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Schwartz' Distributions in Nonlinear Setting: Applications to Differential Equations, Filtering and Optimal Control

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The paper is intended to be of tutorial value for Schwartz' distributions theory in nonlinear setting. Mathematical models are presented for nonlinear systems which admit both standard and impulsive inputs. These models are governed by differential equations in distributions whose meaning is generalized to involve nonlinear, non single-valued operating over distributions. The set of generalized solutions of these differential equations is defined via closure, in a certain topology, of the set of the conventional solutions corresponding to standard integrable inputs. The theory is exemplified by mechanical systems with impulsive phenomena, optimal impulsive feedback synthesis, sampled-data filtering of stochastic and deterministic dynamic systems.

Key words: Nonlinear dynamics; Distribution; Instantaneous impulse response; Generalized solution; Generalized feedback

1 INTRODUCTION

Although the standard Schwartz' distributions theory is now fully understood, however, in a number of key applications, such as dynamic systems and optimal control, the theory calls for further development to admit a nonlinear and, generally speaking, multi-valued impulse response of a system. A typical example of such an application is an open-loop control system for re-orientation of spacecrafts with impulsive jets that optimally combines the speed of response with its precision under integral constraints on the control effort (see, *e.g.*, [11, 17, 25]). The integral constraints applied to the impulsive spacecraft jets indicate that the cumulative power of the jet is limited, whereas the thrust level can attain a very high value during a very short period of time. This allows one to approximate such a power by an impulsive δ-function whereas the system response remains nonlinear in the instantaneous power. In addition, mechanical systems with striking (impact) mechanisms and systems with strong actuation where actuator-induced transients are close to jumps and therefore are much faster than typical responses, belong to the class of systems with nonlinear impulse responses (see, *e.g.*, [3, 31]).

These systems turn out to be highly sensitive to the method of realization of impulsive inputs, especially if the system has significant nonlinearities. Therefore, practical imple-

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mentation of impulsive inputs requires an adequate estimate of the impulse response of a nonlinear system. This implies that Schwartz' distributions, while being modified for the description of general impulsive systems, must accommodate the details of the realization of the impulsive inputs in the nonlinear setting.

Until recently, no appropriate analytical technique has been developed due to difficulty of the rigorous introduction of the impulsive inputs into nonlinear dynamical systems. The existing methods provide adequate description to only a particular class of nonlinear systems with impulsive inputs (see, *e.g.*, [18, 20, 21] and references quoted therein) and generally cannot guarantee precision, stability, and robustness for system responses of discontinuous type. Without touching the entire range of problems associated with the design of impulsive regulators it should be noted that even fundamental concept of generalized feedback has not been introduced yet. Thus, the nonlinear setting for the Schwartz' distributions theory is continuing to be developed.

The present work develops a consistent modeling methodology which gives rise to mathematical models of the nonlinear dynamical systems with nonlinear impulse responses. Modeling accommodates the details of the realization of the impulse input and admits the impulse response to depend upon the manner in which the impulse is implemented. The models are subsequently shown to be extremely suited to describe impulsive phenomena in mechanical systems, to derive filtering equations over sampled-data measurements, to optimally synthesize generalized, particularly, impulsive controllers.

In this paper, the author summarizes his investigation in the area that was originated in the Institute of Control Sciences early in the 1980's. The paper is written to be accessible for a variety of readers. While the adopted style may be a bit awkward for the specialists who plan to study all the details, it will hopefully enhance the tutorial value of the paper.

The paper has the following structure. The nonlinear distribution formalizm is developed in Section 2 for nonlinear dynamic systems with impulsive phenomena. The section consists of two subsections which address deterministic and stochastic systems side by side. The main result stated here is that although the impulse response of a nonlinear system is not unique (even for a certain realization of the stochastic process) and depends upon the impulse implementation, the set of all possible impulse responses may be found from a certain auxiliary system with integrable inputs. Particularly, necessary and sufficient conditions are obtained for the impulse response to be independent on the impulse realization.

In Section 3, Schwartz' distributions in nonlinear setting are used to formally derive filtering equations over sampled-data measurements from those over continuous measurements. In fact, we formalize here, with the help of the nonlinear distribution formalizm, an intuitively clear idea that a sampled-data measurement may be viewed as a time-continuous measurement made for a short time period, whereas a time-continuous measurement may be represented as a continuum of sampled-data measurements. This formalizm is successively applied to Kalman filtering of linear stochastic processes and nonlinear H_∞-output feedback control of affine systems affected by external disturbances.

In Section 4, the distribution-based representation of nonlinear dynamic systems is involved to develop valid models for systems with impulsive feedback control laws and use these models for feedback control synthesis. This section addresses a singular quadratic optimization problem for feedback linearizable systems within the framework of what is further referred to as a generalized control approach in the closed-loop setting. The existing results [9, 27] on optimal control synthesis under degenerate performance criterion

$$J = \int_0^\infty x^T(t)Px(t)\,\mathrm{d}t, \quad P > 0, \tag{1}$$

depending on the state vector $x(t)$ only, replace this singular optimization problem by its regularization through ε-approximation of this criterion given by

$$J = \int_0^\infty [x^T(t)Px(t) + \varepsilon u^T Ru]\mathrm{d}t, \quad \varepsilon > 0, \ R > 0. \tag{2}$$

The optimal control law u corresponding to (1) is then obtained as a limit as $\varepsilon \to 0$ of the optimal control law u_ε corresponding to (2). Since only particular approximation is taken while other approximations are possible as well there is no guarantee that the original performance criterion is minimized by the control law obtained via this procedure.

Unlike the above procedure, the generalized feedback approach developed does not specify a particular approximation of the singular optimization problem and yields the solution to the original problem in itself. Another very important feature of the methodology proposed is its capability to guarantee the asymptotic stability even for higher order distributions in the control law. To the best of our knowledge there is no equivalence of this feature for the methodology with ε-approximation described above.

Section 5 applies the nonlinear distribution formalism to modeling of impact (shock, collisions) dynamics of mechanical systems. As an illustration of the capabilities of this modeling, an instantaneous change of the initial velocity is straightforwardly deduced for the vertical launch of a spacecraft. Finally, Section 6 presents the conclusions.

2 NONLINEAR DIFFERENTIAL EQUATIONS IN DISTRIBUTIONS

2.1 Preliminaries

Some mathematical preliminaries are reviewed to be used subsequently.

A linear continuous functional, mapping the space C_0^k of k-time continuously differentiable functions with compact support into the real line R^1, is referred to as a kth *order distribution*. Thus, the distributions are defined indirectly, by specifying their effect on the test functions. Recall that the support of a function $\varphi(t)$ specified point-wise is the closure of the set $\{t: \varphi(t) \neq 0\}$.

As usual, D_0^* denotes the dual space of zero order distributions, *i.e.*, those defined on the space $D_0 = C_0^0$ of continuous functions with compact support. The dual product $\langle u, \varphi\rangle$ of a distribution $u(t) \in D_0^*$ and a test function $\varphi(t) \in D_0$ is denoted by $\int_{-\infty}^{\infty} u(t)\phi(t)\,\mathrm{d}t$. Alternatively, D_0^* may be viewed as the space of all (Borel) measures $\mathrm{d}\mu(t)$ with locally bounded variation. The dual product is then explicitly defined by the Stieltjes integral $\langle \mathrm{d}\mu, \varphi\rangle = \int_{-\infty}^{\infty} \phi(t)\,\mathrm{d}\mu(t)$.

A sequence of distributions $u_k(t) \in D_0^*$, $k = 1, 2, \ldots$ converges to $u(t) \in D_0^*$ in the weak* topology, *i.e.*,

$$* - \lim_{k\to\infty} u_k(t) = u(t)$$

iff

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} u_k(t)\varphi(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} u(t)\varphi(t)\,\mathrm{d}t$$

for any test function $\varphi(t) \in D_0$.

2.2 Multivalued Impulse Response in Nonlinear Deterministic Systems

In this subsection we deal with affine systems, the dynamics of which are described by a nonlinear differential equation of the form

$$\dot{x}(t) = f(x, t) + b(x, t)u, \quad x(0) = x_0 \tag{3}$$

where $x(t) \in R^n$ is the state vector, $x_0 \in R^n$ is the initial state, $u(t) \in R^m$ is the input, and t is the time variable. Obviously, the smoothness of f and b guarantees locally the existence of a unique trajectory driven by an integrable input $u(t)$. In turn, unbounded inputs can make trajectories of system (3) be arbitrary close to discontinuous ones. In order to admit discontinuous behavior of the system one should extend the description. If $b(x, t)$ is a state-invariant continuous function $b(t)$, it appears that one can rigorously introduce discontinuous solutions into the equation by admitting the input to be a measure-type function (*e.g.*, a δ-pulse). In general, this way is hampered, however, by the irregularity of a product of the impulsive input $u(t)$ and the discontinuous (in t) function $b(x(t), t)$. In order to avoid this we generalize the meaning of the differential equation according to [21]. Other possible approaches to description of impulsive control systems can be found, *e.g.*, in [2, 15, 16, 32].

DEFINITION 1 *A sequence $\{u_k(t)\}$ of integrable inputs, the L_1-norms of which are uniformly bounded, is a generalized system input, if the solutions $x_k(t)$, $k = 1, 2, \ldots$ of* (3), *corresponding to the inputs $u(t) = u_k(t)$, converge to a left-continuous function $x(t)$ for all continuity points $t \geq 0$ of $x(t)$. The function $x(t)$ is referred to as a generalized solution of* (3).

Relating the generalized control input $\{u_k(t)\}$ in the above definition to a peaking sequence of L_1 inputs, which weakly* converges to a δ-pulse, one can define the impulse response of a nonlinear system as the corresponding generalized solution of (3). Generally speaking, different approximations of the δ-pulse result in different generalized solutions. Thus, the impulse response depends upon the implementation of the impulse. In this work, the symbol $x_{\{u_k\}}(t)$ is used to denote the generalized solution of (3) corresponding to a generalized input $\{u_k(t)\}$.

The goal of this subsection is to describe the set

$$X(\gamma, t_0) = \left\{ x_{\{u_k\}}(t_0+) \colon * - \lim_{k\to\infty} u_k(t) = \gamma\delta(t - t_0) \right\}$$

of the instantaneous impulse responses $x_{\{u_k\}}(t_0+) = \lim_{t\to t_0} x_{\{u_k\}}(t)$ of (3) to all possible realizations $\{u_k(t)\}$ of the impulsive input $\gamma\delta(t - t_0)$, $\gamma \in R^m$, $t_0 \geq 0$. We shall prove that $X(\gamma, t_0)$ can be specified by means of the reachability set

$$R(y_0, \gamma, t_0) = \left\{ \eta_w(y_0, 1, t_0) \colon \int_0^1 w(t)\, dt = \gamma \right\}$$

of the trajectories $\eta_w(y_0, t, t_0)$ of the auxiliary dynamical system

$$\dot{\eta} = b(\eta, t_0)w(t), \quad \eta(0) = y_0 \tag{4}$$

with integrable inputs $w(t)$ of fixed integral power $\int_0^1 w(t)\, dt = \gamma$. Denoting the solution of the unforced system

$$\dot{y} = f(y, t), \quad y(0) = x_0 \tag{5}$$

as $y(t)$, we arrive at the result proven in [21].

THEOREM 1 *Let functions* $f(x, t)$, $b(x, t) \in C^1$ *satisfy the linear growth condition in* x. *Then* $X(\gamma, t_0) = R(y(t_0), \gamma, t_0)$. *Moreover, let* y_1 *belong to* $R(y(t_0), \gamma, t_0)$ *and let the auxiliary system* (4) *be driven from the initial state* $y(t_0)$ *to the terminal state* y_1 *for* $t = 1$ *by the admissible input* $w(t)$, *i.e.*, $\eta_w(y(t_0), 1, t_0) = y_1$. *Then the instantaneous impulse response* $x_{\{u_k\}}(t_0+) = y_1$ *of the affine system* (3) *is particularly forced by the generalized input* $\{u_k(t)\}$ *where*

$$u_k(t) = \begin{cases} kw(k(t - t_0)) & \text{if } t \in \left[t_0, t_0 + \frac{1}{k}\right] \quad k = 1, 2, \dots \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

Remark 1 Let the assumptions of Theorem 1 be in force. Then in analogy to the well-known interpretation of the Schwartz' distributions through the sequential approach [1], the set of all generalized solutions can be viewed as the weak* closure of the set of the conventional solutions corresponding to the integrable inputs. Furthermore, any sequence of L_1 inputs, converging in w_2-weak topology, is shown in [28] to be a generalized system input. Recall that the w_2-weak convergence, introduced in [20, p. 18], requires both the weak* convergence of the L_1, inputs $u_k(t)$, $k = 1, 2, \dots$ and the point-wise convergence of their variations $V_k(t) = \int_0^1 \|u_k(s)\| \, ds$ for all $t \geq 0$.

Remark 2 Due to the peaking phenomenon [29], the impulse response of the affine system (3) with no growth condition on the right-hand side can escape to infinity in infinitesimal time.

It is of interest to note that the Frobenius condition

$$\sum_{k=1}^{n} \frac{\partial b_{li}(x, t)}{\partial x_k} b_{kj}(x, t) = \sum_{k=1}^{n} \frac{\partial b_{lj}(x, t)}{\partial x_k} b_{ki}(x, t), \tag{7}$$

imposed on the system function $b(x, t)$ for all $l = 1, \dots, n$, $i, j = 1, \dots, m$ and $\xi \in R^n$, $t \geq 0$, ensures the uniqueness of the impulse response.

THEOREM 2 *Let the assumptions of Theorem* 1 *be satisfied and let system* (3) *be driven by a generalized input* $\{u_k(t)\}$ *such that*

$$* - \lim_{k \to \infty} u_k(t) = \gamma\delta(t - t_0), \quad \gamma \in R^m, \quad t_0 \geq 0.$$

Then, the generalized solution $x(t) = x_{\{u_k\}}(t)$ *of* (3) *does not depend upon a choice of the approximating sequence* $\{u_k(l)\}$ *if and only if the Frobenius condition* (7) *holds.*

Proof of Theorem 2 can be found in [21].

Thus, the Frobenius condition allows one to replace the peak functions (6) in modeling of the nonlinear system (3) by the δ-pulse $\delta(t - t_0)$. If the Frobenius condition (7) holds then [4] the Pfaffian equation

$$\frac{d\xi}{dv} = b(\xi, s), \quad \xi \in R^n, \quad v \in R^m, \tag{8}$$

generated by the matrix function b, is solvable for arbitrary initial conditions $\xi(0) = z \in R^n$ and scalar parameter s, and the equation integrates to the function $\xi(z, v, s)$ regardless of a path $v(t)$ between the initial $v(0) = y_0$ and terminal $v(1) = \gamma$ points. Hence, the reachability set $R(y_0, \gamma, t_0)$ of (8), written in the parametric form (4) with $w(t) = \dot{v}(t)$, consists of the unique point $\xi(y_0, \gamma, t_0)$ and according to Theorem 1 the instantaneous impulse response is uniquely defined as

$$x(t_0+) = \xi(y(t_0), \gamma, t_0) \tag{9}$$

where $y(t)$ satisfies (5). Summarizing, the following result is shown.

THEOREM 3 *Let the assumptions of Theorem* 2 *be satisfied and let the Frobenius condition* (7) *hold. Then the instantaneous impulse response is uniquely determined from the solution of the Pfaffian system* (8) *according to* (9) *with* $y(t)$ *satisfying* (5).

The latter theorem extends the results of [19] in that the system function $b(x, t)$ needs only be of class C^1 rather than C^2. Particularly, in the case where $b(x, t) = b(t)$ is a state-independent function and hence the mild solution of (3) is well-defined, the solution $\xi(z, v, s)$ of the Pfaffian system is found as $\xi(z, v, s) = z + b(s)v$ and by Theorem 3 the instantaneous impulse response $x(0+) = y(t_0) + b(0)\gamma$ is the same as if the conventional mild solution would be under consideration. In general, modeling the nonlinear system (3) subjected to impulsive inputs should follow Theorem 1 to accommodate details of the impulse implementation and the set of all possible instantaneous impulse responses may be found from the auxiliary system (4) with integrable inputs.

It worth noting that for the involutive dynamic systems (3) (*i.e.*, those satisfying the Frobenius condition (7)) the concept of generalized solution is nothing else than that of vibroimpact solution earlier introduced in [10] and [19] to admit unambiguous nonsmooth and, respectively, discontinuous solutions (see also closely related work [13]). In the rest of this section, the concepts of generalized and vibroimpact solutions are extended to stochastic dynamic systems.

2.3 Vibroimpact Solutions of Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing family of σ-algebras $\mathcal{F}_t$, $t \geq 0$. The symbol E stands for the mathematical expectation throughout.

An n-vector Ito stochastic process x_t in question is associated with the probability space $(\Omega, \mathcal{F}, P)$ and governed by

$$dx_t = f(x_t, t)\,dt + b(x_t, t)\,du(t) + \sigma(x_t, t)\,dw_t, \quad x_{t=0} = x_0 \tag{10}$$

where $(w_t, \mathcal{F}_t, t \geq 0)$ is a standard Wiener process with independent components w_t^i, $i = 1, \dots, r$, $du(t)$ is a Stieltjes measure generated by an m-vector function $u(t)$ of bounded variation, matrix functions f, b, σ are of appropriate dimensions, continuous in all arguments and satisfying the Lipschitz condition in x for all t.

If an $\mathcal{F}_0$-measurable initial condition x_0 is fixed *a priori* and $u(t)$ is absolutely continuous then [12] there exists a unique strong solution x_t of the stochastic Ito differential equation (10). In general, when $u(t)$ is a function of bounded variation only, the meaning of (10) is adopted in the sense of generalized solutions.

DEFINITION 2 *A sequence* $\{u_k(t)\}$ *of integrable inputs, the* L_1*-norms of which are uniformly bounded, is a generalized input of the stochastic system* (10), *if the mean square convergence*

$$\lim_{k\to\infty} E(x_t^k - x_t)^2 = \lim_{k\to\infty} \int_\Omega (x_t^k - x_t)^2 P(d\omega) = 0 \tag{11}$$

of the strong solutions x_t^k, $k = 1, 2 \ldots$ *of* (10), *corresponding to the inputs* $u(t) = u_k(t)$, *holds with some a.s. (almost sure) left-continuous function* x_t *for all continuity points* $t \geq 0$ *of* x_t. *The function* x_t *is referred to as a generalized solution of* (10).

Vibroimpact solutions of (10) appear when equivalent generalized inputs $\{u_k^1(t)\}$, $\{u_k^2(t)\}$, *i.e.* such that the sequence

$$u_1^1(t), u_1^2(t), \ldots, u_k^1(t), u_k^2(t), \ldots$$

is fundamental in the weak* topology, generate the same generalized solution.

DEFINITION 3 *The generalized solution* $x(t)$ *of* (10) *is said to be a vibroimpact solution iff it does not depend upon a choice of a generalized input* $\{u_k(t)\}$ *among equivalent inputs.*

The above results, Theorems 1–3, turn out to be in force for the stochastic processes (10), too. The proof is rather technical and follows the same line of reasoning as that in the deterministic case. It is however unreasonably lengthy to be presented here. Instead, the practical utility and constructive abilities of the generalized, particularly, vibroimpact solutions are subsequently studied.

3 NONLINEAR DISTRIBUTION FORMALIZM IN FILTERING OVER SAMPLED-DATA MEASUREMENTS

In this section, we demonstrate how vibroimpact solutions are utilized to formally derive the filtering equations over sampled-data measurements from those over continuous measurements. Based on such a formalizm two specific problems, filtering of linear stochastic processes via sampled-data observations and nonlinear sampled-data measurement $\mathcal{H}_\infty$-control of uncertain dynamic systems, are successively resolved.

3.1 Vibroimpact Solutions in Kalman Filtering

Let the probability space $(\Omega, \mathcal{F}, P)$ be the same as before and let the partially observed $\mathcal{F}_t$-measurable process (x_t, y_t) be governed by the linear stochastic differential equations

$$dx_t = F(t)x_t\,dt + G(t)u(t)\,dt + Q(t)\,dv_t \tag{12}$$

$$dy_t = H(t)x_t\,dt + R(t)\,dw_t \tag{13}$$

where $x_t \in R^n$ is the state to be estimated, $y_t \in R^p$ is the measurement vector, the random initial states $x_0 \in R^n$ and $y_0 \in R^p$ are $\mathcal{F}_t$-measurable, $u(t) \in R^m$ is the system input, also available for measurements, $(v_t, \mathcal{F}_t, t \geq 0)$ and $(w_t, \mathcal{F}_t, t \geq 0)$ are independent standard Wiener processes whose independent components $v_t^j, j = 1, \ldots, s$, w_t^i, $i = 1, \ldots, r$, coupled to those of x_0 and y_0, form a Gaussian system, F, G, Q, H, R are *a priori* known time varying matrices of appropriate dimensions and the matrix

$$N(t) = R(t)R^T(t) \tag{14}$$

is nonsingular for all $t \geq 0$.

The filtering problem is typically to construct from measurements of $y_0^t = \{y_s, s \in [0, t]\}$ the minimum variance n-vector estimate $m_t = \arg\min E[x_t - \tilde{x}_t]^2$ where the minimum is seek over all $\mathcal{F}_t^y$-measurable estimates $\tilde{x}_t \in R^n$ and $\mathcal{F}_t^y$ is the family of σ-algebras generated by y_0^t and completed by zero probability subsets of $\mathcal{F}$.

The solution of the problem is well-known and given by the Kalman-Bucy filter

$$\mathrm{d}m_t = F(t)m_t\,\mathrm{d}t + G(t)u(t)\,\mathrm{d}t + P(t)H^T(t)N^{-1}(t)[\mathrm{d}y_t - H(t)m_t\,\mathrm{d}t], \quad m_0 = E(x_0|\mathcal{F}_0^y) \quad (15)$$

initialized with the conditional expectation $E(x_0|\mathcal{F}_0^y)$ and coupled to the Riccati equation

$$\dot{P} = PF^T + FP - PH^TN^{-1}HP + QQ^T, \quad P(0) = E(x_0 - m_0)(x_0 - m_0)^T \quad (16)$$

on the covariance matrix $P(t) = E(x_t - m_t)(x_t - m_t)^T$.

Along with the continuous measurement filtering, we shall consider its counterpart over sampled-data measurements taken at possibly nonuniformly spaced time instants which have no finite limiting points:

$$y_{t_l} = H(t_l)x_{t_l} + R(t_l)\xi_l, \quad l = 1, 2, \ldots \quad (17)$$

where $\xi_l \in R^p$ is the measurement noise whose independent components, coupled to those of v_t, x_0, form a Gaussian system. Solution of the filtering problem over the sampled-data measurement (17) is also well-known and given by the following differential equations with jumps:

$$\begin{aligned} \mathrm{d}m_t &= F(t)m_t\,\mathrm{d}t + G(t)u(t)\,\mathrm{d}t, \quad m_0 = E(x_0|\mathcal{F}_0^y) \\ m_{t_l}+ &= m_{t_l} - +P(t_l+)H^T(t_l)N^{-1}(t_l)[y_{t_l} - H(t_l)m_{t_l}-] \end{aligned} \quad (18)$$

$$\begin{aligned} \dot{P} &= PF^T + FP + QQ^T, \quad P(0) = E(x_0 - m_0)(x_0 - m_0)^T, \\ P(t_l+) &= P(t_l-)[I + H^T(t_l)N^{-1}(t_l)H(t_l)P(t_l-)], \quad l = 1, 2, \ldots \end{aligned} \quad (19)$$

It is implied, although rigorously unproven, that the solution of the filtering problem over continuous measurements, made for a short time period, approximate that of the sampled-data measurement problem. Our goal is to make such a connection between the above filtering problems transparent by demonstrating that the filtering Eqs. (15) and (16) of the linear stochastic process (12) over the continuous measurement (13) become identical to those over the sampled-data measurement (17) if the matrix function (14) is formally set to

$$N(t) = \left[\sum_{l=1}^{\infty} N(t_l)\delta(t - t_l)\right]^{-1}. \quad (20)$$

Although nonlinear operating over Dirac functions in (20) is senseless from the point of view of the Schwartz' distributions theory the latter relation, however, has a sense while being substituted into the filtering Eqs. (15) and (16). In what follows, the meaning of the equations

$$\begin{aligned} \mathrm{d}m_t &= F(t)m_t\,\mathrm{d}t + G(t)u(t)\,\mathrm{d}t + P(t)H^T(t) \\ &\times \left[\sum_{l=1}^{\infty} N(t_l)\delta(t - t_l)\right][\mathrm{d}y_t - H(t)m_t\,\mathrm{d}t], \quad m_0 = E(x_0|\mathcal{F}_0^y) \end{aligned} \quad (21)$$

$$\dot{P} = PF^T + FP - PH^T\left[\sum_{l=1}^{\infty} N(t_l)\delta(t - t_l)\right]HP + QQ^T, \quad P(0) = E(x_0 - m_0)(x_0 - m_0)^T, \quad (22)$$

thus obtained, is considered in the sense of vibroimpact solutions. It appears that system (21), (22) has a unique vibroimpact solution and this vibroimpact solution coincides with the strong solution of (18), (19).

THEOREM 4 *There exists a unique vibroimpact solution of system* (21), (22) *and this vibroimpact solution is identical to the strong solution of the sampled-data measurement filter Equations* (18) *and* (19) *with jumps.*

The validity of the above theorem can be established by applying the general theory developed in the previous section to the Kalman-Bucy filter given in terms of vibroimpact solutions of Eqs. (21) and (22). In order to provide a deeper insight on the vibroimpact solutions we present here a straightforward proof of the theorem.

Proof of Theorem 4 Let $\{v_k(t)\}_{k=1}^{\infty}$ be an arbitrary sequence of measurable functions, approximating the distribution $v(t) = \sum_{l=1}^{\infty} N(t_l)\delta(t - t_l)$ in the weak* topology:

$$* - \lim_{k\to\infty} v_k(t) = \sum_{l=1}^{\infty} N(t_l)\delta(t - t_l). \tag{23}$$

Then [12] there exists a unique strong solution $\{m_t^k, P_k(t)\}$, of system

$$\mathrm{d}m_t^k = F(t)m_t^k\,\mathrm{d}t + G(t)u(t)\,\mathrm{d}t + P_k(t)H^T(t)v_k(t)[\mathrm{d}y_t - H(t)m_t^k\,\mathrm{d}t], \quad m_0^k = m_0 \tag{24}$$

$$\dot{P}_k = P_kF^T + FP_k - P_kH^Tv_k(t)HP_k + QQ^T, \quad P_k(0) = P(0), \tag{25}$$

obtained from (21), (22) by substituting $v_k(t)$ for $\sum_{l=1}^{\infty} N(t_l)\delta(t - t_l)$. Moreover, this solution is well-defined on $[0, \infty)$ and $P_k(t)$ remains positive definite for all $t \geq 0$.

Our nearest aim is to demonstrate that

$$\lim_{k\to\infty} E(m_t^k - m_t)^2 = 0, \quad \lim_{k\to\infty} P_k(t) = P(t) \tag{26}$$

for some a.s. left-continuous functions m_t, $P(t)$ and all the continuity points $t \geq 0$ of these functions, regardless of a choice of an approximating sequence $\{v_k(t)\}_{k=1}^{\infty}$. The functions m_t, $P(t)$ could then be accepted as the vibroimpact solutions of (21), (22) and the first assertion of the theorem would be proven.

For this purpose, we introduce the functions

$$D_k(t) = P_k^{-1}(t), \quad n_1^k = P_k^{-1}(t)m_t^k, \qquad k = 1, 2, \ldots \tag{27}$$

which in accordance with (24), (25) are strong solutions of the following equations:

$$\begin{aligned}\mathrm{d}n_t^k &= -F^T(t)n_t^k\,\mathrm{d}t + D_k(t)G(t)u(t)\,\mathrm{d}t - D_k(t)Q(t)Q^T(t)n_t^k\,\mathrm{d}t \\ &\quad + H^T(t)v_k(t)\,\mathrm{d}y_t, \quad n_0^k = P^{-1}(0)m_0\end{aligned} \tag{28}$$

$$\dot{D}_k = -D_kF - F^TD_k - D_kQQ^TD_k + H^T(t)v_k(t)H(t), \quad D_k(0) = P^{-1}(0). \tag{29}$$

In turn, by virtue of (23) the strong solutions of (28), (29) converge

$$\lim_{k\to\infty} E(n_t^k - n_t)^2 = 0, \quad \lim_{k\to\infty} D_k(t) = D(t) \tag{30}$$

to the strong solution $(n_t, D(t))$ of the system

$$\mathrm{d}n_t = -F^T(t)n_t\,\mathrm{d}t + D(t)G(t)u(t)\,\mathrm{d}t - D(t)Q(t)Q^T(t)n_t\,\mathrm{d}t + H^T(t_l)\left[\sum_{l=1}^{\infty} N(t_l)\delta(t-t_l)\right]\mathrm{d}y_t, \quad n_0 = P^{-1}(0)m_0 \tag{31}$$

$$\dot{D} = -DF - F^T D - DQQ^T D + H^T(t_l)\left[\sum_{l=1}^{\infty} N(t_l)\delta(t-t_l)\right]H(t_l), \quad D(0) = P^{-1}(0) \tag{32}$$

for all continuity points $t \geq 0$ of $(n_t, D(t))$ as $k \to \infty$. Indeed, in contrast to (21), (22), system (31), (32) contains no multiplication of the singular distribution $\sum_{l=1}^{\infty} N(t_l)\delta(t-t_l)$ by the discontinuous state $(n_t, D(t))$, and it can therefore be viewed in the conventional sense as the system

$$\mathrm{d}n_t = -F^T(t)n_t\,\mathrm{d}t + D(t)G(t)u(t)\,\mathrm{d}t - D(t)Q(t)Q^T(t)n_t\,\mathrm{d}t, \quad n_0 = P^{-1}(0)m_0, \quad n_{t_l+} = n_{t_l-} + H^T(t_l)N(t_l)\,\mathrm{d}y_{t_l}, \quad l = 1, 2, \ldots, \tag{33}$$

$$\dot{D} = -DF - F^T D - DQQ^T D, \quad D(0) = P^{-1}(0), \quad D(t_l+) = D(t_l-) + H^T(t_l)N(t_l)H(t_l) \tag{34}$$

of differential equations with jumps. Thus, the solution of (31), (32) is well-defined on $[0, \infty)$ and the point-wise convergence (30) is concluded for all the continuity points of $(n_t, D(t))$ due to the sequential interpretation (cf. Remark 1) of the strong solution of (31), (32).

Since $D(t)$ is the solution of the differential Riccati equation (34) with positive definite initial conditions and positive definite changes $\Delta D(t_l) = H^T(t_l)R(t_l)R^T(t_l)H(t_l)$ at time instants t_l, $l = 1, 2, \ldots$, it remains positive definite for all $t \geq 0$. Coupled to (27), (30), this results in the limiting relation (26) with

$$m_t = D^{-1}(t)n_t, \quad P(t) = D^{-1}(t), \tag{35}$$

regardless of a choice of an approximating sequence $\{v_k(t)\}_{k=1}^{\infty}$, thereby ensuring the unique vibroimpact solution of (21), (22) to exist.

To complete the proof it remains to note that by inspection (via substituting (35) into (18), (19)) the vibroimpact solution (35) becomes identical to the strong solution of the differential equations with jumps (18), (19). Theorem 4 is thus proven.

Remark 3 By Theorem 4 the sampled-data measurement filter can be interpreted as a limiting result of the pulse-wise continuous measurement filters. Indeed, Definition 3 of vibroimpact solutions ensures such an interpretation. This interpretation, in turn, guarantees robustness of the filter design, regardless either observation model (sampled-data measurements or continuous measurements made for a short time period) is accepted.

3.2 Vibroimpact Solutions in Nonlinear H_∞-Control

The above nonlinear distribution formalizm is subsequently used to derive a solution of the sampled-data measurement H_∞-control problem from that of the continuous measurement H_∞-control problem. A time-varying nonlinear system governed by

$$\dot{x}(t) = f(x(t), t) + g_1(x(t), t)w(t) + g_2(x(t), t)u(t) \tag{36}$$

$$z(t) = h_1(x(t), t) + k_{12}(x(t), t)u(t) \tag{37}$$

$$y_j = h_2(x(\tau_j), \tau_j) + k_{21}(x(\tau_j), \tau_j)w(\tau_j), \quad j = 0, 1, \ldots \tag{38}$$

is under study. Hereinafter, $x \in R^n$ is the state vector, $t \in R^1$ is the time variable, $u \in R^m$ is the control input, $w \in R^r$ is the unknown disturbance, $z \in R^l$ is the unknown output to be controlled, $y = (y_0, y_1, \ldots)$ is the only available measurement on the system with the discrete measurements $y_j \in R^p, j = 0, 1, \ldots$, taken at possibly nonuniformly spaced time instants τ_j, which have no finite limiting points. The functions $f(x, t)$, $g_1(x, t)$, $g_2(x, t)$, $h_1(x, t)$, $h_2(x, t)$, $k_{12}(x, t)$, $k_{21}(x, t)$ are assumed to be continuous in t for all x and twice continuously differentiable in x for all t, whereas their first and second order state derivatives are assumed to be continuous and uniformly bounded in t. It is also assumed that $f(0, t) = 0$, $h_1(0, t) = 0$ and $h_2(0, t) = 0$ for all t.

A causal dynamic feedback compensator

$$u = \kappa(y, t), \tag{39}$$

with internal state $\xi \in R^q$, is said to be a globally (locally) admissible controller if the closed-loop system (36)–(39) is globally (uniformly) asymptotically stable when $w = 0$.

Given a real number $\gamma > 0$, it is said that system (36)–(39) has L_2/l_2-gain less than γ if the response z, resulting from w for initial state $x(t_0) = 0$, $\xi(t_0) = 0$, satisfies

$$\int_{t_0}^{t_1} \|z(t)\|^2 \, dt < \gamma^2 \left[\int_{t_0}^{t_1} \|w(t)\|^2 \, dt + \sum_{\tau_j \in [t_0, t_1]} \|w(\tau_j)\|^2 \right] \tag{40}$$

for all $t_1 > t_0$ and all continuous functions $w(t)$. The right-hand side in (40) should be viewed as a mixed L_2/l_2-norm on the uncertain signals affecting the system and the sampled-data measurements.

The nonlinear H_∞-control problem over sampled-data measurements is to find a globally admissible controller (39) such that L_2/l_2-gain of the closed-loop system (36)–(39) is less than γ. Since the H_∞-norm translated to the continuous and discrete time domains is nothing else than the L_2- and l_2-induced norms, respectively, the above stated problem is a natural generalization of the standard H_∞-control problem. Such a generalization has been demonstrated in [6] to be eminently suited to time-invariant nonlinear systems with time-continuous measurements.

In turn, a local solution to the H_∞-control problem is defined as follows. A locally admissible controller (39) is said to be a local solution of the H_∞-control problem if there exists a neighborhood U of the equilibrium such that inequality (40) is satisfied for all $t_1 > t_0$ and all continuous functions $w(t)$ for which the state trajectory of the closed-loop system starting from the initial point $(x(t_0), \xi(t_0)) = (0, 0)$ remains in U for all $t \subset [t_0, t_1]$.

In the remainder we confine our investigation to seeking a local solution of the nonlinear H_∞-control problem in question. The following assumptions inherited from the standard H_∞-control problem are made throughout

$$\begin{aligned} h_1^T(x,t)k_{12}(x,t) &= 0, \\ k_{12}^T(x,t)k_{12}(x,t) &= I \\ k_{21}(x,t)g_1^T(x,t) &= 0 \\ k_{21}(x(\tau_j),\tau_j)k_{21}^T(x(\tau_j),\tau_j) &= I, \quad j = 0,1,\ldots \end{aligned} \tag{41}$$

Under these assumptions we shall derive a local solution to the H_∞-control problem. Relaxing these assumptions is indeed possible, but it would substantially complicate the formulas to be worked out. The local solution will be derived from that of a nonlinear H_∞-control problem where continuous measurements are available. For this purpose let us represent the sampled-data measurement (38) in the form

$$y(t) = h_2(x(\tau),\tau) + k_{21}(x(\tau),\tau)w(\tau), \tag{42}$$

similar to what is known to describe continuous measurements. The latter assumption in (41) should then be replaced with

$$k_{21}(x(\tau),\tau)k_{21}^T(x(\tau),\tau) = v(t) \tag{43}$$

with some everywhere positive definite matrix function $v(t)$ to subsequently be specified.

Certainly, (43) could be normalized with $v(t) = I$ by adsorbing $v(t)$ into $g_1(x,t)$ and $h_2(x,t)$. However, we prefer not to do so to encompass the sampled-data measurement case by specifying $v(t)$ as (cf. (20))

$$v(t) = \left[\sum_{j=0}^{\infty} \delta(t-\tau_j)I\right]^{-1}. \tag{44}$$

The corresponding nonlinear H_∞-control problem over the continuous measurements (42), (43) with some piece-wise continuous function $v(t)$ is now fully understood (see, *e.g.*, [6, 30]) and its local solution involves the standard differential Riccati equations

$$-\dot{P}_\varepsilon = P_\varepsilon(t)A(t) + A^T(t)P_\varepsilon(t) + C_1^T(t)C_1(t) + P_\varepsilon(t)\left[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T\right](t)P_\varepsilon(t) + \varepsilon I, \tag{45}$$

$$\begin{aligned} \dot{Z}_\varepsilon = {} & \tilde{A}_\varepsilon(t)Z_\varepsilon(t) + Z_\varepsilon(t)\tilde{A}_\varepsilon^T(t) + B_1(t)B_1^T(t) + Z_\varepsilon(t)\left[\frac{1}{\gamma^2}P_\varepsilon B_2B_2^TP_\varepsilon\right] \\ & \times (t)Z_\varepsilon(t) - Z_\varepsilon(t)C_2^T(t)C_2(t)Z_\varepsilon(t)v^{-1}(t) + \varepsilon I, \end{aligned} \tag{46}$$

where $\varepsilon \geq 0$, $A(t) = \partial I(0,t)/\partial x$, $B_i(t) = g_i(0,t)$, $C_i(t) = \partial h_i(0,t)/\partial x$, $i = 1,2$, $\tilde{A}_\varepsilon(t) = A(t) + (1/\gamma^2)B_1(t)B_1^T(t)P_\varepsilon(t)$. The following conditions are introduced to guarantee a solution of the nonlinear continuous measurement feedback H_∞-control problem to exist:

(A1) under $\varepsilon = 0$ there exists a bounded positive semidefinite symmetric solution $P(t)$ of Eq. (45) such that the system

$$\dot{x} = [A - (B_2B_2^T - \gamma^{-2}B_1B_1^T)P](t)x(t) \tag{47}$$

is exponentially stable;

(A2) under $\varepsilon = 0$ there exists a bounded, positive semidefinite, symmetric solution $Z(t)$ of (46) such that the system

$$\dot{x} = [\tilde{A} + \gamma^{-2} ZPB_2B_2^T P](t)x(t) - Z(t)C_2^T(t)C_2(t)x(t)v^{-1}(t) \tag{48}$$

is exponentially stable.

If Conditions (A1) and (A2) are satisfied then [23] there exists $\varepsilon_0 > 0$ such that system (45), (46) has a unique bounded positive definite symmetric solution $(P_\varepsilon(t), Z_\varepsilon(t))$ for each $\varepsilon \in (0, \varepsilon_0)$ and the output feedback

$$u = -g_2^T(\xi, t)P_\varepsilon(t)\xi \tag{49}$$

with the external state $\xi(t)$, governed by

$$\begin{aligned}\dot{\xi} &= f(\xi, t) + \left[\frac{1}{\gamma^2} g_1(\xi, t)g_1^T(\xi, t) - g_2(\xi, t)g_2^T(\xi, t)\right]P_\varepsilon(t)\xi \\ &\quad + Z_\varepsilon(t)C_2^T(t)[y(t) - h_2(\xi, t)]v^{-1}(t),\end{aligned} \tag{50}$$

is a local solution of the continuous measurement feedback H_∞-control problem. Under an appropriate assumption [23], Conditions (A1) and (A2) become not only sufficient but also necessary for a local exponentially stabilizing solution of the problem to exist.

Surprisingly, the same Eqs. (45)–(50) subject to (44) describe a local solution of the sampled-data measurement feedback H_∞-control problem. Just in case some equations, namely, (46), (48), (50) contain the distribution $v^{-1}(t) = \sum_{j=0}^{\infty} \delta(t - \tau_j)I$ and these equations are therefore to be considered in the generalized sense. Since the singular part of the aforementioned distribution is scalar, vibroimpact solutions of (46), (48), (50) are uniquely defined by Theorem 2 for arbitrary initial conditions. Furthermore, by applying Theorem 3 these solutions satisfy the differential equations

$$\begin{aligned}\dot{Z}_\varepsilon &= \tilde{A}_\varepsilon(t)Z_\varepsilon(t) + Z_\varepsilon(t)\tilde{A}_\varepsilon^T(t) + B_1(t)B_1^T(t) \\ &\quad + \gamma^{-2}Z_\varepsilon(t)P_\varepsilon(t)B_2(t)B_2^T(t)P_\varepsilon(t)Z_\varepsilon(t) + \varepsilon I\end{aligned} \tag{51}$$

$$\dot{x} = [\tilde{A} + \gamma^{-2}ZPB_2B_2^T P](t)x(t) \tag{52}$$

$$\dot{\xi} = f(\xi, t) + \left[\frac{1}{\gamma^2} g_1(\xi, t)g_1^T(\xi, t) - g_2(\xi, t)g_2^T(\xi, t)\right]P_\varepsilon(t)\xi \tag{53}$$

for all t but the sampling time moments, whereas their values at $t = \tau_j, j = 0, 1, \ldots$ are found through the relations

$$Z_\varepsilon(\tau_j+) = \mu_j(1), \quad \xi(\tau_j+) = \zeta_j(1), \quad x(\tau_j+) = \lambda_j(1) \tag{54}$$

by solving the auxiliary differential equations

$$\dot{\mu}_j(t) = -\mu_j(t)C_2^T(\tau_j)C_2(\tau_j)\mu_j(t), \quad \mu_j(0) = Z_\varepsilon(\tau_j-), \tag{55}$$

$$\dot{\zeta}_j(t) = \mu_j(t)C_2^T(\tau_j)[y(\tau_j) - h_2(\zeta_j(t), \tau_j)], \quad \zeta_j(0) = \xi(\tau_j-) \tag{56}$$

$$\dot{\lambda}_j(t) = -\mu_j(t)C_2^T(\tau_j)C_2(\tau_j)\lambda_j(t), \quad \lambda_j(0) = x(\tau_j-). \tag{57}$$

Since Eqs. (55) and (57) integrate to

$$\begin{aligned}\mu_j(t) &= Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)t]^{-1}, \\ \lambda_j(t) &= \{I - Z(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)t]^{-1}C_2^T(\tau_j)C_2(\tau_j)t\}x(\tau_j-),\end{aligned}$$

relations (54)–(57) result in

$$Z_\varepsilon(\tau_j+) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)]^{-1} \tag{58}$$

$$x(\tau_j+) = x(\tau_j-) - Z(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z(\tau_j-)]^{-1} \times C_2^T(\tau_j)C_2(\tau_j)x(\tau_j-) \tag{59}$$

$$\xi(\tau_j+) = \zeta_j(1), \quad j = 0, 1, \ldots \tag{60}$$

where $\zeta_j(t)$ satisfies

$$\dot{\zeta}_j(t) = Z_\varepsilon(\tau_j-)[I + C_2^T(\tau_j)C_2(\tau_j)Z_\varepsilon(\tau_j-)t]^{-1} \times C_2^T(\tau_j)[y_j - h_2(\zeta_j(t), \tau_j)], \quad \zeta_j(0) = \xi(\tau_j-). \tag{61}$$

So, the vibroimpact solutions of (46), (50) with the impulsive input (44) coincide with the corresponding solutions of the differential equations (51), (53) with jumps (58), (60), (61) and the same initial conditions. In turn, Condition (A2) is reformulated as follows:

(A2′) under $\varepsilon = 0$ there exists a bounded, positive semidefinite, symmetric solution $Z(t)$ of (51), (58) such that system (52) with jumps (59) is exponentially stable.

Thus, the nonlinear distribution formalizm has allowed us to derive a local solution of the sampled-data measurement H_∞-control problem from that of the continuous measurement H_∞-control problem. Summarizing, the following result is shown.

THEOREM 5 *Let conditions* (A1) *and* (A2′) *be satisfied. Then there exists* $\varepsilon_0 > 0$ *such that system of the differential equations* (45), (51) *with jumps* (58) *has a unique, continuous from the left, bounded, positive definite, symmetric solution* $(P_\varepsilon(t), Z_\varepsilon(t))$ *for each* $\varepsilon \in (0, \varepsilon_0)$ *and the output feedback* (49) *with the external state* $\xi(t)$, *governed by the differential equation* (53) *with jumps* (60), (61) *is a local solution of the sampled-data measurement feedback* H_∞*-control problem. Conversely, conditions* (A1) *and* (A2′) *are satisfied if the problem has a local exponentially stabilizing solution.*

The detailed proof of Theorem 5 can be found in [22]. Remark 3 is valid for the nonlinear H_∞-controller as well.

4 SINGULAR QUADRATIC OPTIMIZATION OF AFFINE CONTROL SYSTEMS

In the present section, a generalized feedback approach is developed to address optimal stabilization of nonlinear systems with no penalty on control inputs. The concept of a generalized feedback is introduced and then applied to a class of nonlinear systems to attain perfect stabilization yielding zero value of the degenerated quadratic performance.

4.1 Problem Statement

The problem considered is to optimally stabilize the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x(0) = x^0, \\ y &= h(x) \end{aligned} \tag{62}$$

with respect to the degenerate cost function

$$J(x^0, u(\cdot)) = \int_0^\infty y^T(t)Qy(t)\,\mathrm{d}t \to \inf \tag{63}$$

where $x \in R^n$, y, $u \in R^m$. Throughout this section, $f(x)$ and columns $g_1(x), \ldots, g_m(x)$ of $g(x)$ are assumed to be vector fields of class C^∞ defined on a simply connected smooth manifold M; $h(x) = (h_1(x), \ldots, h_m(x))^T$ is a C^∞ mapping defined on M; Q is a positive-definite matrix. In most cases, all vector and matrix dimensions are omitted, and all quantities are implied to have compatible dimensions.

As mentioned in introduction, the minimum possible value of the degenerate criterion (63) can not be attained via an integrable control function $u(t)$. Therefore, the concept of solution to the above optimization problem should be modified.

In order to properly state a generalized optimization problem we shall introduce the generalized feedback concept for nonlinear systems (62), (63) which can be viewed as a further development of Definition 1.

DEFINITION 4 *A sequence of smooth feedback (open-loop) control signals $u^k(x, t)$ ($u^k(t)$), $k = 1, 2, \ldots$ is a generalized feedback (open-loop) control input $u(x, t)$ ($u(t)$), if for all initial conditions x^0 there exists a finite limit*

$$\lim_{k\to\infty} J(x^0, u^k(\cdot)) = J(x^0, u(\cdot)) < \infty$$

of the corresponding criterion values.

DEFINITION 5 *A generalized feedback controller $u = \{u^k\}_{k=1}^\infty$ of the nonlinear system* (62), (63) *is said to be admissible iff system* (62) *is semiglobally stabilizable to the origin by means of every subsequence of $\{u^k\}_{k=1}^\infty$.*

Recall that a control system Σ is semiglobally stabilizable to a point p by means of a class $\mathcal{F}$ of smooth feedback control laws [29], if for every bounded subset Ω of the state space of Σ there exists a control law, which belongs to $\mathcal{F}$ and makes p an asymptotically stable equilibrium of the corresponding closed-loop system, with a basin of attraction containing Ω (in terms of [7] Σ is asymptotically stabilizable on compacta).

DEFINITION 6 *If $u = \{u^k\}_{k=1}^\infty$ is a generalized (feedback or open-loop) controller of* (62), (63) *such that there exists a weak* limit $x(t; x^0, u)$ of the solutions $x(t; x^0, u^k)$ of* (62) *under $u = u^k$, then the distribution $x(t; x^0, u)$ is referred to as a generalized solution of* (62) *corresponding to the generalized control u.*

Remark 4 Generally speaking, the use of a generalized (feedback or open-loop) control u does not imply the existence of the corresponding generalized solution of the nonlinear system (62).

Remark 5 For nonlinear systems (62) with open-loop control signals $u(t)$ of measure type the generalized solution coincides with the vibroimpact solution of the same initial-value problem whenever the vibroimpact solution exists.

We are now in a position to state the generalized quadratic optimization problem:

it is required to construct an admissible controller $u(\cdot)$ of the nonlinear system (62), (63) *with a minimum possible value of the cost function* (63).

4.2 Generalized Perfect Regulation

By adding some additional conditions to the ones introduced earlier, we shall demonstrate how to optimally stabilize the system by means of a generalized feedback. In the rest of the presentation we assume that the following conditions hold:

1. system (62) is globally minimum phase and it has a uniform vector relative degree $(r_1, \ldots, r_m)$ with $r = r_1 + \cdots + r_m < n$;
2. the vector fields

$$X_i^j = ad_{\tilde{f}}^{j-1}\tilde{g}_i, \quad 1 \leq j \leq r_i, \quad 1 \leq i \leq m$$

with

$$\tilde{f} = f - \tilde{g}\beta, \quad \tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m) = g\alpha^{-1},$$
$$\alpha(x) = \{\alpha_{ik}(x)\} = \{L_{g_k}L_f^{r_i-1}h_i(x)\},$$
$$\beta(x) = \{L_f^{r_1}h_1(x), \ldots, L_f^{r_m}h_m(x)\}^T, \quad 1 \leq k \leq m$$

are complete

3. $[X_i^j, X_k^s] = 0$ for all $1 \leq k, i \leq m$, $1 \leq j \leq r_i$, $1 \leq s \leq r_k$.

It is well-known [7] that, under Assumptions 2 and 3, system (62) is globally diffeomorphic to a system in the normal form

$$\begin{aligned} \dot{\xi}_1^i = \xi_2^i, \ldots, \dot{\xi}_{r_i-1}^i = \xi_{r_i}^i, \quad \dot{\xi}_{r_i}^i = v_i, \quad 1 \leq i \leq m, \\ \dot{z} = f_0(z, y), \quad y = (\xi_1^1, \ldots, \xi_1^m)^T \end{aligned} \tag{64}$$

where

$$v = (v_1, \ldots, v_m)^T = F(z, \xi_1, \ldots, \xi_m) + G(z, \xi_1, \ldots, \xi_m)u, \tag{65}$$

and matrices $F(z, \xi_1, \ldots, \xi_m)$ and $G(z, \xi_1, \ldots, \xi_m)$ coincide with the functions

$$B(x) = \{L_f^{r_k}h_k(x)\}, \quad A(x) = \{L_{g_k}L_f^{r_i-1}h_i(x)\}, \tag{66}$$

written through the new coordinates

$$z = (z_1, \ldots, z_{n-r})^T, \quad \xi_i = (\xi_1^i, \ldots, \xi_{r_i}^i)^T, \tag{67}$$

which are defined by the following relations

$$\begin{aligned} \xi_j^i(x) = L_f^{j-1}h_i(x), \quad 1 \leq j \leq r_i, \ 1 \leq i \leq m, \\ L_{g_k}z_l(x) = 0, \quad 1 \leq l \leq n-r, \ 1 \leq k \leq m. \end{aligned} \tag{68}$$

The matrix $G(z, \xi_1, \ldots, \xi_m)$ is invertible for all argument values.

It was shown in [29] that, under the minimum phase assumption 1 and with no growth condition on f_0, system (64) can be globally asymptotically stabilized by restricting the dependency of $f_0(z, y)$ on the state $\xi_1, \ldots, \xi_m$ of the linear part of the system (which will be referred to as the "dependence condition"). Moreover, examples given in [26] demonstrate that the dependence condition (guaranteed in our case by Assumption 3) cannot be further relaxed unless a growth condition is imposed on f.

The main result of the section is formulated as follows.

THEOREM 6 *A solution to the singular optimization problem* (62), (63) *is given by the generalized feedback*

$$u^k(x) = A^{-1}(x)v^k(x) - B(x), \quad k = 1, \dots, \tag{69}$$

where

$$\begin{aligned} v_i^k(x) &= -k^{v_{r_i}}\{\xi^i_{r_i}(x) + k^{v_{r_i-1}}\{\xi^i_{r_{i-1}}(x) + \cdots \\ &\quad + k^{v_2}\{\xi^i_2(x) + k^{v_1}\xi^i_1(x)\} + \cdots\}, \quad 1 \le i \le m \\ v_1 &= 1, \quad v_j = 2v_{j-1} + 1, \quad 2 \le j \le r_i. \end{aligned} \tag{70}$$

The generalized feedback control law (69) *provides perfect stabilization of system* (62) *with zero value of the performance criterion* (69).

Proof of Theorem 6 Introducing new variables

$$q^i_1 = \xi^i_i, \; q^i_j = \xi^i_j + k^{v_{j-1}}q^i_{j-1}, \quad 2 \le j \le r_i$$

we rewrite the linear part of Eqs. (64) in terms of the new variables:

$$\begin{aligned} \dot q^i_1 &= q^i_2 - k^{v_1}q^1_1, \\ &\;\;\vdots \\ \dot q^i_{r_{i-1}} &= q^i_{r_i} - k^{v_{r_i-1}}q^i_{r_i^{-1}} + k^{v_{r_i-2}}\{q^i_{r_i-1} - k^{v_{r_i-2}}q^i_{r_i^{-2}} + \cdots \\ &\quad + k^{v_1}\{q^i_2 - k^{v_1}q^i_i\} + \cdots\} \end{aligned} \tag{71}$$

$$\begin{aligned} \dot q^i_{r_i} &= -k^{v_{r_i}}q^i_{r_i} + k^{v_{r_i-1}}\{q^i_{r_i} - k^{v_{r_i-1}}q^i_{r_i-1} + \cdots \\ &\quad + k^{v_1}\{q^i_2 - k^{v_1}q^i_1\} + \cdots\}. \end{aligned} \tag{72}$$

First of all, using mathematical induction with respect to the system's relative degree r we ascertain that the linear closed-loop system (71), (72) is exponentially stable for k sufficiently large.

Indeed, the proof of exponential stability of (71), (72) for $r = 1$ is straightforward. Assuming that this is a property of all systems of the form (71), (72) with relative degree $r < r^*$ we note that hierarchy (70) of large coefficients implies the exponential stability of the closed loop system (71), (72). The slow subsystem (71) is exponentially stable by the induction assumption, and representing Eq. (72) in the singularly perturbed form

$$\varepsilon^{v_{r_i}}\dot q^i_{r_i} = -(1+\varepsilon)q^i_{r_i} + \varepsilon(\varepsilon^{v_{r_i-1}} - 1)q^i_{r_i-1} + \cdots + \varepsilon(\varepsilon^{v_2} - 1)q^i_2 + \varepsilon(\varepsilon^{v_1} - 1)q^i_1 \tag{73}$$

where $\varepsilon = k^{-v_r}$ is small, we conclude that according to [9] exponential stability of the singular perturbed system (71), (73) follows from exponential stability of the slow and fast subsystems (71) and (73), respectively.

Furthermore, due to the presence of small parameter $\varepsilon = k^{-1}$, the closed-loop system (71), (73) contains a fast transient $\gamma_{r_i}\exp\{-k^{v_{r_i}}t\}$, $\gamma_{r_i} = const$ in the variables $q^i_{r_i}$, $1 \le i \le m$, and similarly we establish that absolute values $\lambda^i_j(k)$, $1 \le j \le r_i$ of real parts of eigenvalues of (71), (73) tend to infinity as $k \to \infty$. By [29, Theorem 3.1], this ensures that the class of feedback control laws (69) asymptotically stabilizes system (62) on compacta, and the generalized feedback (69) provides the system stabilization.

Finally, the performance criterion (63) corresponding to (69) turns out to be equivalent to zero since the quadratic integration of the fast transients $\gamma_j^i \exp\{-\lambda_j^i(k)t\}$, $\gamma_j^i = const$, $1 \le i \le m$, $1 \le j \le r_i$ of (71), (72) yields infinitesimal values as $k \to \infty$. Thus, Theorem 6 is proven.

Remark 6 Generalized feedback (70) is a solution to the optimal stabilization problem (63), (64). For all initial conditions

$$\xi_j^i(0) = \zeta_j^i, \quad 1 \le i \le m, \quad 1 \le j \le r_i \tag{74}$$

there exists a generalized solution of (64) corresponding to (70). This generalized solution coincides with the mild solution of (64), (74) under impulsive input

$$v_i = \sum_{j=1}^{r_i} \zeta_j^i \delta^{(r_i - j)}(t), \quad 1 \le i \le m \tag{75}$$

and is governed by the zero dynamics equation

$$\dot{z} = f_0(z, 0), \quad \xi_i = 0.$$

In this sense the impulsive feedback (75) solves the optimal stabilization problem (63) for nonlinear systems in the normal form (64).

Remark 7 The generalized feedback law (69) imposes on the closed-loop system (62) favorable robustness properties against matched external disturbances.

4.3 Example

To this end, we illustrate with the help of an elementary example some of the features of the optimal generalized feedback design. Consider singular quadratic problem

$$J = \int_0^\infty \xi_1^2(t)\, \mathrm{d}t \to \inf$$

for globally minimum phase system in the normal form

$$\dot{z} = \psi(z, \xi_1), \quad \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = v \tag{76}$$

where all variables are scalar and ψ is a smooth function. By Theorem 6 the generalized feedback

$$v_k = -k^3(\xi_2 + k\xi_1) \tag{77}$$

stabilizes system (76) on compacta and provides perfect regulation $J = 0$. In accordance with Remark 6 the generalized feedback (77) can be viewed as that of the impulsive type

$$v = -\xi_2(0)\delta(t) - \xi_1(0)\dot{\delta}(t),$$

and the corresponding generalized solution of (76) becomes identical to the zero dynamics

$$\dot{z} = \psi(z, 0), \quad \xi_1 = 0, \quad \xi_2 = 0, \quad t > 0.$$

5 VIBROIMPACT MODELING OF MECHANICAL SYSTEMS

Now we demonstrate how impact dynamics (collisions, percussions, etc.) of mechanical systems can be treated within the framework of the nonlinear distribution formalizm developed. The vertical launch of a spacecraft, governed by

$$\begin{aligned}\dot{h} &= V, \quad h(0) = 0\\ \dot{V} &= \frac{p - mg - q(h, V)}{m}, \quad V(0) = 0\\ \dot{m} &= -\frac{p}{c}, \quad m(0) = m_0,\end{aligned} \tag{78}$$

serves here as an illustrative example. In the above equations, $h(t)$, $V(t)$, $m(t)$ are, respectively, the altitude, the velocity, and the mass of the rocket at the time moment t, p is the thrust, q is the air resistance, g is the gravity acceleration, c is the specific impulse, m_0 is the initial mass.

System (78) implies a continuous change of the mass $m(t)$, whereas a discrete change $m(0+) - m_0 = \Delta m_0 < 0$, corresponding to an instantaneous fuel combustion in the spacecraft jet, is also possible and it is caused by the impulsive thrust

$$p(t) = -c\Delta m_0 \delta(t). \tag{79}$$

From the physical point of view, the impulsive thrust (79) indicates that the cumulative power of the jet is limited while attaining a very high value during a very short period of time. With this in mind, the meaning of (78) subject to (79) has to be treated in the generalized sense because of the irregularity of the product $c\Delta m_0 \delta(t) m^{-1}(t)$ of the impulsive thrust (79) and the discontinuous function $m^{-1}(t)$ that appears in the right-hand side of the second equation of (78). Since system (78) is driven by the scalar impulsive action (79), the corresponding Frobenius condition is therefore satisfied and by Theorem 2 the system has a unique vibroimpact solution. Apparently, the altitude h has no jump whereas the change Δm_0 of the mass $m(t)$ at the initial time moment is straightforwardly computed by integrating the latter equation of (78) which has no irregular product. In turn, the instantaneous change $\Delta V_0 = V(0+)$ of the velocity is found by applying Theorem 3 through the relation

$$V(0+) = \xi_2(1), \tag{80}$$

where $\xi(v) = (\xi_1(v), \xi_2(v), \xi_3(v))^T$ solves the Cauchy problem

$$\begin{aligned}\frac{d\xi_1}{dv} &= 0, \quad \xi_1(0) = 0\\ \frac{d\xi_2}{dv} &= -\frac{c\Delta m_0}{\xi_3}, \quad \xi_2(0) = 0\\ \frac{d\xi_3}{dv} &= \Delta m_0, \quad \xi_1(0) = m_0.\end{aligned} \tag{81}$$

Since (81) integrates to $\xi_1(v) = 0$, $\xi_2(v) = -c\ln((m_0 + \Delta m_0 v)/m_0)$, $\xi_3(v) = m_0 + \Delta m_0 v$, relation (80) results in the formula

$$V(0+) = -c\ln\left(1 + \frac{\Delta m_0}{m_0}\right), \tag{82}$$

for the instantaneous change of the velocity.

Thus, the spacecraft dynamics enforced by the impulsive jet is modeled by vibroimpact solutions of the nonlinear differential equation (78) with the impulsive input (79). The nonlinear distribution formalizm allows one to qualitatively analyze this dynamics. It is anticipated that the formalizm, while being applied to description of mechanical systems, will provide a deeper insight on many phenomena with an impulsive nature.

6 CONCLUSIONS

Schwartz' distributions theory is developed in nonlinear setting. In order to describe complex dynamic systems with impulsive inputs the meaning of differential equations in distributions is extended. Generalized solutions for these equations are introduced via closure, in a certain topology, of the set of the conventional solutions corresponding to standard integrable inputs. Mathematical models proposed involve nonlinear and, generally speaking, non single-valued operating over distributions. The instantaneous impulse response of a nonlinear system is shown to depend on the impulse realization. The complete integrability of a certain auxiliary system is proven to guarantee the uniqueness of the impulse response. The theory is demonstrated to be eminently suited to sampled-data measurement filtering of stochastic and deterministic dynamic systems, optimal impulsive feedback synthesis, analysis of mechanical systems with impulsive phenomena. Many other problems such as analysis and synthesis of mechanical systems with unilateral constraints are anticipated to effectively be addressed within the framework of the nonlinear distribution formalizm developed.

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