S}_i$ is a reflection group.

Let $\mathrm{O}(\mathbf{E})$ be the orthogonal group of operators $S \in \mathrm{End}(\mathbf{E})$ preserving the bilinear form $\langle -, - \rangle$, in the sense that $\langle Sv, Sw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbf{E}$.

4.1. **Lemma.** *Assume that $q_1 \neq q_2$ (equivalently, $q \neq -1$).*

(a) $S_i \cdot \boldsymbol{\ell}_0 = \boldsymbol{\ell}_0$.
(b) *S_i belongs to $\mathrm{O}(\mathbf{E})$; i.e., $\overline{S}_i^{\mathsf{T}} J \overline{S}_i = J$.*
(c) *$H_n(q_1, q_2)$ is generated, as an algebra, by the S_i.*

Proof. (a) follows immediately from Lemma 3.1, as $\boldsymbol{\ell}_0$ is a sum of S_i-fixed points.

(b) By Lemma 3.1, $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_{-1}$, where $\mathbf{E}_1$, $\mathbf{E}_{-1}$ are the eigenspaces belonging to the S_i-eigenvalues $1, -1$ respectively. By definition, the $\mathbf{e}_i$ are pairwise orthogonal with respect to the form. Notice that the eigenvectors $\mathbf{e}_i + \mathbf{e}_{i+1}$ and $q\mathbf{e}_i - \mathbf{e}_{i+1}$ are also orthogonal. It follows that $\mathbf{E}_1 \perp \mathbf{E}_{-1}$. Given any $v, w \in \mathbf{E}$, write $v = v_1 + v_{-1}$, $w = w_1 + w_{-1}$ (uniquely) where $v_1, w_1 \in \mathbf{E}_1$, $v_{-1}, w_{-1} \in \mathbf{E}_{-1}$. Then

$$\begin{aligned}\langle S_i \cdot v, S_i \cdot w\rangle &= \langle v_1 - v_{-1}, w_1 - w_{-1}\rangle \\ &= \langle v_1, w_1\rangle + \langle v_{-1}, w_{-1}\rangle \\ &= \langle v_1 + v_{-1}, w_1 + w_{-1}\rangle = \langle v, w\rangle.\end{aligned}$$

Thus S_i preserves the form, which implies the result.

(c) This is immediate from the fact that the mapping

$$T_i \mapsto \tfrac{1}{q_1-q_2}\big(2T_i - (q_1+q_2)\big) = S_i$$

is invertible, with inverse given by

$$S_i \mapsto \tfrac{q_1-q_2}{2} S_i + \tfrac{q_1+q_2}{2} = T_i.$$

So any linear combination of products of S_i's is expressible as a linear combination of products of T_i's, and vice versa. □

Now define $\mathbf{e}'_i = q^{-(i-1)/2}\mathbf{e}_i$. Then the basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ is orthonormal with respect to the form; that is,

$$\langle \mathbf{e}'_i, \mathbf{e}'_j \rangle = \delta_{ij}$$

where δ_{ij} is the usual Kronecker delta function. For each $i = 1, \dots, n-1$ we define

$$\mathbf{f}_i = \sqrt{q}\,\mathbf{e}'_i - \mathbf{e}'_{i+1}.$$

Notice that $\mathbf{f}_i$ is a nonzero scalar multiple of the S_i-eigenvector $q\mathbf{e}_i - \mathbf{e}_{i+1}$ in Lemma 3.1, hence is itself an S_i-eigenvector (of eigenvalue -1).

4.2. **Lemma.** *Assume that $q \neq -1$. Then $\overline{S}_i$ is the reflection in the complex hyperplane $H_i = \mathbf{f}_i^{\perp}$, so*

$$S_i \cdot v = v - 2\frac{\langle \mathbf{f}_i, v\rangle}{\langle \mathbf{f}_i, \mathbf{f}_i\rangle}\,\mathbf{f}_i$$

for any $v \in \mathbf{E}$. In particular,

(a) $S_i \cdot \mathbf{e}'_i = \mathbf{e}'_i - \frac{2\sqrt{q}}{1+q}\mathbf{f}_i = \frac{1-q}{1+q}\mathbf{e}'_i + \frac{2\sqrt{q}}{1+q}\mathbf{e}'_{i+1}$.
(b) $S_i \cdot \mathbf{e}'_{i+1} = \mathbf{e}'_{i+1} + \frac{2}{1+q}\mathbf{f}_i = \frac{2\sqrt{q}}{1+q}\mathbf{e}'_i - \frac{1-q}{1+q}\mathbf{e}'_{i+1}$.
(c) $S_i \cdot \mathbf{e}'_j = \mathbf{e}'_j$ *for all* $j \neq i, i+1$.

Proof. The displayed formula is standard [Bou02, Ch. 5, §3]. It can also be verified by direct computation from the definition of S_i and equations (4).

Part (c) is immediate once one notices that the $\mathbf{e}'_j$ for $j \neq i, i+1$ belong to the 1-eigenspace of S_i. Moreover, we have

$$\langle \mathbf{f}_i, \mathbf{f}_i\rangle = 1+q, \quad \langle \mathbf{e}'_i, \mathbf{f}_i\rangle = \sqrt{q}, \quad \langle \mathbf{e}'_{i+1}, \mathbf{f}_i\rangle = -1$$

which gives the formulas in (a), (b). □

For concreteness, the action in Lemma 4.2 of S_i on the orthonormal basis $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is given by the $n \times n$ block matrix

$$\overline{S}_i = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-1-i} \end{bmatrix} \text{ where } Q = \frac{1}{1+q}\begin{bmatrix} 1-q & 2\sqrt{q} \\ 2\sqrt{q} & -(1-q) \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix. Notice that the action of the S_i depends only on $q = -q_2/q_1$ and Q is a 2×2 orthogonal matrix, in the sense that $Q^{\mathsf{T}} Q$ is the 2×2 identity matrix.

5. Twin group

Let W_n be the Weyl group of $\mathrm{GL}(\mathbf{E})$, which we identify with the set of $n \times n$ permutation matrices. Then $W_n \cong \mathfrak{S}_n$, the symmetric group on n letters, generated by $s_1, \dots, s_{n-1}$ subject to the standard Coxeter relations: $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ if $i \neq j$.

Let TW_n be the *twin group* [Kho97] on n strands; that is, the group generated by $t_1, \dots, t_{n-1}$ subject to the defining relations

$$t_i^2 = 1, \quad t_i t_j = t_j t_i \text{ if } |i-j| > 1. \tag{8}$$

We have a quotient mapping $TW_n \twoheadrightarrow W_n$ (with kernel the subgroup generated by all $(t_i t_{i+1})^3$) defined by $t_i \mapsto s_i$.

Since the $\overline{S}_i$ satisfy the defining relations of TW_n, the linear mapping

$$\rho : TW_n \to \mathrm{GL}(\mathbf{E}) \text{ defined by } t_i \mapsto \overline{S}_i$$

is a representation. This is the analogue of the Burau representation adapted to the twin group.

5.1. **Remark.** Unless $q = 1$, the operators $\overline{S}_i$ do not satisfy the braid relations. A calculation from the definition of S_i reveals that

$$S_i S_{i+1} S_i - S_{i+1} S_i S_{i+1} = -\tfrac{(1-q)^2}{(1+q)^2} \, (S_i - S_{i+1})$$

holds in $H_n(q_1, q_2)$. In particular, $\overline{S}_i \overline{S}_{i+1} \overline{S}_i = \overline{S}_{i+1} \overline{S}_i \overline{S}_{i+1}$ if and only if $q = 1$. Thus the representation ρ factors through W_n at $q = 1$, giving the natural n-dimensional permutation representation of W_n; at $q \neq 1$ we have deformed the natural representation away from W_n to a representation of TW_n.

The linear extension of ρ to the group algebra $\mathbb{C}[TW_n]$ factors through $H_n(q_1, q_2)$, via $t_i \mapsto S_i \mapsto \overline{S}_i$. As each $\overline{S}_i$ belongs to $\mathrm{O}(\mathbf{E})$ it follows that the image of the representation ρ is contained in $\mathrm{O}(\mathbf{E})$; in other words, ρ is an orthogonal representation of TW_n. Note that

$$\mathrm{O}(\mathbf{E}) \cong \mathrm{O}_n(\mathbb{C}) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^{\mathsf{T}} A = I\}$$

when the operators are expressed as matrices with respect to the orthonormal basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$.

Since all the $\rho(t_i) = \overline{S}_i$ fix $\boldsymbol{\ell}_0$ by Lemma 4.1, the line $\mathbf{L}$ is isomorphic to $\mathbb{C}$, the trivial $\mathbb{C}[TW_n]$-module.

5.2. **Lemma.** *Suppose that $q \neq -1$. Any $H_n(q_1, q_2)$-module V becomes a $\mathbb{C}[TW_n]$-module by defining $t_i \cdot v = S_i \cdot v$, for any $v \in V$. Its submodule structure is the same, regarded as a module for either algebra.*

Proof. This follows from Lemma 4.1(c) and the fact that the linear transformation (on $H_n(q_1, q_2)$) defined by $T_i \mapsto S_i$ is invertible. □

Assume henceforth that $[n]_q \neq 0$. Combining Proposition 3.3 and Lemma 5.2, we conclude that

$$\mathbf{E} = \mathbf{F} \oplus \mathbf{L} \tag{9}$$

as $\mathbb{C}[TW_n]$-modules, where $\mathbf{L}$ and $\mathbf{F}$ are irreducible $\mathbb{C}[TW_n]$-submodules. Hence

$$\rho(TW_n) \subset \mathrm{O}(\mathbf{L}) \times \mathrm{O}(\mathbf{F}) \tag{10}$$

where $\mathrm{O}(\mathbf{L})$, $\mathrm{O}(\mathbf{F})$ are taken with respect to the restriction of the bilinear form.

5.3. **Lemma.** *Suppose that $q \neq -1$ and $[n]_q \neq 0$. The action of the S_i on $\mathbf{F}$ is given by*

$$S_i \cdot w = w - 2\frac{\langle \mathbf{f}_i, w\rangle}{\langle \mathbf{f}_i, \mathbf{f}_i\rangle}\,\mathbf{f}_i$$

for any $w \in \mathbf{F}$. Hence, S_i acts on $\mathbf{F}$ as reflection in the hyperplane $\mathbf{f}_i^{\perp} = \{w \in \mathbf{F} : \langle \mathbf{f}_i, w\rangle = 0\}$. In particular,

- (a) $S_i \cdot \mathbf{f}_{i-1} = \mathbf{f}_{i-1} + \frac{2\sqrt{q}}{1+q}\mathbf{f}_i$.
- (b) $S_i \cdot \mathbf{f}_i = -\mathbf{f}_i$.
- (c) $S_i \cdot \mathbf{f}_{i+1} = \mathbf{f}_{i+1} + \frac{2\sqrt{q}}{1+q}\mathbf{f}_i$.
- (d) $S_i \cdot \mathbf{f}_j = \mathbf{f}_j$ *for all* $j \neq i-1, i, i+1$.

Hence the S_i-eigenspace of eigenvalue -1 is spanned by $\mathbf{f}_i$, and the S_i-eigenspace of eigenvalue 1 is spanned by

$$\mathbf{f}_{i-1} + \tfrac{\sqrt{q}}{1+q}\mathbf{f}_i,\ \ \mathbf{f}_{i+1} + \tfrac{\sqrt{q}}{1+q}\mathbf{f}_i,\ \ \textit{and } \mathbf{f}_j \textit{ for all } j \neq i-1, i, i+1.$$

Proof. Given $v \in \mathbf{E}$, there exist unique $u \in \mathbf{L}$, $w \in \mathbf{F}$ such that $v = u + w$. As $S_i \cdot u = u$, Lemma 4.1 says that

$$S_i \cdot (u + w) = u + S_i \cdot w.$$

On the other hand, Lemma 4.2 in light of the equality $\langle \mathbf{f}_i, u\rangle = 0$ says that

$$S_i \cdot (u + w) = u + w - 2\frac{\langle \mathbf{f}_i, u + w\rangle}{\langle \mathbf{f}_i, \mathbf{f}_i\rangle}\,\mathbf{f}_i = u + w - 2\frac{\langle \mathbf{f}_i, w\rangle}{\langle \mathbf{f}_i, \mathbf{f}_i\rangle}\,\mathbf{f}_i$$

and the first claim follows by comparing the right hand sides of the two displayed equalities. Formulas (a)–(d) then follow immediately. (They also follow from Lemma 4.2 by a routine calculation.) They in turn imply the claims about the eigenvectors, which implies the final claim. □

Since (under the hypotheses of the lemma) the S_i act as reflections on $\mathbf{F}$, we call it the *reduced* reflection representation. We record the following observation for later use.

5.4. **Lemma.** *Suppose that $q \neq -1$ and $[n]_q \neq 0$. The matrix of the orthogonal projection $\mathbf{E} \twoheadrightarrow \mathbf{L}$ with respect to the orthonormal basis $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is*

$$\frac{1}{[n]_q}\left(q^{(i+j-2)/2}\right)_{i,j=1,\dots,n}$$

and this projection is an endomorphism of $\mathbf{E}$ commuting with the action of $\mathrm{O}(\mathbf{E})$ and thus also with the action of TW_n.

Proof. Let P be the projection operator. Then we have

$$P(\mathbf{e}'_j) = \frac{\langle \boldsymbol{\ell}_0, \mathbf{e}'_j \rangle}{\langle \boldsymbol{\ell}_0, \boldsymbol{\ell}_0 \rangle}\boldsymbol{\ell}_0 = \frac{q^{(j-1)/2}}{[n]_q}\boldsymbol{\ell}_0 = \frac{q^{(j-1)/2}}{[n]_q}\sum_{i=1}^{n} q^{(i-1)/2}\mathbf{e}'_i.$$

This proves the first claim, and the rest follows from Lemma 5.2 and the definitions of the actions. □

6. Density

The main result of this section is the following density result, which is the central technical result of the paper. Recall the "quantum factorial" notation

$$[n]^!_q = [1]_q[2]_q \cdots [n]_q$$

for any positive integer n.

6.1. **Theorem.** *Let $\rho : TW_n \to \mathrm{O}(\mathbf{E})$ be the reflection representation, defined on generators by sending t_i to $\overline{S}_i$ for all $i = 1, \dots, n-1$. Assume that $[n]_q \neq 0$, $[n-2]^!_q \neq 0$, and*

$$q \neq \frac{-\lambda \pm \sqrt{-1-2\lambda}}{1+\lambda}$$

for any $\lambda = \cos(2k\pi/m)$ with $m \in \mathbb{Z}_{\geq 0}$. Then the image $\rho(TW_n)$ is Zariski-dense in $\mathrm{O}(\mathbf{L}) \times \mathrm{O}(\mathbf{F})$. Hence the image of the reduced reflection representation $TW_n \to \mathrm{O}(\mathbf{F})$ is Zariski-dense in $\mathrm{O}(\mathbf{F})$.

6.2. **Remark.** Set $w^{\pm}(\lambda) = (-\lambda \pm \sqrt{-1-2\lambda})/(1+\lambda)$. As $w^+(\lambda)w^-(\lambda) = 1$, we have $w^-(\lambda) = 1/w^+(\lambda)$. If $\lambda = \cos(2k\pi/m)$ for some k, m then λ is a real number satisfying $-1 \leq \lambda \leq 1$. There are two cases to consider:

- (i) If $-1 < \lambda \leq -\frac{1}{2}$ then $w^+(\lambda)$ is real and satisfies $w^+(\lambda) \geq 1$. Also $w^+(\lambda) \to \infty$ as $\lambda \to -1$. In this case $w^-(\lambda) = 1/w^+(\lambda)$ is also real and satisfies $0 < w^-(\lambda) \leq 1$.
- (ii) If $-\frac{1}{2} < \lambda \leq 1$ then $w^+(\lambda)$, $w^-(\lambda)$ are non-real conjugate complex values, so both lie on the unit circle $|z| = 1$ in the complex plane.

In particular, all the hypotheses of Theorem 6.1 hold if q is chosen to be any complex number not on the positive real axis or the unit circle.

The proof of Theorem 6.1 will take up the rest of this section. Here is an outline of the strategy. We know that $\rho(TW_n) \subset \mathrm{O}(\mathbf{L}) \times \mathrm{O}(\mathbf{F})$. Since $\mathrm{O}(\mathbf{L})$ is finite, its Lie algebra is zero and so we have

$$\mathrm{Lie}\,\mathrm{O}(\mathbf{E}) = \mathrm{Lie}\,\mathrm{O}(\mathbf{F}) \cong \mathfrak{so}_{n-1}(\mathbb{C}).$$

Hence the dimension of the Lie algebra of $\mathrm{O}(\mathbf{E})$ is $\dim \mathfrak{so}_{n-1}(\mathbb{C}) = \binom{n-1}{2}$. Put

$$G = \rho(TW_n), \quad \overline{G} = \text{ the Zariski-closure of } G \text{ in } \mathrm{O}(\mathbf{F}).$$

Under the stated assumptions on q, we will use the defining matrices $\overline{S}_i$ to find $\binom{n-1}{2}$ linearly independent elements of $\mathrm{Lie}\,\overline{G}$. This forces $\mathrm{Lie}\,\overline{G} = \mathfrak{so}_{n-1}(\mathbb{C})$ and hence justifies the conclusion $\overline{G} = \mathrm{O}(\mathbf{F})$.

For $i = 1, \dots, n-1$, recall from Lemma 4.2 that $\overline{S}_i = \rho(t_i)$ is given (with respect to the orthonormal basis $\mathbf{e}'_1, \dots, \mathbf{e}'_n$) by the block diagonal matrix

$$\overline{S}_i = \mathrm{diag}(I_{i-1}, Q, I_{n-1-i}), \quad \text{where } Q = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

and $a = (1-q)/(1+q)$, $b = 2\sqrt{q}/(1+q)$. Notice that $a^2 + b^2 = 1$. Since $\overline{S}_i$ squares to 1 and $\overline{S}_i\overline{S}_j = \overline{S}_j\overline{S}_i$ for all $|i-j| > 1$, understanding $G = \rho(TW_n)$ requires studying the products

$$\overline{S}_{i,i+1} := \overline{S}_i\overline{S}_{i+1} \quad \text{for } i = 1, \dots, n-2.$$

A direct calculation shows that

$$\overline{S}_{i,i+1} = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-2-i} \end{bmatrix} \text{ where } R = \begin{bmatrix} a & ab & b^2 \\ b & -a^2 & -ab \\ 0 & b & -a \end{bmatrix}.$$

We need to study the powers of this matrix.

The next step arises from classical geometry in three-dimensional euclidean spaces. Assume for the moment that $q > 0$ is a real number and that our vector space $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ is over the real numbers. We restrict our attention to the three-dimensional subspace

$$V = \mathbb{R}\mathbf{e}'_i \oplus \mathbb{R}\mathbf{e}'_{i+1} \oplus \mathbb{R}\mathbf{e}'_{i+2} \cong \mathbb{R}^3.$$

We know that $\overline{S}_i$ is reflection in $\mathbf{f}_i^{\perp}$ for each i. Let $\theta/2$ be the angle between $\mathbf{f}_i$ and $\mathbf{f}_{i+1}$ and let $\mathbf{n}$ be the unit vector in the direction of the cross-product $\mathbf{f}_i \times \mathbf{f}_{i+1}$. Then $\overline{S}_{i,i+1}$ is the composition of the two reflections, hence a rotation. In fact, it is the rotation in the plane orthogonal to $\mathbf{n}$. By the classical Rodrigueś rotation formula [Rod40], we have

$$R = I + (\sin\theta)N + (1-\cos\theta)N^2$$

where I is the identity matrix, $\mathbf{n} = (n_1, n_2, n_3)$ in local coordinates of $V \cong \mathbb{R}^3$, and where

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

is the "cross-product" matrix representing the linear map $\mathbf{x} \mapsto \mathbf{n} \times \mathbf{x}$ on $V \cong \mathbb{R}^3$. Since $\mathbf{n}$ is proportional to the cross-product

$$\mathbf{f}_i \times \mathbf{f}_{i+1} = (1, \sqrt{q}, q),$$

after normalization we get $\mathbf{n} = [3]_q^{-1/2}(1, \sqrt{q}, q)$ and hence

$$N = \frac{1}{\sqrt{[3]_q}} \begin{bmatrix} 0 & -q & \sqrt{q} \\ q & 0 & -1 \\ -\sqrt{q} & 1 & 0 \end{bmatrix}.$$

It is well known that $N \in \mathfrak{so}(3) = \mathfrak{so}_3(\mathbb{R})$ and that $R = \exp(\theta N)$. Furthermore, N satisfies the relation $N^3 = -N$.

Now we revert to the general case $q \in \mathbb{C}$. Motivated by the above considerations, for each $i = 1, \dots, n-2$, define

$$N_i = \operatorname{diag}(0_{i-1}, N, 0_{n-2-i})$$

where 0_k denotes a $k \times k$ zero matrix. Then $N_i \in \mathfrak{so}_n(\mathbb{C})$. Consider the one-parameter subgroup G_i consisting of all matrices of the form

$$R(\alpha) = \exp(\alpha N_i) = I + (\sin\alpha) N_i + (1 - \cos\alpha) N_i^2 \quad (\alpha \in \mathbb{C}). \tag{11}$$

Note that $R(0) = I$ is the identity matrix; more generally

$$R(2k\pi) = I \quad \text{for all } k \in \mathbb{Z}. \tag{12}$$

The complex sine function is surjective, so for any $z \in \mathbb{C}$, there exists an $\alpha \in \mathbb{C}$ such that $z = \sin\alpha$. As $\sin^2\alpha + \cos^2\alpha = 1$, it follows that $\cos z = \sqrt{1-z^2}$ for the appropriate choice of the square root. Hence, for any $z \in \mathbb{C}$ there is an $\alpha \in \mathbb{C}$ for which

$$R(\alpha) = I + zN_i + (1 - \sqrt{1-z^2}) N_i^2. \tag{13}$$

In this sense, $R(\alpha)$ depends not on α but only on $z = \sin\alpha$ and a choice of the square root. More precisely, we have

$$\alpha = -\sqrt{-1} \log(z\sqrt{-1} + \sqrt{1-z^2})$$

(where $\sqrt{-1}$ is the imaginary unit) and we have already chosen the square root of $1 - z^2$ in choosing α.

6.3. **Lemma.** *Fix any $i = 1, \dots, n-2$. Then $\overline{S}_{i,i+1} \in G_i$. Hence, the cyclic subgroup generated by $\overline{S}_{i,i+1}$ is contained in G_i.*

Proof. The assertion holds if and only if there exists a value $z \in \mathbb{C}$ such that

$$R = I_3 + z\,N + (1 - \sqrt{1-z^2})\,N^2.$$

A calculation shows that

$$z = \frac{2\sqrt{q\,[3]_q}}{[2]_q^2} \text{ and } \sqrt{1-z^2} = \frac{-[2]_{q^2}}{[2]_q^2}$$

gives a solution, which corresponds to

$$\alpha = -\sqrt{-1}\log\left(\frac{2\sqrt{-q\,[3]_q} - [2]_{q^2}}{[2]_q^2}\right).$$

This proves the claim. □

6.4. **Lemma.** *Fix any index i in the range $1, \dots, n-2$. For α as above, we have*

$$\overline{S}_{i,i+1}^{\,k} = R(\alpha)^k = I + zU_{k-1}(\cos\alpha)\,N + (1 - T_k(\cos\alpha))\,N^2$$

for all $k \geq 0$, where T_k, U_k are the Chebyshev polynomials of the first and second kind, respectively.

Proof. This follows from the well known properties of Chebyshev polynomials. As $\overline{S}_{i,i+1} = R(\alpha)$ we have

$$\overline{S}_{i,i+1}^{\,k} = R(\alpha)^k = \exp(k\alpha N) = I + \sin(k\alpha)N + (1 - \cos(k\alpha))N^2$$

and the result follows from the standard trigonometric definition of Chebyshev polynomials (see e.g. [MH03]), which defines the polynomials U_{k-1}, T_k in order that $(\sin\alpha)U_{k-1}(\cos\alpha) = \sin(k\alpha)$ and $T_k(\cos\alpha) = \cos(k\alpha)$ for all nonnegative integers k. □

6.5. **Corollary.** *Fix any index i in the range $1, \dots, n-2$. With $z = \sin\alpha = 2\sqrt{q\,[3]_q}/[2]_q^2$ as above, $\overline{S}_{i,i+1}^{\,k} = I$ if and only if*

$$U_{k-1}(\cos\alpha) = 0 \text{ and } T_k(\cos\alpha) = 1,$$

which happens if and only if $\alpha = 2l\pi/k$ for some $l \in \{0, 1, \dots, k-1\}$.

Proof. The joint equations $U_{k-1}(\cos\alpha) = 0$, $T_k(\cos\alpha) = 1$ are equivalent to the conditions $\sin(k\alpha) = 0$, $\cos(k\alpha) = 1$, which in turn are equivalent to the single equality $\exp(k\alpha\sqrt{-1}) = 1$. The result follows. □

6.6. **Remark.** For all i, the last proof shows that $\overline{S}_{i,i+1} = R(\alpha)$ has order k if and only if α is an integer multiple of $2\pi/k$.

6.7. **Corollary.** *The matrix $\overline{S}_{i,i+1} = R(\alpha)$ has finite order if and only if*

$$q = \frac{-\lambda \pm \sqrt{-1 - 2\lambda}}{1 + \lambda}$$

for some $\lambda = \cos(2l\pi/k)$ with $l \in \{0, 1, \dots, k-1\}$.

Proof. Finite order occurs if and only if $\alpha = 2l\pi/k$ with $l \in \{0, 1, \dots, k-1\}$. For any such α, we have $\cos\alpha = -[2]_{q^2}/[2]_q^2 = -(1 + q^2)/(1 + q)^2$. Since $z = \sin\alpha$,

$$\sqrt{1 - z^2} = \cos\alpha = -(1 + q^2)/(1 + q)^2$$

as previously noted. Setting $\lambda = \cos\alpha = \sqrt{1 - z^2}$ in the above and solving for q gives the result. □

6.8. **Remark.** An interesting question is to characterize the reflection group generated by the $\overline{S}_i$ in case q has one of the values in Corollary 6.7. We do not know whether the group is finite, for example, except when $n = 3$, in which case it is a finite dihedral group.

For all integers i, j with $1 \le i < j \le n-1$, define elements $L_{i,j}$ in $\mathrm{Lie}(\overline{\rho(TW_n)})$ as follows. First set $L_{i,i+1} = \sqrt{[3]_q}\, K_i$. The scaling factor $\sqrt{[3]_q}$ is present for convenience to clear denominators. Next, inductively define $L_{i,j+1} = [L_{i,j}, L_{j,j+1}]$ for $j = i+1, \dots, n-2$. For all $1 \le i < j \le n-1$ set $\overline{e}_{i,j} = e_{i,j} - e_{j,i}$, where $e_{i,j}$ is the matrix unit with a 1 in row i, column j and 0 elsewhere.

6.9. **Lemma.** *The Lie algebra elements $L_{r,s}$ are given by the formula*

$$L_{r,s} = (-1)^t \Big(q^{(t-1)/2}\, [t-1]_q\, \overline{e}_{r,s-1} + q^t\, \overline{e}_{r,s} - q^{1/2}\, [t]_q\, \overline{e}_{r,s+1} - \sum_{i=1}^{t-2} q^{(i+1)/2}\, \overline{e}_{r+i,s-1} + \sum_{j=1}^{t} q^{(j-1)/2}\, \overline{e}_{r+j,s+1} \Big)$$

where $t = s - r$.

Proof. A straightforward induction on t proves the formula. The base case, $t = 1$, is the definition of the elements $L_{r,r+1}$ given above. Then assuming the result for some $L_{r,s}$, the formula for $L_{r,s+1}$ is derived from the definition $L_{r,s+1} = [L_{r,s}, L_{s,s+1}]$ by computing the bracket using the matrix unit commutator formulas $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{ki}$. □

Next we investigate the independence of the elements $L_{r,s}$. It will be helpful to consider a total order on the indexing set

$$\Omega = \{(r, s) \mid 1 \le r < s \le n-1\}.$$

For $(i, j), (k, l) \in \Omega$, define $(i, j) \succ (k, l)$ if either $j > l$ or both $j = l$ and $i < k$. Note that the order $\succ$ first compares column indices. Elements in a given column have greater indices than all those in preceding columns, and elements in the same column are ordered inversely by their row index.

6.10. **Lemma.** *Suppose that $[n-2]^!_q \neq 0$. Then the elements $L_{r,s}$ with $1 \le r < s \le n-1$ are linearly independent.*

Proof. Suppose that $\sum c_{r,s} L_{r,s} = 0$. We will show that each $c_{r,s} = 0$ starting with $c_{1,n-1}$, the term with subscript of maximal order, and proceeding in decreasing order according to the coefficient's subscript and ending at $c_{1,2}$.

Lemma 6.9 implies that the coefficient of $\overline{e}_{1,n}$ in the sum $\sum c_{r,s} L_{r,s}$ is, up to a sign, equal to $\sqrt{q}[n-2]_q\, c_{1,n-1}$. As q and $[n-2]_q$ are assumed to be non-zero it follows that $c_{n-1,1} = 0$. Next, consider the coefficient of $\overline{e}_{2,n}$ in the sum. Since $c_{1,n-1} = 0$ the coefficient of $\overline{e}_{2,n}$ is, up to a sign, $\sqrt{q}\, [n-3]_q\, c_{2,n-1}$. It follows that $c_{2,n-2} = 0$. Moving inductively through the subsequent indices in Ω in descending order, from $(3, n-3)$ down to $(1, 2)$ similarly shows that all $c_{r,s} = 0$. □

6.11. **Remark.** If $[j]_q = 0$ for some $j = 1, \dots n-2$, then the explicit dependence relation

$$L_{1,j+1} + \sum_{i=1}^{j-1} q^{(j-i)/2}\ (L_{1,j+1} + L_{2,j+1}) = 0$$

holds. We do not know whether the image $\rho(TW_n)$ is dense in that case.

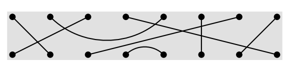
Proof of Theorem 6.1. Lemma 6.10 implies Theorem 6.1 by a dimension comparison, as the $L_{r,s}$ form a set of $\binom{n-1}{2}$ linearly independent elements in the Lie algebra $\operatorname{Lie}\mathrm{O}(\mathbf{E}) \cong \mathfrak{so}_{n-1}(\mathbb{C})$ (see the remarks at the beginning of this section). □

7. The partial Brauer algebra

Recall [Bra37, HR05] that the Brauer algebra $\mathfrak{B}_r(\delta)$ with parameter $\delta \in \mathbb{C}$ is the algebra with basis indexed by the pairings of a finite set

$$\mathbf{r} = \{1, \dots, r, 1', \dots, r'\}$$

of $2r$ elements, where we define a pairing to be a set partition in which all the subsets have cardinality two. Each basis element is typically pictured as an undirected graph on $2r$ vertices, arranged in two rows with the numbers $1, \dots, r$ (resp., $1', \dots, r'$) labeling the top (resp., bottom) row vertices in order from left to right; an edge connects two vertices if and only if they are paired. For example, the diagram



depicts the pairing

$$\{\{1, 2'\}, \{2, 5\}, \{1', 3\}, \{3', 7\}, \{4, 8'\}, \{4', 5'\}\{6, 6'\}, \{7', 8\}\}.$$

The product of two basis elements d_1, d_2 is obtained by stacking d_1 above d_2, identifying (and removing) the middle rows of vertices, and pairing two elements from the remaining two rows if and only if there is a path between them. Closed loops in the middle rows are also removed. If $d_1 \circ d_2$ is the resulting graph, the product $d_1 d_2$ is defined by

$$d_1 d_2 = \delta^N\, d_1 \circ d_2$$

where N is the number of such interior loops. This makes $\mathfrak{B}_r(\delta)$ into an associative algebra. If all edges in a diagram connect one endpoint in the top row to one in the bottom row then the diagram depicts a permutation; the subalgebra spanned by such diagrams is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_r]$ of the symmetric group $\mathfrak{S}_r$.

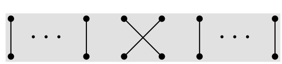
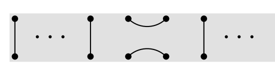
Let $\mathcal{P}\mathfrak{B}_r(\delta, \delta')$ be the two-parameter "partialization" of $\mathfrak{B}_r(\delta)$ as defined in [MM14, Hd14]; cf. also [KM06]. The idea is to allow graphs with pairings and, possibly, a number of isolated vertices. In other words, the subsets of the underlying set partition have cardinality at most two. The resulting (larger) algebra is the *partial Brauer algebra*, also known as the "rook Brauer algebra". The multiplication is again given by stacking, but a second parameter δ' is introduced in order to track the number of non-loops (isolated vertices or connected paths) in the deleted middle row. Multiplication of basis elements is defined by

$$d_1 d_2 = \delta^{N_1} \delta'^{N_2}\, d_1 \circ d_2 \tag{14}$$

where N_1 and N_2 is the number of removed loops and non-loops, respectively. (The use of two parameters in this context goes back at least to Mazorchuk [Maz02].) This makes $\mathcal{PB}_r(\delta, \delta')$ into an associative algebra. It contains $\mathfrak{B}_r(\delta)$ as a subalgebra (the span of the set of diagrams with no isolated points) and also contains the partial permutation algebra $\mathcal{PP}_r(\delta')$, spanned by the partial diagrams in which no two vertices in the same row are ever paired. The algebra $\mathcal{PP}_r(\delta')$ was studied in [DG21]; it is closely related to the symmetric inverse semigroup (also known as the rook monoid) studied by Munn [Mun57, Mun57b], Solomon [Sol02], and others. $\mathcal{PP}_r(\delta')$ is the partialization of $\mathbb{C}[\mathfrak{S}_r]$ in the same sense that $\mathcal{PB}_r(\delta, \delta')$ is the partialization of $\mathfrak{B}_r(\delta)$.

If we set $\delta' = \delta$ then $\mathcal{PB}_r(\delta, \delta)$ is a subalgebra of the partition algebra $\mathcal{P}_r(\delta)$ of [Mar91, Mar94, Mar96, Jon94]. The paper [HR05] provides a convenient comprehensive summary of many basic properties of $\mathcal{P}_r(\delta)$.

7.1. **Theorem** ([KM06, MM14]). *Let s_i, e_i (for $i = 1, \dots, r-1$) and p_j (for $j = 1, \dots, r$) be defined by*

$s_i =$  , $e_i =$  ,

$p_j =$ 

depicting the pairings

$$\{\{i, (i+1)'\}, \{i+1, i'\}\} \cup \{\{j, j'\} : j \neq i, i+1\},$$
$$\{\{i, i+1\}, \{i', (i+1)'\}\} \cup \{\{j, j'\} : j \neq i, i+1\},$$

and the partial pairing

$$\{\{j\}, \{j'\}\} \cup \{\{k, k'\} : k \neq j\}$$

respectively. Then

(a) *The symmetric group algebra $\mathbb{C}[\mathfrak{S}_r]$ is generated by the s_i subject to the defining relations*

$$s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } |i-j| > 1.$$

(b) *The Brauer algebra $\mathfrak{B}_r(\delta)$ is generated by the s_i, e_i for $i = 1, \dots, r-1$ subject to the defining relations*

$$e_i^2 = \delta e_i,\ e_i e_{i\pm1} e_i = e_i,\ e_i e_j = e_j e_i \text{ if } |i-j| > 1,$$
$$s_i e_i = e_i s_i = e_i,\ s_i e_{i\pm1} e_i = s_{i\pm1} e_i,\ e_i e_{i\pm1} s_i = e_i s_{i\pm1},$$
$$e_i s_j = s_j e_i \text{ if } |i-j| > 1.$$

together with the relations in part (a).

(c) *The partial permutation algebra $\mathcal{PP}_r(\delta')$ is generated by the s_i, p_j for $i = 1, \dots, r-1$ and $j = 1, \dots, r$ subject to the defining relations*

$$p_i^2 = \delta' p_i,\ p_i p_j = p_j p_i \text{ if } i \neq j,\ p_i s_i p_i = p_i p_{i+1},$$
$$s_i p_i = p_{i+1} s_i,\ s_i p_j = p_j s_i \text{ if } j \neq i, i+1$$

together with the relations in part (a).

(d) *The partial Brauer algebra $\mathcal{P}\mathfrak{B}_r(\delta, \delta')$ is generated by the s_i, e_i for $i = 1, \dots, r-1$ along with the p_j for $j = 1, \dots, r$ subject to the defining relations*

$$e_i p_i e_i = \delta' e_i,\ e_i p_i p_{i+1} = \delta' e_i p_i,\ p_i p_{i+1} e_i = \delta' p_i e_i,$$
$$p_i e_i p_i = p_i p_{i+1},\ e_i p_i = e_i p_{i+1},\ p_i e_i = p_{i+1} e_i,$$
$$e_i p_j = p_j e_i \text{ if } j \neq i, i+1$$

along with the relations in parts (a), (b), and (c).

7.2. **Remark.** (i) Part (a) of Theorem 7.1 is standard. Presentations similar to those in (b), (c) were known earlier [BW89, Naz96, Sol02]. A slightly different presentation of $\mathcal{PP}_r(\delta')$ was derived in [DG21]; its equivalence with the presentation in part (c) is easy to check. The book [GM09] gives a wealth of results on the semigroups related to the theorem.

(ii) We use e_i for cup-cap diagrams (as is common in the literature) and p_i for projection diagrams. *Warning:* those notations are transposed in [MM14]. The published version of [MM14, Prop. 5.2] includes a redundant relation (see [KM06, Lemma 5.2]) which we have omitted.

(iii) As explained in [Hd14], there are several other interesting subalgebras of $\mathcal{P}\mathfrak{B}_r(\delta, \delta')$; e.g., the Motzkin algebra [BH14].

A special case of the following was observed in [MM14, §6.2]; see [DG21, Cor. 8.7] for a similar result in a different context.

7.3. **Lemma.** *For any $\delta' \neq 0$, $\mathcal{P}\mathfrak{B}_r(\delta, \delta') \cong \mathcal{P}\mathfrak{B}_r(\delta, 1)$ as algebras.*

Proof. The isomorphism is $\mathcal{P}\mathfrak{B}_r(\delta, \delta') \to \mathcal{P}\mathfrak{B}_r(\delta, 1)$ is given by mapping p_j to p_j/δ' for all j. That this defines an isomorphism follows from the defining relations in Theorem 7.1. □

7.4. **Remark.** The representation theory of $\mathcal{P}\mathfrak{B}_r(\delta, \delta')$ is worked out in [MM14], using a Morita equivalence result (cf. [Hd14] for another approach). In particular, it is shown that $\mathcal{P}\mathfrak{B}_r(\delta, \delta')$ is cellular [GL96] and generically semisimple; the cell modules are indexed by the set

$$\Lambda_r = \{\lambda \vdash k : 0 \leq k \leq r\}$$

of partitions of the integers $0, 1, \dots, r$.

8. Schur–Weyl duality

Now we can formulate and prove our main results. We will need the following general fact, which is presumably known. We include a proof since we were unable to find a suitable reference.

8.1. **Lemma.** *Let G be a subgroup of* $\mathrm{GL}(V)$*. Then for any $r \geq 1$, we have the equality* $\mathrm{End}_G(V^{\otimes r}) = \mathrm{End}_{\overline{G}}(V^{\otimes r})$*, where $\overline{G}$ is the Zariski-closure of G in* $\mathrm{GL}(V)$*.*

Proof. It suffices to show that $\mathrm{End}_G(V^{\otimes r}) \subset \mathrm{End}_{\overline{G}}(V^{\otimes r})$, the reverse inclusion being obvious. Fix any $A \in \mathrm{End}_G(V^{\otimes r})$, and define a map $F : \mathrm{End}(V) \to \mathrm{End}(V^{\otimes r})$ by

$$F(g) := A(g \otimes \cdots \otimes g) - (g \otimes \cdots \otimes g)A.$$

Fix a basis $\{v_l\}_{l=1,\dots,n}$ of V, and identify $\mathrm{End}(V) \cong \mathrm{M}_n(\mathbb{C})$ via the basis; similarly identify $\mathrm{End}(V^{\otimes r}) \cong \mathrm{M}_{n^r}(\mathbb{C})$ via the corresponding basis

$$\{v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_r}\}_{j_1,\dots,j_r=1,\dots,n}$$

of $V^{\otimes r}$. Then F vanishes on G, so its matrix coordinate functions $F_{i,j}$ also vanish on G, for any multi-index pair $i = (i_1, i_2, \dots, i_r)$, $j = (j_1, j_2, \dots, j_r)$. But $F_{i,j}$ is a complex-valued polynomial function on $\mathrm{End}(V) = \mathrm{M}_n(\mathbb{C})$ which vanishes on G, hence it vanishes on $\overline{G}$. So F itself vanishes on $\overline{G}$. This implies that

$$A(g \otimes \cdots \otimes g) - (g \otimes \cdots \otimes g)A = 0$$

for any $g \in \overline{G}$. In other words, $A \in \mathrm{End}_{\overline{G}}(V^{\otimes r})$, as required. □

In the following, we always assume that $[n]_q \neq 0$, so that the orthogonal decomposition $\mathbf{E} = \mathbf{F} \oplus \mathbf{L}$ holds. Hence, by Lemma 5.2, Proposition 3.3, and [Che55, p. 88] the module $\mathbf{E}^{\otimes r}$ is semisimple as a $\mathbb{C}[TW_n]$-module. Recall that $\langle \boldsymbol{\ell}_0, \boldsymbol{\ell}_0 \rangle = [n]_q$ and define

$$\mathbf{u}_0 = [n]_q^{-1/2} \boldsymbol{\ell}_0.$$

The assumption $[n]_q \neq 0$ implies that the restriction of the bilinear form to $\mathbf{F}$ is nondegenerate (i.e., $\mathbf{F}$ is a non-isotropic subspace of $\mathbf{E}$). Fix any orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ of $\mathbf{F}$. Then

$$\{\mathbf{u}_0\} \cup \{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}$$

is an orthonormal basis of $\mathbf{E}$ which is compatible with the decomposition $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ in the sense that $\mathbf{L} = \mathbb{C}\mathbf{u}_0$ and $\mathbf{F} = \bigoplus_{j=1}^{n-1} \mathbb{C}\mathbf{u}_j$.

Define linear endomorphisms $\overline{s}_i$, $\overline{e}_i$ (for $i = 1, \dots, r-1$) and $\overline{p}_j$ (for $j = 1, \dots, r$) of $\mathbf{E}^{\otimes r}$ on basis elements $u = \mathbf{u}_{k_1} \otimes \cdots \otimes \mathbf{u}_{k_r}$ (where $k_\alpha \in \{0, \dots, n-1\}$ for all α) as follows:

(i) $\overline{s}_i(u)$ is the same tensor but with the factors in tensor positions $i, i+1$ interchanged.

(ii) $\overline{e}_i(u)$ is $\delta_{k_i,k_{i+1}}$ times the vector obtained by replacing the factors of u in tensor positions $i, i+1$ by $\mathbf{u}_0 \otimes \mathbf{u}_0 + \cdots + \mathbf{u}_{n-1} \otimes \mathbf{u}_{n-1}$.

(iii) $\overline{p}_j(u)$ is the same tensor but with the factor in tensor position j replaced by $\pi(\mathbf{u}_j)$, where π is the orthogonal projection $\mathbf{E} \twoheadrightarrow \mathbf{L}$ in Lemma 5.4.

The endomorphism $\overline{e}_i$ defined in (ii) is independent of the chosen orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}$. Indeed, $\overline{e}_i$ corresponds with the identity map $\mathrm{id}_{\mathbf{E}} \in \mathrm{End}(\mathbf{E})$ under the canonical isomorphism

$$\mathrm{End}(\mathbf{E}) \cong \mathbf{E}^* \otimes \mathbf{E} \cong \mathbf{E} \otimes \mathbf{E}$$

coming from the identification $\mathbf{E}^* \cong \mathbf{E}$ arising from the given bilinear form. It is also possible (see [Hd14]) to define the action diagrammatically by a uniform description of the action of any partial Brauer diagram.

8.2. **Lemma.** *The diagonal action of the twin group TW_n on $\mathbf{E}^{\otimes r}$ commutes with the operators $\overline{s}_i$, $\overline{e}_i$, and $\overline{p}_j$ for $i = 1, \dots, r-1$, $j = 1, \dots, r$.*

Proof. That the diagonal TW_n-action commutes with $\overline{s}_i$ is obvious, and commutativity with $\overline{e}_i$ is clear because the S_i act as orthogonal matrices in $\mathrm{O}(\mathbf{E})$, so we can appeal to the classical fact [Bra37] that the diagonal $\mathrm{O}(\mathbf{E})$-action commutes with $\overline{e}_i$. For the commutativity with $\overline{p}_j$, see Lemma 5.4. □

8.3. **Proposition** ([MM14, Prop. 5.1])**.** *Suppose that $[n]_q \neq 0$. Regard $\mathbf{E} = \mathbf{L} \oplus \mathbf{F}$ as a representation of $\mathrm{O}(\mathbf{F})$ by restriction from $\mathrm{O}(\mathbf{E})$ to $\mathrm{O}(\mathbf{F})$. Then the centralizer algebra $\mathrm{End}_{\mathrm{O}(\mathbf{F})}(\mathbf{E}^{\otimes r})$ for the diagonal action of $\mathrm{O}(\mathbf{F}) \cong \mathrm{O}_{n-1}(\mathbb{C})$ on $\mathbf{E}^{\otimes r}$ is the algebra $Z(r)$ generated by $\overline{s}_i$, $\overline{e}_i$, and $\overline{p}_j$ for $i = 1, \dots, r-1$, $j = 1, \dots, r$.*

Proof. The action of $\mathrm{O}(\mathbf{E})$ on $\mathbf{L}$ is trivial; hence the same is true of its restriction to $\mathrm{O}(\mathbf{F})$. So $\mathbf{L} \cong \mathbb{C}$ as $\mathbb{C}[\mathrm{O}(\mathbf{F})]$-modules. This precisely matches the situation addressed in [MM14, Prop. 5.1]. □

8.4. **Proposition.** *Suppose that $[n]_q \neq 0$. Let $\mathrm{O}(\mathbf{E}) \cong \mathrm{O}_n(\mathbb{C})$ act naturally on $\mathbf{E}$ and diagonally on $\mathbf{E}^{\otimes r}$, with $\mathrm{O}(\mathbf{L}) \times \mathrm{O}(\mathbf{F})$ acting by restriction. Then:*

(a) $\mathrm{End}_{\mathrm{O}(\mathbf{L})\times\mathrm{O}(\mathbf{F})}(\mathbf{E}^{\otimes r}) = Z(r)$.

(b) *For any $\delta' \neq 0$, $Z(r)$ is generated by $\overline{s}_i$, $\overline{e}_i$, and $\delta'\,\overline{p}_j$ for $i = 1, \dots, r-1$, $j = 1, \dots, r$.*

Proof. (a) Since $\mathrm{O}(\mathbf{L}) \cong \{\pm 1\}$ is the cyclic group of order 2, the action of $\mathrm{O}(\mathbf{L}) \times \mathrm{O}(\mathbf{F})$ differs from the action of $\mathrm{O}(\mathbf{F})$ only by signs, so

$$\mathrm{End}_{\mathrm{O}(\mathbf{L})\times\mathrm{O}(\mathbf{F})}(\mathbf{E}^{\otimes r}) = \mathrm{End}_{\mathrm{O}(\mathbf{F})}(\mathbf{E}^{\otimes r}).$$

The result then follows from Proposition 8.3.

(b) Scaling any generator by a nonzero scalar does not affect the centralizer algebra $Z(r)$. □

8.5. **Proposition.** *Assume that* $[n]_q \neq 0$, $[n-2]_q^! \neq 0$, *and that*

$$q \neq \frac{-\lambda \pm \sqrt{-1-2\lambda}}{1+\lambda}$$

for any $\lambda = \cos(2k\pi/m)$ *with* $m \in \mathbb{Z}_{\geq 0}$. *Then* $\mathrm{End}_{TW_n}(\mathbf{E}^{\otimes r}) = Z(r)$.

Proof. Let G be the image of the representation $\rho : TW_n \to \mathrm{GL}(\mathbf{E})$. By Theorem 6.1 and Lemma 8.1 we have

$$\mathrm{End}_{TW_n}(\mathbf{E}^{\otimes r}) = \mathrm{End}_G(\mathbf{E}^{\otimes r}) = \mathrm{End}_{\overline{G}}(\mathbf{E}^{\otimes r})$$

and the result follows from Lemma 8.4(a). □

8.6. **Theorem.** *Assume that* $[n]_q \neq 0$, $[n-2]_q^! \neq 0$, *and that*

$$q \neq \frac{-\lambda \pm \sqrt{-1-2\lambda}}{1+\lambda}$$

for any $\lambda = \cos(2k\pi/m)$ *with* $m \in \mathbb{Z}_{\geq 0}$. *Let* $0 \neq \delta'$ *be a complex number. Regarded as a* $(\mathbb{C}[TW_n], \mathcal{P}\mathfrak{B}_r(n,\delta'))$*-bimodule,* $\mathbf{E}^{\otimes r}$ *satisfies Schur–Weyl duality, in the sense that the enveloping algebra of each action is equal to the full centralizer of the other. Here* TW_n *acts diagonally and the generators* s_i, e_i, p_j *of* $\mathcal{P}\mathfrak{B}_r(n,\delta')$ *act as* $\overline{s}_i$, $\overline{e}_i$, $\delta'\overline{p}_j$ *respectively. Finally, the action of* $\mathcal{P}\mathfrak{B}_r(n,\delta'))$ *is faithful if and only if* $n > r$.

Proof. We just have to put the pieces together. Proposition 8.5 computes the centralizer algebra $Z(r) = \mathrm{End}_{TW_n}(\mathbf{E}^{\otimes r})$ and by verifying that the defining relations in Theorem 7.1 are satisfied by the generators of $Z(r)$, part (c) shows that it is a homomorphic image of the algebra $\mathcal{P}\mathfrak{B}_r(n,\delta')$. This proves one half of Schur–Weyl duality. The other half follows by standard arguments in the theory of semisimple algebras, since the enveloping algebra of the orthogonal group (or its restriction to TW_n) action on the semisimple module $\mathbf{E}^{\otimes r}$ is a semisimple algebra. The final claim follows from the dimension comparison in [MM14, Thm. 5.3(iii)]. □

8.7. **Remark.** There are a number of interesting choices of the scaling parameter δ' in Theorem 8.6:

(i) Taking $\delta' = 1$ recovers an analogue of the Schur–Weyl duality result of [MM14] in which the action of the orthogonal group $\mathrm{O}_{n-1}(\mathbb{C})$ has been replaced by the action of the twin group TW_n. This connects the representations of $\mathcal{P}\mathfrak{B}_r(n,1)$ to those of TW_n.

(ii) If we take $\delta' = n$ in the theorem, the corresponding algebra $\mathcal{P}\mathfrak{B}_r(n,n)$ is contained in the partition algebra $\mathcal{P}_r(n)$ at parameter n.

(iii) Choosing $\delta' = [n]_q$ clears denominators in the corresponding pseudo-projections $[n]_q\overline{p}_j$. By Lemma 5.4, the entries of the matrix of $[n]_q\overline{p}_j$ with respect to the $\{\mathbf{e}'_i\}$-basis depend only on nonnegative integral powers of $\sqrt{q}$.

Recall that the irreducible polynomial representations of $\mathrm{O}_n(\mathbb{C})$ are typically indexed by the set of all partitions λ with not more than n boxes in the first two columns of the corresponding Young diagram. Replacing n by $n-1$ we can index the irreducible polynomial $\mathrm{O}_{n-1}(\mathbb{C})$-modules by the set of partitions λ satisfying the condition $\lambda'_1 + \lambda'_2 \le n-1$, where λ' is the conjugate partition. This gives the following immediate consequence of Theorem 8.6. The set Λ_r is defined in Remark 7.4.

8.8. **Corollary.** *Under the same hypotheses as in Theorem 8.6, we have a decomposition*

$$\mathbf{E}^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_r : \lambda'_1 + \lambda'_2 \le n-1} T^\lambda \otimes B^\lambda$$

as $(\mathbb{C}[TW_n], \mathcal{P}\mathfrak{B}_r(n, \delta'))$*-bimodules, where* T^λ*,* B^λ *are irreducible representations of* TW_n*,* $\mathcal{P}\mathfrak{B}_r(n,\delta')$ *respectively.*

We obtain a second new instance of Schur–Weyl duality involving the twin group (for tensor powers of $\mathbf{F}$) as an immediate consequence of Theorem 6.1 and the classical result of [Bra37].

8.9. **Theorem.** *Let* $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ *be any orthonormal basis of* $\mathbf{F}$ *with respect to the restriction of the bilinear form* $\langle -, - \rangle$*. Assume the hypotheses of Theorem 8.6. Regarded as a* $(\mathbb{C}[TW_n], \mathfrak{B}_r(n-1))$*-bimodule,* $\mathbf{F}^{\otimes r}$ *satisfies Schur–Weyl duality, with the generator* e_i *of* $\mathfrak{B}_r(n-1)$ *acting by*

$$\mathbf{u}_{j_1} \otimes \cdots \mathbf{u}_{j_r} \mapsto \delta_{j_i, j_{i+1}} \sum_{k=1}^{n-1} \mathbf{u}_{j_1} \otimes \cdots \mathbf{u}_{j_{i-1}} \otimes \mathbf{u}_k \otimes \mathbf{u}_k \otimes \mathbf{u}_{j_{i+2}} \otimes \cdots \mathbf{u}_{j_r}.$$

and the s_i *acting by swapping places* $i, i+1$ *as usual, for each* $i = 1, \dots, r-1$*. The action of* $\mathfrak{B}_r(n-1)$ *is faithful if and only if* $n-1 \ge 2r$*.*

Proof. By [Bra37], the actions of $\mathrm{O}_{n-1}(\mathbb{C})$ and $\mathfrak{B}_r(n-1)$ on $\mathbf{F}^{\otimes r}$ commute, where $\mathbf{F}$ is regarded as the (natural) vector representation of $\mathrm{O}_{n-1}(\mathbb{C})$. Let G be the image of the representation $TW_n \to \mathrm{O}(\mathbf{F})$. By Lemma 8.1 we have

$$\mathrm{End}_{TW_n}(\mathbf{F}^{\otimes r}) = \mathrm{End}_{\overline{G}}(\mathbf{F}^{\otimes r}) = \mathrm{End}_{\mathrm{O}_{n-1}(\mathbb{C})}(\mathbf{F}^{\otimes r}).$$

The result now follows by [Bra37]. □

Appendix A. An explicit orthonormal basis of F

The proof of Schur–Weyl duality in Section 8 requires an orthonormal basis of $\mathbf{F}$, which exists by general principles (e.g., [Vin03, Thm. 5.46]). It is sometimes useful to have an explicit orthonormal basis, so we now construct one using the Gram–Schmidt orthogonalization procedure applied to the basis $\{\mathbf{f}_1, \dots, \mathbf{f}_{n-1}\}$. As an application, in Proposition A.5 we obtain a simpler proof of the density conclusion in Theorem 6.1, under somewhat stronger hypotheses.

To begin the orthogonalization procedure, it is useful to observe that

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle = \delta_{ij}[2]_q - (\delta_{i,j+1} + \delta_{i,j-1})\sqrt{q}. \tag{15}$$

In particular, $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = 0$ unless $i = j$ or i, j are adjacent integers. Thus, the matrix of $\langle -, - \rangle$ with respect to the basis $\{\mathbf{f}_i\}$ is a banded tridiagonal $(n-1) \times (n-1)$ matrix of the form

$$A_{n-1} = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}$$

where $a = [2]_q$, $b = c = -\sqrt{q}$. An easy inductive argument shows that

$$\det A_{n-1} = [n]_q. \tag{16}$$

For any $k \le n-1$, let $\mathbf{F}_k = \mathbb{C}\mathbf{f}_1 \oplus \cdots \oplus \mathbb{C}\mathbf{f}_k$ and let A_k be the matrix of the restriction of $\langle -, - \rangle$ to $\mathbf{F}_k$. The matrix A_k is the upper left $k \times k$ submatrix of A_{n-1}. Put $\mathbf{F}_0 = 0$, $d_0 = 1$, and set

$$d_k = \det A_k = [k+1]_q \quad \text{for all } k \ge 1.$$

We obtain the following result from the standard Gram–Schmidt orthogonalization procedure (see e.g. [Vin03, Thm. 5.47]).

A.1. **Lemma.** *Suppose that $d_k = [k+1]_q \neq 0$ for $k = 1, \dots, n-1$ (that is, $[n]_q^! \neq 0$). Then there exists a unique orthogonal basis $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ of $\mathbf{F}$ such that*

$$\mathbf{v}_k \in \mathbf{f}_k + \mathbf{F}_{k-1}, \quad \text{for all } k = 1, \dots, n-1.$$

Furthermore, $\langle \mathbf{v}_k, \mathbf{v}_k \rangle = [k+1]_q/[k]_q$ for all $k = 1, \dots, n-1$.

Under the hypotheses of Lemma A.1, the fact that $\mathbf{v}_j \in \mathbf{f}_j + \mathbf{F}_{j-1}$ in light of (15) immediately implies that

$$\langle \mathbf{f}_i, \mathbf{v}_{i-1} \rangle = -\sqrt{q} \quad \text{and} \quad \langle \mathbf{f}_i, \mathbf{v}_j \rangle = 0 \text{ for all } j < i-1. \tag{17}$$

Thus the Gram–Schmidt formula yields the relation

$$\mathbf{v}_i = \mathbf{f}_i + \frac{\sqrt{q}}{\langle \mathbf{v}_{i-1}, \mathbf{v}_{i-1} \rangle} \mathbf{v}_{i-1} = \mathbf{f}_i + \frac{q^{1/2}[i-1]_q}{[i]_q} \mathbf{v}_{i-1}. \tag{18}$$

This can be used recursively (with $\mathbf{v}_1 = \mathbf{f}_1$) to compute the transition coefficients expressing $\mathbf{v}_i$ as a linear combination of the $\mathbf{f}_j$, but it is slightly simpler to rewrite (18) in the equivalent form

$$[i]_q \mathbf{v}_i = [i]_q \mathbf{f}_i + q^{1/2}[i-1]_q \mathbf{v}_{i-1}. \tag{19}$$

We summarize our conclusions.

A.2. **Lemma.** *Define $\mathbf{v}'_i = [i]_q \mathbf{v}_i$ and suppose that $[n]_q^! \neq 0$. Then $\mathbf{v}'_1, \dots, \mathbf{v}'_{n-1}$ is another orthogonal basis of $\mathbf{F}$ satisfying:*

(a) $\mathbf{v}'_1 = \mathbf{v}_1 = \mathbf{f}_1$.
(b) $\mathbf{v}'_i = [i]_q \mathbf{f}_i + q^{1/2} \mathbf{v}'_{i-1}$ *for all $i = 2, \dots, n-1$.*

Note that $\langle \mathbf{v}'_i, \mathbf{v}'_i \rangle = [i]_q[i+1]_q$ for all i. Solving the recurrence relation in Lemma A.2 gives the explicit formulas

$$\mathbf{v}'_i = \sum_{j=1}^{i} q^{(i-j)/2}[j]_q \mathbf{f}_j = -[i]_q \mathbf{e}'_{i+1} + \sum_{j=1}^{i} q^{(i+j-1)/2} \mathbf{e}'_j \tag{20}$$

expressing the $\mathbf{v}'_i$ in terms of either the $\{\mathbf{f}_j\}$ or the $\{\mathbf{e}'_j\}$.

A.3. **Lemma.** *Suppose that* $[n]^!_q \neq 0$. *Then*

$$\langle \mathbf{f}_i, \mathbf{v}'_{i-1} \rangle = -q^{1/2}[i-1]_q, \quad \langle \mathbf{f}_i, \mathbf{v}'_i \rangle = [i+1]_q$$

and $\langle \mathbf{f}_i, \mathbf{v}'_j \rangle = 0$ *for all* $j \neq i, i-1$.

Proof. The value of $\langle \mathbf{f}_i, \mathbf{v}_j \rangle$ for all $j = 1, \dots, i-1$ was computed in (17), which yields the result in those cases. Lemma A.2(b) implies that when $j = i$,

$$\langle \mathbf{f}_i, \mathbf{v}'_i \rangle = [i]_q \langle \mathbf{f}_i, \mathbf{f}_i \rangle + q^{1/2} \langle \mathbf{f}_i, \mathbf{v}'_{i-1} \rangle = [i]_q[2]_q - q[i-1]_q = [i+1]_q.$$

and similarly when $j = i+1$,

$$\begin{aligned} \langle \mathbf{f}_i, \mathbf{v}'_{i+1} \rangle &= [i+1]_q \langle \mathbf{f}_i, \mathbf{f}_{i+1} \rangle + q^{1/2} \langle \mathbf{f}_i, \mathbf{v}'_i \rangle \\ &= -q^{1/2}[i+1]_q + q^{1/2}[i+1]_q = 0. \end{aligned}$$

By repeating the argument, the above equality inductively implies that $\langle \mathbf{f}_i, \mathbf{v}'_j \rangle = 0$ for any $j > i+1$. □

By scaling the orthogonal basis $\mathbf{v}'_1, \dots, \mathbf{v}'_{n-1}$ we obtain the desired orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ of $\mathbf{F}$, where

$$\mathbf{u}_i = [i]_q^{-1/2}[i+1]_q^{-1/2} \mathbf{v}'_i \tag{21}$$

Now we compute the matrices expressing the action of the operators S_i, defined in (8), with respect to the orthonormal basis $\{\mathbf{u}_j\}$.

A.4. **Lemma.** *Suppose that* $[n]^!_q \neq 0$. *Then* S_i *fixes* $\mathbf{u}_j$ *for all* $j \neq i-1, i$ *and*

$$\begin{aligned} S_i \cdot \mathbf{u}_{i-1} &= a_i \mathbf{u}_{i-1} + b_i \mathbf{u}_i \\ S_i \cdot \mathbf{u}_i &= b_i \mathbf{u}_{i-1} - a_i \mathbf{u}_i \end{aligned}$$

where

$$a_i = \frac{[2]_{q^i}}{[2]_q[i]_q}, \quad b_i = \frac{2\sqrt{q}\,[i+1]_q^{1/2}\,[i-1]_q^{1/2}}{[2]_q[i]_q}.$$

Here $a_i^2 + b_i^2 = 1$. *Moreover,* $a_i = [i]_{q^2}[i]_q^{-2}$ *is an alternative expression for* a_i.

Proof. Observe that we know all the $\langle \mathbf{f}_i, \mathbf{u}_j \rangle$ from Lemma A.3 and the definition of $\mathbf{u}_j$; in particular, $\langle \mathbf{f}_i, \mathbf{u}_j \rangle = 0$ unless j is equal to $i-1$ or i. By Lemma 5.3 we have

$$S_i \cdot \mathbf{u}_j = \mathbf{u}_j - 2\frac{\langle \mathbf{f}_i, \mathbf{u}_j \rangle}{\langle \mathbf{f}_i, \mathbf{f}_i \rangle} \mathbf{f}_i.$$

Hence $S_i \cdot \mathbf{u}_j = \mathbf{u}_j$ if $j \neq i, i-1$. This proves the first claim. It only remains to calculate the a_i, b_i. For example,

$$\begin{aligned} a_i = \langle S_i \cdot \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle &= \langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle - 2\frac{\langle \mathbf{f}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{f}_i, \mathbf{f}_i \rangle} \langle \mathbf{f}_i, \mathbf{u}_{i-1} \rangle \\ &= 1 - 2\frac{[i-1]_q^{-1}[i]_q^{-1} \langle \mathbf{f}_i, \mathbf{v}'_{i-1} \rangle^2}{[2]_q} = 1 - \frac{2q[i-1]_q}{[2]_q[i]_q} = \frac{[2]_{q^i}}{[2]_q[i]_q}. \end{aligned}$$

The calculation of b_i is similar. The proof of the alternative formula for a_i is an easy exercise. □

The matrix of the action of S_i on the orthonormal $\{\mathbf{u}_j\}$-basis is of the block diagonal form $\Delta_i = \operatorname{diag}(I_{i-1}, \Delta'_i, I_{n-i-2})$, where

$$\Delta'_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \Delta'_i = \begin{bmatrix} a_i & b_i \\ b_i & -a_i \end{bmatrix} \text{ if } i > 1.$$

Hence $\Delta_1 \Delta_2 = \operatorname{diag}(I_{i-1}, \Delta'_1 \Delta'_2, I_{n-i-2})$ is a rotation matrix, where

$$\Delta'_1 \Delta'_2 = \begin{bmatrix} -a_2 & -b_2 \\ b_2 & -a_2 \end{bmatrix}.$$

We will also need the diagonal matrices

$$D_i = \operatorname{diag}(1, \dots, 1, -1, 1, \dots, 1)$$

with the -1 appearing in the ith diagonal entry. Note that $D_1 = \Delta_1$ is in G.

We have the following variant of Theorem 6.1 (under slightly stronger hypotheses).

A.5. **Proposition.** *Suppose that $[n]^!_q \neq 0$. Let G be the image of the representation $TW_n \to \mathrm{O}(\mathbf{F})$, and $\overline{G}$ its Zariski-closure in $\mathrm{O}(\mathbf{F})$. Suppose that the matrix $D_i \Delta_{i+1}$ has infinite order for each $i = 1, \dots, n-2$. Then for all $k = 2, \dots, n-1$:*

(a) *$\overline{G}$ contains $\operatorname{diag}(I_{k-2}, \mathrm{O}_2(\mathbb{C}), I_{n-1-k})$.*
(b) *D_k belongs to $\overline{G}$.*

Thus $\overline{G} = \mathrm{O}(\mathbf{F})$.

Proof. As noted above, $D_1 = \Delta_1$ is in G, so $D_1 \Delta_2$ belongs to G. The group generated by $D_1 \Delta_2$ is an infinite cyclic subgroup of the one-parameter group $\operatorname{diag}(\mathrm{SO}_2(\mathbb{C}), I_{n-3})$, so its Zariski-closure is equal to $\operatorname{diag}(\mathrm{SO}_2(\mathbb{C}), I_{n-3})$, and this is contained in $\overline{G}$. It follows that $\operatorname{diag}(\mathrm{O}_2(\mathbb{C}), I_{n-3}) \subset \overline{G}$, proving (a) for $k = 2$. This implies (b) for the case $k = 2$.

For $k > 2$ we may assume by induction that (a), (b) hold for $k-1$. In particular, $D_{k-1} \in \overline{G}$. Repeat the argument to see that the Zariski-closure of the group generated by $D_{k-1}\Delta_k$ is $\mathrm{diag}(I_{k-2}, \mathrm{SO}_2(\mathbb{C}), I_{n-1-k})$, and this is contained in $\overline{G}$. It follows that $\mathrm{diag}(I_{k-2}, \mathrm{O}_2(\mathbb{C}), I_{n-1-k}) \subset \overline{G}$, which proves (a) and (b) by induction.

Let $G(k)$ be the group generated by $\Delta_1, \dots, \Delta_k$. Notice that $G = G(n-1)$. Assume by induction that $\overline{G(k-1)} = \mathrm{diag}(\mathrm{O}_{k-1}(\mathbb{C}), I_{n-k})$. Thus

$$e_{i,j} - e_{j,i} \in \mathrm{Lie}\,\overline{G(k-1)} \text{ for all } 1 \le i < j \le k-1 \tag{22}$$

and $\mathrm{Lie}\,\overline{G(k-1)} \cong \mathfrak{so}_{k-1}(\mathbb{C})$ is contained in $\mathrm{Lie}\,\overline{G(k)}$. By (a) we know that $\mathrm{Lie}\,\overline{G(k)}$ contains $X = e_{k-1,k} - e_{k,k-1}$. By taking commutators of X with the elements in (22) we see that $\mathrm{Lie}\,\overline{G(k)} \cong \mathfrak{so}_k(\mathbb{C})$. By induction, this holds for all k.

Hence, $\mathrm{Lie}\,\overline{G} = \mathrm{Lie}\,\overline{G(n-1)} = \mathfrak{so}_{n-1}(\mathbb{C})$. As $\overline{G} \subset \mathrm{O}_{n-1}(\mathbb{C})$ and contains reflections (elements of determinant -1; e.g., any D_k) it follows that $\overline{G} = \mathrm{O}_{n-1}(\mathbb{C}) \cong \mathrm{O}(\mathbf{F})$. □

A.6. **Remark.** The values of q making $D_i\Delta_{i+1}$ of finite order can be analyzed by means of Chebyshev polynomials, similar to calculations in Section 6. We omit the details.

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